A Method of Solving Quadratic Equations in A Historical Perspective

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The article opens with the earnest tribute and homage to Sreedharacharyya (C 850-950 AD) who succeeded to advance a beautiful and authentic method of obtaining the roots of a quadratic equation. The rule was enunciated with due reference and regard to Sreedharacharyya by the renowned and celebrated Hindu mathematician and astronomer, Bhaskaracharyya II (C 1114-1200 AD) in the fifth chapter of his famous text, 'Vijganintam'.

चतुराहतवर्गसमैः रूपैः पक्षद्वयं गुणयेत्।
अव्यक्तवर्गरूपैर्युक्तौ पक्षौ ततो मूलम्॥

The sloka is written in Sanskrit in Upagita Chhanda. It was the practice in the past in India that the subject matters were expressed in slokas and verses. Incidently, every mathematician in the past was a poet in the true sense of the term. It needs notice that many subtle and intricate points were kept hidden in the slokas, perhaps purposely, leaving them for the preceptor or *guru* to fill up and explain to inquisitive disciples while imparting the knowledge. This was the Indian tradition in the past through which the knowledge of one generation passed over to the next.

The algebraic demonstration of the rule due to Sreedharacharyya is given below with necessary comments in support of the content of the sloka.

We start with the quadratic

$ax^2 + bx + c = 0$ or, $ax^2 + bx = -c$

Multiplying both sides by 4a, i.e., four times the coefficient of the square of unknown x, we have

$4a^2x^2 + 4abx = -4ac$

Adding b^2, i.e., square of the coefficient of unknown x, to both the sides

$4a^2x^2 + 4abx + b^2 = b^2 - 4ac$

or, $(2ax + b)^2 = b^2 - 4ac$

Performing the square root,

$2ax + b = \pm \sqrt{b^2 - 4ac}$

or, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

which gives the roots of the equation by the method of Sreedharacharyya.

There are, however, various methods of solving a quadratic equation, A method different from the methods in vogue, is presented now.

Let $ax^2 + bx + c = 0$, where $a \neq 0$, be the quadratic equation whose solution is sought. We write the equation as

$ax^2 + \{(b/2 + t) + (b/2 - t)\}x + c = 0$

or, $ax^2 + (b/2 + t)x + (b/2 - t)x + c = 0$

or, $ax^2 + (b/2 + t)x + (b/2 - t)x + (b^2/4 - t)/a + c - (b^2/4 - t)/a = 0$

or, $x(ax + b/2 + t) + (b/2 - t)(ax + b/2 + t)/a = (b^2/4 - t)/a - c$

or, $(ax + b/2 + t)\{x + (b/2 - t)/a\} = (b^2/4 - t)/a - c$

or, $(ax + b/2 + t)(ax + b/2 - t) = b^2/4 - t - ac$

Uptil t has not been specified, now we choose t so that $b^2/4 - t^2 - ac = 0$

or $t = \pm \frac{1}{2} \sqrt{b^2 - 4ac}$,

For this choice of t, we have

$(ax + b/2 + t)(ax + b/2 - t) = 0$

or, $ax = -b/2 \pm t$ or, $x = \dfrac{-b \pm \sqrt{b^2 - 4ac}}{2a}$

which are the required roots of the quadratic equation.

In conclusion, the present author desires to put the above method through the following sloka in Anustup Chhanda:

वर्गोनार्धेण युक्त्वा वै वियोज्य च विभाजयेत्।
पदानि चातिरिक्तानि लभन्ते विरतिं यथा॥

TEN BEST QUESTIONS CONTEST

PERMUTATION & COMBINATION

— Animesh, Kankarbagh, Patna

1. A square of n units by n units is divided into n^2 squares each of area 1 sq. unit. Find the number of ways in which 4 points [out of $(n + 1)^2$ vertices of the squares] can be chosen so that they form the vertices of a square.

2. Find the number of all whole numbers formed on the screen of a calculator which can be recognised as numbers with (unique) correct digits when they are read inverted. The greatest number formed on its screen is **999999**

3. India and South Africa play one-day International cricket series until one team wins 4 matches. No match ends in a draw. Find in how many ways the series can be won

4. How many different 7 digit numbers are there the sum of whose digits is even?

5. How many different numbers which are smaller than $2 \cdot 10^8$ and are divisible by 3, can be written by means of the digits 0, 1 and 2?

6. In the n + 1 numbers a, b, c, d, ... be all different and each of them be a prime number. Prove that the number of different factors (other than 1) of the expression $a^m \cdot b \cdot c \cdot d$ ..... is $(m + 1)2^n - 1$.

7. How many sets of 2 and 3 (different) numbers can be formed by using numbers between 0 and 180 (both including) so that 60 is their average?

8. 6 letters are written to 6 persons and addresses to corresponding letters are written on 6 envelopes. In how many ways can the letter be placed in the envelopes so that all the letters are in the wrong envelopes.

9. 6 balls marked as 1, 2, 3, 4, 5 and 6 are kept in a box. Two players A and B start to take out 1 ball at a time from the box one after another without replacing the ball till the game is over. The number marked on the ball is added each time to the previous sum to get the sum of numbers marked on the balls taken out. If this sum is even then 1 point is given to the player taking the ball; if it is odd then 1 point is cut from the same player. The first player to get 2 points is declared winner. At the start of the game the sum is 0. If A starts to take out the ball, then find the number of ways in which the game can be won

10. 4 players A, B, C, D are playing with a ball giving catches to each other. If A starts to give catch to any of players B, C and D; then in how many different sequences the ball can return to A when it is known that a player can't get the ball more than 2 times?

SOLUTIONS

1. No. of squares of area n^2 square units = 1

No. of squares of area $(n - 1)^2$ square units = 2^2

No. of squares of area $(n - 2)^2$ square units = 3^2

....

No. of squares of area 1 square unit = n^2

Adding gives $N_1 = 1^2 + 2^2 + 3^2 + ... + n^2$

$$= \frac{n(n + 1)(2n + 1)}{6}$$

When n is even $\frac{n^2}{2}$

No. of squares of area square units = 1

No. of squares of area $\frac{(n - 2)^2}{2}$ square units = 3^2

No. of squares of area 2 square units = $(n - 1)^2$

Adding gives $N_2 = 1^2 + 3^2 + 5^2 + ... + (n - 1)^2$

$= n(n - 1)(n + 1)/6$

When n is odd

No. squares of area $(n - 1)^2/2$ square units $= 2^2$

No. of squares of area $(n - 3)^2/2$ square units $= 4^2$

No. of square of area $(n - 5)^2/2$ square units $= 6^2$

. . . .

No. of squares of area 2 square units $= (n - 1)^2$

Adding gives $N_2 = 2^2 + 4^2 + 6^2 + ...+ (n - 1)^2$

$= n(n - 1)(n + 1)/6$

$\therefore$ Total no. of squares formed which can be obtained by taking 4 points out of $(n + 1)^2$ points

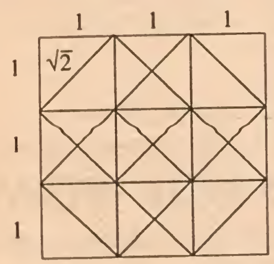
$$= N_1 + N_2$$

$$= \frac{n(n + 1)(2n + 1)}{6} + \frac{n(n - 1)(n + 1)}{6}$$

$$= \frac{n(n + 1)}{6}[2n + 1 + n - 1] = \frac{n(n + 1]3n}{6}$$

$$= \frac{n^2(n + 1)}{2}$$

e.g. Take a square of 3 units by 3 units divided into 9 smaller squares



No. of squares having

(i) 9 sq. units = 1

(ii) 4 sq. units = 4

(iii) 1 sq/ unit = 9

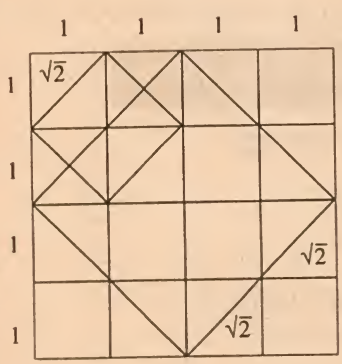
(iv) 2 sq. units = 4

Total = 18

From formula

$$\frac{n^2(n + 1)}{2} = \frac{9 \times 4}{2} = 18$$

Take the square of 4 units by 4 units divided into 16 smaller squares



4 points are to be chosen from 25 points to form a square

No. of squares having

(i) 16 sq. units = 1

(ii) 9 sq. units = 4

(iii) 4 sq. units = 9

(iv) 1 sq. unit = 16

(v) 2 sq. units = 9

(vi) 8 sq. units = 1

Total = 40

From formula $\dfrac{n^2(n + 1)}{2} = \dfrac{16 \times 5}{2} = 40$

2. The number can use digits **0, 1, 2, 5, 6, 8** and **9** because they can be recognised as digits when they are seen inverted.

A number can't begin with 0, therefore all numbers having 0 at unit's digit should not be counted. (When those numbers will be rend inverted they will begin with 0)

No. of digits	Total numbers
1	7
2	$6^2 = 36$
3	$6 \times 7 \times 6 = 252$
4	$6 \times 7^2 \times 6 = 1764$
5	$6 \times 7^3 \times 6 = 12348$
6	$6 \times 7^4 \times 6 = 86436$
	$\therefore$ Total = 100843

3. Taking I for India and S for South Africa. We can arrange I and S to show the wins for India and S. Africa respectively

e.g.: ISSSS means first match is won by India which is followed by 4 wins by S. Africa. This is one way in which series can be won.

Suppose S.Africa wins the series, then last match is always won by S. Africa

	Wins of I	Wins of S	No. of ways
(1)	0	4	1
(2)	1	4	4!/3! = 4
(3)	2	4	5!/2!3! = 10
(4)	3	4	6!/3!3! = 20

$\therefore$ Total no. of ways = 35

In the same no. of ways India can win the series

$\therefore$ Total no. ways in which the series can be won $= 35 \times 2 = 70$

4. Let us consider 10 successive seven digit numbers

$a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, 0$,

$a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, 1$,

....

$a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, 9$

where $a_1, a_2, a_3, a_4, a_5, a_6$ are some digits. We see that half of these 10 numbers. i.e. 5 numbers have an even sum of digits

The first digit a_1 can assume 9 different values and

each of the digits a_2, a_3, a_4, a_5, a_6 can assume 10 different values

The last digit a_7 can assume only 5 different values for which the sum of all digits is even.

$\therefore$ There are $9 \times 10^5 \times 5$

$= 45 \times 10^5$ seven digit numbers the sum of whose digits is even.

5. 12, 21,......, 122222222 are the required numbers We can assume all of them to be nine-digit in the form $a_1\, a_2\, a_3\, a_4\, a_5\, a_6\, a_7\, a_8\, a_9$ and can use 0 for a_1; a_1 and a_0, a_1 a_2 & a_3.... and so on to get 8 digit, 7 digit, 6 digit numbers etc.

a_1 can assume one of the 2 values of 0 or 1

$a_2, a_3, a_4, a_5, a_6, a_7, a_8$ can assume any of 3 values 0, 1, 2.

The number for which $a_1 = a_2 = a_3 = a_4 = a_5 = a_6 = a_7 = a_8 = a_9 = 0$ must be eliminated.

The sum of first 8 digits i.e. $a_1 + a_2 + a_3 + ... + a_8$ can be in the form of $3n - 2$ or $3n - 1$ or $3n$. In each case a_9 can be chosen from 0, 1, 2 in only 1 way so that the sum of all 9 digits is equal to $3n$.

$\therefore$ Total numbers $= 2 \times 3^7 \times 1 - 1$

$= 4374 - 1 = 4373$

6. There are n prime factors (b, c, d....) other than a. First we find the total no. of factors not involving any powers of a

No. of prime factors in the factor of expression	No. of ways
1	${}^nC_1$
2	${}^nC_2$
.	.
.	.
n	${}^nC_n$

$\therefore$ Total $= {}^nC_1 + {}^nC_2 + ... + {}^nC_n = 2^n - 1$

Now we consider the factors having the powers of a.

Factor of expression

$= a^x \times$ product of y prime numbers

where $1 \le x \le m$ and $0 \le y \le n$

$\therefore$ There are m values of a^x and ${}^nC_y$ values of product of y prime numbers.

$\therefore$ Total no. of factors having the powers of a

$= m[{}^nC_0 + {}^nC_1 + {}^nC_2 + ... {}^nC_n] = m \cdot 2^n$

$\therefore$ Total no. of factors of the expression =

$= m \cdot 2^n + 2^n - 1 = (m + 1)2^n - 1$

7.(i) *Set of 2 numbers*

Let a and b be 2 numbers

$\therefore \dfrac{a + b}{2} = 60 \Rightarrow a + b = 120$

a and b both can't be equal to or greater than 60

($\because$ 60 can't be used twice)

Let $0 \le a \le 59$ and $61 \le b \le 120$

The total no. of ways in which

a can be chosen $= {}^{60}C_1 = 60$

The value of b depends on the value of a and there is 1 value of b corresponding to 1 of a

$\therefore$ Total no. of sets having 2 numbers = 60

(ii) Set of 3 numbers

Let a, b, c be the 3 numbers

Then $\dfrac{a + b + c}{3} = 60 \Rightarrow a + b + c = 180$

Case I

Let $0 \le a \le 59$, $0 \le b \le 59$ and $c > 60$

a can be chosen in ${}^{60}C_1 = 60$ ways

b can be chosen in ${}^{59}C_1 = 59$ ways

($\therefore$ b can't use the value of a

$\therefore$ no. of ways in which

a and b can be chosen $= 60 \times 59$

Now $1 \le a + b \le 117$ and there is only one value of c for 1 value of a + b so that a + b + c = 180

$\therefore$ no. of ways in which

a, b, c can be chosen $= 60 \times 59$

case II

a = 60 $\qquad \therefore$ b + c = 120

The no. of ways in which b and c can assume values = 60 (from (i)

$\therefore$ no. of ways in which a, b, c can be chosen = 60

Case III

$61 \le a \le 90$, $61 \le b \le 90$ and $c < 60$

a can assume values in ${}^{30}C_1 = 30$ ways

b can assume values in ${}^{29}C_1 = 29$ ways

the value of c depends on the value of a and b

$\therefore$ no. of ways in which

a, b, c can be chosen $= 30 \times 29$

$\therefore$ Total no. of ways in which sets of 3 numbers can

be chosen = $60 \times 59 + 60 + 30 \times 29 = 30[2 \times 59 + 2 + 29] = 30[118 + 2 + 29] = 30 \times 149 = 4470$

$\therefore$ Total no. of ways in which sets of 2 and 3 numbers can be chosen = $4470 + 60 = 4530$

8. Let's consider the general case of n letters $L_1, L_2, L_3, L_4, \ldots$ and n corresponding envelopes $E_1, E_2, E_3, E_4, \ldots$. Let f(n) denote the number of ways of putting all n letters in wrong envelopes, so that L_1 is not in E_1, L_2 is not in E_2 and so on. L_1 can be put in any one of the remining (n - 1) envelopes E_2, E_3, E_4 etc. Suppose L_1 is put in the envelope E_x; then 2 cases arise,

(1) L_x may be put in E_1. Then we've to see that the remaining (n - 2) letters are put in the wrong envelopes. The number of ways of doing this is f(n - 2).

(2) L_x is not put in E_1. Then we've to see that (n - 1) letters $L_2, L_3, L_4, \ldots, L_x$ are put into wrong envelopes. The number of ways of doing this is f(n - 1).

Hence when L_1 is put in E_x, there are f(n - 2) + f(n - 1) ways of putting the letters into the wrong envelopes. But L_1 can be put $E_2, E_3, E_4 \ldots$ instead of E_x, thus giving (n - 1) ways.

$\therefore f(n) = (n - 1)[f(n - 2) + f(n - 1)]$ where $f(1) = 0$ and $f(2) = 1$

$\therefore f(3) = 2[f(1) + f(2)] = 2[0 + 1] = 2$ $\quad f(4) = 3[f(2) + f(3)] = 3[1 + 2] = 9$

$f(5) = 4[f(3) + f(4)] = 4[2 + 9] = 44$ $\quad f(6) = 5[f(4) + f(5)] = 5[9 + 44] = 5 \times 53 = 265$

9. Denoting A_1, B_1, A_2 and B_2 for their taking out the ball, a chart is made to denote the winner

		A_1	B_1	A_2	B_2	No. of ways
(1)	*points* number on the ball *sum*	1 Even (1 of 3) Even	1 Even 1 of (2) Even	0 Odd (1 of 2) Odd	2 odd (1 of 3) Even	${}^3C_1 \times {}^2C_1 \times {}^3C_1 \times {}^2C_1$ = 36
(2)	*points* number on the ball *sum*	-1 Odd (1 of 3) Odd	1 Odd (1 of 2) Even	0 Even (1 of 3) Even	2 Even (1 of 2) Even	${}^3C_1 \times {}^2C_1 \times {}^3C_1 \times {}^2C_1$ = 36
(3)	*points* number on the ball *sum*	1 Even (1 of 3) Even	-1 Odd (1 of 3) Odd	2 Odd (1 of 2) Even	—	${}^3C_1 \times {}^3C_1 \times {}^2C_1$ = 18
(4)	*points* number on the ball *sum*	1 Even (1 of 3) Even	1 Even (1 of 2) Even	2 Even (1 of 1) Even	—	${}^3C_1 \times {}^2C_1 \times 1$ = 6

$\therefore$ Total Number of ways in which the game can be won when A starts the game = 36 + 36 + 18 + 6 = 96

10. We denote N_1 for the player who gets the ball from A. N_2 is the next player to get the ball and N_3 is the last of B, C and D to get the ball at least once.

First throw is always from A and last; always to A

No. of throws	Sequence	No. of Ways
2	N_1	${}^3P_1 = 3$

No. of throws	Sequence	No. of ways
3	N_1N_2	$^3P_2 = 3!/1! = 6$
4	$N_1\ N_2\ \square$ N_1 or N_3	$^3P_2 \times {}^2C_1 = 12$
5	$N_1\ N_2\ \square\ \square$ N_3 or N_1 N_2 or ○ 3rd throw is to N_3 or N_1, 4th throw is N_2 or one of N_3 and N_1 who didn't get the ball in 3rd throw	$^3P_2 \times {}^2C_1 \times {}^2C_1$ $= 6 \times 4 = 24$
6	(i) $N_1\ N_2\ N_3\ \square\ N_3$ N_1 or N_2 (ii) $N_1\ N_2\ \square\ \square\ \square$ N_1 or N_3 N_2 or ○ ○	$^3P_3 \times 2!$ $= (3!/0!) \times 2! = 12$ $^3P_2 \times {}^2C_1 \times {}^2C_1$ $= (3!/1!) \times 4 = 24$
7	(i) $N_1\ N_2\ N_1\ N_3\ N_2\ N_3$ (ii) $N_1\ N_2\ N_3\ \square\ \square\ \square$ N_1 or N_2 N_3 or ○ ○	$^3P_3 = 3! = 6$ $^3P_3 \times {}^2C_1 \times {}^2C_1$ $= 3! \times 4 = 24$
	Total	$3+6+12+24+12+24+24+6$ $= 111$

THEY SAID IT

"I keep six honest serving-men
(They taught me all I knew);
Their names are What and Why and When
And How and Where and who."
— RUDYARD KIPLING

"For every ailment under the sun,
There is a remedy, or there is none,
If there be one, try to find it,
If there be none, never mind it.
— ARISTOTLE

By : M.V.R.P. Shastry, Srikakulam, AP

25 BEST QUESTIONS

ELLIPSE

— Md. Motiur Rahman

1. The eccentric angles of two points on an ellipse are ϕ and ϕ' and their join intersects the major axis at a distance c from the centre. If 2a be its major axis prove that

$$\frac{c-a}{c+a} = \tan\frac{\phi}{2}\tan\frac{\phi'}{2}$$

2. Prove that the locus of the foot of perpendicular drawn from the centre on any tangent to the ellipse; whose semi major and minor axes are a and b respectively, is the curve $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$.

3. If a normal at the end of a latus rectum passes through an end of a minor axes, prove that $e^4 + e^2 - 1 = 0$.

4. The tangent and the normal at the point P on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ meets its major axis in T and T′. If TT′ = a, prove that the eccentric angle θ of P is given by $e^2\cos^2\theta + \cos\theta - 1 = 0$ where e is the eccentricity of the ellipse.

5. Show that the locus of the foot of the perpendicular on a varying tangent to an ellipse from either of its foci is a concentric circle.

6. A tangent to an ellipse is cut by the tangents at the ends of the major axis in points T and T′. Prove that the circle on TT′ as diameter, will pass through the foci.

7. Prove that the perpendicular from the focus of an ellipse whose centre is C, on the tangent at any point P will meet the line CP on the directrix.

8. Prove that the line joining two points on an ellipse, the difference of whose eccentric angles is constant, touches another ellipse.

9. If P be the point of contact of the tangent $y = mx + \sqrt{a^2m^2 + b^2}$ to the ellipse $(x^2/a^2) + (y^2/b^2) = 1$, prove that $\tan^{-1}(-b/am)$ is the eccentric angle of P

10. Show that tangents drawn at those points of the ellipse $(x^2/a) + (y^2/b) = a + b$ where it is cut by any tangent to $(x^2/a^2) + (y^2/b^2) = 1$ intersect at right angles.

11. If two concentric ellipses be such that the foci of one lie on the other and if e and e′ be their eccentricities, show that their axes are inclined at the angle $\cos^{-1}\frac{\sqrt{e^2 + e'^2 - 1}}{ee'}$

12. Prove that the normals at the four points where the lines $(lx/a) + (my/b) = 1$ and $(x/la) + (y/mb) = -1$ cut the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ are concurrent

13. Tangents are drawn from an external point T(h, k) to the ellipse. $(x^2/a^2) + (y^2/b^2) = 1$ at P and Q. If s be any focus, prove that

$$\frac{ST^2}{SP \cdot SQ} = \frac{h^2}{a^2} + \frac{h^2}{b^2}$$

14. The foci of an ellipse are S and S′. If the normal at any point P on the ellipse intersects the line SS′ at G, prove that $PG^2 = SP \cdot S'P(1 - e^2)$ where e is the eccentricity of the ellipse.

15. The eccentric angle of any point P measured from the semi major axis CA is ϕ. If S be the focus nearest to A, and $\angle ASP = \theta$, prove that

$$\tan\theta/2 = \sqrt{\frac{1+e}{1-e}}\tan(\phi/2)$$

16. If α, β, γ be the three points on the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ show that the area of the triangle

formed by the tangents at these points is

$$ab \tan\frac{\alpha - \beta}{2} \tan\frac{\beta - \gamma}{2} \tan\frac{\gamma - \alpha}{2}$$

17. If the normal at the points on an ellipse whose eccentric angles are $\alpha, \beta, \gamma, \delta$ be the concurrent, then prove that $\Sigma \cos\alpha \ \Sigma \sec\alpha = 4$

18. Prove that the equation of the locus of the points of intersection of the tangent at one end of a focal chord of an ellipse with the normal at the other end is $\dfrac{x^2}{a^2} + \dfrac{b^2y^2}{(2a^2 - b^2)^2} = 1$

19. PSQ and PHR are focal chords of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ where S and H are the foci. The tangents at Q and R meet at T. Show that the locus of T as P moves round the ellipse is

$$(1 + e^2)^2\frac{x^2}{a^2} + (1 - e^2)^2\frac{y^2}{b^2} = (1 + e^2)^2$$

where e is the eccentricity of the ellipse

20. Show that the angles which the normals from any point (f, g) to the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ make with the x-axis are given by $(f\sin\theta - \cos\theta)^2 \cdot (a^2\cos^2\theta + b^2\sin^2\theta) = \frac{1}{4}(a^2 - b^2)\sin^2 2\theta$

21. Prove that the eccentricity of the ellipse $(x^2/a^2) + (y^2/b^2) = 1$ is given by $2\cot w = e^2/\sqrt{1 - e^2}\,\sin 2\theta$ where w is one of the angles between the normals at the points whose eccentric angles are θ and $\pi/2 + \theta$

22. From any point (h, k) two tangents OP and OQ are drawn to the ellipse $(x^2/a^2) + (y^2/b^2) = 1$. If C be the centre of the ellipse prove that the area of the DCPQ is $\dfrac{a^2b^2\sqrt{b^2h^2 + a^2k^2 - a^2b^2}}{b^2h^2 + a^2k^2}$

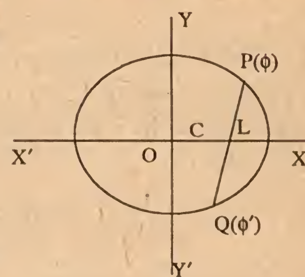
23. Show that the length of the focal chord of the ellipse. $(x^2/a^2) + (y^2/b^2) = 1$ which makes an angle θ with the major axis is $\dfrac{2ab^2}{a^2\sin^2\theta + b^2\cos^2\theta}$

24. Find the condition that the area of a triangle inscribed in an ellipse is greatest.

25. If three of the sides of a quadrilateral inscribed in an ellipse are parallel respectively to three given straight lines, show that fourth side will also be parallel to a fixed straight line

SOLUTIONS

1. Let P and Q be two points on the ellipse whose eccentric angles are ϕ and ϕ'. Thus equation of PQ is



$$\frac{x}{a}\cos\frac{\phi + \phi'}{2} + \frac{y}{b}\sin\frac{\phi + \phi'}{2} = \cos\frac{\phi - \phi'}{2} \quad \text{.......(1)}$$

If PQ intersects at L, then it is given that OL = c. So coordinates of L are (c, 0) (1) passes through (c, 0)

$$\frac{c}{a}\cos\frac{\phi + \phi'}{2} = \cos\frac{\phi - \phi'}{2}$$

$$\text{or } \frac{c}{a} = \frac{\cos\frac{\phi - \phi'}{2}}{\cos\frac{\phi + \phi'}{2}}$$

By componendo and dividendo we have

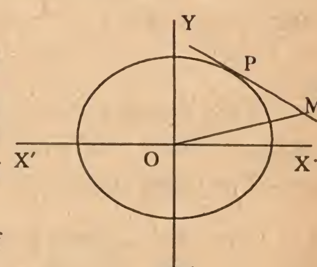
$$\frac{c - a}{c + a} = \frac{\cos\frac{\phi - \phi'}{2} - \cos\frac{\phi + \phi'}{2}}{\cos\frac{\phi - \phi'}{2} + \cos\frac{\phi + \phi'}{2}}$$

$$\text{or } \frac{c - a}{c + a} = \frac{2\sin(\phi/2)\sin(\phi'/2)}{2\cos(\phi/2)\cos(\phi'/2)}$$

$$\text{or } \frac{c - a}{c + a} = \tan(\phi/2)\tan(\phi'/2)$$

2. Equation of the ellipse is

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$



Let PM be the tangent at P to the ellipse. From O we draw OM ⊥ar to PT. Let coordinates of M be (x_1, y_1)

Then equation of tangent PM is

$$y = mx + \sqrt{a^2m^2 + b^2} \quad \text{.... (1)}$$

(1) passes through (x_1, y_1)

$y_1 = mx_1 + \sqrt{a^2m^2 + b^2}$

or $(y_1 - mx_1)^2 = a^2m^2 + b^2$ .... (2)

m_1 of OM $= y_1/x_1$

Since PM is ⊥ar to OM $mm_1 = -1$

$m\frac{y_1}{x_1} = -1 \qquad m = -\frac{x_1}{y_1}$

Thus from (2)

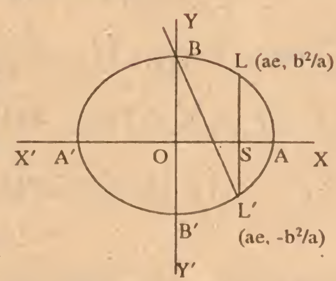
$\left(y_1 + \frac{x_1^2}{y_1}\right)^2 = a^2\frac{x_1^2}{y_1^2} + b^2$

$(x_1^2 + y_1^2)^2 = a^2x_1^2 + b^2y_1^2$

Locus of (x_1, y_1) is $(x^2 + y^2)^2 = a^2x^2 + b^2y^2$

3. Let equation of the ellipse be

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Let LSL′ be the latus rectum of the ellipse. Equation of normal at L′ is

$$\frac{y + \frac{b^2}{a}}{-\frac{b^2}{ab^2}} = \frac{x - ae}{\frac{ae}{a^2}}$$

or $\frac{ay + b^2}{-1} = \frac{(x - ae)a}{e}$

or $aey + b^2e = -xa + a^2e$

It passes through (0, b) ∴ $aeb + b^2e = a^2e$

or $ab + b^2 = a^2 \qquad$ or $ab + a^2(1 - e^2) = a^2$

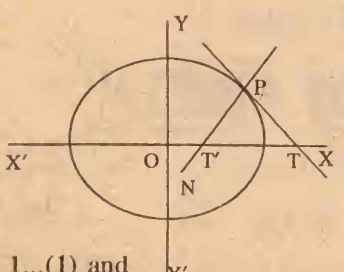
or $b + a(1 - e^2) = a$ or $(b/a) + 1 - e^2 = 1$

or $b/a = e^2 \qquad$ or $b^2/a^2 = e^4$

or $\frac{a^2(1 - e^2)}{a^2} = e^4$

or $1 - e^2 = e^4 \qquad$ or $e^4 + e^2 - 1 = 0$

4. Let PT and PN be the tangent and normal at P to the ellipse. Let coordinates of P be $(a\cos\theta, b\sin\theta)$. Thus equations of PT and PN are



$\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$...(1) and

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$ ... (2)

since (1) and (2) meet the major axis y = 0 in T and T′, so that coordinates of T and T′ are

$(a\sec\theta, 0)$ and $\left\{\frac{(a^2 - b^2)}{a}\cos\theta, 0\right\}$

Given TT′ = a $\qquad$ or CT - CT′ = a

or $a\sec\theta - \frac{(a^2 - b^2)}{a}\cos\theta = a$

or $a^2 - (a^2 - b^2)\cos^2\theta = a^2\cos\theta$

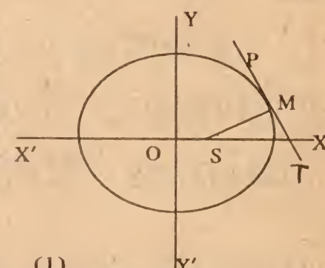
or $a^2 - a^2e^2\cos^2\theta = a^2\cos\theta$

or $1 - e^2\cos^2\theta = \cos\theta$

or $e^2\cos^2\theta + \cos\theta - 1 = 0$

5. Let equation of the ellipse be

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Let PT be the tangent to the ellipse at P. Let S be the focus. From S we draw SM ⊥ar to PT.

Equation of PT is

$y = mx + \sqrt{a^2m^2 + b^2}$ .... (1)

Any line ⊥ar to (1) and through the focus S(ae, 0) is

$y - 0 = \frac{-1}{m}(x - ae)$

$my + x - ae = 0$

or $my + x = ae = \sqrt{a^2 - b^2}$ .... (2)

From (1) $y - mx = \sqrt{a^2m^2 + b^2}$ .... (3)

Squaring and adding we get

$(my + x)^2 + (y - mx)^2 = a^2 - b^2 + a^2m^2 + b^2$

or $m^2y^2 + x^2 + 2mxy + y^2 + m^2x^2 - 2mxy = a^2(1 + m^2)$

or $(x^2 + y^2) + m^2(x^2 + y^2) = a^2(1 + m^2)$

or $(x^2 + y^2)(1 + m^2) = a^2(1 + m^2)$

or $x^2 + y^2 = a^2$

which is a circle. Whose centre is (0, 0)

6. Let the equation of the ellipse be

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ ... (1)

Any tangent to the ellipse (1) is

$y = mx + \sqrt{a^2m^2 + b^2}$ ..... (2)

Tangents at the extremities of the major axis are

$x = a$ ... (3) and $x = -a$ ... (4)

solving (2), (3) and (2), (4) we will get coordinates of T and T′ as $\{a, am + \sqrt{a^2m^2 + b^2}\}$ and $\{-a, -am + \sqrt{a^2m^2 + b^2}\}$

Thus equation of the cirlce drawn having TT′ as diameter is

$$(x - a)(x + a) + \{y - am - \sqrt{a^2m^2 + b^2}\}\{y + am - \sqrt{a^2m^2 + b^2}\} = 0$$

If it passes through ($\pm ae$, 0) then

$a^2e^2 - a^2 - (a^2m^2 - a^2m^2 + b^2) = 0$

or $a^2e^2 = a^2 - b^2$

or $a^2e^2 = a^2 - a^2(1 - e^2) = a^2e^2$ which is true

7. Let equation of the ellipse be

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ ... (1)}$$

Let coordinates of P be ($a\cos\theta$, $b\sin\theta$). Thus equation of tangent at P is

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1 \text{ .. (2)}$$

Any line ⊥ar to (2) and through the focus S(ae, 0) is

$$y - 0 = \frac{a}{b}\tan\theta(x - ae)$$

or $yb\cos\theta = ax\sin\theta - a^2e\sin\theta$ .. (3)

Also equation of CP is

$$\frac{y - 0}{0 - b\sin\theta} = \frac{x - 0}{0 - a\cos\theta}$$

$ay\cos\theta = bx\sin\theta$ .. (4)

solving (3) and (4), we get

$$b\cos\theta\,\frac{bx\sin\theta}{a\cos\theta} = ax\sin\theta - a^2e\sin\theta$$

or $b^2x\sin\theta = a^2x\sin\theta - a^3e\sin\theta$

or $b^2x = a^2x - a^3e$

or $x(a^2 - b^2) = a^3e = \dfrac{a^2e^2 \cdot a}{e}$

or $x(a^2 - b^2) = \dfrac{(a^2 - b^2)a}{e}$ or $x = \dfrac{a}{e}$

Which is the equation of the directrix

8. Let P and Q be two points on the ellipse. Whose eccentric angles are α and β.

Given $\alpha - \beta$ = const. = k(Let)

Thus equation of line PQ is

$$\frac{x}{a}\cos\frac{\alpha + \beta}{2} + \frac{y}{b}\sin\frac{\alpha + \beta}{2} = \cos\frac{\alpha - \beta}{2} = \cos\frac{k}{2}$$

$= k_1$ (Let)

Which is a tangent to the ellipse

$$\frac{x^2}{a^2k_1^2} + \frac{y^2}{b^2k_1^2} = 1$$

at the point whose eccentric angle is $\dfrac{\alpha + \beta}{2}$

9. Let eccentric angle of P be θ. Thus coordinates of P are ($a\cos\theta$, $b\sin\theta$)

since P lies on $y = mx + \sqrt{a^2m^2 + b^2}$

$\therefore b\sin\theta = ma\cos\theta + \sqrt{a^2m^2 + b^2}$

or $(b\sin\theta - am\cos\theta)^2 = a^2m^2 + b^2$

or $b^2\sin^2\theta + a^2m^2\cos^2\theta - 2amb\sin\theta\cos\theta = a^2m^2 + b^2$

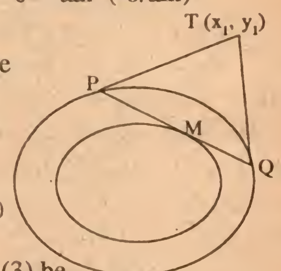
or $a^2m^2 - a^2m^2\cos^2\theta + b^2 - b^2\sin^2\theta + 2amb\sin\theta\cos\theta$

or $a^2m^2\sin^2\theta + b^2\cos^2\theta + 2amb\sin\theta\cos\theta = 0$

or $(am\sin\theta + b\cos\theta)^2 = 0$

$am\sin\theta + b\cos\theta = 0$ $\quad$ $am\sin\theta = -b\cos\theta$

$\tan\theta = -b/am$ $\quad$ $\theta = \tan^{-1}(-b/am)$



10. The given ellipses are

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1 \text{ .. (1) and}$$

$$\frac{x^2}{a(a + b)} + \frac{y^2}{b(a + b)} = 1 \text{ .. (2)}$$

Let

$mx - y + \sqrt{a^2m^2 + b^2} = 0$ ... (3) be the equation of tangent PMQ to the ellipes (1) intersecting (2) at P and Q

If the tangents at P and Q meet at $T(x_1, y_1)$; then we are to prove that ∠PTQ is a right angle.

Now PQ is a chord of contact for ellipse (2). So equation of PQ is

$$\frac{xx_1}{a(a + b)} + \frac{yy_1}{b(a + b)} = 1 \text{ ... (4)}$$

since (3) and (4) represents the same straight line so comparing them we get

$$\frac{m}{\frac{x_1}{a(a + b)}} = \frac{-1}{\frac{y_1}{b(a + b)}} = \frac{-\sqrt{a^2m^2 + b^2}}{1}$$

$$\frac{m(a + b)a}{x_1} = \frac{-b(a + b)}{y_1} = -\sqrt{a^2m^2 + b^2}$$

$$x_1 = \frac{-ma(a + b)}{\sqrt{a^2m^2 + b^2}},\quad y_1 = \frac{b(a + b)}{\sqrt{a^2m^2 + b^2}} \quad \text{... (5)}$$

$\therefore$ The equation of the pair of tangents from (x_1, y_1) to the ellipse

$\frac{x^2}{a(a+b)} + \frac{y^2}{b(a+b)} = 1$ is

$$\left[\frac{x^2}{a(a+b)} + \frac{y^2}{a(a+b)} - 1\right]\left[\frac{x_1^2}{a(a+b)} + \frac{y_1^2}{a(a+b)} - 1\right]$$

$$= \left[\frac{xx_1}{a(a+b)} + \frac{yy_1}{b(a+b)} - 1\right]^2$$

These tangents will be at right angles if coeff. of x^2 + coeff of $y^2 = 0$

or $\frac{x_1^2}{ab(a+b)^2} + \frac{y_1^2}{ab(a+b)^2} - \frac{1}{a(a+b)} - \frac{1}{b(a+b)} = 0$

or $\frac{(x_1^2 + y_1^2)}{ab(a+b)^2} = \frac{a+b}{ab(a+b)}$

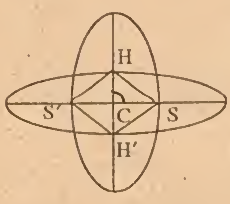
or $x_1^2 + y_1^2 = (a+b)^2$ which is true from (5)

11. Let S and S′ be the foci of one ellipse and H and H′ the other, C being their common centre

Then SHS′H′ is a parallelogram and since

$SH + S'H = HS' + H'S' = 2a$

Since the sum of focal distances of any point on an ellipse is equal to its major axis which is 2a.



Then $CS = ae$, $CH = ae'$

Let θ be the angle between thier axes

$SH^2 = a^2e^2 + a^2e'^2 - 2a^2ee'\cos\theta$

$HS'^2 = a^2e^2 + a^2e'^2 + 2a^2ee'\cos\theta$

Now $2a = SH + S'H$

$4a^2 = SH^2 + HS'^2 + 2SH.S'H$

$= 2a^2(e^2 + e'^2) + 2a^2\sqrt{\{(e^2 + e'^2 - 2ee'\cos\theta) \times (e^2 + e'^2 + 2ee'\cos\theta)}$

$2a^2(2 - e^2 - e'^2) = 2a^2\sqrt{(e^2 + e'^2)^2 - 4e^2e'^2\cos^2\theta}$

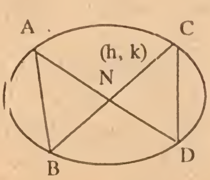
or $4 + (e^2 + e'^2)^2 - 4(e^2 + e'^2) = (e^2 + e'^2)^2 - 4e^2e'^2\cos^2\theta$

$1 - e^2 - e'^2 = -e^2e'^2\cos^2\theta$

$e^2e'^2\cos^2\theta = e^2 + e'^2 - 1$

$\cos\theta = \frac{\sqrt{e^2 + e'^2 - 1}}{ee'}$

12. Let the normals at the points where the line $\frac{lx}{a} + \frac{my}{b} = 1$ cut the ellipse meet in (h, k)



Let the line joining the feet of the other two normals from (h, k) have for its equation $px + qy = 1$

Then the general equation of a curve through the intersection of the ellipse and the two lines is

$$\left(\frac{x^2}{a^2} + \frac{y^2}{b^2} - 1\right) + \lambda\left(\frac{lx}{a} + \frac{my}{b} - 1\right) \times (px + qy - 1) = 0 \quad \dots (1)$$

But the four points A, B, C, D also lie on the hyperbola

$(a^2 - b^2)xy + kb^2x - ha^2y = 0$ .. (2)

Hence for some value of arbitrary constant λ, curves (1) and (2) must be identical. This is possible only when coefficient of x^2 and y^2 and the constant term in (1) must vanish separately.

$\frac{1}{a^2} + \frac{\lambda pl}{a} = 0$, $\frac{1}{b^2} + \frac{\lambda mq}{b} = 0$, $\lambda - 1 = 0$

$\lambda = 1$, $1 + pla = 0 = 1 + qmb$

$p = -\frac{1}{la}$, $q = -\frac{1}{mb}$

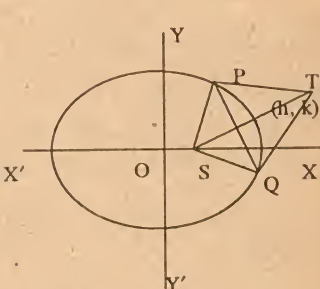
Therefore the line $px + qy = 1$ become

$\frac{-x}{la} - \frac{y}{mb} = 1$ or $\frac{x}{la} + \frac{y}{mb} = -1$

13. Equation of the chord of contact PQ is

$\frac{xh}{a^2} + \frac{yk}{b^2} = 1$ .. (1)

The equation of the ellipse is



$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$.... (2)

From (1) $\frac{yk}{b^2} = 1 - \frac{xh}{a^2}$

$y = \frac{b^2}{k}\left(1 - \frac{xh}{a^2}\right)$

From (1)

$\frac{x^2h}{a^2} + \frac{1}{b^2}\frac{b^4}{k^2}\left(1 - \frac{xh}{a^2}\right)^2 = 1$

or $\frac{x^2}{a^2} + \frac{b^2}{k^2}\frac{(a^2 - xh)^2}{a^4} = 1$

or $a^4k^2x^2 + b^2(a^4 + x^2h^2 - 2xha^2) = a^4k^2$

or $x^2(a^2k^2 + b^2h^2) - 2a^2b^2xh + a^4(b^2 - k^2) = 0$

Let x_1 and x_2 be its roots

$x_1 + x_2 = \frac{2a^2b^2h}{a^2k^2 + b^2h^2}$ $\quad x_1x_2 = \frac{a^4(b^2 - k^2)}{a^2k^2 + b^2h^2}$

Now $ST^2 = (h - ae)^2 + k^2$

Also $SP \cdot SQ = (a - ex_1)(a - ex_2)$

$$= a^2 - ae(x_1 + x_2) + e^2x_1x_2$$

$$= a^2 - ae\,\frac{2a^2b^2h}{a^2k^2 + b^2h^2} + \frac{e^2a^4(b^2 - k^2)}{a^2k^2 + b^2h^2}$$

$$= \frac{a^2}{a^2k^2 + b^2h^2}[a^2k^2 + b^2h^2 - 2aehb^2 + a^2e^2(b^2 - k^2)]$$

$$= \frac{a^2}{a^2k^2 + b^2h^2}[b^2h^2 + a^2k^2 - 2aehb^2 + a^2e^2b^2 - a^2e^2k^2]$$

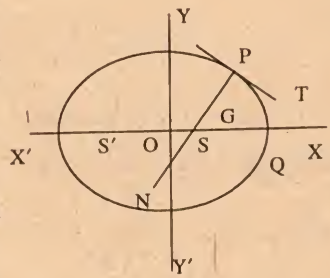
$$= \frac{a^2}{a^2k^2 + b^2h^2}[b^2h^2 + a^2e^2 - 2aeh] + a^2k^2(1 - e^2)]$$

$$= \frac{a^2}{a^2k^2 + b^2h^2}[b^2(h - ae)^2 + b^2k^2]$$

$$SPSQ = \frac{a^2b^2\ ST^2}{a^2k^2 + b^2h^2}$$

$$\frac{ST^2}{SPSQ} = \frac{a^2k^2 + b^2h^2}{a^2b^2} = \frac{k^2}{b^2} + \frac{h^2}{a^2}$$

14. Let coordinates of P be $(a\cos\theta, b\sin\theta)$



Thus equation of normal PN at P is

$$ax\sec\theta - by\,\mathrm{cose}\theta = a^2 - b^2$$

Its intersection with x-axis are given by

$$ax\sec\theta = a^2 - b^2 = a^2e^2$$

$$x = ae^2\cos\theta$$

Thus the coordinates of G are $(ae^2\cos\theta, 0)$

$$PG^2 = (a\cos\theta - ae^2\cos\theta)^2 + (b\sin\theta - 0)^2$$

$$= a^2(1 - e^2)^2\cos^2\theta + a^2(1 - e^2)\sin^2\theta$$

$$= a^2(1 - e^2)[\sin^2\theta + (1 - e)^2\cos^2\theta]$$

$$= a^2(1 - e^2)(1 - e^2\cos^2\theta) \quad \dots (1)$$

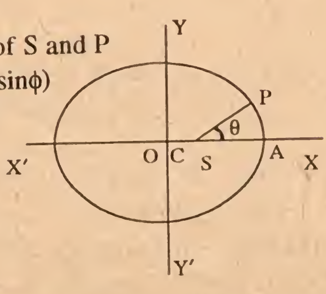
Also $SP \cdot S'P = (a + ae\cos\theta)(a - ae\cos\theta)$

$$= a^2(1 - e^2\cos^2\theta) \quad \dots (2)$$

Hence from (1) and (2) we get

$$PG^2 = (1 - e^2)\,SP \cdot S'P$$

15. The coordinates of S and P are $(ae, 0)$ and $(a\cos\phi, b\sin\phi)$



$\therefore \tan\theta$ = slope of SP

$$= \frac{b\sin\phi - 0}{a(\cos\phi - e)}$$

$$= \frac{\sqrt{1 - e^2}\sin\phi}{\cos\phi - e}$$

$$\sin\theta\cos\phi - e\sin\theta = \cos\theta\sin\phi\sqrt{1 - e^2}$$

or $$\frac{2\tan\theta/2}{1 + \tan^2\theta/2}\,\frac{1 - \tan^2\phi/2}{1 + \tan^2\phi/2} - \frac{e^2\tan\theta/2}{1 + \tan^2\theta/2}$$

$$= \sqrt{1 - e^2}\,\frac{2\tan\phi/2}{1 + \tan^2\phi/2} \times \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2}$$

or $\tan\theta/2\,(1 - \tan^2\phi/2) - e\tan\theta/2(1 + \tan^2\phi/2)$
$= \sqrt{1 - e^2}\tan\phi/2(1 - \tan^2\theta/2)$

or $(1 - e)\tan\theta/2 - (1 + e)(\tan\theta/2\tan^2\phi/2 -$
$\sqrt{1 - e^2}\tan\phi/2(1 - \tan^2\phi/2)$

or $(1 + e)\tan^2\phi/2 + \sqrt{1 - e^2}\tan\phi/2\,(\cot\phi/2 - \tan\phi/2)$
$- (1 - e) = 0$

or $(1 + e)\tan^2\phi/2 + \sqrt{1 - e^2}\tan\phi/2\cot\theta/2$
$- \sqrt{1 - e^2}\tan\phi/2\tan\theta/2 - (1 - e) = 0$

or $\sqrt{1 + e}\tan\phi/2\,[\sqrt{1 + e}\tan\phi/2 - \sqrt{1 - e}\tan\theta/2] +$
$\sqrt{1 - e}\cot\theta/2\,[\sqrt{1 + e}\tan\phi/2 - \sqrt{1 - e}\tan\theta/2] = 0$

or $[\sqrt{1 + e}\tan\phi/2 - \sqrt{1 - e}\tan\theta/2] \times [\sqrt{1 + e}\tan\phi/2$
$+ \sqrt{1 - e}\cot\theta/2] = 0$

As ϕ and θ lies between 0 and π

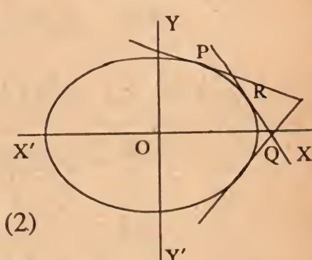
$\tan\alpha/2 > 0$ and $\tan\theta/2 > 0$

so the second factor cannot be zero

Hence $\sqrt{1 + e}\tan\phi/2 - \sqrt{1 - e}\tan\theta/2 = 0$

or $$\tan\theta/2 = \sqrt{\frac{1 + e}{1 - e}}\ \tan\phi/2$$

16. The equation of tangents at the points β and γ to the ellipse



$$\frac{x^2}{a^2} + \frac{x^2}{b^2} = 1 \text{ are}$$

$$\frac{x}{a}\cos\beta + \frac{y}{b}\sin\beta = 1 \quad \dots (1)$$

$$\frac{x}{a}\cos\gamma + \frac{y}{b}\sin\gamma = 1 \quad .. (2)$$

$$\frac{x/a}{-\sin\beta + \sin\gamma} = \frac{y/b}{-\cos\gamma + \cos\beta} = \frac{1}{\cos\beta\sin\gamma - \sin\beta\cos\gamma}$$

$$\frac{x/a}{\sin\beta - \sin\gamma} = \frac{y/b}{\cos\gamma - \cos\beta} = \frac{1}{\sin(\beta - \gamma)}$$

$$\frac{x/a}{2\cos\frac{\beta + \gamma}{2}\sin\frac{\beta - \gamma}{2}} = \frac{x/b}{2\sin\frac{\beta + \gamma}{2}\sin\frac{\beta - \gamma}{2}}$$

$$= \frac{1}{2\sin\frac{\beta - \gamma}{2}\cos\frac{\beta - \gamma}{2}}$$

$$\frac{x}{a} = \frac{\cos(\beta+\gamma)/2}{\cos(\beta-\gamma)/2}, \quad \frac{y}{b} = \frac{\sin(\beta+\gamma)/2}{\cos(\beta-\gamma)/2}$$

Thus the point of intersection of the tangents at (1) and (2) is

$$\left\{\frac{a(\cos(\beta+\gamma)/2)}{\cos(\beta-\gamma)/2}, \frac{b(\sin(\beta+\gamma)/2)}{\cos(\beta-\gamma)/2}\right\}$$

which are coordinates of one of the vertices of the triangle. Similarly for other vertices

Hence the required area of the triangle is

$$= \tfrac{1}{2} ab \begin{vmatrix} \dfrac{\cos\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}} & \dfrac{\sin\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}} & 1 \\ \dfrac{\cos\frac{\gamma+\alpha}{2}}{\cos\frac{\gamma-\alpha}{2}} & \dfrac{\sin\frac{\gamma+\alpha}{2}}{\cos\frac{\gamma-\alpha}{2}} & 1 \\ \dfrac{\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} & \dfrac{\sin\frac{\alpha+\beta}{2}}{\cos\frac{\alpha-\beta}{2}} & 1 \end{vmatrix}$$

$$= \frac{ab}{2\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2}} \times$$

$$\begin{vmatrix} \cos\frac{\beta+\gamma}{2} & \sin\frac{\beta+\gamma}{2} & \cos\frac{\beta-\gamma}{2} \\ \cos\frac{\gamma+\alpha}{2} & \sin\frac{\gamma+\alpha}{2} & \cos\frac{\gamma-\alpha}{2} \\ \cos\frac{\alpha+\beta}{2} & \sin\frac{\alpha+\beta}{2} & \cos\frac{\alpha-\beta}{2} \end{vmatrix}$$

$$= \frac{ab}{2\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2}} \left[\cos\frac{\beta+\gamma}{2} \times\right.$$

$$\left\{\sin\frac{\gamma+\alpha}{2}\cos\frac{\alpha-\beta}{2} - \cos\frac{\gamma-\alpha}{2}\sin\frac{\alpha+\beta}{2}\right\}$$

$$- \sin\frac{\beta+\gamma}{2}\left\{\cos\frac{\gamma+\alpha}{2}\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta}{2}\right.$$

$$\left.\left.\cos\frac{\gamma-\alpha}{2}\right\} + \cos\frac{\beta-\gamma}{2}\sin\frac{-\gamma+\beta}{2}\right]$$

$$= \frac{ab}{2\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}\cos\frac{\alpha-\beta}{2}} \left[\cos\frac{\alpha-\beta}{2}\right.$$

$$\left.\sin\frac{\alpha-\beta}{2} - \cos\frac{\gamma-\alpha}{2}\sin\frac{\alpha-\gamma}{2} + \tfrac{1}{2}\sin(\beta-\gamma)\right]$$

$$= \frac{ab}{2\cdot 2\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}} [\sin(\alpha-\beta) +$$

$$\sin(\beta-\gamma) + \sin(\gamma-\alpha)]$$

$$= \frac{ab}{4\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}} [\sin(\alpha-\beta) +$$

$$2\sin\frac{\beta-\alpha}{2}\cos\frac{\beta+\alpha-2\gamma}{2}]$$

$$= \frac{ab\sin(\alpha-\beta)/2}{2\cos\frac{\alpha-\beta}{2}\cos\frac{\beta-\gamma}{2}\cos\frac{\gamma-\alpha}{2}} 2\sin\frac{\alpha-\gamma}{2}$$

$$\sin\frac{\beta-\gamma}{2}$$

$$= ab\tan\frac{\alpha-\beta}{2}\tan\frac{\beta-\gamma}{2}\tan\frac{\gamma-\alpha}{2}$$

17. Equation of the normal at any point (x', y') on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is

$$\frac{a^2(x - x')}{x'} = \frac{b^2(y - y')}{y'}$$

If it passes through a given point (h, k) then

$a^2(h - x')y' = x'b^2(k - y')$

or $y'[a^2(h - x') + x'b^2] = x'^2b^2k$ ... (1)

But $\frac{x'^2}{a} + \frac{y'^2}{b^2} = 1$

or $y'^2 = \frac{b^2}{a^2}(a^2 - x'^2)$

From (1), we get $y_1^2[a^2(h - x') + x'b^2]^2 = x'^4b^4k^2$

or $\frac{b^2}{a^2}(a^2 - x'^2)[a^4(h^2 + x'^2 - 2hx') + x'^2b^4 + 2b^2x'a^2(h - x')] = x^4b^4k^2$

or $(a^2 - x'^2)[a^4h^2 + a^4x'^2 - 2ha^4x' + x'^2b^4 + 2b^2a^2hx' - 2b^2a^2x'^2] = x'^4a^2b^2k^2$

or $a^6h^2 + a^6x'^2 - 2ha^6x' + a^2b^4x'^2 + 2a^4b^2hx' - 2b^2a^4x'^2 - a^4x'^2h^2 - a^4x'^4 + 2ha^4x'^3 - x'^4b^4 - 2b^2a^2hx'^3 + 2b^2a^2x'^4 = x'^4a^2b^2k^2$

or $x'^4(a^4 + b^4 - 2a^2b^2) - 2ha^2x'^3(a^2 - b^2) + x'^2(a^6 - a^2b^4 + 2b^2a^4) - 2x'ha^4(a^2 - b^2) + a^6h^2 = 0$

The four value of x' are the abscissae of the four points on the ellipse the normals at which passes

through (h, k)
Let x_1, x_2, x_3, x_4 be the four value of x' correpsonding to the points whose eccentric angle are $\alpha, \beta, \gamma, \delta$

$$\Sigma x_1 = \frac{2ha^2}{a^2 - b^2}, \qquad \Sigma x_1x_2x_3 = \frac{2a^4h}{a^2 - b^2}$$

$$x_1x_2x_3x_4 = \frac{a^6h^2}{(a^2 - b^2)^2}$$

$$\Sigma \frac{1}{x_1} = \frac{1}{x_1} + \frac{1}{x_2} + \frac{1}{x_3} + \frac{1}{x_4}$$

$$= \frac{\Sigma x_1x_2x_3}{x_1x_2x_3x_4} = \frac{2a^4h}{a^2 - b^2}\ \frac{(a^2 - b^2)^2}{a^6h^2}$$

$$= \frac{2(a^2 - b^2)}{a^2h}$$

$$\text{Now } \Sigma x_1 \Sigma \frac{1}{x_1} = \frac{2a^2h}{a^2 - b^2}\ \frac{2(a^2 - b^2)}{a^2h} = 4$$

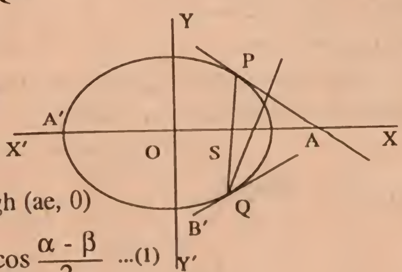
$$\text{or } \Sigma\, a\cos\alpha\, \Sigma \frac{1}{a\cos\alpha} = 4$$

$$\Sigma \cos\alpha\, \Sigma \sec\alpha = 4$$

18. Let α and β be the eccentric angles of the ends of a focal chord of the ellipse.

Equation of PQ is

$$\frac{x}{a}\cos\frac{\alpha + \beta}{2} + \frac{y}{b}\sin\frac{\alpha + \beta}{2} = \cos\frac{\alpha - \beta}{2}$$



It passes through (ae, 0)

$$e\cos\frac{\alpha + \beta}{2} = \cos\frac{\alpha - \beta}{2} \quad ...(1)$$

Equation of tangent at α is

$$\frac{x}{a}\cos\alpha + \frac{y}{b}\sin\alpha = 1 \quad ..... (2)$$

Equation of normal at β is

$$xa\sec\beta - yb\,\mathrm{cosec}\beta = a^2 - b^2 = c^2 \text{ (Let)}$$

Thus the point of intesection of (2) and (3) is given by

$$\frac{x}{\frac{c^2}{b}\sin\beta + \frac{b}{\sin\beta}} = \frac{y}{\frac{a}{\cos\beta} - \frac{c^2}{a}\cos\alpha}$$

$$= \frac{-1}{\frac{-b\cos\alpha}{a\sin\beta} + \frac{a\sin\alpha}{b\cos\beta}}$$

$$\text{or } \frac{xb\sin\beta}{c^2\sin^2\beta + b^2} = \frac{y\cos\beta}{a^2 - c^2\cos\alpha\cos\beta}$$

$$= \frac{ab\sin\beta\cos\beta}{b^2\cos\alpha\cos\beta + a^2\sin\alpha\sin\beta} \quad ..... (4)$$

From (1), we get

$a^2e^2[1 + \cos(\alpha + \beta)] = a^2[1 + \cos(\alpha - \beta)]$

or $a^2 - a^2e^2 = a^2[e^2\cos(\alpha + \beta) - \cos(\alpha - \beta)]$

or $b^2 = a^2e^2\cos(\alpha + \beta) - a^2\cos(\alpha - \beta)$

or $b^2 = (a^2 - b^2)\cos(\alpha + \beta) - a^2\cos(\alpha - \beta)$

or $b^2 = (a^2 - b^2)(\cos\alpha\cos\beta - \sin\alpha\sin\beta) - a^2(\cos\alpha\cos\beta + \sin\alpha\sin\beta)$

or $b^2 = \cos\alpha\cos\beta[a^2 - b^2 - a^2] - \sin\alpha\sin\beta[a^2 - b^2 + a^2]$

or $b^2 = -b^2\cos\alpha\cos\beta - (2a^2 - b^2)\sin\alpha\sin\beta$

or $b^2 + (a^2 - b^2)\sin\alpha\sin\beta = -(a^2\sin\alpha\sin\beta + b^2\cos\alpha\cos\beta)$

or $b^2 + c^2\sin\alpha\sin\beta = -(a^2\sin\alpha\sin\beta + b^2\cos\alpha\cos\beta)$

Hence from (4)

$$\frac{xb\sin\beta}{b^2 + c^2\sin\alpha\sin\beta} = \frac{ya\cos\beta}{a^2 - c^2\cos\alpha\cos\beta}$$

$$= \frac{ab\sin\beta\cos\beta}{-(b^2 + c^2\sin\alpha\sin\beta)}$$

$\therefore x = -a\cos\beta$ .... (5)

Hence from (3)

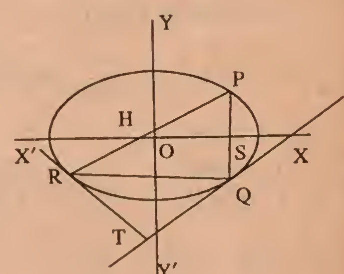
$$-yb\,\mathrm{cosec}\beta = c^2 + \frac{a^2\cos\beta}{\cos\beta} = c^2 + a^2 = 2a^2 - b^2$$

$$y = \frac{-2a^2 - b^2}{b}\sin\beta \quad ... (6)$$

Eliminating β from (5) and (6) we get

$$\frac{x^2}{a^2} + \frac{b^2y^2}{(2a^2 - b^2)} = 1$$

19. Let coordinates of T be (x_1, y_1). Let eccentric angles of P, Q, R be ϕ, ϕ_1 and ϕ_2. Thus equation of chords PSQ and PHR are



$$\frac{x}{a}\cos\frac{\phi + \phi_1}{2} + \frac{y}{b}\sin\frac{\phi + \phi_1}{2} = \cos\frac{\phi - \phi_1}{2} \quad ... (1)$$

and $\frac{x}{a}\cos\frac{\phi + \phi_2}{2} + \frac{y}{b}\sin\frac{\phi + \phi_2}{2} = \cos\frac{\phi - \phi_2}{2}$ ..(2)

(1) passes through (ae, 0)

$$e\cos\frac{\phi + \phi_1}{2} = \cos\frac{\phi - \phi_1}{2}$$

(2) passes through (-ae, 0)

$$-e\cos\frac{\phi + \phi_2}{2} = \cos\frac{\phi - \phi_2}{2}$$

$$\therefore \frac{1}{-e} = \frac{\cos\frac{\phi + \phi_2}{2}}{\cos\frac{\phi - \phi_2}{2}}$$

By componendo and Dividendo

$$\frac{1 - e}{1 + e} = \frac{\cos\frac{\phi + \phi_2}{2} + \cos\frac{\phi - \phi_2}{2}}{\cos\frac{\phi + \phi_2}{2} - \cos\frac{\phi - \phi_2}{2}}$$

or $\frac{1 - e}{1 + e} = \frac{2\cos\phi/2\cos\phi_2/2}{2\sin\phi/2\sin\phi_2/2}$

or $\frac{1 - e}{1 + e} = -\cot\phi/2\cot\phi_2/2$

or $\cot\phi_2/2 = -\frac{1 - e}{1 + e}\tan\phi/2$

$\tan\phi_2/2 = -\frac{1 + e}{1 - e}\cot\phi/2$

Similarly $\tan\phi_1/2 = -\frac{1 - e}{1 + e}\cot\phi/2$

The equation QR is

$$\frac{x}{a}\cos\frac{\phi_1 + \phi_2}{2} + \frac{y}{b}\sin\frac{\phi_1 + \phi_2}{2} = \cos\frac{\phi_1 - \phi_2}{2}$$

Dividing throughout by $\cos(\phi_1/2)\cos(\phi_2/2)$, we get

$$\frac{x}{a}(1 - \tan\phi_1/2\tan\phi_2/2) + \frac{y}{b}(\tan\phi_1/2 + \tan\phi_2/2) = 1 + \tan\phi_1/2\tan\phi_2/2$$

or $\frac{x}{a}(1 - \cot^2\phi/2) + \frac{y}{b}\left[-\cot\phi/2\frac{2(1 + e^2)}{1 - e^2}\right] = 1 + \cot^2\phi/2$

or $\frac{x}{a}\frac{(1 - \cot^2\phi/2)}{1 + \cot^2\phi/2} - \frac{1 + e^2}{1 - e^2}\frac{2\cot(\phi/2)\,y}{1 + \cot^2(\phi/2)\,b} = 1$

or $-\frac{x}{a}\cos\phi - \frac{1 + e^2}{1 - e^2}\sin\phi\frac{y}{b} = 1$

or $\frac{x}{a}(1 - e^2)\cos\phi + \frac{y}{b}(1 + e^2)\sin\phi = -(1 - e^2)$ ... (3)

But QR is also chord of contact

Thus equation of QR is

$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ .... (4)

Comparing the coefficients in (3) and (4) we get

$$\frac{(1 - e^2)\cos\phi}{x_1/a} = \frac{(1 + e^2)\sin\phi}{y_1/b} = -(1 - e^2)$$

$\cos^2\phi = \frac{x_1^2}{a^2}$, $\sin^2\phi = \frac{y_1^2}{b^2}\frac{(1 - e^2)^2}{(1 + e^2)^2}$

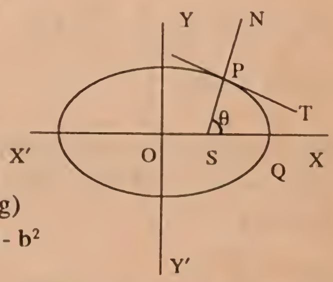
$\cos^2\phi + \sin^2\phi = 1$

or $\frac{x_1^2}{a^2} + \frac{y_1^2(1 - e^2)^2}{b^2(1 + e^2)^2} = 1$

Locus of (x_1, y_1) is

$$\frac{x^2}{a^2}(1 + e^2)^2 + (1 - e^2)^2\frac{y^2}{b^2} = 1$$

20. Let the normal PN at the point P whose eccentric angle is ϕ makes angle θ with x axis.



Thus equation of PN is

$ax\sec\phi - by\,\mathrm{cosec}\phi = a^2 - b^2$ .... (1)

$\tan\theta = m$ of the normal

$= \frac{a\sec\phi}{b\,\mathrm{cosec}\phi} = \frac{a}{b}\tan\phi$

(1) passes through (f, g)

$fa\sec\phi - gb\,\mathrm{cosec}\phi = a^2 - b^2$

squaring we get

$f^2a^2\sec^2\phi + g^2b^2\mathrm{cosec}^2\phi - 2abfg\sec\phi\,\mathrm{cosec}\phi = (a^2 - b^2)^2$

or $f^2a^2(1 + \tan^2\phi) + g^2b^2(1 + \cot^2\phi) - 2fgab(\tan\phi + \cot\phi) = (a^2 - b^2)^2$

or $f^2a^2\left(1 + \frac{b^2\tan^2\theta}{a^2}\right) + g^2b^2\left(1 + \frac{a^2}{b^2\tan^2\theta}\right) - 2fgab\left(\frac{b\tan\theta}{a} + \frac{a}{b\tan\theta}\right) = (a^2 - b^2)^2$

or $\frac{f^2(a^2\cos^2\theta + b^2\sin^2\theta)}{\cos^2\theta} + \frac{g^2(a^2\cos^2\theta + b^2\sin^2\theta)}{\sin^2\theta}$

$\frac{2fg(b^2\sin^2\theta + a^2\cos^2\theta)}{\sin\theta\cos\theta} = (a^2 - b^2)^2$

or $(a^2\cos^2\theta + b^2\sin^2\theta)(f^2\sin^2\theta + g^2\cos^2\theta) - 2fg\sin\theta\cos\theta = (a^2 - b^2)^2\sin^2\theta\cos^2\theta$

or $(a^2\cos^2\theta + b^2\sin^2\theta)(f\sin\theta - g\cos\theta)^2 = (a^2 - b^2)^2\sin^2\theta\cos^2\theta$

21. The equation of the normal at two given points θ and $\pi/2 + \theta$ are

$ax\sec\theta - by\,\mathrm{cosec}\theta = a^2 - b^2$ ... (1)

and $ax\sec(\pi/2 + \theta) - by\text{cosec}(\pi/2 + \theta) = a^2 - b^2$ .. (2)

Their slopes are $m_1 = \frac{a}{b}\tan\theta$

and $m_2 = -\frac{a}{b}\cot\theta$

$\therefore \tan w = \frac{m_2 - m_1}{1 + m_2 m_1}$

$= \frac{-\frac{a}{b}(\cot\theta + \tan\theta)}{1 - (a^2/b^2)}$

$= \frac{ab}{(a^2 - b^2)\sin\theta\cos\theta}$

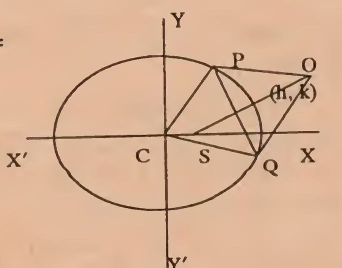
$\tan w = \frac{2ab}{(a^2 - b^2)\sin\theta 2\theta}$

or $\frac{1}{\cot w} = \frac{2ab}{(a^2 - b^2)\sin 2\theta}$

or $\frac{2\cot w}{\sin 2\theta} = \frac{a^2 - b^2}{ab} = \frac{a^2 - a^2(1 - e^2)}{a\, a\sqrt{1 - e^2}}$

$= \frac{e^2}{\sqrt{1 - e^2}}$

22. Let ϕ_1 and ϕ_2 be the eccentric angles of the points P and Q.



Then area of ΔCPQ = $\frac{1}{2}[a\cos\phi_1 b\sin\phi_1 - a\cos\phi_2 b\sin\phi_1]$

$= \frac{1}{2}ab\sin(\phi_1 - \phi_2)$

Also equation of PQ is

$\frac{x}{a}\cos\frac{\phi_1 + \phi_2}{2} + \frac{y}{b}\sin\frac{\phi_1 + \phi_2}{2} = \cos\frac{\phi_1 - \phi_2}{2}$ ... (1)

But PQ is the chord of contact

thus equation of PQ is

$\frac{xh}{a^2} + \frac{yk}{b^2} = 1$ ... (2)

comparing (1) and (2) we get

$\frac{h}{a\cos(\phi_1 + \phi_2)/2} = \frac{k}{b\sin(\phi_1 + \phi_2)/2} = \frac{1}{\cos(\phi_1 - \phi_2)/2}$

$\frac{h}{a} = \frac{\cos(\phi_1 + \phi_2)/2}{\cos(\phi_1 - \phi_2)/2}$ $\quad \frac{k}{b} = \frac{\sin(\phi_1 + \phi_2)/2}{\cos(\phi_1 - \phi_2)/2}$

$\frac{h^2}{a^2}\cos^2\frac{\phi_1 - \phi_2}{2} + \frac{k^2}{b^2}\cos^2\frac{\phi_1 - \phi_2}{2} = 1$

$\left(\frac{h^2}{a^2} + \frac{k^2}{b^2}\right)\cos^2\frac{\phi_1 - \phi_2}{2} = 1$

$\frac{b^2h^2 + a^2k^2}{a^2b^2}\cos^2\frac{\phi_1 - \phi_2}{2} = 1$

$\cos^2\frac{\phi_1 - \phi_2}{2} = \frac{a^2b^2}{b^2h^2 + a^2k^2}$

$\therefore \sin^2\frac{\phi_1 - \phi_2}{2} = 1 - \frac{a^2b^2}{b^2h^2 + a^2k^2}$

$= \frac{b^2h^2 + a^2k^2 - a^2b^2}{a^2k^2 + b^2h^2}$

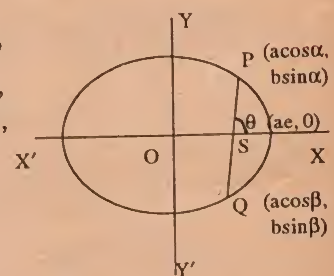
Thus area of $\Delta CPQ = ab\sin\frac{\phi_1 - \phi_2}{2}\cos\frac{\phi_1 - \phi_2}{2}$

$= \frac{ab\sqrt{b^2h^2 + a^2k^2 - a^2b^2}}{\sqrt{b^2h^2 + a^2k^2}} \frac{ab}{\sqrt{b^2h^2 + a^2k^2}}$

$= \frac{a^2b^2\sqrt{b^2h^2 + a^2k^2 - a^2b^2}}{b^2h^2 + a^2k^2}$

23. Let PQ be a focal chord of the ellipse

$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



Let coordinates of P and Q be $(a\cos\alpha, b\sin\alpha)$ and $(a\cos\beta, b\sin\beta)$

Equation of PQ is

$\frac{x}{a}\cos\frac{\alpha + \beta}{2} + \frac{y}{b}\sin\frac{\alpha + \beta}{2} = \cos\frac{\alpha - \beta}{2}$ ... (1)

It passes through (ae, 0)

$e\cos\frac{\alpha + \beta}{2} = \cos\frac{\alpha - \beta}{2}$ ... (2)

since (1) makes angle θ with x-axis

$\tan\theta = -\frac{b}{a}\cot\frac{\alpha + \beta}{2}$

Now $PQ = \sqrt{a^2(\cos\alpha - \cos\beta)^2 + b^2(\sin\alpha - \sin\beta)^2}$

$= \sqrt{a^2 4\sin^2\frac{\alpha+\beta}{2}\sin^2\frac{\alpha-\beta}{2} + 4b^2\cos^2\frac{\alpha+\beta}{2}\sin^2\frac{\alpha-\beta}{2}}$

$= 2\sin\frac{\alpha-\beta}{2}\sqrt{a^2\sin^2\frac{\alpha+\beta}{2} + b^2\cos^2\frac{\alpha+\beta}{2}}$

$= 2\sin\frac{\alpha-\beta}{2}\sqrt{a^2\sin^2\frac{\alpha+\beta}{2} + a^2(1 - e^2)\cos^2\frac{\alpha+\beta}{2}}$

$= 2a\sin\frac{\alpha-\beta}{2}\sqrt{1 - e^2\cos^2\frac{\alpha+\beta}{2}}$

$= 2a\sin\frac{\alpha-\beta}{2}\sqrt{1 - e^2\cos^2\frac{\alpha-\beta}{2}} = 2a\sin^2\frac{\alpha-\beta}{2}$

Also $\dfrac{2ab^2}{a^2\sin^2\theta + b^2\cos^2\theta}$

$$= \frac{2ab^2}{\left[a^2 \dfrac{\sin^2\theta}{\cos^2\theta} + b^2\right]\cos^2\theta}$$

$$= \frac{2ab^2\sec^2\theta}{a^2\tan^2\theta + b^2}$$

$$= \frac{2ab^2\sec^2\theta}{b^2 + b^2\cot^2(\alpha+\beta)/2} = 2a\sec^2\theta\,\sin^2\frac{\alpha+\beta}{2}$$

$$= 2a\sin^2\frac{\alpha+\beta}{2}\,[1 + \tan^2\theta]$$

$$= 2a\sin^2\frac{\alpha+\beta}{2}\left[1 + \frac{b^2}{a^2}\cot^2\frac{\alpha+\beta}{2}\right]$$

$$= 2a\sin^2\frac{\alpha+\beta}{2}\left[1 + \frac{b^2\cos^2(\alpha+\beta/2)}{a^2\sin^2(\alpha+\beta/2)}\right]$$

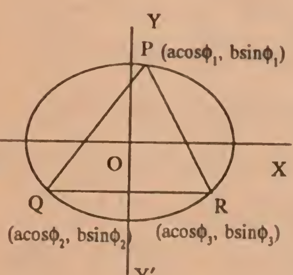
$$= \frac{2a\sin^2(\alpha+\beta/2)}{a^2\sin^2(\alpha+\beta/2)}\left[a^2\sin^2\frac{\alpha+\beta}{2} + b^2\cos^2\frac{\alpha+\beta}{2}\right]$$

$$= \frac{2}{a}\left[a^2\sin^2\frac{\alpha+\beta}{2} + a^2(1 - e^2)\cos^2\frac{\alpha+\beta}{2}\right]$$

$$= 2a\left(1 - e^2\cos^2\frac{\alpha+\beta}{2}\right) \text{ From (2)}$$

$$= 2a\sin^2\frac{\alpha-\beta}{2} = PQ$$

24. Let PQR be a Δ inscribed in the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$



Let eccentric angles of P, Q and R be ϕ_1, ϕ_2 and ϕ_3. Let P′, Q′, R′ be the corresponding points on the auxiliary circle.

Then area of $\Delta PQR = \frac{1}{2}\begin{vmatrix} a\cos\phi_1 & b\sin\phi_1 & 1 \\ a\cos\phi_2 & b\sin\phi_2 & 1 \\ a\cos\phi_3 & b\sin\phi_3 & 1 \end{vmatrix}$

And area of $\Delta P'Q'R' = \frac{1}{2}\begin{vmatrix} a\cos\phi_1 & a\sin\phi_1 & 1 \\ a\cos\phi_2 & a\sin\phi_2 & 1 \\ a\cos\phi_3 & a\sin\phi_3 & 1 \end{vmatrix}$

Hence $\dfrac{\Delta PQR}{\Delta P'Q'R'} = \dfrac{b}{a}$ = constant

Therefore ΔPQR is greatest when $\Delta P'Q'R'$ is greatest But $\Delta P'Q'R'$ is greatest when $\Delta P'Q'R'$ is an equilateral triangle and in that case

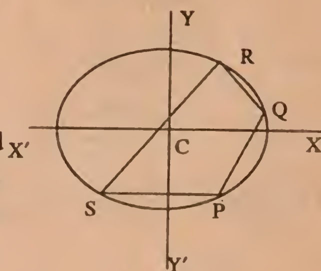
$$\phi_1 \sim \phi_2 = \phi_2 \sim \phi_3 = \phi_3 \sim \phi_1 = \frac{2\pi}{3}$$

$\therefore$ ΔPQR is greatest when eccentric angles of its vertices are of the type

$$\alpha,\ \alpha + \frac{2\pi}{3},\ \alpha + \frac{4\pi}{3}$$

25. Let PQRS be a quadrilateral inscribed in the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1.$$



Let PQ, QR and RS be the three sides parallel to the given lines.

Equation of PQ is

$$\frac{x}{a}\cos\frac{\alpha+\beta}{2} + \frac{y}{b}\sin\frac{\alpha+\beta}{2} = \cos\frac{\alpha-\beta}{2}$$

Its slope is $-\dfrac{b}{a}\cot\dfrac{\alpha+\beta}{2}$ which is a constant by hypothesis

$\therefore$ $\alpha + \beta$ = constant = 2λ say

similarly $\beta + \gamma$ = constant = 2μ say

$\gamma + \delta$ = constant = 2γ say

Now the equation of SP is

$$\frac{x}{a}\cos\frac{\alpha+\delta}{2} + \frac{y}{b}\sin\frac{\alpha+\delta}{2} = \cos\frac{\alpha-\delta}{2}$$

Its slope $m = -\dfrac{b}{a}\cot\dfrac{\alpha+\delta}{2}$

But $\alpha + \delta = (\alpha + \beta) + (\gamma + \delta) - (\beta + \gamma)$

$= 2\lambda + 2v - 2\mu$ = constant

Hence the slope of the fourth side PS is constant. Hence the fourth side is also parallel to a fixed straight line.

✻

INTERNATIONAL MATH OLYMPIAD '95 PROBLEMS

First Day 4½ hrs

1. Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y. The line XY meets BC at the point Z. Let P be a point on the line XY different from Z. The line CP intersects the circle with diameter AC at the points C and M, and the line BP intersects the circle with diameter BD at the points B and N. Prove that the lines AM, DN and XY are concurrent.

2. Let a, b, and c be positive real numbers such that abc = 1. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(c+a)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}$$

3. Determine all integers $n > 3$ for which there exist n points $A_1, A_2, ..., A_n$ in the plane, the real numbers $r_1, r_2, ..., r_n$ satisfying the following two conditions:
(i) no three of the points $A_1, A_2, ..., A_n$ lie on a line;
(ii) for each triple i, j, k ($1 \leq i < j < k \leq n$) the triangle $A_iA_jA_k$ has area equal to $r_i + r_j + r_k$.

Second Day 4 ½ hrs.

4. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, ...,$ x_{1995} satisfying the two conditions:
(i) $x_0 = x_{1995}$;
(ii) $x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$ for each i = 1, 2, ..., 1995

5. Let ABCDEF be a convex hexagon with AB = BC = CD, DE = EF = FA, and $\angle BCD = \angle EFA = 60^0$. Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120^0$. Prove that $AG + GB + GH + DH + HE \geq CF$

6. Let p be an odd prime number. Find the number of subsets A of the set {1, 2, ..., 2p} such that
(i) A has exactly p elements, and
(ii) the sum of all the elements in A is divisible by p.

Contd. from page No... 6

given by
(a) d = 2 (b) d = 1
(c) d = 3 (d) none of these

27. The limiting position of the point of intersection of the straight lines 3x + 5y = 1 and $(2 + c)x + 5c^2y = 1$ as c tends to 1 is:
(a) (1/2, -1/10) (b) (1/5, -1/25)
(c) (3/8, -1/40) (d) none of these

28. The value of $\int_0^\pi \sin(n + 1/2)x/\sin x/2 \, dx$ is:
(a) $\pi/2$ (b) 0
(c) π (d) 2π

29. $ln\ (2^{1/3} + 1/3^{1/3})^x$, if the ratio the seventh term from the beginning to the seventh term from the end is 1/6, then x is:
(a) 9 (b) 6
(c) 12 (d) none of these

30. The area bounded by the curves $y = x^2/4a$ and $y = [8a^3]/(x^2 + 4a^2)$ is:
(a) $a^2(2\pi - 4/3)$ (b) $a^2(\pi - 4/3)$
(c) $a^2(2\pi + 4/3)$ (d) $a^2(\pi + 4/3)$ ■

floated the Vidyasagar scheme under which it gives soft loans to students for pursing higher studies here or as partial help to go to foreign universities. The bank does not take the address of the educational institution for reference work but the home address of the student. Often the home property it held collateral, "to be used as a threat than anything else". So far, the recovery rate has been cent percent.

The National Loan Scheme will hold no collateral but, says a member of the sub-committee, "the catches will be in-built. For instance, if a student defaults with more than two instalments, he will be disqualified from the scheme and will have to return the principal amount, immediately. Again, every student will have to renew his or her loan annually so the past payment performance is a major guiding factor for the renewal."

The scheme, modelled on the American system of subsidising higher education, has been welcomed by educational institutions and prospective students. "In this age of spiraling fees for higher education, these loans will surely sell like hot cakes," says Gaurav Mehta, a BSc graduate who is hoping to get into "either computers or management". The Delhi-based Indian Institute of technology's director US Raju says, "This scheme is definitely welcome. With the cost of higher education going up, we have to look at different fronts for generating resources."

Welcoming the scheme, the University of Delhi vice-chancellor, V R Mehta says, "Our university is a member of this committee and we have tried to impress upon them that the scheme should include loans for subjects in humanities and social sciences." On the insistence of the university, the subcommittee has agreed to include Education and Social Work students as being eligible for the loan.

Chaturvedi, however, emphatically says the scheme will be restricted to professional courses in the first instance, "where the potential for employment is high". The government has identified 50 institutior including the highly coveted IITs and IIMs who students can avail of the National Loan Schem Says Chaturvedi, "About 10 to 20 per cent of studen joining courses in these 50 institutions are expect to benefit from this scheme. This could mean up 50,000 students. If the scheme is successful, it m be extended to other institutions, including priva ones.

Will the scheme push up fees in profession educational institutions? While members of t subcommittee rule out any direct linkage, observe say the loan scheme will surely tempt institutions hike up fees, knowing that the more promising studen have the option to borrow and pay. "Perhaps t future will see similar bankable schemes where studen will only be able to defer payments rather th simply enjoy heavy government subsidy," says sub-committee member. But as Mehta conclude "Ultimately whatever you do in the education sect is an investment in the future of India and in o people.

BRAIN-TWISTER

Q : What is the maximum number of acute angles that any convex polygon can have?

Sol: Three. In a convex polygon, each interior angle is either acute, right, or obtuse. An interior angle of measure more than 180 would contradict the convexity of the polygon.

In a convex polygon, the sum of the measures of the exterior angles is equal to 360 degrees. The angle exterior to an acute interior angle is an obtuse angle. The sum of measures of four or more obtuse angles exceeds 360. Therefore, the maximum number of obtuse exterior angles, or acute interior angles.

PROBLEMS OF THE MONTH

COMPLEX NUMBERS

Contributed by : Alok Kumar Singh, Lucknow and Pranjal Goswami, Orissa

1. Find the cube root of -8. Express the result in the form a + ib.

Soln.: The trigonometric form of-8

$= -8 + 0i = 8(\cos\pi + i\sin\pi)$

$-8 = 8[\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)]$

where n = integer including zero.

Its cube root

$= [8\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)]^{1/3}$

$= 2\left[\cos\dfrac{(2n\pi + \pi)}{3} + i\sin\dfrac{(2n\pi + \pi)}{3}\right]$

putting n = 0, 1, 2,

Let the roots are x_0, x_1, x_2

$x_0 = 2\left(\cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3}\right) = 2\left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right) = 1 + i\sqrt{3}$

$x_1 = 2(\cos\pi + i\sin\pi) = 2(-1 + 0 \cdot i) = -2$

$x_2 = 2\left(\cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}\right) = 2\left(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right) = 1 - i\sqrt{3}$

$\therefore$ The required roots are $1 + i\sqrt{3}$, -2, $1 - i\sqrt{3}$.

2. Find the roots of polynomial $z^6 + 2z^3 + 1$ and factor it.

Soln.: $z^6 + 2z^3 + 1 = (z^3 + 1)^2$

the roots of this polynomial are cube roots of -1

$z^3 = -1 = \cos\pi + i\sin\pi$

$z^3 = \cos(2n\pi + \pi) + i\sin(2n\pi + \pi)$

$z = [\cos(2n\pi + \pi) + i\sin(2n\pi + \pi)]^{1/3}$

$= \cos\dfrac{(2n\pi + \pi)}{3} + i\sin\dfrac{(2n\pi + \pi)}{3}$

putting n = 0, 1, 2, the roots are z_0, z_1, z_2

$z_0 = \cos\dfrac{\pi}{3} + i\sin\dfrac{\pi}{3} = \dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}$

$z_1 = \cos\pi + i\sin\pi = -1$

$z_2 = \cos\dfrac{5\pi}{3} + i\sin\dfrac{5\pi}{3}$

$= \cos\dfrac{\pi}{3} - i\sin\dfrac{\pi}{3} = \dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}$

Now factor of $z^6 + 2z^3 + 1$

$(z^3 + 1)^2 = \{(z - z_0)(z - z_1)(z - z_2)\}^2$

$= \left\{(z + 1)\left(z - \left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)\right)\left(z - \left(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)\right)\right\}^2$

$= (z + 1)^2\left\{z - \left(\dfrac{1}{2} + i\dfrac{\sqrt{3}}{2}\right)\right\}^2\left\{z - \left(\dfrac{1}{2} - i\dfrac{\sqrt{3}}{2}\right)\right\}^2$

$= (z + 1)^2(z^2 - z + 1)^2$

$z^6 + 2z^3 + 1 = (z + 1)^2(z^2 - z + 1)^2$

3. Dividing f(z) by z - i we obtain the remainder 1 - i and dividng it by z + i we get the remainder 1 + i. Then find the remainder upon the division of f(z) by $z^2 - 1$.

Soln.: Let g(z) be quotient and az + b be remainder upon division of f(z) by $z^2 + 1$

$f(z) = g(z)(z^2 + 1) + az + b$

$= g(z)(z - i)(z + i) + az + b$

Now, according to question dividing f(2) by z - i the remainder is (1 - i)

$\therefore f(i) = g(z)(1 - i)(i + i) + ai + b$

$= 0 + ai + b$ from remainder theorem

$ai + b = 1 - i$

Equating real and imaginary parts

$a = -1$

$b = 1$

Again dividing f(z) by z + i remainder is 1 + i

$f(-i) = g(z)(-i - i)(-i + i) + a(-i) + b)$
$= 0 - ai + b$
$-ai + b = 1 + i$
$a = -1, b = 1$
So the required remainder is $-z + 1$.

4. Find all the roots of equation $z^n = (z - 1)^n$ and show that all of them lie on a line which is parallel to the imaginary axis of the Arg and plane.

Soln.: $z^n = (z - 1)^n$

$w^n = 1$ where $w = \dfrac{z}{z-1}$

$\Rightarrow w^n = \cos 2r\pi + i\sin(2r\pi)$ where r is integer

$w = (\cos 2r\pi + i\sin 2r\pi)^{1/n}$ including zero

$w = \cos\dfrac{2r\pi}{n} + i\sin\dfrac{2r\pi}{n}$

$\dfrac{z}{z-1} = \cos\dfrac{2r\pi}{n} + i\sin\dfrac{2r\pi}{n}$

Taking moduli both sides

$\dfrac{|z|}{|z-1|} = \cos\dfrac{2r\pi}{n} + i\sin\dfrac{2r\pi}{n}$

$\dfrac{|z|}{|z-1|} = \left[\cos^2\dfrac{2r\pi}{n} + \sin^2\dfrac{2r\pi}{n}\right]^{1/2}$

$\dfrac{|z|}{|z-1|} = 1$ $\quad |z| = |z - 1|$

Now putting $|z| = |x + iy|$
$|x + iy| = |x - 1 + iy|$
$\sqrt{x^2 + y^2} = \sqrt{(x-1)^2 + y^2}$
$x^2 + y^2 = x^2 - 2x + 1 + y^2$
$2x - 1 = 0 \quad \Rightarrow x = 1/2$

Hence the roots of the given equation lie on the line parallel to the imaginary axis

5. If n is positive integer and $(1 + x)^n = C_0 + C_1x + C_2x^2 \ldots\ldots + C_nx^n$. Then prove that
$C_0 + C_4 + C_8 \ldots = 2^{n-2} + 2^{(n/2)-1}\cos\dfrac{n\pi}{4}$

Soln.:

$(1 + x)^n = C_0 + C_1x + C_2x^2 + \ldots\ldots + C_nx^n$

If $x = 1$ Then

$2^n = C_0 + C_1 + C_2 + C_3 \ldots\ldots C_n$ .... (1)

If $x = -1$

$0 = C_0 - C_1 + C_2 - C_3 + \ldots + (-1)^nC_n$ .... (2)

Now adding eq. (1) and (2)

$2(C_0 + C_2 + C_4 \ldots\ldots) = 2^n$

$C_0 + C_2 + C_4 \ldots\ldots = \dfrac{2^n}{2}$

$C_0 + C_2 + C_4 \ldots = 2^{n-1}$ .... (3)

If $x = i$ Then

$(1 + i)^n = C_0 + iC_1 - iC_2 - iC_3 + \ldots\ldots$

But $(1 + i)^n = 2^{n/2}\left\{\cos\dfrac{n\pi}{4} + i\sin\dfrac{n\pi}{4}\right\}$

$\therefore 2^{n/2}\left\{\cos\dfrac{n\pi}{4} + i\sin\dfrac{n\pi}{4}\right\} = C_0 + iC_1 - C_2 - iC_3$ .... (4)

Equating real part from eq (4)

$C_0 - C_2 + C_4 + \ldots = 2^{n/2}\cos\dfrac{n\pi}{4}$ .... (5)

Adding Eq(3) and Eq (5)

$2(C_0 + C_4 + C_8 + \ldots\ldots) = 2^{n-1} + 2^{n/2}\cos\dfrac{n\pi}{4}$

or $C_0 + C_4 + C_8 + \ldots = 2^{n-2} + 2^{(n/2)-1}\cos\dfrac{n\pi}{4}$

6. If $\left(1 + i\dfrac{x}{a}\right)\left(1 + i\dfrac{x}{b}\right)\left(1 + i\dfrac{x}{c}\right) = A + iB$ Then prove that $\left(1 + i\dfrac{x^2}{a^2}\right)\left(1 + i\dfrac{x^2}{b^2}\right)\left(1 + i\dfrac{x^2}{c^2}\right) = A^2 + B^2$

Soln.:

If $1 + i\dfrac{x}{a} = r(\cos\theta + i\sin\theta)$

$r\cos\theta = 1,\ r\sin\theta = \dfrac{x}{a}$

$\therefore r^2 = 1 + \dfrac{x^2}{a^2}$, $\tan\theta = \dfrac{x}{a}$

$\therefore \left(1 + \dfrac{ix}{a}\right) = \left(1 + \dfrac{x^2}{a^2}\right)^{1/2}(\cos\theta + i\sin\theta)$

Similarly $\left(1 + i\dfrac{x}{b}\right) = \left(1 + \dfrac{x^2}{b^2}\right)^{1/2}(\cos\phi + i\sin\phi)$

$\left(1 + i\dfrac{x}{c}\right) = \left(1 + \dfrac{x^2}{c^2}\right)^{1/2}(\cos\psi + i\sin\psi)$

Now $\left(1 + i\dfrac{x}{a}\right)\left(1 + i\dfrac{x}{b}\right)\left(1 + i\dfrac{x}{c}\right)$

$= \left(1 + \dfrac{x^2}{a^2}\right)^{1/2}\left(1 + \dfrac{x^2}{b^2}\right)^{1/2}\left(1 + \dfrac{x^2}{c^2}\right)^{1/2}$

$\{\cos(\theta + \phi + \psi - i\sin(\theta + \phi + \psi\}$

$A + iB = \left(1 + \dfrac{x^2}{a^2}\right)^{1/2}\left(1 + \dfrac{x^2}{b^2}\right)^{1/2}\left(1 + \dfrac{x^2}{c^2}\right)^{1/2}$

$\{\cos(\theta + \phi + \psi) - i\sin(\theta + \phi + \psi)\}$

Now $A - iB = \left(1 + \dfrac{x^2}{a^2}\right)^{1/2}\left(1 + \dfrac{x^2}{b^2}\right)^{1/2}\left(1 + \dfrac{x^2}{c^2}\right)^{1/2}$

$$\{\cos(\theta + \phi + \psi) - i\sin(\theta + \phi + \psi)\}$$

$$(A + iB)(A - iB) = \left(1 + \frac{x^2}{a^2}\right)\left(1 + \frac{x^2}{b^2}\right)\left(1 + \frac{x^2}{c^2}\right)$$

$$\{\cos^2(\theta + \phi + \psi) - i^2\sin(\theta + \phi + \psi)\}$$

$$A + B^2 = \left(1 + \frac{x^2}{a^2}\right)\left(1 + \frac{x^2}{b^2}\right)\left(1 + \frac{x^2}{c^2}\right)$$

$$\{\cos^2(\theta + \phi + \psi) - \sin^2(\theta + \phi + \psi)\}$$

$$A^2 + B^2 = \left(1 + \frac{x^2}{a^2}\right)\left(1 + \frac{x^2}{b^2}\right)\left(1 + \frac{x^2}{c^2}\right)$$

7. Seperate $\dfrac{e^{i\beta}}{1 - ke^{i\alpha}}$ into real and imaginary parts.

Soln.: $\dfrac{e^{i\beta}}{1 - ke^{i\alpha}}$

multiplying the numerator and denominator by complex conjugate of the denominator) we have

$$\frac{e^{i\beta}}{1 - ke^{i\alpha}} = \frac{e^{i\beta}}{1 - ke^{i\alpha}} \times \frac{1 - ke^{-i\alpha}}{1 - ke^{-i\alpha}}$$

$$= \frac{e^{i\beta} - ke^{i(\beta - \alpha)}}{1 - k(e^{i\alpha} + e^{-i\alpha}) + k^2}$$

$$\left\{\begin{array}{l}\text{since } e^{i\theta} = \cos\theta + i\sin\theta \\ \text{and } \cos\theta = \dfrac{e^{i\theta} + e^{-i\theta}}{2}\end{array}\right\}$$

$$\frac{e^{i\beta}}{1 - ke^{i\alpha}} = \frac{(\cos\beta + i\sin\beta) - k\{\cos(\beta - \alpha) + i\sin(\beta - \alpha)\}}{1 - 2k\cos\alpha + k^2}$$

$$= \frac{\{\cos\beta - k\cos(\beta - \alpha)\} + i\{\sin\beta - \sin(\beta - \alpha)\}}{1 - 2k\cos\alpha + k^2}$$

$$= \left\{\frac{\cos\beta - k\cos(\beta - \alpha)}{1 - 2k\cos\alpha + k^2}\right\} + i\left\{\frac{\sin\beta - \sin(\beta - \alpha)}{1 - 2k\cos\alpha + k^2}\right\}$$

8. If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$ then prove that $\theta = \dfrac{n\pi}{2} + \dfrac{\pi}{4}$

Soln.: $2\theta = (\theta + i\phi) + (\theta - i\phi)$ .... (1)

If $\tan(\theta + i\phi) = \cos\alpha + i\sin\alpha$

Then $\tan(\theta - i\phi) = \cos\alpha - i\sin\alpha$

$\therefore \tan 2\theta = \tan[(\theta + i\phi) + (\theta - i\phi)]$

$$\tan 2\theta = \frac{\tan(\theta + i\phi) + \tan(\theta - i\phi)}{1 - \tan(\theta - i\phi)\tan(\theta - i\phi)}$$

$$\tan 2\theta = \frac{\cos\alpha + i\sin\alpha + \cos\alpha - i\sin\alpha}{1 - [\cos\alpha + i\sin\alpha)(\cos\alpha - i\sin\alpha)}$$

$$\tan 2\theta = \frac{2\cos\alpha}{1 - (\cos^2\alpha + \sin^2\alpha)}$$

$$\tan 2\theta = \frac{2\cos\alpha}{1 - 1} \qquad \tan 2\theta = \frac{2\cos\alpha}{0}$$

$$\tan 2\theta = \infty = \tan\frac{\pi}{2}$$

$$2\theta = n\pi + \frac{\pi}{2}$$

$$\theta = \frac{n\pi}{2} + \frac{\pi}{4}$$

9. Show that complex conugate of $(-1 + i)^7$ is $-8(1 - i)$

Soln.: Let $z = (-1 + i)^7$

$= (-1 + i)[(-1 + i)^2]^3$

$z = (-1 + i)(1 + i^2 - 2i)^3$

$z = (-1 + i)(1 - 1 - 2i)^3$

$z = (-1 + i)(-2i)^3$ $\quad z = (-1 + i)(+8i)$

$z = 8(-i + i^2)$ $\quad z = 8(-i - 1)$

$z = 8(-1 - i)$

Now conjugate of z is

$\bar{z} = 8(-1 + i) = -8(1 - i)$.

10. If $|z_1| = 13$ and $z_2 = 3 + 4i$. Find the greatest and least value of $|z_1 + z_2|$

Soln.:

$|z_1 + z_2| \le |z_1| + |z_2|$

Here $|z_1| = 13$ and $|z_2| = \sqrt{(3)^2 + (4)^2}$

$|z_2| = \sqrt{9 + 16} \quad = \sqrt{25}$

$|z_2| = 5$

$|z_1 + z_2| \le |z_1| + |z_2| \le 13 + 5 \le 18$

Thus the greatest value of $|z_1 + z_2|$ is 18.

For least value we have

$|z_1 + z_2| \ge |z_1| - |z_2|$

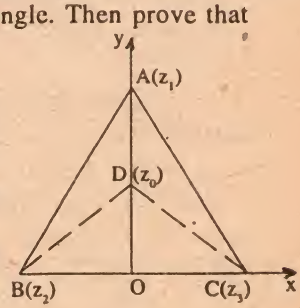
$|z_1 + z_2| \ge 13 - 5 \ge 8$

The least value of $|z_1 + z_2|$ is 8

11. Let the complex numbers z_1, z_2 and z_3 be the vertices of an equilateral triangle. Let z_0 be the circumcentre of the triangle. Then prove that $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$.

Soln.:

Let z_1, z_2 and z_3 be the vertices A, B and C respectively. $D(z_0)$ be the circumcentre of ΔABC. Taking OX and OY as coordinate axes,



OB = OC (in magnitude)

If AB = BC = CA = 2a then

$OB = z_2 = -a$; $OC = z_3 = a$

and $OA = \frac{2a}{2}\tan 60^0 = a\sqrt{3}$ $\quad \therefore z_1 = a\sqrt{3}$

Now, $OD = OC\tan 30^0 = a\sqrt{3}$ $\quad \therefore z_0 = \frac{a}{\sqrt{3}} i$

$\therefore z_1^2 + z_2^2 + z_3^2 = (a\sqrt{3}i)^2 + (-a)^2 + (a)^2$
$= -3a^2 + a^2 + a^2 = -a^2$

and $3z_0^2 = 3\left(\frac{ai}{\sqrt{3}}\right)^2 = \frac{3a^2i^2}{3} = -a^2$

From these two results we get $z_1^2 + z_2^2 + z_3^2 = 3z_0^2$

12. If z_1 and z_2 are two non-zero complex numbers such that $|z_1 + z_2| = |z_1| + |z_2|$, then $Arg(z_1) - Arg(z_2) = ?$

Soln.: Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_1) = r_1e^{i\theta_1}$

and $z_2 = r_2ei\theta_2$

Then $|z_1| = r_1$; $|z_2| = r_2$, $Arg(z_1) = \theta_1$

and $Arg(z_2) = \theta_2$

$|z_1 + z_2| = |(r_1\cos\theta_1 + r_2\cos\theta_2 + i(r_1\sin\theta_1 + r_2\sin\theta_2)|$
$= \sqrt{\{(r_1\cos\theta_1 + r_2\cos\theta_2)^2 + (r_1\sin\theta_1 + r_2\sin\theta_2)^2\}}$
$= \sqrt{\{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)\}}$

Now, $|z_1 + z_2|^2 = \{|z_1| + |z_2|\}^2$

$\Rightarrow \{r_1^2 + r_2^2 + 2r_1r_2\cos(\theta_1 - \theta_2)\} = (r_1 + r_2)^2$

$\Rightarrow \cos(\theta_1 - \theta_2) = 1 \Rightarrow \theta_1 - \theta_2 = 0$

$\Rightarrow Arg(z_1) - Arg(z_2) = 0$

13. Express $\frac{1}{1 - \cos\theta + 2i\sin\theta}$ in the from A + iB

Soln.: $\frac{1}{1 - \cos\theta + 2i\sin\theta}$

$$= \frac{1}{2\sin^2 ½\theta + 2i(2\sin½\theta\cos½\theta)}$$

$$= \frac{1}{2\sin½\theta(\sin½\theta + 2i\cos½\theta)}$$

$$= \frac{\sin½\theta - 2i\cos½\theta}{2\sin½\theta(\sin½\theta + 2i\cos½\theta)(\sin½\theta - 2i\cos½\theta)}$$

$$= \frac{\sin½\theta - 2i\cos½\theta}{2\sin½\theta(\sin^2½\theta + 4\cos^2½\theta)}$$

$$= \frac{\sin½\theta - 2i\cos½\theta}{2\sin½\theta(1 + 3\cos^2½\theta)}$$

$$= \left[\frac{1}{2(1 + 3\cos^2½\theta)}\right] - i\left[\frac{\cot½\theta}{1 + 3\cos^2½\theta}\right]$$

14. If 1, w, w^2 be the cube roots of unity, prove that $(1 + w)(1 + w^2)(1 + w^4)(1 + w^8)$ .... 2n factors = 1.

Soln.: We have

$(1 + w)(1 + w^2)(1 + w^4)(1 + w^8)$ ... 2n factors
$= [(1 + w)(1 + w^2)][(1 + w^4)(1 + w^8)]$ ... n factors
$= [(1 + w)(1 + w^2)][(1 + w)(1 + w^2)]$ ... n factors
$= [(1 + w)(1 + w^2)]^n$
$= [(-w^2)(-w)]^n = (w^3)^n = 1^n = 1$

15. Find the real values of x and y so that

$$\frac{(1 + i)x - 2i}{3 + i} + \frac{(2 - 3i)y + i}{3 - i} = i$$

Soln:

$$\frac{(1 + i)(x - 2i)}{3 + i} + \frac{(2 - 3i)y + i}{3 - i} = i$$

multiplying both sides by $(3 + i)(3 - i)$, we get

$\{(1 + i)(x - 2i)\}(3 - i) + \{(2 - 3i)y + i\}(3 + i)$
$= i(3 + i)(3 - i)$

$\Rightarrow \{x + i(x - 2)\}(3 - i) + \{2y + i(1 - 3y\}(3 + i)$
$= i(9 - i^2)$

$\Rightarrow \{3x + (x - 2)\} + i(-x + 3x - 6) + \{6y - (1 - 3y)\}$
$+ i(2y + 3 - 9y\} = i(9 + 1)$

$\Rightarrow (4x - 2) + i(2x - 6) + (9y - 1) + i(-7y + 3) = 10i$

$\Rightarrow (4x + 9y - 3) + i(2x - 7y - 3) = 10i$

Equating real and imaginary parts, we get

$4x + 9y - 3 = 0$, $2x - 7y - 3 = 10$

Solving these equations, we get $x = 3$, $= -1$

$\therefore x = 3, y = -1$

16. If $(1 + x + x^2)^n = a_0 + a_1x + a_2x^2 + ... + a_rx^r + ... + a_{2n}x^{2n}$. Then show that

$a_0 + a_3 + a_6 + ... = a_1 + a_4 + a_7 + ...$
$= a_2 + a_5 + a_8 + ...$

Soln.: Let x = w. Then

$(1 + w + w^2)^n = a_0 + a_1w + a_2w^2 + ... + a_rw^r + ... + a_{2n}w^{2n}$

$\Rightarrow 0 = (a_0 + a_3w^3 + a_6w^6 + ...) + (a_1w + a_4w^4 + a_7w^7 + ...) + (a_2w^2 + a_5w^5 + ...)$

But $w^3 = w^6 = w^9 = ... = 1$,

$w^4 = w^7 = w^{10} = ... = w$

and $w^5 = w^8 = w^{11} = ... w^2$

$\therefore (a_0 + a_3 + a_6 + ...) + w(a_1 + a_4 + a_7 + ...) + w^2(a_2 + a_5 + a_8 + ...) = 0$

$\Rightarrow a_0 + a_3 + a_6 + ... = 0,$

$a_1 + a_4 + a_7 + \ldots = 0$ $\quad [\because w \neq 0, w^2 \neq 0]$
$a_2 + a_5 + a_8 + \ldots = 0$

17. Show that the cube root of unity lie on the unit circle and divide the circumference into three equal parts from z = 1.

Soln.: Let $z = 1 = r(\cos\theta + i\sin\theta)$

$\therefore$ $r\cos\theta = 0$

or, squaring and adding we get, r = 1

$\tan\theta = 0 = 0^0$

$\therefore$ $z = 1 = 1(\cos 0^0 + i\sin 0^0)$

Using the formula

$$r^{1/n}\left[\frac{\cos 2k\pi + \theta}{n} + i\,\frac{\sin 2k\pi + \theta}{n}\right]$$

we have $1^{1/3}\left[\dfrac{\cos 2k\pi + 0^0}{3} + \dfrac{i\sin 2k\pi + 0^0}{3}\right]$

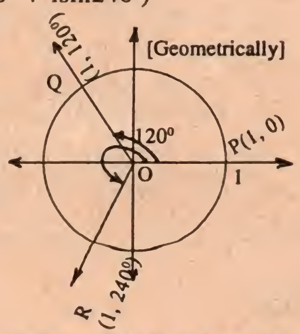
Putting k = 0, 1, 2

at $k = 0 = \alpha = 1\ (\cos 0^0 + i\sin 0^0)$

at $k = 1 = \beta = 1\ (\cos 120^0 + i\sin 120^0)$

at $k = 2 = \delta = 1(\cos 240^0 + i\sin 240^0)$

Taking 'O' as centre and drawing a circle, with radius = 1 unit. Drawing the angles 0^0, 120^0 and 240^0 we see that the cube root of unit circle and divide it in three equal parts.



18. If z_1, z_2, z_3 are vertices of an equilateral triangle. Show that $z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$.

Soln.: Let ABC be an equilateral triangle

i.e. $AB^2 = BC^2 = CA^2$

$\therefore$ $|z_1 - z_2|^2 = |z_2 - z_3|^2$

$= |z_3 - z_1|^2$

$\Rightarrow (z_1 - z_2)(\overline{z_1 - z_2})$

$= (z_2 - z_3)$

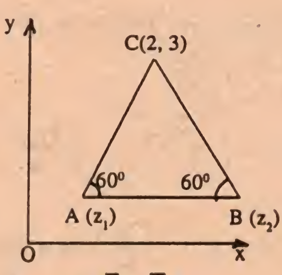
$(\overline{z_2 - z_3})$

$= (z_3 - z_1)(\overline{z_3 - z_1})$

$\Rightarrow (z_1 - z_2)(\bar{z}_1 - \bar{z}_2)$

$= (z_2 - z_3)\ (\bar{z}_2 - \bar{z}_3)$

$= (z_3 - z_1)(\bar{z}_3 - \bar{z}_1)$



Now $(z_1 - z_2)(\bar{z}_1 - \bar{z}_2) = (z_2 - z_3)(\bar{z}_2 - \bar{z}_3)$

$$\Rightarrow \frac{z_1 - z_2}{\bar{z}_2 - \bar{z}_3} = \frac{z_2 - z_3}{\bar{z}_1 - \bar{z}_2}$$

and $(z_2 - z_3)(\bar{z}_2 - \bar{z}_3) = (z_3 - z_1)(\bar{z}_3 - \bar{z}_1)$

$$\Rightarrow \frac{z_2 - z_3}{\bar{z}_3 - \bar{z}_1} = \frac{z_3 - z_1}{\bar{z}_2 - \bar{z}_3} \qquad \ldots\ldots \text{(i)}$$

$$\text{Again } \frac{z_1 - z_2}{\bar{z}_2 - \bar{z}_3} = \frac{z_2 - z_3}{\bar{z}_1 - \bar{z}_2} = \frac{z_1 - z_2 + z_2 - z_3}{\bar{z}_2 - \bar{z}_3 + \bar{z}_1 - \bar{z}_2}$$

$$= \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3}$$

$$\text{or } \frac{z_1 - z_2}{\bar{z}_2 - \bar{z}_3} = \frac{z_1 - z_3}{\bar{z}_1 - \bar{z}_3} \qquad \ldots\ldots \text{(ii)}$$

From (i) and (ii) we get $(z_1 - z_2)(z_2 - z_3) = (z_1 - z_3)^2$

$\Rightarrow z_1z_2 - z_2^2 - z_1z_3 + z_2z_3 = z_1^2 + z_3^2 - 2z_1z_3$

$\Rightarrow z_1^2 + z_2^2 + z_3^2 = z_1z_2 + z_2z_3 + z_3z_1$

19. If $|z| = 1$, then prove that $\left(\dfrac{z - 1}{z + 1}\right)$ is purely imaginary. What will be your conclusion if z = 1.

Soln.: Let $z = x + iy$ then $|z| = \sqrt{(x^2 + y^2)}$

$\therefore$ $|z| = 1 \Rightarrow (x^2 + y^2) = 1$

$$\text{Now } \left(\frac{z - 1}{z + 1}\right) = \frac{x + iy - 1}{x + iy + 1} = \frac{(x - 1) + iy}{(x + 1) + iy}$$

$$= \frac{[(x - 1) + iy][(x + 1) - iy]}{[(x + 1) + iy][(x + 1) - iy]}$$

$$= \frac{(x^2 + y^2 - 1) + 2iy}{(x + 1)^2 + y}$$

$$= \frac{2iy}{2(1 + x)} = \frac{iy}{1 + x},$$

which is purely imaginary $\quad [\because (x^2 + y^2) = 1]$

Again, $z = 1 \Rightarrow x + iy = 1 = 1 + 0\cdot i$

$\therefore$ $x = 1, y = 0$

$$\therefore \text{ In this case, } \frac{z - 1}{z + 1} = \frac{x + iy - 1}{x + iy + 1}$$

$$= \frac{1 + i \cdot 0 - 1}{1 + i \cdot 0 + 1} = 0$$

which is purely imaginary. So in this case also, $\dfrac{z - 1}{z + 1}$ is a pure imaginary number.

20. Prove that

$$\cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right) = \left(\frac{1 + \sin\alpha + i\cos\alpha}{1 + \sin\alpha - i\cos\alpha}\right)^n$$

Soln.: $\left(\dfrac{1 + \sin\alpha + i\cos\alpha}{1 + \sin\alpha - i\cos\alpha}\right)^n$

$$= \left[\frac{1 + \cos(\pi/2 - \alpha) + i\sin(\pi/2 - \alpha)}{1 + \cos(\pi/2 - \alpha) - i\sin(\pi/2 - \alpha)}\right]^n$$

$$= \left[\frac{2\cos^2(\pi/4 - \alpha/2) + 2i\sin(\pi/4 - \alpha/2)\cos(\pi/4 - \alpha/2)}{2\cos^2(\pi/4 - \alpha/2) - 2i\sin(\pi/4 - \alpha/2)\cos(\pi/4 - \alpha/2)}\right]^n$$

$$= \left[\frac{\cos(\pi/4 - \alpha/2) + i\sin(\pi/4 - \alpha/2)}{\cos(\pi/4 - \alpha/2) - i\sin(\pi/4 - \alpha/2)}\right]^n$$

$$= [\cos(\pi/2 - \alpha) + i\sin(\pi/2 - \alpha)]^n$$

$$\left[\because \frac{\cos x + i\sin x}{\cos x + i\sin x} = (\cos x + i\sin x)^2 = \cos 2x + i\sin 2x\right]$$

$$= \cos\left(\frac{n\pi}{2} - n\alpha\right) + i\sin\left(\frac{n\pi}{2} - n\alpha\right) = \text{L.H.S.}$$

21. For any two real numbers m and p, show that

$$e^{2mi\cot^{-1}p}\left(\frac{pi + 1}{pi - 1}\right)^m = 1.$$

Soln.: $e^{2i\cot^{-1}p} = e^{i2\tan^{-1}1/p}$

$$= \cos(2\tan^{-1}1/p) + i \cdot \sin(2\tan^{-1}1/p)$$

$$= \cos\left(\cos^{-1}\frac{1 - (1/p^2)}{1 + (1/p^2)}\right) + i \cdot \sin\left(\sin^{-1}\frac{2/p}{1 + (1/p^2)}\right)$$

$$= \frac{p^2 - 1}{p^2 + 1} + i \cdot \frac{2p}{p^2 + 1}$$

$$= \frac{p^2 - 1 + 2ip}{p^2 + 1} = \frac{(p + i)^2}{(p + i)(p - i)}$$

$$= \frac{p + i}{p - i} = \frac{pi + i^2}{pi - i^2} = \frac{pi - 1}{pi + 1}$$

$$\therefore e^{2i\cot^{-1}p} = \frac{pi - 1}{pi + 1}$$

or, $e^{2i\cot^{-1}p} \cdot \dfrac{pi + 1}{pi - 1} = 1$

Raising both sides to the power m,

$$e^{2mi\cot^{-1}p} \cdot \left[\frac{pi + 1}{pi - 1}\right]^m = 1^m = 1$$

__Alternate Solution__

We have $\dfrac{pi + 1}{pi - 1} = \dfrac{p + 1/i}{p - 1/i} = \dfrac{p - i}{p + i}$

Now put $p + i = r(\cos\theta + i\sin\theta)$

Then $p = r\cos\theta$, $1 = r\sin\theta$

$\therefore \cot\theta = p$, or $\theta = \cot^{-1}p$

We have $p + i = r(\cos\theta + i\sin\theta) = re^{i\theta}$

$\therefore p - i = r(\cos\theta - i\sin\theta) = re^{-i\theta}$

$\therefore$ By substitution $\dfrac{pi + 1}{pi - 1} = \dfrac{re^{-i\theta}}{re^{i\theta}} = \dfrac{1}{e^{2i\theta}}$

$$\therefore e^{2i\theta} \cdot \left\{\frac{pi + 1}{pi - 1}\right\} = 1$$

Raising both sides to the power m

$$e^{2im\theta} \cdot \left\{\frac{pi + 1}{pi - 1}\right\}^m = 1$$

or, $e^{2mi\cot^{-1}p}\left\{\dfrac{pi + 1}{pi - 1}\right\}^m = 1$

22. Show that the complex number a + ib whose modulus is equal to unit can be represented as

$$a + ib = \frac{c + i}{c - i},$$

where c is a real number ($b \neq 0$).

Soln.:

$\because |a + ib| = 1$, we can take

$|a + ib| = \cos\theta + i\sin\theta$

$$= \frac{1 - \tan^2\theta/2}{1 + \tan^2\theta/2} + i \cdot \frac{2\tan\theta/2}{1 + \tan^2\theta/2}$$

$$= \frac{1 - \tan^2\theta/2 + 2i\tan\theta/2}{1 + \tan^2\theta/2}$$

$$= \frac{(1 + i\tan^2\theta/2)^2}{(1 + i\tan\theta/2)(1 - i\tan\theta/2)}$$

$$= \frac{(1 + i\tan\theta/2)}{1 - i\tan\theta/2} = \frac{\cot\theta/2 + i}{\cot\theta/2 - i}$$

$= \dfrac{c + i}{c - i}$ where $c = \cot\theta/2$ which is a real number.

__Aliter__

Let a + ib be represented as $\dfrac{c + i}{c - i}$ .... (1)

so that $a + ib = \dfrac{c^2 - 1 + 2ic}{c^2 + 1}$

Then equating the real parts $a = \dfrac{c^2 - 1}{c^2 + 1}$

Applying componendo and dividendo, we have

$$\frac{1 + a}{1 - a} = \frac{ec^2}{2}$$

$$\therefore c = \sqrt{\frac{1 + a}{1 - a}} = \frac{1 + a}{\sqrt{1 - a^2}} = \frac{1 + a}{b}$$

$\because (a^2 + b^2 = 1)$

Since $b \neq 0$, $\dfrac{1 + a}{b}$ is a real number.

Substituting this value of c in (1)

$$a + ib = \frac{\frac{1+a}{b} + i}{\frac{1+a}{b} - i}$$

Thus $a + ib = \frac{c+i}{c-i}$ where $c = \frac{1+a}{b}$

23. Simplify the expression

$$\tan\left[i\log\frac{a-ib}{a+ib}\right]$$

Soln.: Put $\log(a + ib) = s + it$

Then $a + ib = e^{s+it} = e^s \cdot e^{it}$

$= e^s(\cos t + i\sin t)$

$\therefore e^s = |a + ib| = \sqrt{a^2 + b^2}$

and $t = \arg(a + ib) = \tan^{-1}b/a$

$\therefore \log[a + ib] = s + it$

$= \log\sqrt{a^2 + b^2} + i\tan^{-1}b/a$

$\therefore \log(a - ib) = \log\sqrt{a^2 + b^2} - i\tan^{-1} b/a$

$\therefore \log(a - ib) - \log(a + ib) = -2i\tan^{-1}b/a$

i.e. $\log\left(\frac{a-ib}{a+ib}\right) = -2i\tan^{-1}b/a$

$\therefore i\log\left(\frac{a-ib}{a+ib}\right) = 2\tan^{-1}b/a$

Now taking tangent on both sides

$\tan\left[i\log\frac{a-ib}{a+ib}\right] = \tan[2\tan^{-1}b/a]$

$= \frac{2b/a}{1 - b^2/a^2} = \frac{2ab}{a^2 - b^2}$.

24. Assume that $A_i(i = 1, 2, \ldots n)$ are the vertices of a regular n-gon inscribed in a circle of radius unity.

Find :- $|A_1A_2|^2 + |A_1A_3|^2 + \ldots|A_1A_n|^2$

Soln.: With origin as the centre of the circle of radius unity, Let $z_1, z_2, z_3 \ldots z_n$ represent the vertices $A_1, A_2, \ldots A_n$ of the n-gon. Then $z_2 = z_1e^{2\pi i/n}$. Since $\angle A_2OA_1 = 2\pi/n$.

Similarly if $z_3, z_4, \ldots z_n$ are the other vertices in order, then $z_3 = z_1e^{4\pi i/n}$, $z_4 = z_1e^{6\pi i/n}$ etc.

Now $|A_1A_r|^2 = |z_1 - z_r|^2$

$= |z_1 - z_1e^{2(r-1)\pi i/n}|^2$

$= |z_1|^2\, |1 - e^{2(r-1)\pi i/n}|^2$

$= 1 \cdot \left|1 - \cos 2\frac{(r-1)\pi}{n} - i \cdot \sin 2\frac{(r-1)\pi}{n}\right|^2$

$\because |z_1| = 1 =$ radius of the unit circle

$= \left(1 - \cos 2\frac{(r-1)\pi}{n}\right)^2 + \sin 2\frac{2 \cdot (r-1)\pi}{n}$

$= 2 - \frac{2\cos 2(r-1)\pi}{n}$

Hence $\sum_{r=2}^{n}|A_1 A_r|^2 = 2(n - 1) - 2\sum_{r=2}^{n}\left[\cos\frac{2 \cdot (r-1)\pi}{n}\right]$

Let $s = \sum_{r=2}^{n}\cos\frac{2(r-1)\pi}{n}$

$= \cos\frac{2\pi}{n} + \cos\frac{4\pi}{n} + \cos\frac{6\pi}{n} + \ldots\cos\frac{2(n-1)\pi}{n}$

Then $2\sin\frac{\pi}{n} \cdot s = 2\sin\frac{\pi}{n} \cdot \cos\frac{2\pi}{n} + 2\sin\frac{\pi}{n} \cdot \cos\frac{4\pi}{n}$

$+ \ldots\ldots 2\sin\frac{\pi}{n} \cdot \cos\frac{2(n-1)\pi}{n}$

$= \left(\sin\frac{3\pi}{n} - \sin\frac{\pi}{n}\right) + \left(\sin\frac{5\pi}{n} - \sin\frac{3\pi}{n}\right) + \ldots\ldots$

$+ \left(\sin\frac{2n-1}{n}\pi - \sin\frac{2n-3}{n}\pi\right)$

$= \sin\frac{2n-1}{n}\pi - \sin\frac{\pi}{n} = 2\cos\pi\,\sin\left(\pi - \frac{\pi}{n}\right)$

$= -2\sin\frac{\pi}{n}$ $\qquad \therefore s = -1$

Hence $\sum_{r=2}^{n}|A_1A_r|^2 = 2(n - 1) - 2(-1) = 2n.$

25. If $(1 + x)^n = P_0 + P_1x + P_2x^2 + P_3x^3 + \ldots$ Find the value of $P_0 + P_3 + P_6 + P_9 + \ldots\ldots$

Soln.

$(1 + x)^n = P_0 + P_1x + P_2x^2 + P_3x^3$

$(1 + wx)^n = P_0 + P_1wx + P_2w^2x^2 + P_3x^3$

$(1 + w^2x)^n = P_0 + P_1w^2x + P_2x^2 \cdot w^4 + P_3w^6x^3$

Adding we have

$(1 + x)^n + (1 + wx)^n + (1 + w^2x)^n$

$= 3P_0 + P_1(1 + w + w^2)x + P_2(1 + w + w^2)x^2 + 3P_3x^3$

$= 3(P_0 + P_3x^3 + P_6x^6)$

Putting $x = 1$, we have

$\therefore \frac{1}{3}[2^n + (1 + w)^n + (1 + w^2)^n] = P_0 + P_3 + P_6$

Now $1 + w = -w^2$ and $1 + w^2 = -w$

$\therefore \frac{1}{3}[2^n + (-1)^nw^{2n} + (-1)^nw^n]$

$= \frac{1}{3}[2^n + (-1)^n(w^{2n} + w^n)]$

Now $w = \cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}$

$w^n = \cos\frac{2n\pi}{3} + i\sin\frac{2n\pi}{3}$

$$w^{2n} = \cos\frac{4n\pi}{3} + i\sin\frac{4n\pi}{3}$$

$$w^n + w^{2n} = \left(\cos\frac{2\pi n}{3} + \cos\frac{4\pi n}{3}\right) + i\cdot\left(\sin\frac{2n\pi}{3} + \sin\frac{4n\pi}{3}\right)$$

$$= \cos\frac{2\pi n}{3} + \cos\left(2n\pi - \frac{2n\pi}{3}\right) + i\left(2\sin n\pi \cdot \cos\frac{n\pi}{3}\right)$$

$$= \cos\frac{2\pi n}{3} + \cos\frac{2\pi n}{3} + 0 = 2\cos\frac{2n\pi}{3}$$

$\therefore$ The value is $\frac{1}{3}\left[2^n + (-1)^n 2\cos\frac{2n\pi}{3}\right]$

26. The equation of any circle in the z-plane can be written as $z\bar{z} + az + a\bar{z} + b = 0$ where 'a' is a complex constant and 'b' is a real constant. A, B, C, D are the points z_1, z_2, z_3, z_4 respectively. Show that the points A, B, C, D are concyclic if $z_1z_2 + z_3z_4 = 0$ and $z_1 + z_2 = 0$.

Soln.: The equation of any circle in the z-plane can be written as

$z\bar{z} + az + \bar{a}\bar{z} + b = 0$ where 'a' is a complex constant.

Now let the circle through A, B, C have equation

$z\bar{z} + az + \bar{a}\bar{z} + b = 0$.

Since z_1 and $z_2 = -z_1$ are points on this circle,

$z_1\bar{z}_1 + az_1 + \bar{a}\bar{z}_1 + b = 0$ .... (1)

$z_1\bar{z}_1 - az_1 - \bar{a}\bar{z}_1 + b = 0$ .... (2)

From (1) and (2), $z_1\bar{z}_1 + b = 0$ .... (3)

$az_1 + \bar{a}\bar{z}_1 = 0$ .... (4)

Since z_3 is a point on the circle, we have

$z_3\bar{z}_3 + az_3 + \bar{a}\bar{z}_3 + b = 0$ .... (5)

But $z_3z_4 = -z_1z_2$ (Given)

$= (-z_1)(-z_1) = z_1^2$

$z_3 = \frac{z_1^2}{z_4}$ and $\bar{z}_3\bar{z}_4 = (\bar{z}_1)^2$

$\bar{z}_3 = \frac{(\bar{z}_1)^2}{\bar{z}_4}$

$\therefore$ By substitution (5) becomes

$$\frac{z_1^2}{z_4} \cdot \frac{(\bar{z}_1)^2}{\bar{z}_4} + a\frac{z_1^2}{z_4} + \bar{a} \cdot \frac{(\bar{z}_1)^2}{\bar{z}_4} + b = 0 \quad .... (6)$$

From (3) we have $(z_1\bar{z}_1)^2 = b^2$

From (4) we have $az_1 = -\bar{a}\bar{z}_1$

$\Rightarrow az_1z_1 = -\bar{a} \cdot \bar{z}_1 \cdot z_1 = -\bar{a} \cdot (-b) = \bar{a}b$

similarly $\bar{a}(\bar{z}_1)^2 = ab$.

by substitution, (6) becomes

$$\frac{b^2}{z_4\bar{z}_4} + \frac{\bar{a}b}{z_4} + \frac{ab}{\bar{z}_4} + b = 0$$

Removing b throughout,

$$\frac{b}{z_4\bar{z}_4} + \frac{\bar{a}}{z_4} + \frac{a}{\bar{z}_4} + 1 = 0$$

Multiplying by $z_4\bar{z}_4$, we get

$b + \bar{a}\bar{z}_4 + az_4 + z_4\bar{z}_4 = 0$

i.e $z_4\bar{z}_4 + az_4 + \bar{a}\bar{z}_4 + b = 0$

This shows that z_4 lies on the circle through A, B C. Then A, B, C, D are concyclic.

27. If A and B have affixes z_1 and z_2 then we define the complex slope of the line AB as

$$\mu = \frac{z_1 - z_2}{\bar{z}_1 - \bar{z}_2}$$

Prove that two lines in the Argand plane with complex slopes μ_1 and μ_2 are perpendicular if and only if $\mu_2 + \mu_1 = 0$.

Soln.: If the line joining z_1 and z_2 is perpendicular to the line joining z_3 and z_4.

Then $\arg \frac{z_4 - z_3}{z_2 - z_1} = \frac{\pi}{2}$

Therefore $\frac{z_4 - z_3}{z_2 - z_1}$ is imaginary

$$\therefore \frac{z_4 - z_3}{z_2 - z_1} + \frac{\bar{z}_4 - \bar{z}_3}{\bar{z}_2 - \bar{z}_1} = 0$$

$$\frac{z_4 - z_3}{\bar{z}_4 - \bar{z}_3} + \frac{z_2 - z_1}{\bar{z}_2 - \bar{z}_1} = 0.$$

28. Let z_1, z_2 be any two complex numbers and a, b real numbers $(a^2 + b^2 \neq 0)$.

Prove the inequalities

$$|z_1|^2 + |z_2|^2 - |z_1^2 + z_2^2| \leq \frac{2|az_1 + bz_2|^2}{a^2 + b^2} \leq |z_1|^2 + |z_2|^2 + |z_1^2 + z_2^2|$$

Soln.:

Introduce an auxillary angle α such that $\tan\alpha = \frac{b}{a}$

Now $\frac{2|az_1 + bz_2|^2}{a^2 + b^2} = 2|z_1\cos\alpha + z_2\sin\alpha|^2$

Let $z_1 = r_1(\cos\theta_1 + i\sin\theta_2)$

$z_2 = r_2(\cos\theta_2 + i\sin\theta_2)$

Then $2|z_1\cos\alpha + z_2\sin\alpha|^2$

$= r_1^2 + r_2^2 + (r_1^2 - r_2^2)\cos2\alpha + 2r_1r_2\cos(\theta_1 - \theta_2)\sin2\alpha$

$= A + B\cos2\alpha + C\sin2\alpha$

where $A = r_1^2 + r_2^2$; $B = r_1^2 - r_2^2$,
$C = 2r_1r_2\cos(\theta_1 - \theta_2)$
The minimum value of $A + B\cos2\alpha + C\sin2\alpha$
is $A - \sqrt{B^2 + C^2}$
and maximum value is $A + \sqrt{B^2 + C^2}$
Hence the result follows

29. Show that all the roots of the equation

$$\left(\frac{1 + ix}{1 - ix}\right)^n = \frac{1 + ia}{1 - ia}, \quad a \in R \text{ are real and distinct.}$$

Soln.:
Let $1 + ia = r\cos\theta + i\cdot\sin\theta = re^{i\theta}$
Then $1 - ia = r(\cos\theta - i\cdot\sin\theta) = re^{-i\theta}$

$$\therefore \frac{1 + ia}{1 - ia} = e^{2i\theta} = \cos2\theta + i\cdot\sin2\theta$$

where $\theta = \tan^{-1}a$

∴ The given equation becomes

$$\left\{\frac{1 + ix}{1 - ix}\right\}^n = \cos2\theta + i\sin2\theta$$

$= \cos(2r\pi + 2\theta) + i\cdot\sin(2r\pi + 2\theta)$ in general form

$$\therefore \frac{1 + ix}{1 - ix} = \{\cos(2r\pi + 2\theta) + i\sin(2r\pi + 2\theta)\}^{1/n}$$

where r is any integer incluing zero

$$= \cos\frac{2r\pi + 2\theta}{n} + i\cdot\sin\frac{2r\pi + 2\theta}{n}$$

$$\therefore ix = \frac{\cos\dfrac{2r\pi + 2\theta}{n} + i\cdot\sin\dfrac{2r\pi + 2\theta}{n} - 1}{\cos\dfrac{2r\pi + 2\theta}{n} + i\cdot\sin\dfrac{2r\pi + 2\theta}{n} + 1}$$

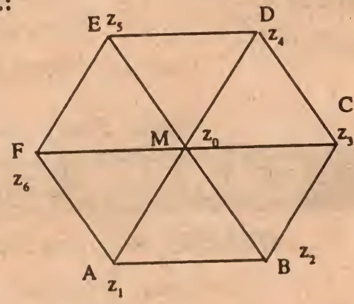
$$= \frac{2i\cdot\sin\left(\frac{r\pi + \theta}{n}\right)\cos\left(\frac{r\pi + \theta}{n}\right) - 2\sin^2\left(\frac{r\pi + \theta}{n}\right)}{2\cos^2\left(\frac{r\pi + \theta}{n}\right) + 2i\cdot\sin\left(\frac{r\pi + \theta}{n}\right)\cdot\cos\left(\frac{r\pi + \theta}{n}\right)}$$

$$\therefore x = \frac{2\sin\left(\frac{r\pi + \theta}{n}\right)\cos\left(\frac{r\pi + \theta}{n}\right) + 2i\cdot\sin^2\left(\frac{r\pi + \theta}{n}\right)}{2\cos^2\left(\frac{r\pi + \theta}{n}\right) + 2i\cdot\sin\left(\frac{r\pi + \theta}{n}\right)\cdot\cos\left(\frac{r\pi + \theta}{n}\right)}$$

$$= \tan\left(\frac{r\pi + \theta}{n}\right). \text{ where } r_i = 0, 1, 2, \ldots\ n - 1$$

These n values of x are real and distinct. Thus all the roots of the given equation are real and distinct.

30. If the complex numbers $z_1, z_2, z_3, z_4, z_5, z_6$ represent the vertices of a regular hexagon in a complex plane and z_0 represents the centre of the hexagon
Prove that $z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2 = 6z_0^2$
Soln.:



Since ACE is an equilateral triangle, we have
$z_1^2 + z_3^2 + z_5^2 = z_1z_3 + z_3z_5 + z_5z_1$
(Property of equilateral triangles)
Since BDF is an equilateral triangle, we have
$z_2^2 + z_4^2 + z_6^2 = z_2z_4 + z_4z_6 + z_6z_2$
By addition

$$z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2 = z_1z_3 + z_3z_5 + z_5z_1 + z_2z_4 + z_4z_6 + z_6z_2 = x$$

i.e $z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2 = x$
and $z_1z_3 + z_3z_5 + z_5z_1 + z_2z_4 + z_4z_6 + z_6z_2 = x$
From fig, we have

$$z_0 + z_1 = z_2 + z_6$$
$$z_0 + z_2 = z_1 + z_3$$
$$z_0 + z_3 = z_2 + z_4$$
$$z_0 + z_4 = z_3 + z_5$$
$$z_0 + z_5 = z_4 + z_6$$
$$z_0 + z_6 = z_5 + z_1$$

Squaring each individual equality and adding

$$6z_0^2 + z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2 + 2z_0(z_1 + z_2 + z_3 + z_4 + z_5 + z_6)$$
$$= 2(z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2) + 2(z_2z_6 + z_1z_3 + z_2z_4 + z_3z_5 + z_4z_6 + z_5z_1)$$

$6z_0^2 + x + 2z_0(6z_0) = 2x + 2x = 4x$
$18z_0^2 = 3x$
$x = 6z_0^2$
i.e. $z_1^2 + z_2^2 + z_3^2 + z_4^2 + z_5^2 + z_6^2 = 6z_0^2$.

INTERNATIONAL CONTEST PROBLEMS

1. *Let $f(x) = x^n + 5x^{n-1} + 3$, where $n > 1$ is an integer. Prove that $f(x)$ cannot be expressed as a product of two polynomials, each of which has all its coefficients integers and degree at least 1.*

Soln.: We first prove a generalization of Einstein's criterion for irreducibility:

Theorem : Let $f(x) = a_0 + a_1x + a_2x^2 + ... + a_nx^n$ be a polynomial with $a_i \in Z$ for $i = 1, 2...n$. Suppose that

(a) $\gcd(a_0, a_1, ..., a_n) = 1$

(b) There exists an integer $k > 0$ such that f has no factors of degree k or less

(c) There exists a prime p such that p divides $a_0, a_1, ..., a_{n-k-1}$ and p^2 does not divide a_0

Then f is irreducible over Z

Proof : If $k \geq (n/2)$, f is clearly irreducible. We can assume that $k < (n/2)$. Suppose that $f(x) = (b_0 + b_1x + ... + b_mx^m)(c_0 + c_1x + ... + c_lx^l)$ where $m + l = n$, $l, m > k$. Note that $m = n - l < n - k$, $l = n - m < n - k$. Comparing coefficients we get

$$a_0 = b_0c_0$$
$$a_1 = b_0c_1 + b_1c_0$$
$$a_2 = b_0c_2 + b_1c_1 + b_2c_0$$
$$......$$
$$a_{n-k-1} = b_0c_{n-k-1} + b_1c_{n-k-2} + ... + b_{n-k-1}c_0$$

Since p divides a_0 either b_0 is divisible by p or c_0 is divisible by p. Also p^2 does not divide a_0. Hence p does not divide both c_0 and b_0. For definiteness let us assume that p divides b_0. From the above relations between coefficients it follows that p divides $b_1, b_2, ... b_{n-k-1}$. Since $m < n - k$ it follows that p divides b_i for $i = 0, 1, ..., b_m \Rightarrow$ p divides a_i for all i, a contradiction. Thus f is irreducible.

Let $f(x) = x^n + 5x^{n-1} + 3$. We apply the theorem with $p = 3$, $k = 1$. It is enough to show that f nas no linear factors. This is equivalent to showing that $f(x) = 0$ has no rational roots. Suppose that r/s where $r, s \in Z$, $\gcd(r, s) = 1$ is a root of $f(x) = 0$. We have $r^n + 5r^{n-1}s + 3s^n = 0 \Rightarrow$ r divides 3 and $s = \pm 1$. Thus the only possible rational roots are ± 3. Clearly neither 3 nor -3 satisfies $f(x) = 0$. Thus f has no rational roots. This completes the proof.

2. *Let D be a point inside the acute-angled triangle ABC such that $\angle ADB = \angle ACB + 90^0$ and $AC.BD = AD.BC$.*

(a) Calculate the value of the ratio $\dfrac{AB.CD}{AC.BD}$

(b) Prove that the tangents at C to the circumcircles of triangles ACD and BCD are perpendicular.

Soln.: At D draw a perpendicular to BD and let E be the point on this perpendicular such that DE = BD. Join CD, BE and AE.

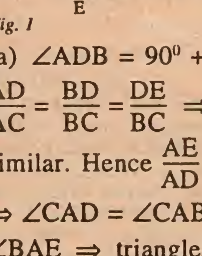


Fig. 1

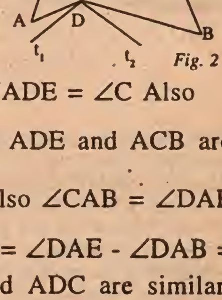


Fig. 2

(a) $\angle ADB = 90^0 + \angle C \Rightarrow \angle ADE = \angle C$ Also $\dfrac{AD}{AC} = \dfrac{BD}{BC} = \dfrac{DE}{BC} \Rightarrow$ triangles ADE and ACB are similar. Hence $\dfrac{AE}{AD} = \dfrac{AB}{AC}$. Also $\angle CAB = \angle DAE \Rightarrow \angle CAD = \angle CAB - \angle DAB = \angle DAE - \angle DAB = \angle BAE \Rightarrow$ triangles AEB and ADC are similar. Hence $\dfrac{AC}{AB} = \dfrac{CD}{BE} = \dfrac{CD}{\sqrt{2}BD} \Rightarrow \dfrac{AB \cdot CD}{AC \cdot BD} = \sqrt{2}$

(b) Since the circumcircles of the triangles ACD and BCD meet at C and D, the angles between the tangents at C to the circles is equal to the angle between the tangents at D to the circles. Let t_1, t_2 be the tangents at D to the circles ACD, BCD respectively. We have $\angle ACB = \angle(AD, t_1)$ and $\angle BCD = \angle(BD, t_2)$

$$\begin{aligned}
\angle ACB + 90^0 &= \angle ADB \\
&= \angle(AD, t_1) + \angle(t_1, t_2) + \angle(t_2, BD) \\
&= \angle ACD + \angle(t_1, t_2) + \angle DCB \\
&= \angle ACB + \angle(t_1, t_2)
\end{aligned}$$

$\Rightarrow \angle(t_1, t_2) = 90^0$.

3. *On an infinite chessboard, a game is played as follows: At the start, n^2 pieces are arranged on the chessboard in an $n \times n$ block of adjoining squares, one piece in each square. A move in the game is a jump in a horizontal or vertical direction over an adjacent occupied square to an unoccupied square immediately beyond. The piece which has been jumped over is then removed.*

Find those values of n for which the game can end with only one piece remaining on the board.

Soln.: We prove that the game ends with only one piece on the board if and only if n is not a multiple of 3.

Four pieces in an 'L' shape can be reduced to one piece as in Fig. 1 below:

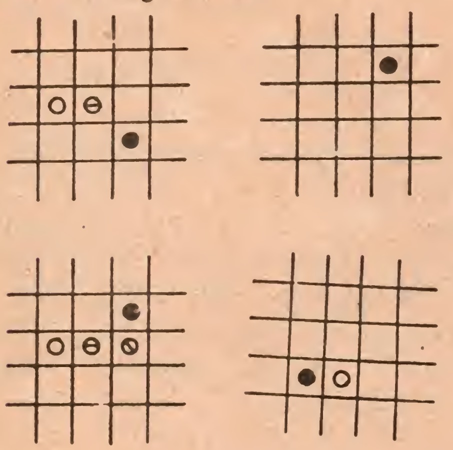


Fig. 1

Using the above repeatedly, we can reduce a $n \times n$ square to $(n - 3) \times (n - 3)$ as in Fig. 2:

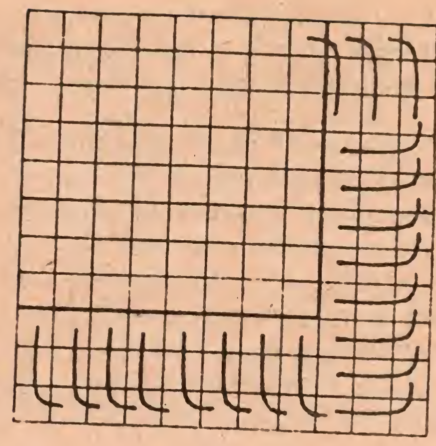


Fig. 2

If n is not a multiple of 3, the above procedure ends with either a 1×1 square or with a 2×2 square. Clearly a 2×2 square can be reduced to one piece. Thus when n is not a multiple of 3, the game ends with only one square on the board.

We will now show that if n is a multiple of 3, the game can not end with only one piece on the board. Name the individual squares with 1, 2, 3 as in Fig. 3:

1	2	3	1	2	3	1	2	3	1
2	3	1	2	3	1	2	3	1	2
3	1	2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2	3	1
2	3	1	2	3	1	2	3	1	2
3	1	2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2	3	1
2	3	1	2	3	1	2	3	1	2
3	1	2	3	1	2	3	1	2	3
1	2	3	1	2	3	1	2	3	1

Fig. 3

Let a_i be the number of pieces in squares labelled i, i = 1, 2, 3. Since n is divisible by 3, we have $a_1 = a_2 = a_3 = a$. Note that each jump over a piece reduces the configuration from (a_1, a_2, a_3) to either $(a_1 + 1, a_2 - 1, a_3 - 1)$ or $(a_1 - 1, a_2 - 1, a_3 + 1)$. Thus for the game to end with one piece, say, on a square labelled 1, there must be positive integers x, y, z such that (a, a, a) - x(1, -1, -1) - y(-1, -1, +1) -

$z(-1, +1, -1) = (1, 0, 0) \Rightarrow x - y - z = a - 1, -x - y + z = a \Rightarrow 2x - 2z = -1$, a contradiction. Thus the game cannot end with one piece on the board.

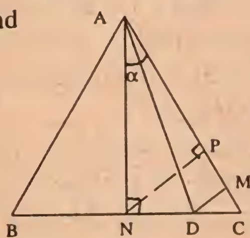
4. *For three points P, Q, R in the plane, we define m(PQR) to be the minimum of the lengths of the altitudes of the triangle PQR (where m(PQR) = 0 if P, Q, R are collinear). Let A, B, C be given points in the plane. Prove that, for any point X in the plane, m(ABC) ≤ m(ABX) + m(AXC) + m(XBC).*

Soln: We first prove:

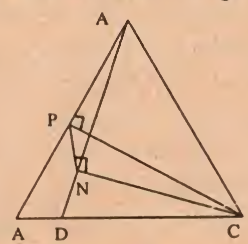
Lemma 1 : Let D be any point on BC. $m(ADC) \leq m(ABC)$

Proof : We distinguish two cases:

Case 1 $\angle ADC \geq 90^0$. Let AN be the altitude through A and



Case : 1



Case : 2

$NP \perp AC$. Let $DM \perp AC$. Clearly, $m(ADC) = DM$. If $m(ABC) = h_b$, the altitude through B, then $\frac{DM}{h_b} = \frac{CD}{BC} < 1 \Rightarrow m(ADC) \leq m(ABC)$. If $m(ABC) = h_a$, the altitude through A, then $h_a > NP \geq DM \Rightarrow m(ADC) \leq m(ABC)$. Finally, let $m(ABC) = h_c$. Let $\angle NAC = \alpha$. AB is the largest side of ABC and $\angle C > 60^0$ and $\angle A < 90^0$. Also since $\angle B < \angle C$, we have $BN \geq NC$ and $\angle A \geq 2\alpha \Rightarrow h_c \geq AC \sin 2\alpha$.

Also

$DM \leq NP = AN\sin\alpha$

$= AC\sin\alpha\cos\alpha = AC \frac{\sin 2\alpha}{2} < h_c$

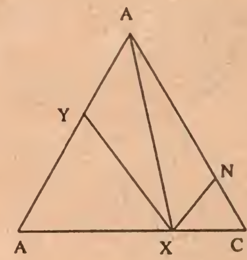
Then $m(ADC) \leq m(ABC)$.

Case 2 : $\angle ADC < 90^0$. Let $DM \perp AC$, $CN \parallel AD$. If $m(ABC) = h_a$, since h_a is an altitude of triangle ADC, it trivially follows that $m(ADC) \leq m(ABC)$. If $m(ABC) = h_c$, $\angle A < 90^0$. If CP is the altitude of ABC through C then APNC is cyclic and $\angle PNC$ is obtuse. Thus $PC = h_c > CN \geq m(ADC)$. If $m(ABC) = h_b$, then

$\frac{h_b}{DM} = \frac{BC}{CD} > 1 \Rightarrow m(ABC) = h_b \geq DM \geq m(ADC)$

Lemma 2 Let X be any point on BC . $m(ABC) \leq m(AXB) + m(AXC)$

Proof : We can assume that $\angle AXC \geq 90^0$. Let $XN \perp AC$ and $XY \parallel AC$.



$m(AXB \geq m(BXY)$

$= m(ABC) \left(\frac{h_b - XN}{h_b}\right)$

$= m(ABC) - m(ABC) \frac{XN}{h_b}$

$\geq m(ABC) - XN$

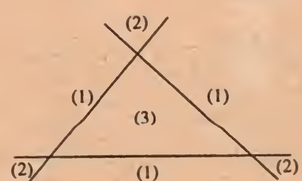
$= m(ABC) - m(AXC)$

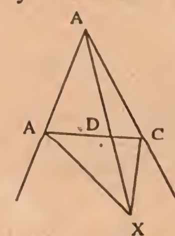
Extending the sides of the triangle ABC, the plane can be divided into seven regions. Label the regions as follows.

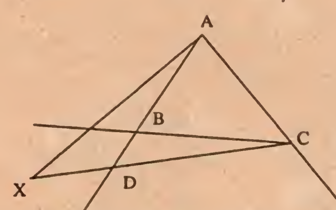
If the point X is in region 1, by Lemma 1 above, we have

$m(ABX) + m(ACX) \geq m(ABD) + m(ACD)$

$\geq m(ABC)$ by Lemma 2







If X lies in region 2 then $m(AXC) \geq m(ADC) \geq m(ABC)$ by Lemma 1.

If X lies in region 3, $\angle AXC$, $\angle BXC$, $\angle CXA$ are all obtuse. Hence $m(AXB) = 2S(AXB)/AB$, $m(BXC) = 2S(BXC)/BC$, $m(CXA) = 2S(CXA)/CA$ where S(AXB) is the area of the triangle AXB etc. If AB is the largest side of ABC we have

$m(ABC) = \frac{2S(ABC)}{AB}$

$= \frac{2S(AXB)}{AB} + \frac{2S(BXC)}{BC} + \frac{2S(CXA)}{CA}$

$\leq m(AXB) + m(BXC) + m(CXA)$

5. *Let $N = \{1, 2, 3, \dots\}$. Determine whether or not there exists a function $f: N \to N$ such that*
$f(1) = 2$
$f(f(n)) = f(n) + n$, for all $n \in N$
and $f(n) < f(n + 1)$ for all $n \in N$.

Soln. : Let $a = \frac{1+\sqrt{5}}{2}$. We have $a^2 = a + 1$. Put $g(n) = an$, for $n \in N^* = \{1, 2, 3, \dots\}$. $g(g(n)) = ag(n) = a^2 n = an + n = g(n) + n$. Define $f(n) = [g(n) + \frac{1}{2}]$ where $[x]$ denotes the largest integer $\leq x$. We show that f has the desired properties.

(1) $f(1) = \left[g(1) + \frac{1}{2}\right] = \left[a + \frac{1}{2}\right] = \left[\frac{1+\sqrt{5}}{2} + \frac{1}{2}\right]$
$= 2$

(2) $g(n + 1) = a(n + 1) > an + 1 = g(n) + 1$
$\Rightarrow g(n + 1) + \frac{1}{2} > g(n) + \frac{1}{2} + 1$
$\Rightarrow f(n + 1) \geq f(n) + 1$

(3) $|f(n) - g(n)| < \frac{1}{2}$ for all $n \geq 1$.

Since $g(g(n)) = g(n) + n$, we have

$$
\begin{aligned}
|f(f(n)) - f(n) - n| &= |g(g(n)) - f(f(n)) + f(n) - g(n)| \\
&= |g(g(n)) - g(f(n)) + g(f(n)) - f(f(n)) + f(n) - g(n)| \\
&= |ag(n) - af(n) + g(f(n)) - f(f(n)) + f(n) - g(n)| \\
&= |(a - 1)(g(n) - f(n)) + g(f(n)) - f(f(n))| \\
&\leq (a - 1)|g(n) - f(n)| + |g(f(n)) - f(f(n))| \\
&\leq \frac{1}{2}(a - 1) + \frac{1}{2} = \frac{a}{2} < 1
\end{aligned}
$$

$\Rightarrow |f(f(n)) - f(n) - n|$ is a positive integer less than 1
$\Rightarrow f(f(n)) = f(n) + n$, for all $n \geq 1$. This completes the proof.

6. *Let $n > 1$ be an integer. There are n lamps $L_0, L_1, \dots, L_{n-1}$ arranged in a circle. Each lamp is either ON or OFF. A sequence of steps $S_0, S_1, \dots, S_i, \dots$ is carried out. Step S_j affects the state of L_j only (leaving the state of all other lamps unaltered) as follows:*
If L_{j-1} is ON, S_j changes the state of L_j from ON to OFF or from OFF to ON;
if L_{j-1} is OFF, S_j leaves the state of L_j unchanged.
The lamps are labelled mod n, that is,
$L_{-1} = L_{n-1}$, $L_0 = L_n$, $L_1 = L_{n+1}$ etc.
Initially all lamps are ON. Show that
(a) there is a positive integer $M(n)$ such that after $M(n)$ steps all the lamps are ON again;
(b) if n has the form 2^k then all the lamps are On after $n^2 - 1$ steps;
(c) if n has the form $2^k + 1$ then all the lamps are ON after $n^2 - n + 1$ steps.

Soln.: Let t_i denote the status of the i - th lamp. We put $t_i = 0$ if L_i is OFF and $t_i = 1$ if L_i is ON. On the set $\{(t_0, t_1, \dots, t_{n-1}) : t_i \in Z_2\}$, define
$R(t_0, t_1, \dots, t_{n-1}) = (t_1, t_2, \dots, t_{n-1}, t_0)$
R is a rotation of the lamps in the anticlockwise direction. The S_j changes the status of the lamp L_j. Clearly
$S_j(t_0, t_1, \dots, t_{n-1}) = (t_0, t_1, \dots, t_{j'-1}, t_{j'-1} + t_{j'}, \dots, t_{n-1})$
where $j' = j(\text{mod } n)$. It is easy to verify that

$S_j = R^{-j}S_0R^j$, where $R^j = \overbrace{R \cdot R \cdot \dots \cdot R}^{j \text{ times}}$

Put $S = RS_0$. Then $S(t_0, \dots, S_{n-1}) = (t_1, t_2, \dots, t_{n-1} + t_0)$. Since R is a rotation of the lamps, it is enough to show that there exists an integer $M(n)$ such that $S^{M(n)}$ is the identity map.

Consider the polynomial ring $Z_2[X]$. For $P, Q \in Z_2[X]$, we write $P \equiv Q$ if $P - Q$ is divisible by the polynomial $X^n + X^{n-1} + 1$. For $t = (t_0, t_1, \dots, t_{n-1})$, define
$P_t(X) = t_{n-2} + t_{n-3}X + \dots + t_0X^{n-2} + t_{n-1}X^{n-1}$
It is easy to see that
$P_{s(t)}(X) = t_{n-1} + t_{n-2}X + \dots + t_1X^{n-2} + (t_{n-1} + t_0)X^{n-1}$

$$
\begin{aligned}
XP_t(X) - P_{S(t)}(X) &= t_0X^{n-1} + t_{n-1}X^n - t_{n-1} \\
&\quad - (t_{n-1} + t_0)X^{n-1} \\
&\equiv t_{n-1}X^n + t_{n-1}X^{n-1} + t_{n-1} \equiv 0
\end{aligned}
$$

Thus $P_{S(t)}(X)$ can be identified with $XP_t(X)$ and hence $P_{S^n(t)}(X) \equiv X^n P_t(X)$. Clearly the number of polynomials of degree atmost n and with coefficients in Z_2 in finite. Thus there exists integers l, m with $l > m$ and $X^l \equiv x^m \Rightarrow X^{l-m} \equiv 1 \Rightarrow P_{S^{l-m}(t)}(X) \equiv X^{l-m}P_t(X) \equiv P_t(X)$. Thus S^{l-m} leaves the status of lamps invariant and if all the lamps were ON in the beginning, they would remain ON after S^{l-m}. This completes the proof of (a).

(b) If $n = 2^k$, we need to show that $X^{n^2-1} \equiv 1$
$X^n \equiv X^{n-1} + 1 \Rightarrow X^{n^2} \equiv (X^{n-1} + 1)^n$

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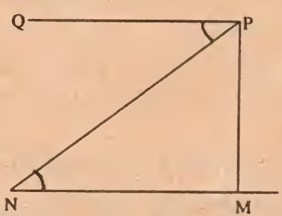
NOTES FILE

Height and Distance

— *Atul Nath Tripathi*

Definition :- One part of trigonometry is to find the distance between points; or the heights of objects without actually measuring these distances or these height is known as 'Height and Distance'.

Angle of Elevation:- The angle MNP is called the 'angle of elevation' of the point P as seen from N.



Angle of Depression :- The angle QPN is known as the 'angle of depression' of the point N as seen from P.

Example 1 :- A person wants to know the height of a tree which stands on a horizontal plane, at a point on this plane he finds the angle of elevation of the top of the tree to be 45^0, on walking 30 metres towards the tree he finds the corresponding angle to be 60^0; deduce the height of the tree and the distance of the person from the base of the tree.

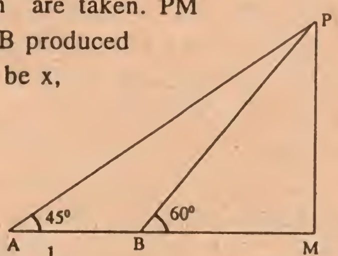
Soln.: Let P be the top of the tree and A and B the two points at which the angles of elevation are taken. PM perpendicular to AB produced (draw) and let MP be x,

Given that:-

AB = 30 metres

$\angle MAP = 45^0$

and $\angle MBP = 60^0$



then; $\frac{AM}{x} = \cot 60^0 = \frac{1}{\sqrt{3}}$

Hence AM = x and BM = $\frac{x}{\sqrt{3}}$

$\therefore 30 = AM - BM = x - \frac{x}{\sqrt{3}} = \frac{x\sqrt{3} - 1}{\sqrt{3}}$

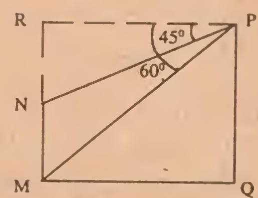
$\therefore x = \frac{30\sqrt{3}}{\sqrt{3} - 1} = \frac{30\sqrt{3}(\sqrt{3} + 1)}{3 - 1} = 15(3 + \sqrt{3})$

$= 15(3 + 1.73205....] = 71$ metres

AM = x; so that both of the required distances are equal to 71 metres.

Example 2 :- From the top of a cliff; 160 metres high, the angles of depression of the top and bottom of a tower are observed 45^0 and 60^0. Find the height of the tower.

Soln.: Let P be the point of observation and QP the height of the cliff; and let NM be the tower. Draw PR horizontally.



So that $\angle RPN = 45^0$ and $\angle RPM = 60^0$

Let x metres be the height of the tower and MN to meet PR at E.

So that NR = QP - x = 160 - x

Since $\angle PMQ = \angle MPR = 60^0$

$\therefore MQ = PQ \cot PMQ = 160 \cot 60^0 = \frac{160}{\sqrt{3}}$

Also $\frac{160 - x}{MQ} = \frac{NR}{RP} = \tan 45^0 = 1$

$\Rightarrow MQ = 160 - x$

$\therefore 160 - x = \frac{160}{\sqrt{3}}$ $\qquad \therefore -x\sqrt{3} = 160 - \sqrt{3} \cdot 160$

$\Rightarrow x\sqrt{3} = 160(\sqrt{3} - 1)$

$\Rightarrow x = \frac{160(\sqrt{3} - 1)}{\sqrt{3}}$

$\Rightarrow x = 67.5$ metres.

Some Geometrical Properties:-

(i) Any perpendicular to a plane is perpendicular to every line lying in the plane.

(ii) In a triangle the internal bisector of the angle

aspect of hotel management, food preparation, service, maintenance, accounting, engineering, interior design, and sales and marketing in the hotel premises.

The Oberoi School of Hotel Management has recently rechristened itself as the Oberoi Centre of Learning and Development. According to Bernard Martyris, vice-president, training and manpower, Oberoi Group of Hotels. "This shift from training to learning is a result of change in focus from entry level executive training to overall learning continuously imparted to all employees of the organisation." Forevery level of promotion, the employee would undergo learning to facilitate his role change and add value to his enhanced responsibility as well as individual career development. According to Martyris, "The focus is going to be on people instead of the product, personalisation instead of standardisation, value-seeking instead of profit-seeking and growth through quality rather than volume."

The Oberoi school offers four key programmes to graduates and hotel management students: The three-year hotel operation and management training programme; the two-year house-keeping executive training programme; the three-year senior kitchen training programme; and the two-year Mercury Travels management training programme, recently included in the curriculum. The school demand for trainees varies between 80 and 100, depending on their manpower requirement. Selection to these various programmes is through campus recruitment. The Oberoi centre does not advertise.

Nirulas, the fast food chain of restaurants with branches all over Delhi, have their own training cell at NOIDA on the outskirts of the capital. They take 25 to 30 trainees every year from hotel management institutes for training ranging from nine to 18 months.

In the government sector, the National Council for Hotel Management and Catering Technology at New Delhi organises a three year diploma programme at 18 centres in the country.

Most training programmes by hotel groups offer stipends ranging from Rs, 4,000 to Rs 7,500 On successful completion of training, students are absorbed in managerial positions for frontline operations with starting salaries ranging between Rs. 5,000 and Rs. 11,000 plus perks. Of course, there is no stopping bright and hardworking young men and women. As Asani says, "In the hotel industry the sky is the limit. Since professionals manage the company, depending on their merit and competence, they can reach the corporate track to the top — to the position of GM and above." The current vice president of Welcomgroup Maurya in Delhi began his career as a management trainee with the same group. ■

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$$\equiv X^{n(n-1)} + 1, \text{ since } \binom{2^k}{i} \text{ is even for } i \neq 0,\ 2^k$$

$$\Rightarrow 1 \equiv X^{n^2} + X^{n^2-n}$$
$$\equiv X^{n^2-n}(X^n + 1)$$
$$\equiv X^{n^2-n}(X^n + 1)$$
$$\equiv X^{n^2-n}X^{n-1} \equiv X^{n^2-1}$$

(c) If $n = 2^k + 1$, we need to show that

$$X^{n^2-n+1} \equiv 1$$

$$X^{n^2-1} \equiv (X^{n+1})^{n-1} \equiv (X \cdot X^n)^{n-1}$$
$$\equiv (X(X^{n-1} + 1))^{n-1} \equiv (X + X^n)^{n-1}$$
$$\equiv X^{n-1} + X^{n^2-n}$$
$$\equiv X^{n^2-n} + X^{n^2-1} \equiv X^{n-1}$$
$$\equiv X^{n^2-n}(X^{n-1} + 1) \equiv X^{n-1}$$
$$\equiv X^{n^2-n} \cdot X^n \equiv X^{n-1}$$
$$\equiv X^{n^2-n+1} \equiv 1$$

CAREER FOR YOU

HOTEL MANAGEMENT

The hospitality industry is on the fast track. By the turn of the century, it hopes to have another 70,000 rooms to meet the target of five million tourists. To keep peace with this expected rate of growth, the hospitality industry will require 70,000 trained personnel by 1997, as against the current base of 31,000.

Little wonder then that this shortage of skilled staff ensures jobs for all hotel management students. Says Lalit Nirula, honorary secretary of the Federation of Hotel and Restaurant Associations of India. "There is not a single person coming out of the 18 catering institutes who goes without a job."

And as supply of trained manpower increases, what is going to count more in this sector is quality service. Says Nakul Anand, vice president and general manager, Welcomgroup Maurya Sheraton Hotel and Towers, "The need for quality hotel personnel is going to increase by four to five times in the next 10 years."

Since hotels are service organisation, those who have initiative and a charming personality, great stamina for hard work and the ability to remain calm under trying situations are the right people for this career. According to Anil Asani, training manager with Welcomgroup Maurya sheraton Hotel and Towers. "Those who want to join hotels should have aptitude for the industry, good inter-personal skills, an outgoing personality and be willing to serve people."

Training : Though the Oberoi group was the first to establish its training school in 1966, all major hotels chains have since introduced in-house training facilities. The Welcomgroup, for instance, has its own Graduate School of Hotel Administration at Manipal, Karnataka. This is, perhaps, the only institute which offers a three-year bachelor's degree course in hotel management. Further, the welcomgroup conducts a two-year in house training programme for hotel management graduates (also for their recruits from Manipal) to train as frontline managers.

Their training institute at Manipal takes 100 trainees per year. Admission announcements are normally advertised in April after the 10 plus two exams. Established in collaboration with ITC-Welcomgroup by the Dr. G.M.A Pai Foundation the institute is affiliated to Mangalore University. Eligibility is at least 50 per cent marks in aggregate in the 10 plus two exam in any discipline.

The Taj Group of Hotels have their three-year training programme at the Indian Institute of Hotel Management, Aurangabad, where students with a minimum of 45 per cent marks in class 12 are eligible to apply.

In addition, since the last two years, the Taj Group have started a two-year graduate management training programme which imparts skills in every

INTERNATIONAL OLYMPIAD PROBLEMS

1. For what values of the variable x does the following inequality hold:

$$\frac{4x^2}{(1 - \sqrt{1 + 2x})^2} < 2x + 9?$$

Soln.: The left side of the given inequality is defined if $x \neq 0$ and real $x \geq -½$. Assume that x satisfies these conditions. We multiply numerator and denominator of the left member by $(1 + \sqrt{1 + 2x})^2$ and obtain the equivalent inequality

$$(1 + \sqrt{1 + 2x})^2 < 2x + 9$$

Putting $y = \sqrt{1 + 2x}$, we have $(1 + y)^2 < y^2 + 8$, which simplifies to $y < 7/2$. Hence $1 + 2x < 49/4$, and $x < 45/8$. The complete solution of the problem thefore consists of all x satisfying

$$-½ \leq x < 45/8, \; x \neq 0.$$

2. Solve the system of equations:

$$x + y + z = a$$
$$x^2 + y^2 + z^2 = b^2$$
$$xy = z^2.$$

where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z (the solutions of the system) are distinct positive numbers.

Soln.: *First solution :* The given equations are

$$x + y + z = a \qquad ...(1)$$
$$x^2 + y^2 + z^2 = b^2 \qquad ...(2)$$
$$xy = z^2 \qquad ...(3)$$

In (2) we substitute xy for z^2 from (3), obtaining

$$x^2 + xy + y^2 = b^2 \qquad ...(4)$$

We rewrite (1) in the form $(x + y) - a = -z$, square it, substitute xy for z^2 from (3) and obtain

$$x^2 + xy + y^2 - 2a(x + y) = -a^2 \qquad ...(5)$$

We subtract (5) from (4) and find that

$$x + y = (a^2 + b^2)/2a; \qquad ...(6)$$

hence $z = a - (x + y) = (a^2 - b^2)/2a$ ...(7)

By (3) and (7) $xy = (a^2 - b^2)^2/4a^2$ ...(8)

We solve the system (6) and (8) by considering x and y as roots of a quadratic equation with coefficients determined by the sum (6) and the product (8) of the roots:

$$w^2 - \frac{a^2 + b^2}{2a} w + \frac{(a^2 - b^2)}{4a^2} = 0 \qquad ...(9)$$

The requirement that x, y, z be positive implies, by (1), that $a > 0$, and by (7), that $a^2 > b^2$. Moreover, the requirement that x and y be real and distinct forces the discriminant of (9),

$$\Delta = \frac{1}{4a^2}(3a^2 - b^2)(3b^2 - a^2),$$

to be the positive. Since $3a^2 - b^2 > 0$, it follows that $3b^2 - a^2 > 0$, so that $3b^2 > a^2$. All these inequalities can be summarized as

$$|b| < a < \sqrt{3}\,|b|,$$

and if they hold, the roots of (9),

$$x = \frac{a^2 + b^2}{4a} \pm \frac{\sqrt{\Delta}}{2}, \qquad y = \frac{a^2 + b^2}{4a} \mp \frac{\sqrt{\Delta}}{2}$$

together with their geometric mean, $z = \sqrt{xy} = \dfrac{a^2 - b^2}{2a}$, yield positive distinct solutions of the problem. (z is distinct from x and y since the geometric mean of two distinct numbers lies between them.)

Our knowledge of the relations between the roots and the coefficinets of a quadratic equation led us to use the sum (6) and the product (8) of x and y to write the quadratic equation (9) satisfied by x and y. The symmetry in x, y, z of the given equations (1) and (2) encourages us to try solving the problem by regarding x, y, z as roots of a cubic, as follows :

Second solution : Let x, y, z be the roots of

$$p^3 + s_1p^2 + s_2p + s_3 = 0 \qquad ...(10)$$

then $x + y + z = -s_1$, $xy + yz + xz = s_2$, $xyz = -s_3$.

Using (1) and (2) and (3), we obtain

$s_1 = -a$, $(x + y + z)^2 = a^2 = b^2 + 2s_2$, $xyz = z^3 = -s_3$, respectively, so that

$s_1 = -a$, $s_2 = ½ (a^2 - b^2)$, $s_3 = -z^3$.

The cubic (10) may therefore be written as

$$p^3 - ap^2 + ½ (a^2 - b^2)p - z^3 = 0 \qquad ...(11)$$

Since z is a root, we find on substituting p = z into (11), that

$$-az^2 + ½ (a^2 - b^2)z = -z[az - ½ (a^2 - b^2)] = 0,$$

so $z = 0$ or $z = (a^2 - b^2)/2a$. If $z = 0$, then the corresponding solution x, y, z would not consist of positive numbers, as required. But if $z = (a^2 - b^2)/2a$, then by (1) $x + y = a - z = (a^2 + b^2)/2a$, and by (3), $xy = (a^2 - b^2)^2/4a^2$. We solve these for x and y by the method given in the first solution.

Note: The necessity of the conditions

$$|b| < a < \sqrt{3}\,|b|$$

for x, y, z postive and distinct may be seen in these two ways:

(i) The distance from the plane $x + y + z = a$ to the origin is $a/\sqrt{3}$, and the radius of the sphere $x^2 + y^2 + z^2 = b^2$ about the origin is $|b|$. These surfaces have points in common if and only if $|b| \ge a/\sqrt{3}$. If equality holds, the sphere is tangent to the plane at a point with equal coordinates, so $0 < a < \sqrt{3}\,|b|$ is necessary for x, y, z distinct and positive.

(ii) To the two positive numbers x and y, apply the arithmetic-geometric mean inequality

$$½ (x + y) \ge \sqrt{xy},$$

using relation (6) and (8) above. The result is $(a^2 + b^2)/4a > a^2b^2/2a$, where the inequality is strict since $x \ne y$. This simplifies to $a^2 < 3b^2$, or since a is positive, to $0 < a < \sqrt{3}\,|b|$.

3. Find all real roots of the equation

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x,$$

where p is real parameter

Soln.: If $p < 0$, we have

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} \ge \sqrt{x^2 - p} > x.$$

So in order for the equation to have a solution, we must have $p \ge 0$. Now write it in the form

$$2\sqrt{x^2 - 1} = x - \sqrt{x^2 - p}$$

and square, obtaining

$$2x^2 + p - 4 = -2x\sqrt{x^2 - p}$$

Squaring again, and solving for x^2, we get

$$x^2 = \frac{(p - 4)^2}{8(2 - p)}.$$

Hence in order for a solution to exist, we must have $0 \le p < 2$, and then the only possible solution is $x = (4 - p)/\sqrt{8(2 - p)}$, we obtain

$$|3p - 4| + 2p = 4 - p; \qquad \text{i.e., } |3p - 4| = -(3p - 4)$$

Clearly this holds if and only if $3p - 4 \le 0$, i.e. $p \le 4/3$. Therefore the equation has solutions only when $0 \le p \le 4/3$, and then $x = (4 - p)/\sqrt{8(2 - p)}$.

4. Determine all values x in the interval $0 \le x \le 2\pi$ which satisfy the inequality

$$2\cos x \le |\sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x}| \le \sqrt{2}$$

Soln.: We observe that the inequality between the middle and right members holds for all x, since the absolute value fo the difference of two positive numbers is at most equal to the larger. Moreover, the inequality between the left and the middle members certainly holds for those x for which $\cos x \le 0$, i.e. for $\pi/2 \le x \le 3\pi/2$. To find all other x for which it holds, we note that if $\cos x \ge 0$, the left inequality is equivalent to the one obtained by squaring both sides, i.e.

$$4\cos^2 x \le 1 + \sin 2x - 2\sqrt{1 - \sin^2 2x} + 1 - \sin 2x$$
$$= 2 - 2|\cos 2x|.$$

Since $\cos^2 x = \dfrac{1 + \cos 2x}{2}$, this is equivalent to

$$2 + 2\cos 2x \le 2 - |\cos 2x|,$$

or finally $|\cos 2x| \le -\cos 2x$

Evidently this holds if and only if $\cos 2x \le 0$, i.e. when x is in either of the intervals $[\pi/4, 3\pi/4]$ or $[4\pi/4, 7\pi/4]$. Combining these intervals with the earlier one $[\pi/2, 3\pi/2]$, we find that the complete solution set is $[\pi/4, 7\pi/4]$.

5. Let a, b, c be the lengths of the sides of a triangle and α, β, γ, repsectively, the angles opposite these sides. Prove that

$$a + b = \tan\frac{\gamma}{2}(a \tan\alpha + b \tan\beta)$$

the triangle is isosceles.

Soln.: In the given equation, we replace tanx by sinx/cosx; then multiply both sides by

$\cos\alpha \cos\beta \cos\frac{\gamma}{2}$. The result is

$$(a+b)\cos\alpha\cos\beta\cos\frac{\gamma}{2} = a\sin\alpha\cos\beta\sin\frac{\gamma}{2} + b\sin\beta\cos\alpha\sin\frac{\gamma}{2},$$

which is equivalent to

$$a\cos\beta\left(\cos\alpha\cos\frac{\gamma}{2} - \sin\alpha\sin\frac{\gamma}{2}\right) + b\cos\alpha\left(\cos\beta\cos\frac{\gamma}{2} - \sin\beta\sin\frac{\gamma}{2}\right) = 0$$

This, in turn, is equivalent to

$$a\cos\beta\cos\left(\alpha + \frac{\gamma}{2}\right) + b\cos\alpha\cos\left(\beta + \frac{\gamma}{2}\right) = 0$$

Since $\alpha + \frac{\gamma}{2} + \beta + \frac{\gamma}{2} = \alpha + \beta + \gamma = \pi$, we have

$$\cos\left(\beta + \frac{\gamma}{2}\right) = -\cos\left(\alpha + \frac{\gamma}{2}\right)$$

and therefore $(a\cos\beta - b\cos\alpha)\cos\left(\alpha + \frac{\gamma}{2}\right) = 0$

If $\cos(\alpha + \gamma/2) = 0$, then $\alpha + \gamma/2 = \pi/2$, so $\beta + \gamma/2 = \pi/2$, and we conclude that $\alpha = \beta$. If $a\cos\beta - b\cos\alpha = 0$, then we use the law of sines $a\sin\beta = b\sin\alpha$ and divide it by $a\cos\beta = b\cos\alpha$ to deduce $\tan\alpha = \tan\beta$. From this it follows that $\alpha = \beta$, and the triangle is isosceles.

6. Let f be a real-valued function defined for all real numbers x such that, for some positive constant a, the equation

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - [f(x)]^2}$$

holds for all x.

(a) Prove that the function f is periodic (i.e, there exists a postive number b such that $f(x + b) = f(x)$ for all x).

(b) For $a = 1$, give an example of a non-constant function with the required properties.

Soln.:

(a) The given equation shows that $f(x + a) \geq ½$, and so, $f(x) \geq ½$ for all x. Hence if we put $g(x) = f(x) - ½$, we have $g(x) \geq 0$ for all x. The given functional equation now becomes

$$g(x+a) = \sqrt{g(x) + ½ - [g(x)]^2 - g(x) - ¼} = \sqrt{¼ - [g(x)]^2}.$$

Squaring we get

$[g(x+a)]^2 = ¼ - [g(x)]^2$ for all x, ...(1)

and hence also $[g(x+2a)]^2 = ¼ - [g(x+a)]^2$

these two imply $[g(x) + 2a)]^2 = [g(x)]^2$. Since $g(x) \geq 0$ for all x, we can take square roots to get $g(x + 2a) = g(x)$, whence $f(x + 2a) - ½ = f(x) - ½$, and $f(x + 2a) = f(x)$ for all x.

This shows that f(x) is periodic with period 2a.

(b) To find all solutions, we set $h(x) = 4[g(x)]^2 - ½$ and writes (1) in the form

$h(x + a) = -h(x)$ ...(2)

Conversely, if $h(x) \geq ½$ and satisfies (2), then g(x) satisfies (1)

An example for $a = 1$ is furnished by the function

$$h(x) = \sin^2\frac{\pi}{2}x - \frac{1}{2}$$

which satisfies (2) with $a = 1$. For this h,

$$g(x) = \frac{1}{2}|\sin(\pi x/2)| \text{ and}$$

$$f(x) = \frac{1}{2}\left|\sin\frac{\pi}{2}x\right| + \frac{1}{2}$$

Acutally h(x) can be defined arbitrarily in $0 \leq x < a$ subject only to the condition $|h(x)| \leq ½$; then (2) defines h for all x.

7. Let a and b be real numbers for which the equation $x^4 + ax^3 + bx^2 + ax + 1 = 0$ has at least one real solution. For all such pairs (a, b), find the minimum value of $a^2 + b^2$.

Soln.: First consider the equation

$x + \frac{1}{x} = y$, for y real ...(1)

This equivalent to $x^2 - yx + 1 = 0$, a quadratic in x which has real roots if and only if its discriminant $y^2 - 4$ is ≥ 0. i.e. if and only if $|y| \geq 2$.

Now to solve $x^4 + ax^3 + bx^2 + ax + 1 = 0$, we divide it by x^2, then set $y = x + (1/x)$ obtaining $y^2 + ay + (b - 2) = 0$. The roots of this quadratic equation are

$$y = x + \frac{1}{x} = \frac{-a \pm \sqrt{a^2 - 4(b-2)}}{2}$$

and the condition that at least one of them be ≥ 2 in absolute value is then that

$$|a| + \sqrt{a^2 - 4(b-2)} \geq 4.$$

This is equivalent to $\sqrt{a^2 - 4(b-2)} \geq 4 - |a|$. After squaring both sides and subtracting a^2, we have $8|a| \geq 8 + 4b$; dividing this by 4 and squaring again

yields $4a^2 \geq b^2 + 4b + 4$. Adding $4b^2$ to both sides, we have $4a^2 + 4b^2 \geq 5b^2 + 4b + 4$, so

$$a^2 + b^2 \geq \frac{5}{4}\left(b^2 + \frac{4}{5}b + \frac{4}{5}\right)$$

$$= \frac{5}{4}\left(b + \frac{2}{5}\right)^2 + \frac{4}{5}$$

The minimum value of the right member occurs when $b = -2/5$ and is $4/5$. Thus the minimum value of $a^2 + b^2$ is $4/5$.

8. Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

where a, b, c, d are arbitrary positive numbers.

Soln.: Clearly

$$S > \frac{a}{a+b+c+d} + \frac{b}{a+b+c+d} + \frac{c}{a+b+c+d} + \frac{d}{a+b+c+d} = 1$$

Suppose for definiteness that d is the largest of the four numbers a, b, c, d. Then

$$S \leq \frac{a}{a+b+c} + \frac{b}{a+b+c} + \frac{c}{b+c+a} + \frac{d}{a+c+d}$$

$$= 1 + \frac{d}{a+c+d} < 1 + 1 = 2$$

Then $1 < S < 2$.

We shall show next that S actually assumes all values in the open interval (1, 2). First we observe that S varies continuously with the positive numbers a, b, c, d. Hence if we can show that S takes values arbitrarily close to the endpoints 1 and 2 of this interval, it will follow that S assumes all values between 1 and 2.

Set $a = 1$, $b = \varepsilon$, $c = d = \varepsilon^2$, where $\varepsilon > 0$. Then

$$S = \frac{1}{1+\varepsilon+\varepsilon^2} + \frac{\varepsilon}{1+\varepsilon+\varepsilon^2} + \frac{\varepsilon^2}{\varepsilon+\varepsilon^2} + \frac{\varepsilon^2}{1+2\varepsilon^2}$$

As $\varepsilon \to 0$, the first term approaches 1, the others approach 0, so $S \to 1$. Next, set $a = c = 1$, $b = d = e$; then

$$S = \frac{1}{1+2\varepsilon} + \frac{\varepsilon}{2+\varepsilon} + \frac{1}{1+2\varepsilon} + \frac{\varepsilon}{2+\varepsilon}$$

As $\varepsilon \to 0$, the first and third fractions approach 1, while the second and fourth approach 0; hence $S \to 2$.

9. A rectangular box can be filled completely with unit cubes. If one places as many cubes as possible, each with volume 2, in the box, so that their edges are parallel to the edges of the box, one can fill exactly 40% of the box. Determine the possible dimensions of all such boxes.

Soln.:

Since unit cubes fill the box completely, each of its dimensions is a natural number, say a_1, a_2 and a_3. The volume of the box, then, is $a_1a_2a_3$. Now let b_i be the largest number of cubes with volume 2, hence with edge length $2^{1/3}$, that can be placed along the edge of length a_i of our box. Thus the integer b_i satisfies the inequalities

$$a_i - \sqrt[3]{2} < \sqrt[3]{2}b_i \leq a_i,$$

i.e.

$$\frac{a_i}{\sqrt[3]{2}} - 1 < b_i \leq \frac{a_i}{\sqrt[3]{2}} \qquad ...(1)$$

In other words, bi is the largest integer not exceeding $\frac{a_i}{\sqrt[3]{2}}$, and this is usually denoted by brackets:

$$b_i = [a_i \sqrt[3]{2}] \qquad ...(2)$$

The volume occupied by these cubes is $\sqrt[3]{3}b_1 \cdot \sqrt[3]{2}b_2 \cdot \sqrt[3]{2}b_3$, and this, we are told, is 40% of the box volume. Thus $2b_1b_2b_3 = (2/5)a_1a_2a_3$, whence

$$\frac{a_1a_2a_3}{b_1b_2b_3} = 5 \qquad ...(3)$$

To find all positive integers a_1, a_2, a_3 subject to conditions (2) and (3), we observe that $a_i > 1$ (otherwise $b_i = 0$) and tabulate a few values of a, $b = [a/\sqrt[3]{2}]$, and a/b:

a	2	3	4	5	6	7	8	...
b	1	2	3	3	4	5	6	...
a/b	2	1.5	1.33...	1.66...	1.5	1.4	1.33...	....

We claim that for $a > 8$, $a/b < 1.5$. To prove this we need only use the estimate $b > (a/\sqrt[3]{2}) - 1$, see

(1), from which it follows that

$$\frac{a}{b} < \frac{a}{\frac{a}{\sqrt[3]{2}} - 1} = \frac{\sqrt[3]{2}}{1 - \frac{\sqrt[3]{2}}{a}}$$

As a increases, the denominator in the last member increases, hence the fraction decreases

$$\text{For } a \geq 8,\ \frac{a}{b} < \frac{\sqrt[3]{2}}{1 - \frac{\sqrt[3]{2}}{8}} = \frac{1.26}{1 - \frac{1.26}{8}} < 1.5 \qquad \ldots(4)$$

This together with the table above yields that

$$\text{for } a \geq 3,\ \frac{a}{b} \leq \frac{5}{3} \qquad \ldots.(5)$$

If a_1, a_2, a_3 were at least 3,
then $a_1a_2a_3/b_1b_2b_3 \leq (5/3)^2$ would violate (3); so we conclude that at least one of the a's, say a_1, must be 2. Consequently by (3)

$$\frac{a_2}{b_2}\,\frac{a_3}{b_3} = \frac{5}{2} \qquad \ldots(6)$$

Next we show that both other dimensions of our box are greater than 2. We accomplish this by demonstrating that for i = 2, 3, $a_i/b_i < 2$, from which $a_i > 2$ will follow. By (1), $b_i \leq a_i/\sqrt[3]{3}$, so $b_i/a_i \leq 1/\sqrt[3]{2}$; and by (6), with i = 3,

$$\frac{a_2}{b_2} = \frac{5}{2}\,\frac{b_3}{a_3} \leq \frac{5}{2\sqrt[3]{2}} < 2.$$

Similarly, with i = 2, (1) and (6) yield $a_3/b_3 < 2$. Finally, we note from the table and (4) that $a/b \leq 3/2$ unless a = 2 or a = 5. Since $(3/2)^2 = 9/5 < 5/2$, we see that (6) would be violated unless one of a_2, a_3, say a_2, is 5. Then $a_2/b_2 = 5/3$ and (6) yields

$$\frac{5}{3}\cdot\frac{a_3}{b_3} = \frac{5}{2}, \qquad \frac{a_3}{b_3} = \frac{3}{2}.$$

Thus a_3 is either 3 or 6. So the dimentions of the box are either $2 \times 3 \times 5$ or $2 \times 5 \times 6$.

10. Four real constants, a, b, A, B are given, and
$f(\theta) = 1 - a\cos\theta - b\sin\theta - A\cos 2\theta - B\sin 2\theta$.
Prove that if $f(\theta) \geq 0$ for all real θ, then
$a^2 + b^2 \leq 2$ and $A^2 + B^2 \leq 1$.

Soln.: Let $\sqrt{a^2 + b^2} = r$. Then $(a/r)^2 + (b/r)^2 = 1$, and there is an angle α such that

$$\frac{a}{r} = \cos\alpha, \quad \frac{b}{r} = \sin\alpha$$

We use these expressions to write

$$a\cos\theta + b\sin\theta = r\left(\frac{a}{r}\cos\theta + \frac{b}{r}\sin\theta\right)$$
$$= r(\cos\alpha\cos\theta + \sin\alpha\sin\theta) = r\cos(\theta - \alpha).$$

Similarly, letting $\sqrt{A^2 + B^2} = R$, and $A/R = \cos 2\beta$, $B/R = \sin 2\beta$ we find

$$A\cos 2\theta + B\sin 2\theta = R\left(\frac{A}{R}\cos 2\theta + \frac{B}{R}\sin 2\theta\right)$$
$$= R(\cos 2\beta\cos 2\theta + \sin 2\beta\sin 2\theta)$$
$$= R\cos 2(\theta - \beta)$$

Now f may be written in the form

$$f(\theta) - 1 - r\cos(\theta - \alpha) - R\cos 2(\theta - \beta) \qquad \ldots(1)$$

For $\theta = \alpha + 45^0$ and $\theta = \alpha - 45^0$, (1) yields

$$f(\alpha + 45^0) = 1 - r/\sqrt{2} - R\cos 2(\alpha - \beta + 45^0) \ldots(2)$$

and

$$f(\alpha - 45^0) = 1 - r/\sqrt{2} - R\cos 2(\alpha - \beta - 45^0) \quad \ldots(3)$$

If $r > \sqrt{2}$, then $1 - r/\sqrt{2} < 0$; and since the angles $2(\alpha - \beta) + 90^0$ and $2(\alpha - \beta) - 90^0$ differ by 180^0, their cosines have opposite signs. Therefore one of the expressions

$$R\cos 2(\alpha - \beta + 45^0), \qquad R\cos 2(\alpha - \beta - 45^0)$$

is positive, so that the right side of one of the equations (2), (3) is negative. This means that at least one of the values $f(\alpha + 45^0)$, $f(\alpha - 45^0)$ is negative, contrary to the hypothesis. We conclude that

$r^2 = a^2 + b^2 \leq 2$.

Similarly, we evaluate f at β and at β + π:

$$f(\beta) = 1 - r\cos(\beta - \alpha) - R\cos 0$$
$$= 1 - r\cos(\beta - \alpha) - R \qquad \ldots(4)$$

and

$$f(\beta + \pi) = 1 - r\cos(\beta - \alpha + \pi) - R\cos 2\pi$$
$$= 1 - r\cos(\beta - \alpha + \pi) - R \qquad \ldots(5)$$

If R > 1, then 1 - R < 0; and since β - α and β - α + π differ by π their cosines have opposite signs. It follows that the right side of one of the equations (4), (5) is negative, i.e. that at least one of the values $f(\beta)$, $f(\beta + \pi) < 0$, contrary to the hypothesis. Hence $R^2 = A^2 + B^2 \leq 1$.

■

5 Challenging Problems

Differential Calculus

1. Prove that $f(x) = |x|$ is continuous at $x = 0$ but not differentiable at $x = 0$. Where $|x|$ means the numerical value of the absolute value of x.

Soln.:

We have $f(0) = |0| = 0$

$$f(0+0) = \operatorname*{Lt}_{h\to 0} f(0+h) = \operatorname*{Lt}_{h\to 0} f(h)$$

$$= \operatorname*{Lt}_{h\to 0} |h| = 0$$

and $f(0-0) = \operatorname*{Lt}_{h\to 0} f(0-h) = \operatorname*{Lt}_{h\to 0} |-h| = 0$

$$f(0+0) = f(0-0) = f(0)$$

Hence f is continuous at $x = 0$.

Also we have

$$R\ f'(0) = \operatorname*{Lt}_{h\to 0} \frac{f(0+h) - f(0)}{h}$$

$$\operatorname*{Lt}_{h\to 0} \frac{f(h) - f(0)}{h} = \operatorname*{Lt}_{h\to 0} \frac{|h| - 0}{h} = \frac{h}{h} = 1$$

(h being +ve)

$\therefore\ R\ f'(0) = 1$

and $L\ f'(0) = \operatorname*{Lt}_{h\to 0} \dfrac{f(0-h) - f(0)}{0-h}$

$$\operatorname*{Lt}_{h\to 0} \frac{f(-h) - f(0)}{-h} = \operatorname*{Lt}_{h\to 0} \frac{|-h| - 0}{-h} = -1$$

(h being +ve)

Since $R\ f'(0) + L\ f'(0)$

$f(x)$ is not differentiable at $x = 0$.

2. Discuss the applicability of Rolle's Theorem to

$f(x) = \log\left[\dfrac{x^2 + ab}{(a+b)x}\right]$ in the interval $[a, b]$.

Soln.:

Given $f(x) = \log\left[\dfrac{x^2 + ab}{(a+b)x}\right]$

$$f(a) = \log\left[\frac{a^2 + ab}{a^2 + ab}\right] = \log 1 = 0$$

$$f(b) = \log 1 = 0$$

Thus $f(a) = f(b) = 0$

Also $R\ f'(x) = \operatorname*{Lt}_{h\to 0} \dfrac{f(x+h) - f(x)}{h}$

$$\operatorname*{Lt}_{h\to 0} \frac{1}{h}\left[\log\left\{\frac{(x+h)^2 + ab}{(a+b)(x+h)}\right\} - \log\left[\frac{x^2 + ab}{(a+b)x}\right]\right]$$

$$\operatorname*{Lt}_{h\to 0} \frac{1}{h}\left[\log\left\{1 + \frac{2hx + h^2}{x^2 + ab}\right\} - \log\left[1 + \frac{h}{x}\right]\right]$$

$$= \frac{2x}{x^2 + ab} - \frac{1}{x}$$

$$\left[\text{since } \log(1+y) = y - \frac{y^2}{2} + \ldots\right]$$

Again $L\ f'(x) = \operatorname*{Lt}_{h\to 0} \dfrac{f(x-h) - f(x)}{-h}$

$$\operatorname*{Lt}_{h\to 0} \frac{1}{-h}\left[\frac{2x(-h) + h^2}{x^2 + ab} + \frac{h}{x}\right] \text{ Replace } h \text{ by } -h$$

$$\operatorname*{Lt}_{h\to 0} \left[\frac{2x}{x^2 + ab} - \frac{1}{x}\right]$$

Since $R\ f'(x) = L\ f'(x)$, $f(x)$ is differentiable for all values of x in $[ab]$. This implies $f(x)$ is continuous for all values of x in $[ab]$. Thus the conditions of Rolle's theorem are satisfied. Hence $f'(x) = 0$ for at least one value of x is in open interval $(a\ b)$.

Now $f'(x) = 0$

$$\frac{2x}{ab + x^2} - \frac{1}{x} = 0$$

$$2x^2 - (x^2 + ab) = 0 \qquad x = \sqrt{ab}.$$

which is the geometric mean of 'a' and 'b' in the open interval (a, b). Hence Rolle's Theorem is verified.

3. If $f(x)$ and $g(x)$, and $h(x)$ are functions such that

(i) all the three are continuous in $[a\ b]$

(ii) all the three are differentiable in $(a\ b)$ Then

$$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} = 0 \quad \text{where } a < c < b.$$

Soln.:

Let us consider the function

$$\phi(x) = \begin{vmatrix} f(x) & g(x) & h(x) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & h(a) \end{vmatrix} \quad ...(1)$$

On expanding the determinant we see that the function $f(x)$ is of the form $A\,f(x_0) + Bg(x) + Ch(x)$ where A, B, C are some real numbers.

In view of the condition (i) and (ii), $f(x)$ is continuous on $[ab]$ and differentiable on (ab) also $f(a) = f(b) = 0$ because two rows of a determinant become identical. The $f(x)$ satisfies all the conditions of Rolles theorem. Hence there exist a value C lying between a and b such that $f'(c) = 0$.

i.e $$\begin{vmatrix} f'(c) & g'(c) & h'(c) \\ f(b) & g(b) & h(b) \\ f(a) & g(a) & -h(a) \end{vmatrix} = 0 \quad ...(2)$$

Note : If we take $h(x) = k$ a constant. Then $h'(c) = 0$; and $h(a) = h(b) = k$.

Then (2) becomes $$\begin{vmatrix} f'(c) & g'(c) & 0 \\ f(b) & g(b) & k \\ f(a) & g(a) & k \end{vmatrix} = 0$$

On simplification given us

mena value theorem

$$\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)} \quad a < c < b$$

Note 2 : If we take $h(x) = k, g(x) = x$

then $h'(x) = 0, h(a) = h(b) = k$

$g'(x) = 1, g(b) = b, g(a) = a$. Then (2) becomes

$$\begin{vmatrix} f'(c) & 1 & 0 \\ f(b) & b & k \\ f(a) & a & k \end{vmatrix} = 0$$

which on simplification given Lagranses mean value theorem.

$$\frac{f(b) - f(a)}{b - a} = f'(c) \quad \text{where } a < c < b$$

Note 3 : If we take $h|x| = k, g(x) = x$

$f(a) = f(b)$ then (2) becomes

$$\begin{vmatrix} f'(c) & 1 & 0 \\ f(a) & b & k \\ f(a) & a & k \end{vmatrix} = 0$$

which on simplification gives Rolles Theorem $f'(c) = 0$ where $a < c < b$.

4. Prove that $\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}0.6 < \frac{\pi}{6} + \frac{1}{8}$

Soln.:

Let $f(x) = \sin^{-1}x \quad [ab] \quad$ where $a > 0, b < \pi/2$

f is minimum and derivable on $[a\ b]$

also $f'(x) = \frac{1}{\sqrt{1-x^2}} \ \forall\ x \in (a\ b)$

By Mean value theorem $C \in (a\ b)$

such that $\frac{f(b) - f(a)}{b - a} = f'(c)$

$$= \frac{\sin^{-1}b - \sin^{-1}a}{b - a} = \frac{1}{\sqrt{1 - c^2}} \quad a<c<b$$

But $a < c < b$

$$\frac{1}{\sqrt{1 - a^2}} < \frac{c}{\sqrt{1 - c^2}} < \frac{1}{\sqrt{1 - a^2}}$$

put $b = 3/5, a = 1/2$

$$\frac{\sqrt{3}}{15} < \sin^{-1}\frac{3}{5} - \sin^{-1}\frac{1}{2} < \frac{1}{8}$$

$$\frac{\sqrt{3}}{15} < \sin^{-1}0.6 - \frac{\pi}{6} < \frac{1}{8}$$

$$\frac{\pi}{6} + \frac{\sqrt{3}}{15} < \sin^{-1}0.6 < \frac{\pi}{6} + \frac{1}{8}.$$

5. Find 'c' of Cauchy's mean value theorem for $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ in $[a\ b]$ where $0<a<b$.

Soln.:

Since x^n, is continuous on R^+, f, g, are continuous on $[a\ b]\ CR^+$

since $f'(x) = \frac{1}{2\sqrt{x}}$, $g'(x) = \frac{-1}{2x^{3/2}}$

f, g are derivable on $(a\ b)$ CR+

further $g'(x) \neq 0\ \forall\ x \in (a\ b)\ C\ R^+$

$\therefore\ \exists\ c \in (a\ b)$ such that $\frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

$$\frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = \frac{1/2\sqrt{c}}{-1/c^3/2}$$

$c = \sqrt{ab}$ since $a > 0, b > 0$, $\sqrt{ab}$ is in.

we have $a < \sqrt{ab} < b$. Hence $c = \sqrt{ab} \in (a\ b)$.

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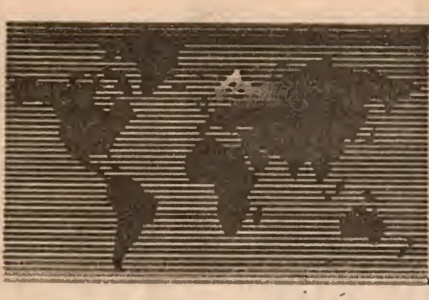


INTERNATIONAL MATH OLYMPIAD PROBLEMS

1. Let x be a real number with $0 < x < \pi$. Prove that, for all natural numbers n, the sum

$$\sin x + \frac{\sin 3x}{3} + \frac{\sin 5x}{5} + \ldots\ldots + \frac{\sin(2n-1)x}{2n-1}$$

is positive.

Soln.: We use mathematical induction. Let

$$S_n(x) = \sum_{k=1}^{n} \frac{\sin(2k-1)x}{(2k-1)}$$

$S_1(x) = \sin x > 0$ for $x \in (0, \pi)$. Thus the proposed inequality is true for $n = 1$. Let $S_r(x) > 0$ for $r = 1, 2, \ldots\ldots, n - 1$. We will deduce that $S_n(x) > 0$ for $x \in (0, \pi)$. Suppose that $S_n(x_0) \le 0$ for some $x_0 \in (0, \pi)$, and that $S_n(x)$ attains its minimum at $x = x_0$. Hence $\frac{d}{dx}[S_n(x)]_{x=x_0} = 0$.

That is

$$S'_n(x_0) = \sum_{k=1}^{n} \cos((2k-1)x_0) = 0,$$

so that

$$2\sin x_0 S'_n(x_0) = \sum_{k=1}^{n} 2\cos((2k-1)x_0)\sin x_0,$$

$$= \sum_{k=1}^{n} [\sin(2kx_0) - \sin((2k-2)x_0)]$$

$$= \sin 2nx_0.$$

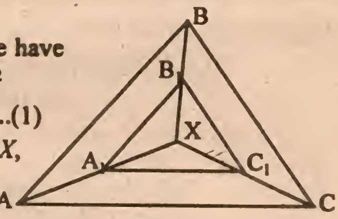
Thus $S'_n(x_0) = \dfrac{\sin 2nx_0}{2\sin x_0} = 0$ implying $2nx_0 = 0$. Hence

$$x_0 \in \left\{\frac{\pi}{2n}, \frac{2\pi}{2n}, \frac{3\pi}{2n}, \ldots\ldots\ldots\ldots, \frac{(2n-1)}{2n}\right\}.$$

It is easily verified that at each of these values $S_n(x_0) > 0$, a contradiction. Hence $S_n(x) > 0$ for $x \in (0, \pi)$.

2. Let A, B and C be non-collinear points. Prove that there is a unique point X in the plane of ABC such that $XA^2 + XB^2 + AB^2 = XB^2 + XC^2 + BC^2 = XC^2 + XA^2 + CA^2$.

First solution :



From the hypothesis we have

$$AX^2 + AB^2 = CX^2 + CB^2 \qquad \ldots(1)$$

If B_1 is the midpoint of BX, applying the first theorem of the median in the triangles ΔABX, ΔCBX we get

$$2AB_1^2 + 2BB_1^2 = 2CB_1^2 + 2BB_1^2 \quad \text{or}$$

$$AB_1 = CB_1 \qquad \ldots\ldots(2)$$

This indicates that the perpendicular bisector of the side AC passes through the point B_1. Let A_1, C_1 be the midpoints of AX and CX, respectively.

Similarly, we obtain that the perpendicular bisectors of BC and AB pass through the midpoints A_1 and C_1 respectively. .....(3)

Furthermore we obtain $AB \parallel A_1B_1$, $AC \parallel A_1C_1$ and $BC \parallel B_1C_1$. .....(4)

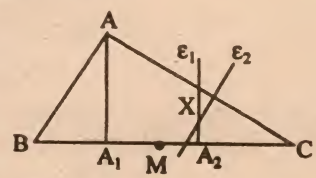
From (3) and (4) we get the circumcentre O of ABC is the orthocentre H_1 of $A_1B_1C_1$. .....(5)

Also from (4) the triangles ABC and $A_1B_1C_1$ are similar with X the centre of similarity and ratio $\frac{1}{2}$. ....(6)

So, their orthocenrtres H and H_1 lie in the same straight line with the point X and $HH_1 = H_1X$. .....(7)

Combining (5) and (7) we get $HO = OX$; that is the point X is known (constant), because X is symmetric to H with respect to the orthocentre O of ABC.

Second solution



The conditions of the problem are equivalent to the system of equations

$$XB^2 - XC^2 = AC^2 - AB^2 \qquad \ldots\ldots.(1)$$

$$XC^2 - XA^2 = BA^2 - BC^2 \qquad \ldots\ldots(2)$$

Now, taking equation (1) gives a locus of points X sat-

isfying the condition. The relation reminds us of the second theorem of the median in a triangle.

Let AA_1, XA_2 be the altitudes of the triangles ABC and XBC respectively on side BC (extended). Let M be the midpoint of the side BC. If we suppose

$$AB \le AC \le BC, \qquad .....(3)$$

for illustration, we get

$$XC \le XB \le XA,$$

and furthermore the point M lies between the points A_1 and A_2 ......(4)

But $\quad XB^2 - XC^2 = 2BC \cdot MA_2$ and

$$AC^2 - AB^2 = 2BC \cdot MA_1.$$

Hence $MA_1 = MA_2$ and A_2 is a constant point on BC because it is symmetric to A_1 with respect to the midpoint M.

Consequently, if (1) holds, the point X lies on the line ε_1 perpendicular to BC at A_2. Similarly, if (2) holds, the point X lies on the line ε_2 perpendicular to AC at B_2 (where $BB_1 \perp AC$ and $AB_1 = CB_2$).

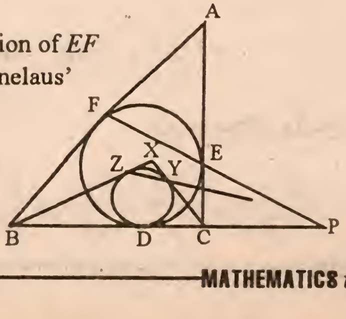
Hence, the required point X lies at the intersection of ε_1 and ε_2.

3. Four dice are thrown. What is the chance that the product of the numbers equals 36?

Soln.: There are four different kinds of outcomes in which the product is 36: each of {1, 1, 6, 6} and {2, 2, 3, 3} can occur in $\frac{4!}{2!\,2!} = 6$ ways; (1, 4, 3, 3} can occur in $\frac{4!}{2!} = 12$ ways; and {1, 2, 3, 6} can occur in 4! = 24 ways. Hence the probability that the product equals 36 is $\frac{48}{6^4} = \frac{1}{27}$.

4. The incircle of ABC touches BC, CA and AB at D, E and F respectively. X is a point inside ABC such that the incircle of XBC touches BC at D also, and touches CX and XB at Y and Z respectively. Prove that $EFZY$ is a cyclic quadrilateral.

Soln.:



Let P be the intersection of EF with BC. Then by Menelaus' Theorem we have

$$\frac{BP}{PC} \cdot \frac{CE}{EA} \cdot \frac{AF}{FB} = 1 \qquad ....(1)$$

Since $CE = CD$, $EA = AF$, and $FB = BD$, we get

$$\frac{BP}{PC} \cdot \frac{CD}{BD} = 1$$

so that

$$\frac{BP}{PC} = \frac{BD}{CD} \qquad .....(2)$$

Since $XZ = XY$, $BZ = BD$ and $CY = CD$, we have from (2)

$$\frac{BP}{PC} \cdot \frac{CY}{YX} \cdot \frac{XZ}{ZB} = \frac{BD}{CD} \cdot \frac{CD}{YX} \cdot \frac{XY}{BD} = 1$$

Hence by Menelaus' Theorem P, Z and Y are collinear.

Since $PF \cdot PE = PD^2$ and $PZ \cdot PY = PD^2$ we have

$$PF \cdot PE = PZ \cdot PY.$$

Hence $EFZY$ is a cyclic quadrilateral.

Comment : If $AB = AC$ then $BD = DC$ and then it can easily be proved that AD is the perpendicular bisector of EF and YZ so that $EFZY$ is an isosceles trapezoid, and is a cyclic trapezoid.

5. For non-negative integers n, r the binomial coefficient $\binom{n}{r}$ denotes the number of combinations of n objects chosen r at a time, with the convention that $\binom{n}{0} = 1$ and $\binom{n}{r} = 0$ if $n < r$. Prove the identity

$$\sum_{d=1}^{\infty} \binom{n-r+1}{d}\binom{r-1}{d-1} = \binom{n}{r}$$

for all integers n, r with $1 \le r \le n$.

Soln.: We use a combinatorial argument to establish the obviously equivalent identity

$$\sum_{d=1}^{k} \binom{n-r+1}{d}\binom{r-1}{r-d} = \binom{n}{r} \qquad .....(*)$$

where $k = \min\{r, n - r + 1\}$. It clearly suffices to demonstrate that the left hand side of (i) counts the number of ways of selecting r objects from n distinct objects (without replacements). Let $|S_2| = r - 1$. For each fixed $d = 1, 2, \ldots, k$, any selection of d objects from S_1 $(S \setminus S_2)$ together with any selection of $r - d$ objects from S_2 would yield a selection of r objects from S. The total number of such electrons is $\binom{n-r+1}{d}\binom{r-1}{r-d}$. Conversely, each selection of r objects from S clearly must arise in this manner. Summing over $d = 1, 2, \ldots$ (*) follows. ■

YOU ASK – WE ANSWER

1. Define the sequence of functions $f_0, f_1, f_2, ...$ by

$$f_0(x) = 8, \text{ for all } x \in R,$$

$$f_{n+1}(x) = \sqrt{x^2 + 6f_n(x)}$$

for $n = 0, 1, 2,....$ and for all $x \in R$. For every positive integer n, solve the equation $f_n(x) = 2x$.

– Sanjay Sharma, Hissar.

Soln.: Since $f_n(x)$ is positive, $f_n(x) = 2x$ has only positive solutions. We show that, for each $n, f_n(x) = 2x$ has a solution $x = 4$. Since $f_1(x) = \sqrt{x^2 + 48}$, $x = 4$ is a solution of $f_2(x) = 2x$. Now $f_{n+1}(4) = \sqrt{4^2 + 6f_n(4)}$ $= \sqrt{4^2 + 6 \cdot 8} = 8 = 2 \cdot 4$, which completes the inductive step. Next, induction on n gives us that for each n, $\frac{f_n(x)}{x}$, decreases as x increases in $(0, \infty)$. It follows that $f_n(x) = 2x$ has the unique solution $x = 4$.

2. Prove that, for every natural k, the number $(k^3)!$ is divisible by $(k!)^{k^2 + k + 1}$.

– Vivek Dixit, Obra.

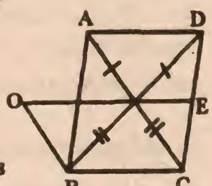
Soln.: Applying the well known fact that $(ab)!$ is divisible by $(a!)^b \cdot b!$ yields $(k^3)! = (k \cdot k^2)!$ is divisible by $(k!)^{k^2} \cdot (k^2)!$ and $(k^2)! = (k \cdot k)!$ is divisible by $(k!)^{k+1}$ from which the required result follows immediately.

3. Segments AC and BD intersect in point P so that $PA = PD$, $PB = PC$. Let O be the circumcentre of triangle PAB. Prove that lines OP and CD are perpendicular.

– Sanjeev Sahni, Jamshedpur

Soln.: Because $PA = PD$, $PB = PC$ and $\angle APB = \angle DPC$ we get $\Delta PAB \equiv \Delta PDC$, so that

$$\angle BAP = \angle CDP. \quad .... (1)$$



At least one of $\angle PAB$ and $\angle PBA$ is acute, so we may assume without loss of generality that $\angle PAB$ is acute. Since O is the circumcentre of ΔPAB we get $OB = OP$ and $\angle BOP = 2\angle BAP$, so that

$$\angle OPB = 90° - \tfrac{1}{2}\angle BOP = 90° - \angle BAP \quad ....(2)$$

Let E be the intersection of OP with CD. Then

$$\angle EPD = \angle OPB \quad ....(3)$$

From (1), (2) and (3) we have

$$\angle EPD = 90° - \angle CDP.$$

Thus $\angle EPD + \angle EDP = \angle EPD + \angle CDP = 90°$. Therefore $OP \perp CD$.

Comment : Generally if A, B, C, D are concyclic, we have $OP \perp CD$ and this theorem is an extension of Brahmagupta's theorem.

4. Determine all functions f defined on the set of positive rational numbers, taking values in the same set, which satisfy for every positive rational number x the conditions.

$$f(x + 1) = f(x) + 1 \text{ and } f(x^3) = (f(x))^3.$$

– Rajan Kohli, Ludhiana

Soln.: Let N and Q^+ denote the set of positive integers and the set of positive rational numbers respectively. We show that $f(x) = x$, for all $x \in Q^+$, is the only function satisfying the given conditions. First of all, by the first condition and an easy induction we see that $f(x + n) = f(x) + n$, for all $x \in Q^+$, and for all $n \in N$. Now for arbitrary $\frac{p}{q} \in Q^+$, where $p, q \in N$, we have

$$f\left(\left(\frac{p}{q} = q^2\right)^3\right) = f\left(\frac{p^3}{q^3} + 3p^2 + 3pq^3 + q^6\right)$$

$$= f\left(\left(\frac{p}{q}\right)^3\right) + 3p^2 + 3pq^3 + q^6 \quad ....(1)$$

On the other hand

$$f\left(\left(\frac{p}{q} + q^2\right)^3\right) = f\left(\left(\frac{p}{q} + q^2\right)\right)^3 = \left(f\left(\frac{p}{q}\right) + q^2\right)^3$$

Twenty Five Challenging Problems

TRIGONOMETRY

1. If $\cot\alpha + \tan\alpha = m$ and $\dfrac{1}{\cos\alpha} - \cos\alpha = n$, then

(a) $m(mn^2)^{1/3} - n(nm^2)^{1/3} = 1$

(b) $n(mn^2)^{1/3} - m(nm^2)^{1/3} = 1$

(c) $m(m^2n)^{1/3} - n(mn^2)^{1/3} = 1$

(d) $n(m^2n)^{1/3} - m(mn^2)^{1/3} = 1$.

Soln.: $\cot\alpha + \tan\alpha = m \Rightarrow 1 + \tan^2\alpha = m\tan\alpha$

$$\Rightarrow \sec^2\alpha = m\tan\alpha \qquad ...(1)$$

and $\dfrac{1}{\cos\alpha} - \cos\alpha = n \Rightarrow \sec^2\alpha - 1 = n\sec\alpha$

$\Rightarrow \tan^2\alpha = n\sec\alpha \Rightarrow \tan^4\alpha = n^2\sec^2\alpha$

Now from (1) $\tan^4\alpha = n^2m\tan\alpha$

$\Rightarrow \tan^3\alpha = n^2m \Rightarrow (n^2m)^{1/3} = \tan\alpha$

and $\sec^2\alpha = m(n^2m)^{1/3}$

Now $\sec^2\alpha - \tan^2\alpha = 1 \Rightarrow m(n^2m)^{1/3} - (n^2m)^{2/3} = 1$

$\Rightarrow m(mn^2)^{1/3} - n(nm^2)^{1/3} = 1$ $\quad\therefore$ **Ans : (a).**

2. $(m + 2)\sin\theta + (2m - 1)\cos\theta = 2m + 1$, if

(a) $\tan\theta = \frac{3}{4}$ (b) $\tan\theta = \frac{4}{3}$

(c) $\tan\theta = \dfrac{2m}{m^2-1}$ (d) $\tan\theta = \dfrac{2m}{m^2+1}$.

Soln.: The given relation can be written as

$(m + 2)\tan\theta + (2m - 1) = (2m + 1)\sec\theta$

$\Rightarrow (m+2)^2\tan^2\theta + 2(m+2)(2m-1)\tan\theta + (2m-1)^2$

$= (2m + 1)^2(1 + \tan^2\theta)$

$\Rightarrow [(m+2)^2 - (2m+1)^2]\tan^2\theta + 2(m+2)(2m-1)\tan\theta$

$+ (2m - 1)^2 - (2m + 1)^2 = 0$

$\Rightarrow 3(1 - m^2)\tan^2\theta + (4m^2 + 6m - 4)\tan\theta - 8m = 0$

$\Rightarrow (3\tan\theta - 4)[(1 - m^2)\tan\theta + 2m] = 0$

i.e. $\tan\theta = \frac{4}{3}$ or $\tan\theta = \dfrac{2m}{m^2-1}$ $\therefore$ **Ans : (b)** and **(c)**

3. If $x\cos\alpha + y\sin\alpha = x\cos\beta + y\sin\beta = 2a$ then

(a) $\cos\alpha + \cos\beta = \dfrac{4ax}{x^2+y^2}$

(b) $\sin\alpha\cdot\sin\beta = \dfrac{4a^2-x^2}{x^2+y^2}$

(c) $\sin\alpha + \sin\beta = \dfrac{4ay}{x^2+y^2}$

(d) $\cos\alpha\cdot\cos\beta = \dfrac{4a^2-y^2}{x^2+y^2}$.

Soln.: From given relations we find that α and β are the roots of the equation

$$x\cos\theta + y\sin\theta = 2a \qquad ...(1)$$

$\Rightarrow (x\cos\theta - 2a)^2 = (-y\sin\theta)^2$

$\Rightarrow x^2\cos^2\theta - 4ax\cos\theta + 4a^2 = y^2\sin^2\theta = y^2(1 - \cos^2\theta)$

$\Rightarrow (x^2 + y^2)\cos^2\theta - 4ax\cos\theta + 4a^2 - y^2 = 0$

This being a quadratic in $\cos\theta$, has two roots $\cos\alpha$ and $\cos\beta$ such that $\cos\alpha + \cos\beta = \dfrac{4ax}{x^2+y^2}$ and

$$\cos\alpha\cdot\cos\beta = \frac{4a^2-y^2}{x^2+y^2}.$$

Similarly, we can write (1) as a quadratic in $\sin\theta$ giving two values $\sin\alpha$ and $\sin\beta$ such that

$$\sin\alpha + \sin\beta = \frac{4ay}{x^2+y^2} \text{ and } \sin\alpha\cdot\sin\beta = \frac{4a^2-x^2}{x^2+y^2}.$$

$\therefore$ **Ans : (a), (b), (c) and (d).**

4. The value of $\cos y\cos\left(\dfrac{\pi}{2}-x\right) - \cos\left(\dfrac{\pi}{2}-y\right)\cdot\cos x + \sin y\cdot\cos\left(\dfrac{\pi}{2}-x\right) + \cos x\cdot\sin\left(\dfrac{\pi}{2}-y\right)$ is zero if

(a) $x = 0$ (b) $y = 0$

(c) $x = y$ (d) $n\pi - \dfrac{\pi}{4} + y \quad (n \in I)$.

Soln.: The given expression is equal to

$\cos y\sin x - \sin y\cos x + \sin y\sin x + \cos x\cos y$

$= \sin(x - y) + \cos(x - y)$

which is zero if

$$\sqrt{2}\left[\frac{1}{\sqrt{2}}\sin(x-y) + \frac{1}{\sqrt{2}}\cos(x-y)\right] = 0$$

$$\Rightarrow \sin\left[\frac{\pi}{4}+(x-y)\right] = 0 \Rightarrow \left[\frac{\pi}{4}+(x-y)\right] = n\pi$$

or $x - y = n\pi - \dfrac{\pi}{4}$. Thus **Ans : (d).**

5. If $\tan x = \dfrac{2b}{a-c} (a \neq c)$, $y = a\cos^2 x + 2b\sin x\cos x + c\cos^2 x$ and $z = a\sin^2 x - 2b\sin x\cos x + c\cos^2 x$ then,

(a) $y = z$ (b) $y + z = a + c$

(c) $y - z = a - c$ (d) $y - z = (a - c)^2 + 4b^2$.

Soln.: Adding the expressions for y and z we get

$$y + z = a(\cos^2 x - \sin^2 x) + 4b\sin x\cos x + c(\sin^2 x - \cos^2 x)$$
$$= a\cos 2x + 2b\sin 2x - c\cos 2x$$
$$= (a - c)\cos 2x + 2b\sin 2x.$$

Now, $\cos 2x = \dfrac{1-\tan^2 x}{1+\tan^2 x} = \dfrac{(a-c)^2 - 4b^2}{(a-c)^2 + 4b^2}$

and $\sin 2x = \dfrac{2\tan x}{1+\tan^2 x} = \dfrac{4b(a-c)}{(a-c)^2+4b^2}$

$$\Rightarrow y - z = \frac{(a-c)[(a-c)^2 - 4b^2] + 8b^2(a-c)}{(a-c)^2+4b^2}$$

$$= \frac{(a-c)[(a-c)^7 + 4b^2]}{(a-c)^2+4b^2} = a - c. \quad \therefore \textbf{Ans: (b) and (c).}$$

6. If A lies in the second quadrant and $3\tan A + 4 = 0$, the value of $2\cot A - 5\cos A + \sin A$ is

(a) -53/10 (b) 23/10

(c) 37/10 (d) 7/10.

Soln.: From $3\tan A + 4 = 0$ we get $\tan A = -4/3$

so that $\sin A = \pm\dfrac{\tan A}{\sqrt{1+\tan^2 A}} = \pm\dfrac{-4/5}{\sqrt{1+16/9}} = \dfrac{4}{5}$

[$\because$ $\sin A$ is +ve in quandrant II]

$\cos A = \pm\dfrac{1}{\sqrt{1+\tan^2 A}} = -\dfrac{3}{5}$

[$\because$ $\cos A$ is -ve in quadrant II]

Hence $2\cot A - 5\cos A + \sin A$

$$= 2\left(-\frac{3}{4}\right) - 5\left(-\frac{3}{5}\right) + \frac{4}{5} = \frac{23}{10}. \quad \therefore \textbf{Ans : (b)}$$

7. The values of θ lying between $\theta = 0$ and $\theta = \pi/2$ and satisfying the equation

$$\begin{vmatrix} 1+\sin^2\theta & \cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & 1+\cos^2\theta & 4\sin 4\theta \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} \text{ are}$$

(a) $\dfrac{7\pi}{24}$ (b) $\dfrac{5\pi}{24}$

(c) $\dfrac{11\pi}{24}$ (d) $\dfrac{\pi}{24}$.

Soln : Applying $R_1 \to R_1 - R_3$ and $R_2 \to R_2 - R_3$ on the RHS, the given equation can be written as,

$$\begin{vmatrix} 1 & 0 & -1 \\ 0 & 1 & -1 \\ \sin^2\theta & \cos^2\theta & 1+4\sin 4\theta \end{vmatrix} = 0$$

Expanding the LHS along R_1 we get,

$$1 + 4\sin 4\theta + \cos^2\theta + \sin^2\theta = 0$$
$$4\sin 4\theta = -2 \Rightarrow \sin 4\theta = -\frac{1}{2}$$
$$4\theta = \frac{7\pi}{6} \text{ or } \frac{11\pi}{6}$$
$$\left[\because 0 < \theta < \frac{\pi}{2} \Rightarrow 0 < 4\theta < 2\pi\right]$$
$$\theta = \frac{7\pi}{24} \text{ or } \frac{11\pi}{24}$$

$\therefore$ **Ans : (a) and (c).**

8. The value of the determinant

$$\begin{vmatrix} 1 & a & a^2 \\ \cos(n-1)x & \cos nx & \cos(n+1)x \\ \sin(n-1)x & \sin nx & \sin(n+1)x \end{vmatrix} \quad a \neq \pm 1$$

is zero if

(a) $\sin x = 0$ (b) $\cos x = 0$

(c) $a = 0$ (d) $\cos x = \dfrac{1+a^2}{2a}$.

Soln.: Applying $C_1 \to C_1 + C_3 - 2\cos x\, C_2$, the given determinant is equal to

$$\begin{vmatrix} 1+a^2-2a\cos x & a & a^2 \\ 0 & \cos nx & \cos(n+1)x \\ 0 & \sin nx & \sin(n+1)x \end{vmatrix}$$

$$= (1 + a^2 - 2a\cos x)[\cos nx \sin(n+1)x - \sin nx\cos(n+1)x]$$
$$= (1 + a^2 - 2a\cos x)\sin(n + 1 - n)x$$
$$= (1 + a^2 - 2a\cos x)\sin x$$

which is zero if $\sin x = 0$ or $\cos x = \dfrac{1+a^2}{2a}$ As $a \neq \pm 1$

$\left|\dfrac{1+a^2}{2a}\right| > 1$, $\therefore \cos x = \dfrac{1+a^2}{2a}$ is not possible.

$\therefore$ **Ans : (a)**

9. If $y = \dfrac{\sqrt{1-\sin 4A} + 1}{\sqrt{1+\sin 4A} - 1}$; then one of the values of y is

(i) -tanA (b) cotA

(c) $\tan\left(\dfrac{\pi}{4}+A\right)$ (d) $-\cot\left(\dfrac{\pi}{4}+A\right)$

Soln.: $y = \dfrac{\sqrt{(\cos 2A - \sin 2A)^2} + 1}{\sqrt{(\cos 2A + \sin 2A)^2} - 1}$

$$\Rightarrow y = \frac{\pm(\cos 2A - \sin 2A) + 1}{\pm(\cos 2A + \sin 2A) - 1}$$

which gives us four values of y, say y_1, y_2, y_3 and y_4. We have

$$y_1 = \frac{\cos 2A - \sin 2A + 1}{\cos 2A + \sin 2A - 1} = \frac{(1+\cos 2A) - \sin 2A}{(\cos 2A - 1) + \sin 2A}$$

$$= \frac{2\cos^2 A - 2\sin A\cos A}{-2\sin^2 A + 2\sin A\cos A} = \frac{\cos A(\cos A - \sin A)}{\sin A(\cos A - \sin A)}$$

$$= \cot A.$$

$$y_2 = \frac{-(\cos 2A - \sin 2A) + 1}{-(\cos 2A + \sin 2A) - 1} = \frac{(1-\cos 2A) + \sin 2A}{-(1+\cos 2A) - \sin 2A}$$

$$= \frac{2\sin^2 A + 2\sin A\cos A}{-2\cos^2 A - 2\sin A\cos A} = -\tan A.$$

$$y_3 = \frac{(\cos 2A - \sin 2A) + 1}{-(\cos 2A + \sin 2A) - 1} = \frac{(1+\cos 2A) - \sin 2A}{-(1+\cos 2A) - \sin 2A}$$

$$= \frac{2\cos^2 A - 2\sin A\cos A}{-2\cos^2 A - 2\sin A\cos A} = \frac{\cos A - \sin A}{\cos A + \sin A}$$

$$= -\frac{1-\tan A}{1+\tan A} = -\tan\left(\frac{\pi}{4} - A\right) = -\cot\left(\frac{\pi}{4} + A\right).$$

$$y_4 = \frac{-(\cos 2A - \sin 2A) + 1}{(\cos 2A + \sin 2A) - 1} = \frac{(1-\cos 2A) + \sin 2A}{-(1-\cos 2A) + \sin 2A}$$

$$= \frac{2\sin^2 A + 2\sin A\cos A}{-2\sin^2 A + 2\sin A\cos A} = \frac{\cos A + \sin A}{\cos A - \sin A}$$

$$= \frac{1+\tan A}{1-\tan A} = \tan\left(\frac{\pi}{4} + A\right)$$

$\therefore$ **Ans : (a), (b), (c) and (d).**

10. $(2\sqrt{3}+4)\sin x + 4\cos x$ lies in the interval.

(a) (-4, 4) (b) $(-2\sqrt{5}, 2\sqrt{5})$

(c) $(-2+\sqrt{5}, 2+\sqrt{5})$

(d) $(-2(2+\sqrt{5}), 2(2+\sqrt{5}))$.

Soln: The given expression is equal to

$$2\left[\left(\sqrt{3}+2\right)\sin x + 2\cos x\right].$$

Put $\sqrt{3}+2 = r\cos\theta$ and $2 = r\sin\theta$, so that

$$r^2 = (\sqrt{3}+2)^2 + 2^2 = 11 + 4\sqrt{3}.$$

Then the expression can be written as

$2(r\cos\theta \sin x + r\sin\theta \cos x) = 2r\sin(\theta + x) = y$ (say)

since $11 + 4\sqrt{3} < 9 + 4\sqrt{5}$ we have

$$\sqrt{11+4\sqrt{3}} < \sqrt{9+4\sqrt{5}} \Rightarrow \sqrt{11+4\sqrt{3}} < 2+\sqrt{5} \quad \text{.... (1)}$$

Also, since $-1 \le \sin(\theta + x) \le 1$

$$-2r \le \sin(\theta + x) \le 2r$$

$$\Rightarrow -2\sqrt{11+4\sqrt{3}} \le y \le 2\sqrt{11+4\sqrt{3}}$$

$$\Rightarrow -2(2+\sqrt{5}) < y < 2(2+\sqrt{5}) \quad \text{[from (1)]}$$

$\therefore$ **Ans : (d).**

11. If $\tan\alpha = \dfrac{1}{7}$, $\sin\beta = \dfrac{1}{\sqrt{10}}$ where $0 < \alpha, \beta < \dfrac{\pi}{2}$ then 2β is equal to

(a) $\dfrac{\pi}{4} - \alpha$ (b) $\dfrac{3\pi}{4} - \alpha$

(c) $\dfrac{\pi}{8} - \dfrac{\alpha}{2}$ (d) $\dfrac{3\pi}{8} - \dfrac{\alpha}{2}$.

Soln.:

$$\sin\beta = \frac{1}{\sqrt{10}} \Rightarrow \cos\beta = \sqrt{1 - \frac{1}{10}} = \frac{3}{\sqrt{10}}$$

$$\Rightarrow \tan\beta = \frac{1}{3} \qquad \tan 2\beta = \frac{2\tan\beta}{1-\tan^2\beta} = \frac{2\cdot\frac{1}{3}}{1-\frac{1}{9}} = \frac{3}{4}$$

$$\tan(\alpha + 2\beta) = \frac{\tan\alpha + \tan 2\beta}{1 - \tan\alpha\tan 2\beta} = \frac{\frac{1}{7}+\frac{3}{4}}{1-\frac{1}{7}\cdot\frac{3}{4}} = \frac{25}{25} = 1.$$

Since $0 < \beta < \dfrac{\pi}{2}$ and $\tan 2\beta = \dfrac{3}{4} > 0$, we get $0 < 2\beta < \dfrac{\pi}{2}$. Also $0 < \alpha < \dfrac{\pi}{2}$.

Hence, $0 < \alpha + 2\beta < \pi$ and $\tan(\alpha + 2\beta) = 1$ so that $\alpha + 2\beta = \dfrac{\pi}{4} \Rightarrow 2\beta = \dfrac{\pi}{4} - \alpha$. $\therefore$ **Ans : (a)**

12. If $(x - a)\cos\theta + y\sin\theta = (x - a)\cos\phi + y\sin\phi = a$ and $\tan\left(\dfrac{\theta}{2}\right) - \tan\left(\dfrac{\phi}{2}\right) = 2b$, then

(a) $y^2 = 2ax(1 - b^2)x^2$ (b) $\tan\frac{\phi}{2} = \frac{1}{x}(y - bx)$

(c) $y^2 = 2bx - (1 - a^2)x^2$ (d) $\tan\frac{\theta}{2} = \frac{1}{x}(y + bx)$.

Soln.: Let $\tan\frac{\theta}{2} = \alpha$ and $\tan\frac{\phi}{2} = \beta$ so that $\alpha - \beta = 2b$.

Also $\cos\theta = \frac{1-\tan^2\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{1-\alpha^2}{1+\alpha^2}$

$\sin\theta = \frac{2\tan\frac{\theta}{2}}{1+\tan^2\frac{\theta}{2}} = \frac{2\alpha}{1+\alpha^2}$

Similarly, $\cos\phi = \frac{1-\beta^2}{1+\beta^2}$ and $\sin\beta = \frac{2\beta}{1+\beta^2}$.

Thus from the given relations we have

$$(x-a)\frac{1-\alpha^2}{1+\alpha^2} + y\left(\frac{2\alpha}{1+\alpha^2}\right) = a$$

$$\Rightarrow x\alpha^2 - 2y\alpha + 2a - x = 0$$

Similarly $\quad x\beta^2 - 2y\beta + 2a - x = 0$

We see that α and β are the roots of the equation

$$xz^2 - 2yz + 2a - x = 0.$$

We see that α and β are the roots of the equation $xz^2 - 2yz + 2a - x = 0$, so that

$$\alpha+\beta = \frac{2y}{x} \text{ and } \alpha\beta = \frac{(2a-x)}{x}.$$

Now $\quad (\alpha + \beta)^2 = (\alpha - \beta)^2 + 4\alpha\beta$, we get

$$\left(\frac{2y}{\lambda}\right)^2 = (2b)^2 + \frac{4(2a-x)}{x}$$

$$\Rightarrow y^2 = 2ax - (1 - b^2)x^2$$

Also $\alpha+\beta = \frac{2y}{x}$ and $\alpha - \beta = 2b$, we get

$$\alpha = \frac{y}{x} + b \text{ and } \beta = \frac{y}{x} - b$$

$$\Rightarrow \tan\frac{\theta}{2} = \frac{1}{x}(y+bx) \text{ and } \tan\frac{\phi}{2} = \frac{1}{x}(y-bx)$$

$\therefore$ **Ans. (a), (b)** and **(d)**

13. The value of $\tan 1^\circ \tan 2^\circ \ldots \tan 89^\circ$ is ....

Soln.: The given expression can be written as

$(\tan 1^\circ \tan 89^\circ)(\tan 2^\circ \tan 88^\circ) \ldots (\tan 44^\circ \tan 46^\circ)\cdot\tan 45^\circ$

$= (\tan 1^\circ \cot 1^\circ)(\tan 2^\circ \cot 2^\circ) \ldots (\tan 44^\circ \cot 44^\circ)\cdot\tan 45^\circ$

$= 1$. **(Ans.)**

14. If $0 < \alpha, \beta < \pi$ and $\cos\alpha + \cos\beta - \cos(\alpha + \beta) = \frac{3}{2}$ then $\alpha =$ .... and $\beta =$ ....

Soln.: The given equation can be written as

$$2\cos\frac{\alpha+\beta}{2}\cdot\cos\frac{\alpha-\beta}{2} - \left(2\cos^2\frac{\alpha+\beta}{2} - 1\right) = \frac{3}{2}$$

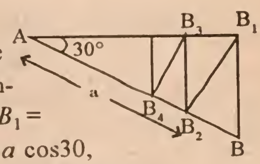
$$\Rightarrow 4\cos^2\frac{\alpha+\beta}{2} - 4\cos\frac{\alpha-\beta}{2}\cdot\cos\frac{\alpha+\beta}{2} + 1 = 0$$

$$\Rightarrow \cos\frac{\alpha+\beta}{2} = \frac{1}{8}\left(4\cos\frac{\alpha-\beta}{2} \pm \sqrt{16\cos^2\frac{\alpha-\beta}{2} - 16}\right) \quad \ldots\ldots (1)$$

Since the radicand is equal to $-16\sin^2\left(\frac{\alpha-\beta}{2}\right)$ and $\cos\left[\frac{\alpha+\beta}{2}\right]$ is real, we have $\sin\left(\frac{\alpha-\beta}{2}\right) = 0$.

Also, since $0 < \alpha, \beta < \pi$, we have $\alpha = \beta$. Therefore, from (1) we get $\cos\alpha = \frac{1}{2}$, so that $\alpha = \beta = \frac{\pi}{3}$.

15. Two rays are drawn through a point A at an angle of 30°. A point B is taken on one of them is at a distance a from the point A. A perpendicular is drawn from the point B to the other ray, and another perpendicular is drawn from its foot to AB. The length of the resulting infinite polynomial line is...........



Soln.: Let $B_1, B_2, B_3, B_4, \ldots$ be the feet of the respective perpendiculars. Then $BB_1 = a\sin\theta$, $BB_1 = a\sin 30$ [$\because \theta = 30^\circ$]. $AB_1 = a\cos 30$, $B_1B_2 = a\cos 30 \times \sin 30$, $AB_2 = a^2\cos^2 30$, $B_1B_3 = a^2\cos^2 30\sin 30$, $AB_3 = a\cos^3 30$ and so on. Therefore, the length of the infinite polynomial line is $BB_1 + B_1B_2 + B_2B_3 + \ldots$

$= a\sin 30 + a\sin 30\cos 30 + a\sin 30\cos^2 30 + \ldots$

$$= \frac{a}{2}\left[1 + \frac{\sqrt{3}}{2} + \left(\frac{\sqrt{3}}{2}\right)^2 + \ldots\right] = \frac{a}{2}\cdot\frac{1}{1-\frac{\sqrt{3}}{2}}$$

$$= \frac{a}{2-\sqrt{3}} = a(2+\sqrt{3}).$$

16. If $\sin^2 A = x$, then $\sin A \cdot \sin 2A \cdot \sin 3A \cdot \sin 4A = ?$

Soln.: The given expression is equal to

$= \sin A \cdot 2\sin A \cdot \cos A\,(3\sin A - 4\sin^3 A)\cdot 2\sin 2A\cos 2A$

$= 2\sin^2 A\cdot\cos A\cdot\sin A(3 - 4\sin^2 A)\,4\sin A\cdot\cos A(1 - 2\sin^2 A)$

$= 8\sin^4 A\cdot\cos^2 A(3 - 4\sin^2 A)(1 - 2\sin^2 A)$

$= 8\sin^4 A\cdot(1 - \sin^2 A)(3 - 4\sin^2 A)(1 - 2\sin^2 A)$

$= 8x^2(1 - x)(3 - 4x)(1 - 2x) = 8x^2[3 - 13x + 18x^2 - 8x^3]$

$\mathbf{= 24x^2 - 104x^3 + 144x^4 - 64x^5}$

17. If $\cos\theta = \dfrac{\sin\beta}{\sin\alpha}$, $\cos\phi = \dfrac{\sin\gamma}{\sin\alpha}$ and $\cos(\theta - \phi) = \sin\beta \cdot \sin\gamma$, then

$$\tan^2\alpha - \tan^2\beta - \tan^2\gamma = \dots\dots$$

Soln.: From the third relation we have

$\cos\theta \cdot \cos\phi + \sin\theta \cdot \sin\phi = \sin\beta \cdot \sin\gamma$

$\Rightarrow \sin^2\theta \cdot \sin^2\phi = (\cos\theta \cdot \cos\phi - \sin\beta \cdot \sin\gamma)^2$

$$\Rightarrow \left(1 - \frac{\sin^2\beta}{\sin^2\alpha}\right)\left(1 - \frac{\sin^2\gamma}{\sin^2\alpha}\right) = \left(\frac{\sin\beta \cdot \sin\gamma}{\sin^2\alpha} - \sin\beta \cdot \sin\gamma\right)^2$$

[From the Ist and IInd relations]

$$\Rightarrow (\sin^2\alpha - \sin^2\beta)(\sin^2\alpha - \sin^2\gamma) = \sin^2\beta\sin^2\gamma(1 - \sin^2\alpha)^2$$

$$\Rightarrow \sin^4\alpha(1 - \sin^2\beta \cdot \sin^2\gamma) - \sin^2\alpha(\sin^2\beta + \sin^2\gamma - 2\sin^2\beta \cdot \sin^2\gamma) = 0$$

$$\therefore \sin^2\alpha = \frac{\sin^2\beta + \sin^2\gamma - 2\sin^2\beta \cdot \sin^2\gamma}{1 - \sin^2\beta \cdot \sin^2\gamma}$$

and $$\cos^2\alpha = \frac{1 - \sin^2\beta - \sin^2\gamma + \sin^2\beta \cdot \sin^2\gamma}{1 - \sin^2\beta \cdot \sin^2\gamma}$$

$$\Rightarrow \tan^2\alpha = \frac{\sin^2\beta - \sin^2\beta \cdot \sin^2\gamma + \sin^2\gamma - \sin^2\beta \cdot \sin^2\gamma}{\cos^2\beta - \sin^2\gamma(1 - \sin^2\beta)}$$

$$= \frac{\sin^2\beta \cdot \cos^2\gamma + \cos^2\beta \cdot \sin^2\gamma}{\cos^2\beta \cdot \cos^2\gamma} = \tan^2\beta + \tan^2\gamma$$

$\Rightarrow \tan^2\alpha - \tan^2\beta - \tan^2\gamma = \mathbf{0}$. **(Ans.)**

18. If $\tan\beta = \dfrac{n\sin\alpha \cdot \cos\alpha}{1 - n\sin^2\alpha}$, then prove that

$$\tan(\alpha - \beta) = 1 - n\tan\alpha \cdot$$

Soln.:

$$\tan\beta = \frac{n\tan\alpha}{1 + \tan^2\alpha - n\tan^2\alpha} = \frac{n\tan\alpha}{1 + (1 - n)\tan^2\alpha}$$

$$\Rightarrow \tan(\alpha - \beta) = \frac{\tan\alpha - \dfrac{n\tan\alpha}{1 + (1 - n)\tan^2\alpha}}{1 + \dfrac{\tan\alpha \cdot n \cdot \tan\alpha}{1 + (1 - n)\tan^2\alpha}}$$

$$= \frac{\tan\alpha + (1 - n)\tan^3\alpha - n\tan\alpha}{1 + (1 - n)\tan^2\alpha + n\tan^2\alpha}$$

$$= \frac{(1 - n)\tan\alpha(1 + \tan^2\alpha)}{1 + \tan^2\alpha} = (1 - n)\tan\alpha.$$

19. If $\sin\alpha + \cos\alpha = m$, then,

is $\sin^6\alpha + \cos^6\alpha = \dfrac{4 - 3(m^2 - 1)^2}{4}$ true for all real values of m.

Soln.: False. It is not true for all values of m.

$\sin\alpha + \cos\alpha = m \Rightarrow 1 + 2\sin\alpha \cdot \cos\alpha = m^2$

$\Rightarrow \sin 2\alpha = m^2 - 1$

which is possible if $m^2 - 1 \le 1$ or $m^2 \le 2$. Again

$$\sin^6\alpha + \cos^6\alpha = (\sin^2\alpha + \cos^2\alpha)^3 - 3\sin^2\alpha \cdot \cos^2\alpha(\sin^2\alpha + \cos^2\alpha)$$

$$= 1 - 3\sin^2\alpha \cdot \cos^2\alpha = 1 - \frac{3}{4}(\sin 2\alpha)^2$$

$$= 1 - \frac{3}{4}(m^2 - 1)^2 = \frac{4 - 3(m^2 - 1)^2}{4}.$$

But this is true only when $m^2 \le 2$.

20. Given that $\dfrac{\sin^4\alpha}{a} + \dfrac{\cos^4\alpha}{b} = \dfrac{1}{a + b}$, prove that

$$\frac{\sin^8\alpha}{a^3} + \frac{\cos^8\alpha}{b^3} = \frac{1}{(a + b)^3}.$$

Soln.: We are given that $(a + b)\left(\dfrac{\sin^4\alpha}{a} + \dfrac{\cos^4\alpha}{b}\right) = 1$

$$\Rightarrow \sin^4\alpha + \cos^4\alpha + \frac{b}{a}\sin^4\alpha + \frac{a}{b}\cos^4\beta = 1$$

But $1 = (\sin^2\alpha + \cos^2\alpha)^2$

$$\therefore \frac{b}{a}\sin^4\alpha + \frac{a}{b}\cos^2\alpha - 2a\sin^2\alpha \cdot \cos^2\alpha = 0$$

$$\Rightarrow \left(\sqrt{\frac{b}{a}}\sin^2\alpha - \sqrt{\frac{a}{b}}\cos^2\alpha\right)^2 = 0$$

$$\Rightarrow \frac{b}{a}\sin^4\alpha = \frac{a}{b}\cos^4\alpha$$

$$\Rightarrow \frac{\sin^4\alpha}{a^2} = \frac{\cos^4\alpha}{b^2} = k \text{ (say)}.$$

Since $\dfrac{\sin^4\alpha}{a} + \dfrac{\cos^4\alpha}{b} = \dfrac{1}{a + b}$, we obtain

$$ak + bk = \frac{1}{a + b} \Rightarrow k = \frac{1}{(a + b)^2}$$

$$\therefore \frac{\sin^8\alpha}{a^3} + \frac{\cos^8\alpha}{b^3} = \frac{(a^2k)^2}{a^3} + \frac{(b^2k)^2}{b^3} = ak^2 + bk^2$$

$$= (a + b)k^2 = (a + b)\frac{1}{(a + b)^4} = \frac{1}{(a + b)^3}.$$

21. If $\dfrac{ax}{\cos\theta}+\dfrac{by}{\sin\theta}=a^2-b^2$ and

$\dfrac{ax\sin\theta}{\cos^2\theta}-\dfrac{by\cos\theta}{\sin^2\theta}=0,$ show that

$$(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}.$$

Soln.: From second relation we have

$$ax\sin^3\theta = by\cos^3\theta$$

$$i.e.\ \frac{\cos^3\theta}{ax} = \frac{\sin^3\theta}{by} = k^3(\text{say})$$

$$\Rightarrow \cos\theta = k(ax)^{1/3} \text{ and } \sin\theta = k(by)^{1/3}$$

Squaring and adding we get

$$\cos^2\theta + \sin^2\theta = k^2(ax)^{2/3} + k^2(by)^{2/3}$$

$$\Rightarrow k^2 = \frac{1}{(ax)^{2/3}+(by)^{2/3}} \quad .....(1)$$

Substituting the values of $\sin\theta$ and $\cos\theta$ in the first given relation, we have $\dfrac{ax}{k(ax)^{1/3}} + \dfrac{by}{k(by)^{1/3}} = a^2-b^2$

$$\frac{1}{k}\left[(ax)^{2/3}+(by)^{2/3}\right] = a^2-b^2$$

$(ax)^{2/3}+(by)^{2/3}=k(a^2-b^2)$ *i.e* $k = \dfrac{(ax)^{2/3}+(by)^{2/3}}{a^2-b^2}$

$$\Rightarrow \frac{1}{(ax)^{2/3}+(by)^{2/3}} = \frac{\left[(ax)^{2/3}+(by)^{2/3}\right]^2}{(a^2-b^2)^2} \text{ from 1}$$

$$(a^2-b^2)^2 = \left((ax)^{2/3}+(by)^{2/3}\right)^3$$

i.e. $(ax)^{2/3}+(by)^{2/3}=(a^2-b^2)^{2/3}$.

22. If $\cos x = \tan y$, $\cos y = \tan z$ and $\cos z = \tan x$, Prove that $\sin x = \sin y = 2\sin 18° = \sin z$.

Soln.: We have $\cos^2 x - \tan^2 y = \sec^2 y - 1 = \cot^2 z - 1$

$$\Rightarrow 1+\cos^2 x = \cot^2 z = \frac{\cos^2 z}{1-\cos^2 z} = \frac{\tan^2 x}{1-\tan^2 x}$$

$$2-\sin^2 x = \frac{\sin^2 x}{\cos^2 x-\sin^2 x} = \frac{\sin^2 x}{1-2\sin^2 x}$$

$$\Rightarrow (2-\sin^2 x)(1-2\sin^2 x) = \sin^2 x$$

$$\Rightarrow 2\sin^4 x - 6\sin^2 x + 2 = 0$$

$$\Rightarrow \sin^4 x - 3\sin^2 x + 1 = 0$$

$$\therefore \sin^2 x = \frac{3\pm\sqrt{9-4}}{2} = \frac{1}{2}(3\pm\sqrt{5})$$

Since $\frac{1}{2}(3+\sqrt{5})>1$, we get $\sin^2 x=\frac{1}{2}(3-\sqrt{5})$

$$\Rightarrow \sin^2 x = \frac{1}{4}(6-2\sqrt{5}) = \frac{1}{4}(\sqrt{5}-1)^2$$

$$\Rightarrow \sin x = \frac{\sqrt{5}-1}{2} = 2\sin 18°$$

Similarly $\sin y = 2\sin 18°$ and $\sin z = 2\sin 18°$.

23. If $P_n = \cos^n\theta + \sin^n\theta$, Prove that $2P_6 - 3P_4 + 1 = 0$ and hence or otherwise prove that

$$6P_{10} - 15P_8 + 10P_6 - 1 = 0.$$

Soln.:

$2P_6 - 3P_4 + 1 = 2(\cos^6\theta + \sin^6\theta) - 3(\cos^4\theta + \sin^4\theta) + 1$

$= 2[(\cos^2\theta + \sin^2\theta)^3 - 3\sin^2\theta\cdot\cos^2\theta(\cos^2\theta + \sin^2\theta)]$

$\quad - 3[(\cos^2\theta + \sin^2\theta)^2 - 2\sin^2\theta\cdot\cos^2\theta] + 1$

$= 2[1 - 3\sin^2\theta\cdot\cos^2\theta] - 3(1 - 2\sin^2\theta\cdot\cos^2\theta) + 1 = 0.$

$\therefore\ 6P_{10} - 15P_8 + 10P_6 - 1$

$= 6(P_{10} - P_8) - 9(P_8 - P_6) + (P_6 - P_4) + P_4 - P_2$ $\quad (\because\ P_2 = 1)$

$= -\sin^2\theta\cdot\cos^2\theta(6P_6 - 9P_4 + P_2 + P_0)$

$= -3\sin^2\theta\cdot\cos^2\theta(2P_6 - 3P_4) - \sin^2\theta\cdot\cos^2\theta(1 + 2)$ $\quad (\because P_0 = 2)$

$= -3\sin^2\theta\cdot\cos^2\theta(-1) - 3\sin^2\theta\cdot\cos^2\theta = 0$

$[\because 2P_6 - 3P_4 + 1 = 0$ as proved$]$.

24. Prove that

$$\frac{1}{2\sin x}(\operatorname{cosec} 2x - \operatorname{cosec} 4x) = \frac{\cos 3x}{\sin 2x\cdot\sin 4x}$$

and hence find the sum upto n terms of the series

$$\frac{\cos 3x}{\sin 2x\cdot\sin 4x}+\frac{\cos 5x}{\sin 4x\cdot\sin 6x}+\frac{\cos 7x}{\sin 6x\cdot\sin 8x}+....$$

Soln.: L.H.S. $= \dfrac{1}{2\sin x\cdot\sin 2x} - \dfrac{1}{2\sin x\cdot\sin 4x}$

$$= \frac{\sin 4x - \sin 2x}{2\sin x\cdot\sin 2x\cdot\sin 4x}$$

$$= \frac{2\cos 3x}{2\sin x\cdot\sin 2x\cdot\sin 4x} = \frac{\cos 3x}{\sin 2x\cdot\sin 4x} = R.H.S.$$

Now,

$$\frac{\cos 3x}{\sin 2x\cdot\sin 4x}+\frac{\cos 5x}{\sin 4x\cdot\sin 6x}+\frac{\cos 7x}{\sin 6x\cdot\sin 8x}+....$$

$$=\frac{1}{2\sin x}[\operatorname{cosec}2x - \operatorname{cosec}4x + \operatorname{cosec}4x - \operatorname{cosec}6x + ... + \operatorname{cosec}2nx - \operatorname{cosec}(2n+2)x]$$

$$= \frac{1}{2\sin x}[\operatorname{cosec} 2x - \operatorname{cosec}(2n+2)x].$$

25. Prove that $4\sin 27° = (5+\sqrt{5})^{1/2} - (3-\sqrt{5})^{1/2}$

Soln.: We know $\sin 54° = \cos 36° = \dfrac{\sqrt{5}+1}{4}$

$$\Rightarrow \cos 54° = \sqrt{1-\left(\frac{\sqrt{5}+1}{4}\right)^2} = \frac{1}{4}\sqrt{10-2\sqrt{5}}$$

$$\Rightarrow 1 - 2\sin^2 27° = \frac{1}{4}\sqrt{10-2\sqrt{5}}$$

$$\Rightarrow 2\sin^2 27^\circ = 1 - \frac{1}{4}\sqrt{10-2\sqrt{5}}$$

$$16\sin^2 27^\circ = 8-2\sqrt{10-2\sqrt{5}}$$
$$= [(5+\sqrt{5})^{1/2}-(3-\sqrt{5})^{1/2}]^2$$
$$4\sin 27^\circ = [(5+\sqrt{5})^{1/2}-(3-\sqrt{5})^{1/2}]$$

[$\because$ $\sin 27^\circ$ is +ve].

PROBLEMS FOR PRACTICE

1. If A, B and C are angles of a triangle, show that

$$\tan^2\frac{A}{2} + \tan^2\frac{B}{2} + \tan^2\frac{C}{2} \geq 1.$$

2. If A, B and C are angles of a triangle such that A is obtuse, prove that $\tan B \cdot \tan C < 1$.

3. Prove that $\sin\frac{2\pi}{7} + \sin\frac{4\pi}{7} + \sin\frac{8\pi}{7} = \frac{\sqrt{7}}{2}$.

4. $\cot\theta\cdot\cot 2\theta + \cot 2\theta\cdot\cot 3\theta + 2 = \cot\theta(\cot\theta - \cot 3\theta)$.

5. If $\tan\left(\frac{\pi}{4}+\frac{y}{2}\right) = \tan^3\left(\frac{\pi}{4}+\frac{x}{2}\right)$, then prove that

$$\sin y = \sin x\cdot\frac{3+\sin^2 x}{1+3\sin^2 x}.$$

6. Prove that

$$\frac{1+\sin A}{\cos A}+\frac{\cos B}{1-\sin B} = \frac{2\sin A - 2\sin B}{\sin(A-B)+\cos A-\cos B}.$$

7. If $m\sin(\alpha+\beta) = \cos(\alpha-\beta)$,

prove that $\frac{1}{1-m\sin^2\alpha} + \frac{1}{1-m\sin^2\beta} = \frac{2}{1-m^2}$.

8. Prove that $x^2 - x\cos(A+B) + 1$ is a factor of $2x^4 + 4x^3\sin A\cdot\sin B - x^2(\cos 2A + \cos 2B) + 4x\cos A\cdot\cos B - 2$ and also find other factor(s).

[*Ans.:* $2x^2 + 2x\cos(A-B) - 2$].

9. Prove that $\sin\theta\cdot\sec 3\theta = \frac{1}{2}(\tan 3\theta - \tan\theta)$ and hence or otherwise find the sum to n terms of series $\sin\theta\cdot\sec 3\theta + \sin(3\theta)\cdot\sec(3^2\theta) + \sin(3^2\theta)\cdot\sec(3^3\theta)+\ldots$

[*Ans .:* $\frac{1}{2}(\tan 3^n\theta - \tan\theta)$].

10. If $a\sin^2\theta + b\cos^2\theta = m$, $b\sin^2\phi + a\cos^2\phi = n$ and $a\tan\theta = b\tan\phi$ then $\frac{1}{n}+\frac{1}{m} = \frac{1}{a}+\frac{1}{b}$.

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Test Your CALCULUS for JEE & Other Engineering Entrance.

Time : 90 min. *Marks : 40*

State True/False (4 × 1 = 4 marks)

1. Greatest and least values of the function

$$f(x) = \tan^{-1}x \text{ are } \frac{\pi}{2} \text{ and } -\frac{\pi}{2} \text{ respectively.}$$

2. $f(x) = \tan x$ has infinite critical points.

3. All the critical points of the function $f(x) = \sin x$ are stationary and extremum points.

4. $f(x) = \sin^{-1}x$ has no extremum.

Fill in the blanks

5. If $f(x) = 1, \quad x > 0$
$= a, \quad x = 0$
$= -1, \quad x < 0$ then

(i) $x = 0$ is a point of maxima when $a \in$ ....

(ii) $x = 0$ is a point of minima when $a \in$ ....

(iii) $x = 0$ is not an extremum point when $a \in$ ....

(3 marks)

6. Given $f(x) = x^2, \ x \neq 0$
$= a, \quad x = 0$ then

(i) $x = 0$ is a point of extremum when $a \in$ ....

(ii) $x = 0$ is a stationary point when $a \in$ ....

(2 marks)

7. If $f(x) = \sin x, x \neq 0$
$= a, \quad x = 0$

then $x = 0$ is a critical point when $a \in$ ....

(2 marks)

8. If $f(x)$ is differentiable and $f'(4) = 5$ then find

$$\lim_{x \to 2} \frac{f(4) - f(x^2)}{x - 2}.$$ (3 marks)

9. If $f(x) = a \ln x + bx^2 + x$ has extremum at $x = 1$ and $x = 2$ then find a and b and test for extremum at $x = 1$ and $x = 2$. (4 marks)

10. Test the following function for extremum

$$f(x) = x^2 + x + 1, \ x < 1$$
$$= x^2 - 4x + 6, x \geq 1.$$ (4 marks)

11. If $2f(x+y) = f(x)f(y) - f(x) - f(y) + 3 \ \ \forall \ x, y$
$f'(0) = 6$ and $f(1) = 2e^3 + 1$,
then test the differentiability of $f(x)$ and find $f'(1)$.

(4 marks)

12. Differentiate $f(x) = x^x$ by definition. (3 marks)

13. Find the greatest and least value of the function $f(x) = e^{-x}(x^2 + x - 5)$ in the interval $[-4, 4]$. (3 marks)

14. Given $f(x) = \sin x - \sin^{-1}x$, test $x = 0$ for extremum.

(3 marks)

15. If $f(x) = [\sin^{-1}(\sqrt[3]{x})]^3$, find $f'(x)$. (5 marks)

ANSWERS

1. F 2. F 3. T 4 T

5.(i) $(1, \infty)$ (ii) $(-\infty, -1)$ (iii) $[-1, -1]$

6.(i) R (ii) $[0]$

7. $R - [0]$ 8. -20

9. $a = -\frac{2}{3}, \ b = -\frac{1}{6}, \ x = 1$ is minima, $x = 2$ is maxima

10. Maxima at $x = 1$, minima at $x = -\frac{1}{2}, 2$

11. $f(x)$ is diff. $\forall \ x, f'(1) = 6e^3$

12. $f'(x) = x^x(\ln x + 1)$

13. $7e^4, -3e^2$ 14. no extremum

15. $$f'(x) = \left[\sin^{-1}\left(\sqrt[3]{x}\right)\right]^2 \cdot \frac{x^{-\frac{2}{3}}}{\sqrt{1 - x^{\frac{2}{3}}}}, \ x \in (-1, 0) \cup (0, 1)$$
$$= 1, \ x = 0$$

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since any odd number has the form $2^m k - 1$ for some smaller odd number k. Take $a_i = 2^i((2^m - 1)k - 1)$ for $i = 0, 1, \ldots, m - 1$. Then $2a_i + 1 = 2^{i+1}(2^m - 1)k - (2^{i+1} - 1)$ and $a_i + 1 = 2^i(2^m - 1)k - (2^i - 1)$. So the product of the $(2a_i + 1)$'s divided by the product of the $(a_i + 1)$'s is $2^m(2^m - 1)k - (2^m - 1)$ divided by $(2^m - 1)k$, or $(2^m k - 1)/k$. Thus if we take these a_is together with those giving k, we get $2^m k - 1$, which completes the induction.

4. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.

Soln.: Answer: $(a, b) = (11, 1), (49, 1)$ or $(7k^2, 7k)$.

If $a < b$, then $b \geq a + 1$, so $ab^2 + b + 7 > ab^2 + b \geq (a + 1)(ab + 1) = a^2b + a + ab \geq a^2b + a + b$. So there can be no solutions with $a < b$. Assume then that $a \geq b$. Let k = the integer $(a^2b + a + b)/(ab^2 + b + 7)$. We have $(a/b + 1/b)(ab^2 + b + 7) = ab^2 + a + ab + 7a/b + 7/b + 1 > ab^2 + a + b$. So $k < a/b + 1/b$. Now if $b \geq 3$, then $(b - 7/b) > 0$ and hence $(a/b - 1/b)(ab^2 + b + 7) = ab^2 + a - a(b - 7/b) - 1 - 7/b < ab^2 + a < ab^2 + a + b$. Hence either $b = 1$ or 2 or $k > a/b - 1/b$.

If $a/b - 1/b < k < a/b + 1/b$, then $a - 1 < kb < a + 1$. Hence $a = kb$. This gives the solution $(a, b) = (7k^2, 7k)$.

It remains to consider $b = 1$ and 2. If $b = 1$, then $a + 8$ divides $a^2 + a + 1$ and hence also $a(a + 8) - (a^2 + a + 1) = 7a - 1$, and hence also $7(a + 8) - (7a - 1) = 57$. The only factors bigger than 8 are 19 and 57, so $a = 11$ or 49. It is easy to check that $(a, b) = (11, 1)$ and $(49, 1)$ are indeed solutions.

If $b = 2$, then $4a + 9$ divides $2a^2 + a + 2$, and hence also $a(4a + 9) - 2(2a^2 + a + 2) = 7a - 4$, and hence also $7(4a + 9) - 4(7a - 4) = 79$. The only factor greater than 9 is 79, but that gives $a = 35/2$ which is not integral. Hence there are no solutions for $b = 2$.

5. Let I be the incenter of triangle ABC. Let the incircle of ABC touch the sides BC, CA and AB at K, L and M respectively. The line through B parallel to MK meets the lines LM and LK at R and S, respectively. Prove that angle RIS is acute.

Soln.: We show that $RI^2 + SI^2 - RS^2 > 0$. The result then follows from the cosine rule.

BI is perpendicular to MK and hence also to RS. So $IR^2 = BR^2 + BI^2$ and $IS^2 = BI^2 + BS^2$. Obviously $RS = RB + BS$, so $RS^2 = BR^2 + BS^2 + 2\,BR \cdot BS$. Hence $RI^2 + SI^2 - RS^2 = 2BI^2 - 2\,BR \cdot BS$. Consider the ΔBRS.

The angles at B and M are $90 - \dfrac{B}{2}$ and $90 - \dfrac{A}{2}$, so the angle at R is $90 - \dfrac{C}{2}$. Hence $\dfrac{BR}{BM} = \dfrac{\cos\frac{A}{2}}{\cos\frac{C}{2}}$ (using the sine rule). Similarly, considering the

$$\Delta BKS, \quad \frac{BS}{BK} = \frac{\cos\frac{C}{2}}{\cos\frac{A}{2}}.$$

So $BR \cdot BS = BM \cdot BK = BK^2$. Hence $RI^2 + SI^2 - RS^2 = 2(BI^2 - BK^2) = 2IK^2 > 0$.

6. Consider all functions f from the set N of all positive integers into itself satisfying $f(t^2f(s)) = s(f(t))^2$ for all s and t in N. Determine the least possible value of $f(1998)$.

Soln.: Let $f(1) = k$. Then $f(kt^2) = f(t)^2$ and $f(f(t)) = k^2t$. Also $f(kt)^2 = 1 \cdot f(kt)^2 = f(k^3t^2) = f(1^2f(f(kt^2))) = k^2f(kt^2) = k^2f(t)^2$. Hence $f(kt) = kf(t)$.

By an easy induction $k^n f(t^{n+1}) = f(t)^{n+1}$. But this implies that k divides $f(t)$. For suppose the highest power of a prime p dividing k is $a > b$, the highest power of p dividing $f(t)$. Then $a > b(1 + 1/n)$ for some integer n. But then $na > (n + 1)b$, so k^n does not divide $f(t)^{n+1}$. Contradiction.

Let $g(t) = f(t)/k$. Then $f(t^2f(s)) = f(t^2kg(s)) = kf(t^2g(s)) = k^2g(t^2g(s))$, whilst $s\,f(t)^2 = k^2s\,f(t)^2$. So $g(t^2g(s)) = s\,g(t)^2$. Hence g is also a function satisfying the conditions which evidently has smaller values than f (for $k > 1$). It also satisfies $g(1) = 1$. Since we want the smallest possible value of $f(1998)$ we may restrict attention to functions f satisfying $f(1) = 1$.

Thus we have $f(f(t) = t$ and $f(t^2) = f(t)^2$. Hence $f(st)^2 = f(s^2t^2) = f(s^2f(f(t^2))) = f(s)^2f(t^2) = f(s)^2f(t)^2$.

So $f(st) = f(s)\,f(t)$.

Suppose p is a prime and $f(p) = m \cdot n$. Then $f(m)f(n) = f(mn) = f(f(p)) = p$, so one of $f(m)$, $f(n) = 1$. But if $f(m) = 1$, then $m = f(f(m)) = f(1) = 1$. So $f(p)$ is prime. If $f(p) = q$, then $f(q) = p$.

Now we may define f arbitarily on the primes subject only to the conditions that each f(prime) is prime and that if $f(p) = q$, then $f(q) = p$. For suppose that $s = p_1^{a_1} \ldots p_r^{a_r}$ and that $f(p_i) = q_i$. If t has any additional prime factors not included in the q_i, then we may add additional p's to the expression for s so that they are included (taking the additional a's to be zero). So suppose $t = q_1^{b_1} \ldots q_r^{b_r}$. Then $t^2f(s) = q_1^{2b_1 + a_1} \ldots q_r^{2b_r + a_r}$ and hence $f(t^2f(s)) = p_1^{2b_1 + a_1} \ldots p_r^{2b_r + a_r} = s\,f(t)^2$.

We want the minimum possible value of $f(1998)$. Now $1998 = 2 \cdot 3^3 \cdot 37$, so we achieve the minimum value by taking $f(2) = 3$, $f(3) = 2$, $f(37) = 5$ (and $f(37) = 5$). This gives $f(1998) = 3 \cdot 2^3 \cdot 5 = 120$. ■

39th International Mathematical Olympiad, Taipei, 1998

1. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.

Soln.: Let AC and BD meet at X. Let H, K be the feet of the perpendiculars from P to AC, BD respectively. We wish to express the areas of ABP and CDP in terms of more tractable triangles. There are essentially two different configurations possible. In the first, we have area PAB = area ABX + area PAX + area PBX, and area PCD = area CDX – area PCX – area PDX. So if the areas being equal is equivalent to: area ABX – area CDX + area PAX + area PCX + area PBX + area PDX. ABX and CDX are right-angled, so we may write their areas as $\frac{AX \cdot BX}{2}$ and $\frac{CX \cdot DX}{2}$. We may also put $AX = AM - MX = AM - PN$, $BX = BN - PM$, $CX = CM + PN$, $DX = DN + PM$. The other triangles combine in pairs to give area ACP + area $BDP = AC \cdot PM + BD \cdot PN$. This leads, after some cancellation to $AM \cdot BN = CM \cdot DN$. There is a similar configuration with the roles of AB and CD reversed. The second configuration is area PAB = area ABX + area PAX – PBX, area PCD = area CDX + area PDX – area PCX. In this case $AX = AM + PN$, $BX = BN - PM$, $CX = CM - PN$, $DX = DN + PM$. But we end up with the same result: $AM \cdot BN = CM \cdot DN$.

Now if $ABCD$ is cyclic, then it follows immediately that P is the center of the circumcircle and $AM = CM$, $BN = DN$. Hence the areas of PAB and PCD are equal. Conversely, suppose the areas are equal. If $PA > PC$, then $AM > CM$. But since $PA = PB$ and $PC = PD$ (by construction), $PB > PD$, so $BN > DN$. So $AM \cdot BN > CM \cdot DN$. Contradiction. So PA is not greater than PC. Similarly it cannot be less. Hence $PA = PC$. But that implies $PA = PB = PC = PD$, so $ABCD$ is cyclic.

2. In a competition, there are a contestants and b judges, where b ≥ 3 is an old integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $\frac{k}{a} \geq \frac{b-1}{2b}$.

Soln.: Let us count the number N of triples (judge, judge, contestant) for which the two judges are distinct and rate the contestant the same. There are $\frac{b(b-1)}{2}$ pairs of judges in total and each pair rates at most k contestants the same, so $N \leq \frac{kb(b-1)}{2}$.

Now consider a fixed contestant X and count the number of pairs of judges rating X the same. Suppose x judges pass X, then there are $\frac{x(x-1)}{2}$ pairs who pass X and $\frac{(b-x)(b-x-1)}{2}$ who fail X, so a total of $\frac{(x(x-1)(b-x-1))}{2} = \frac{(2x^2-2bx+b^2-b)}{2}$ pairs rate X the same. But $\frac{(x(x-1)+(b-x)(b-x-1))}{2} = \frac{(2x^2-2bx+b^2-b)}{2} = \left(\frac{x-b}{2}\right)^2 + \frac{b^2}{4} - \frac{b}{2} \geq \frac{b^2}{4} - \frac{b}{2} \geq \frac{b^2}{4} - \frac{b}{2} = \frac{(b-1)^2}{4} - \frac{1}{4}$. But $\frac{(b-1)^2}{4}$ is an integer (since b is odd), so the number of pairs rating X the same is at least $\frac{(b-1)^2}{4}$. Hence $N \geq \frac{a(b-1)^2}{4}$. Putting the two inequalities together gives $\frac{k}{a} \geq \frac{(b-1)}{2b}$.

3. For any positive integer n, let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $\frac{d(n^2)}{d(n)} = k$ for some n.

Soln.: Let $n = p_1{}^{a_1} \ldots p_r{}^{a_r}$.

Then $d(n) = (a_1 + 1)(a_2 + 1) \ldots (a_r + 1)$, and $d(n^2) = (2a_1 + 1)(2a_2 + 1) \ldots (2a_r + 1)$. So the a_i must be chosen so that $(2a_1 + 1)(2a_2 + 1) \ldots (2a_r + 1) = k(a_1 + 1)(a_2 + 1) \ldots (a_r + 1)$. Since all $(2a_i + 1)$ are odd, this clearly implies that k must be odd. We show that conversely, given any odd k, we can find a_i. We use a form of induction on k. First, it is true for $k = 1$ (take $n = 1$). Second, we show that if it is true for k, then it is true for $2^m k - 1$. That is sufficient,

$= \text{coeff. of } x^k \text{ in } (1-x)^{-n} = {}^{n+k-1}C_k$

So the probability that the sum of the digits is k

$$= \frac{{}^{n+k-1}C_k}{10^n - 1}.$$

(ii) Probability that the sum of the digits is almost 9

$$= \sum_{k=1}^{9} \frac{{}^{n+k-1}C_k}{10^n - 1} = \frac{1}{10^n - 1}\left({}^{n}C_1 + {}^{n+1}C_2 + \ldots + {}^{n+8}C_9\right)$$

$= \frac{1}{10^n - 1}({}^{n+9}C_9 - 1)$, using the result of 9(a).

10. Number of elements in A is n

Total number of subsets of A is 2^n

The number of ways of selecting 2 subsets $= {}^{2^n}C_2$.

This is the total number of ways for the event in question.

No. of r-element subsets of A $(a \le r \le n) = {}^nC_r$

But ${}^nC_0 = 1$, ${}^nC_n = 1$ and ${}^nC_r \ge 2$ for $1 \le r \le n-1$

so there are $n-1$ groups having atleast two members

The number of ways in which 2 subsets of A belong to the same group $= {}^{{}^nC_1}C_2 + {}^{{}^nC_2}C_2 + \ldots + {}^{{}^nC_{n-1}}C_2$

$$= \sum_{r=1}^{n-1} {}^{{}^nC_r}C_2$$

This is the number of ways favourable to the event.

Hence the required probability $= \dfrac{\left(\sum_{r=1}^{n-1} {}^{{}^nC_r}C_2\right)}{{}^{2^n}C_2}$.

■

... contd. from page 27.

gained nearly a stone in weight.... There has never been any sign of any diminuation in his extraordinary mathematical talents. He has produced less, naturally, during his illness but the quality has been the same. ....

He will return to India with a scientific standing and reputation such as no Indian has enjoyed before, and I am confident that India will regard him as the treasure he is. His natural simplicity and modesty has never been affected in the least by success - indeed all that is wanted is to get him to realise that he really is a success.

Ramanujan sailed to India on 27 February 1919 arriving on 13 March. However his health was very poor and, despite medical treatment, he died there the following year.

The letters Ramanujan wrote to Hardy in 1913 had contained many fascinating results. Ramanujan worked out the *Riemann series*, the *elliptic integrals*, *hypergeometric series* and *functional equations* of the *zeta function*. On the other hand he had only a vague idea of what constitutes a mathematical proof. Despite many brilliant results, some of his theorems on prime numbers were completely wrong.

Ramanujan independently discovered results of Gauss, Kummer and others on hypergeometric series. Ramanujan's own work on partial sums and products of hypergeometric series have led to major development in the topic. Perhaps his most famous work was on the number $p(n)$ of partitions of an integer n into summands. MacMahon had produced tables of the value of $p(n)$ for small numbers n, and Ramanujan used this numerical data to conjecture some remarkable properties some of which he proved using elliptic functions. Other were only proved after Ramanujan's death.

In a joint paper with Hardy, Ramanujan gave an asymptotic formula for $p(n)$. It had the remarkable property that it appeared to give the correct value of $p(n)$, and this was later proved by Rademacher.

Ramanujan left a number of unpublished notebooks filled with theorems that mathematicians have continued to study. G N Watson, Mason Professor of Pure Mathematics at Birmingham from 1918 to 1951 published 14 papers under the general title Theorems stated by Ramanujan and in all he published nearly 30 papers which were inspired by Ramanujan's work. Hardy passed on to Watson the large number of manuscripts of Ramanujan that he had, both written before 1914 and some written in Ramanujan's last year in India before his death.

■

$$= \frac{2a}{b}(a^2 - b^2) > 0$$

$\therefore f(x)$ has a local minimum at $x = \sin^{-1}\frac{b}{a}$

and $x = \pi - \sin^{-1}\frac{b}{a}$.

The minimum value $= a^2\left(\frac{b}{a}\right) + b^2\left(\frac{a}{b}\right) = 2ab$

(at both places).

Case III : Let $a = b$

Then $f'(x) = 0$ at $x = \frac{\pi}{2}$ only.

At this point, $f''(x) = -a^2 + b^2 = 0$

We now go for $f'''(x)$ and $f^{(4)}(x)$. From (1)

$$f'''(x) = -a^2\cos x - 3b^2\operatorname{cosec}^3 x\cot x - 2b^3\operatorname{cosec}^3 x\cot x - b^2\cot^3 x\operatorname{cosec} x$$

$$= -a^2\cos x - 6b^2\operatorname{cosec}^3 x\cot x + b^2\operatorname{cosec} x\cot x$$

which is 0 at $x = \frac{\pi}{2}$

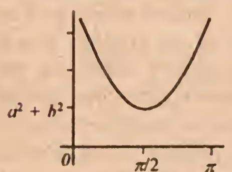
and $f^{(4)}(x) = a^2\sin x + 18b^2\operatorname{cosec}^3 x\cot^2 x + 6b^2\operatorname{cosec}^5 x - b^2\operatorname{cosec}^3 x - b^2\operatorname{cosec} x\cot^2 x$

We have $f^{(4)}\left(\frac{\pi}{2}\right) = a^2 + 6b^2 - b^2 = a^2 + 5b^2 = 6a^2$

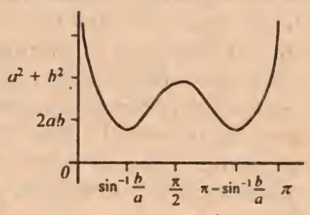
which is positive.

$\therefore f(x)$ has local minimum at $x = \frac{\pi}{2}$

and the minimum value $= a^2 + b^2$.



$a \le b$



$a > b$

8. We observe that $x^2 - 3x + 3$ is positive for all $x \in R$

It attains the minimum value $\frac{4ac - b^2}{4a} = \frac{3}{4}$ at $x = \frac{3}{2}$

and the graph is a parabola.

$x^2 - 3x + 3 = 1$ when $x^2 - 3x + 2 = 0$,

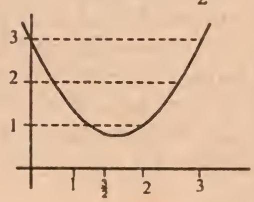
i.e. when $x = 1, 2$

$x^2 - 3x + 3 = 2$ when $x^2 - 3x + 1 = 0$, *i.e* $x = \frac{3 \pm \sqrt{5}}{2}$

$x^2 - 3x + 3 = 3$ when $x^2 - 3x = 0$,

i.e. $x = 0, 3$.

The function $x^2 - 3x + 3$ has the following graph:



Thus, $[x^2 - 3x + 3] =$

$$\begin{cases} 0 & \text{for } 1 < x < 2 \\ 1 & \text{for } \frac{3-\sqrt{5}}{2} < x \le 1 \text{ and } 2 \le x < \frac{3+\sqrt{5}}{2} \\ 2 & \text{for } 0 < x \le \frac{3-\sqrt{5}}{2} \text{ and } \frac{3+\sqrt{5}}{2} \le x < 3 \end{cases}$$

Hence $\int_0^3 [x^2 - 3x + 3]dx =$

$$\int_1^2 0\,dx + \int_{\frac{3-\sqrt{5}}{2}}^1 1\,dx + \int_2^{\frac{3+\sqrt{5}}{2}} 1\,dx + \int_0^{\frac{3-\sqrt{5}}{2}} 2dx + \int_{\frac{3+\sqrt{5}}{2}}^3 2dx$$

$$= 0 + \left(1 - \frac{3-\sqrt{5}}{2}\right) + \left(\frac{3+\sqrt{5}}{2} - 2\right) + 2\left(\frac{3-\sqrt{5}}{2}\right) + 2\left(3 - \frac{3+\sqrt{5}}{2}\right)$$

$$= 5 - \sqrt{5}.$$

9(a). We use induction on m. Let $m = 0$.

We know that ${}^nC_{r-1} + {}^nC_r = {}^{n+1}C_r \Rightarrow {}^nC_r = {}^{n+1}C_r - {}^nC_{r-1}$.

So the result is true for $m = 0$.

Assume the result for m.

i.e. ${}^nC_r + {}^{n+1}C_{r+1} + ... + {}^{n+m}C_{r+m}$

$$= {}^{n+m+1}C_{r+m} - {}^nC_{r-1}$$

Adding ${}^{n+m+1}C_{r+m+1}$ on both sides,

$${}^nC_r + {}^{n+1}C_{r+1} + ... + {}^{n+m+1}C_{r+m+1} =$$
$${}^{n+m+1}C_{r+m} - {}^nC_{r-1} + {}^{n+m+1}C_{r+m+1}$$
$$= {}^{n+m+2}C_{r+m+1} - {}^nC_{r-1}$$

$\therefore$ The result is true for $m + 1$

$\therefore$ By induction, the result follows for any non-negative integer m.

(b) (i) Number of pages in the book $10^n - 1 = 999...$ (n times)

We express each page number as an $n-$ bit decimal number by pre-fixing suitable number of zeros.

For example, 1 is written as .000...01 (0 repeated n–1times)

We observe that the sum of the digits in a page number is k if the sum of the digits in the corresponding n bit decimal number is k.

$\therefore$ Number of pages in which sum of the digits is k

= No. of the non-negative integral solutions of the equation $x_1 + x_2 + ... + x_n = k$

= coeff. of x^k in $(1 + x + x^2 + ... + x^k)^n$

= coeff. of x^k in $(1 - x^{k+1})^n (1 - x)^{-n}$

(2) *there are results which, so far as I know, are new and interesting, but interesting rather from their curiosity and apparent difficulty than their importance;*
(3) *there are results which appear to be new and important...*
Ramanujan was delighted with Hardy's reply and when he wrote again he said:-

I have found a friend in you who views my labours sympathetically. ... I am already a half starving man. To preserve my brains I want food and this is my first consideration. Any sympathetic letter from you will be helpful to me here to get a scholarship either from the university or from the government.

Indeed the University of Madras did give Ramanujan a scholarship in May 1913 for two years and, in 1914, Hardy brought Ramanujan to Trinity College, Cambridge, to begin an extraordinary collaboration. Setting this up was not an easy matter.

Ramanujan was an orthodox Brahmin and so was a strict vegetarian. His religion should have prevented him from travelling but this difficulty was overcome, partly by the work of E H Neville who was a colleague of Hardy's at Trinity College and who met with Ramanujan while lecturing in India.

Ramanujan sailed from India on 17 March 1914. It was a calm voyage except for three days on which Ramanujan was seasick. He arrived in London on 14 April 1914 and was met by Neville. After four days in London they went to Cambridge and Ramanujan spent a couple of weeks in Neville's home before moving into rooms in Trinity College on 30th April. Right from the beginning, however, he had problems with his diet. The outbreak of World War I made obtaining special items of food harder and it was not long before Ramanujan had health problems.

Right from the start Ramanujan's collaboration with Hardy led to important results. Hardy was, however, unsure how to approach the problem of Ramanujan's lack of formal education. He wrote:-
What was to be done in the way of teaching him modern mathematics? The limitations of his knowledge were as startling as its profundity.

Littlewood was asked to help teach Ramanujan rigorous mathematical methods. However he said:-
... that it was extremely difficult because every time some matter, which it was thought that Ramanujan needed to know, was mentioned, Ramanujan's response was an avalanche of original ideas which made it almost impossible for Littlewood to persist in his original intention.

The war soon took Littlewood away on war duty but Hardy remained in Cambridge to work with Ramanujan. Even in his first winter in England, Ramanujan was ill and he wrote in March 1915 that he had been ill due to the winter weather and had not been able to publish anything for five months. What he did publish was the work he did in England, the decision having been made that the results he had obtained while in India, many of which he had communicated to Hardy in his letters, would not be published until the war had ended.

On 16 March 1916 Ramanujan graduated from Cambridge with a Bachelor of Science by Research (the degree was called a Ph.D. from 1920). He had been allowed to enrol in June 1914 despite not having the proper qualifications. Ramanujan's dissertation was on Highly composite numbers and consisted of seven of his papers published in England.

Ramanujan fell seriously ill in 1917 and his doctors feared that he would die. He did improve a little by September but spent most of his time in various nursing homes. In February 1918 Hardy wrote:-

Batty Shaw found out, what other doctors did not know, that he had undergone an operation about four years ago. His worst theory was that this had really been for the removal of a malignant growth, wrongly diagnosed. In view of the fact that Ramanujan is no worse than six months ago, he has now abandoned this theory - the other doctors never gave it any support. Tubercle has been the provisionally accepted theory, apart from this, since the original idea of gastric ulcer was given up. ... Like all Indians he is fatalistic, and it is terribly hard to get him to take care of himself.

On 18 February 1918 Ramanujan was elected a fellow of the Cambridge Philosophical Society and then three days later, the greatest honour that he would receive, his name appeared on the list for election as a fellow of the Royal Society of London.

He had been proposed by an impressive list of mathematicians, namely Hardy, MacMahon, Grace, Larmor, Bromwich, Hobson, Baker, Littlewood, Nicholson, Young, Whittaker, Forsyth and Whitehead. His election as a fellow of the Royal Society was confirmed on 2 May 1918, then on 10 October 1918 he was elected a Fellow of Trinity College Cambridge, the fellowship to run for six years.

The honours which were bestowed on Ramanujan seemed to help his health improve a little and he renewed his efforts at producing mathematics. By the end of November 1918 Ramanujan's health had greatly improved. Hardy wrote in a letter:-

I think we may now hope that he has turned to corner, and is on the road to a real recovery. His temperature has ceased to be irregular, and he has

... contd. to page 32.

10 Selected Problems

By : Ch. Satyanarayana Murthy & Dr. A.Sita Ram Murti
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1. Solve the equation

$$(4+\sqrt{15})^{2x^2-3}+(4-\sqrt{15})^{2x^2-3}=7\sqrt{10}.$$

2. In a ΔABC, prove that $\tan A : \tan B : \tan C = p : q : r$ iff $a : b : c = \sqrt{p(q+r)} : \sqrt{q(r+p)} : \sqrt{r(p+q)}$. Deduce that a^2, b^2, c^2 are in AP iff $\tan A$, $\tan B$, $\tan C$ are in HP.

3. If p, q, r are of the same sign, show that

$$\cot^{-1}\frac{pq+1}{p-q}+\cot^{-1}\frac{qr+1}{q-r}+\cot^{-1}\frac{rp+1}{r-p}=\pi \text{ or } 2\pi.$$

4. In a triangle ABC, the vertices A and B are (x_1, y_1) and (x_2, y_2). Prove that the circumcentre of triangle ABC is $\left(\frac{x_1+x_2\pm\cot C(y_1-y_2)}{2}, \frac{y_1+y_2\mp\cot C(x_1-x_2)}{2}\right)$

Hence obtain 1) the third vertex of an equilateral triangle two of whose vertices are (x_1, y_1) and (x_2, y_2). 2) the right angle vertex of a right-angled isosceles triangle when the ends of the hypotenuse are (x_1, y_1) and (x_2, y_2).

5.(a) Prove that a circle is orthogonal to every point of the circle, regarded as a point circle.

(b) All the circles passing through $A(3, 1)$ and intersecting the circle $x^2 + y^2 + 4x - 6y - \frac{3}{2} = 0$ orthogonally, pass through another point B. Find B.

6. For an ellipse, prove that there are exactly two points of the curve the normals at which pass through any focus.

7. Discuss the local maxima or minima of the function $a^2 \sin x + b^2 \operatorname{cosec} x$ where $x \in (0, \pi)$.

8. Evaluate the integral $\int_0^3 [x^2-3x+3]dx$ where [ ] denotes the greatest integer function.

9.(a) Prove that ${}^nC_r + {}^{n+1}C_{r+1} + {}^{n+2}C_{r+2} + \dots + {}^{n+m}C_{r+m} = {}^{n+m+1}C_{r+m} - {}^nC_{r-1}$ for any non-negative integer m.

(b) A book contains $10^n - 1$ pages. If a page is selected at random, find the probability that the sum of the digits of its number is (i) k, where $1 \le k \le 9$ (ii) atmost 9.

10. A set A has n elements $(n > 1)$. Subsets of A having the same number of elements are grouped together. If two subsets of A are chosen at random, find the probability that they belong to the same group.

SOLUTIONS

1. We have $(4+\sqrt{15})(4-\sqrt{15}) = 1$

put $(4+\sqrt{15})^{2x^2-3} = t$. Then $(4-\sqrt{15})^{2x^2-3} = \frac{1}{t}$.

$$\Rightarrow t + \frac{1}{t} = 7\sqrt{10} \qquad \Rightarrow t^2 - 7\sqrt{10}\,t + 1 = 0$$

$$\therefore t = \frac{7\sqrt{10}\pm\sqrt{490-4}}{2} = \frac{(\sqrt{5}\pm\sqrt{3})^3}{2\sqrt{2}}.$$

$$= \left\{\left(\frac{\sqrt{5}\pm\sqrt{3}}{\sqrt{2}}\right)^2\right\}^{\frac{3}{2}} = \left(\frac{8\pm2\sqrt{15}}{2}\right)^{\frac{3}{2}} = (4\pm\sqrt{15})^{3/2}$$

$= (4+\sqrt{15})^{\pm3/2}$. Hence $(4+\sqrt{15})^{2x^2-3} = (4+\sqrt{15})^{\pm3/2}$

$\therefore 2x^2 - 3 = \pm\frac{3}{2}$ so $x^2 = \frac{9}{4}$ or $\frac{3}{4}$

$\Rightarrow x = \pm\frac{3}{2}$ or $\pm\frac{\sqrt{3}}{2}$.

2. $\tan A : \tan B : \tan C = p : q : r$

$$\Leftrightarrow \frac{\tan A}{p} = \frac{\tan B}{q} = \frac{\tan C}{r}$$

$$\Leftrightarrow \frac{\sin A}{p\cos A} = \frac{\sin B}{q\cos B} = \frac{\sin C}{r\cos C}$$

$$\Leftrightarrow \frac{a/2R}{p\frac{(b^2+c^2-a^2)}{2bc}} = \frac{b/2R}{q\frac{(c^2+a^2-b^2)}{2ca}} = \frac{c/2R}{r\frac{(a^2+b^2-c^2)}{2ab}}$$

$$\Leftrightarrow \frac{abc}{R}\cdot\frac{1}{p(b^2+c^2-a^2)} = \frac{abc}{R}\cdot\frac{1}{q(c^2+a^2-b^2)}$$

$$= \frac{abc}{R}\cdot\frac{1}{r(a^2+b^2-c^2)}$$

$$\Leftrightarrow \frac{1}{p(b^2+c^2-a^2)} = \frac{1}{q(c^2+a^2-b^2)} = \frac{1}{r(a^2+b^2-c^2)}$$

$$\Leftrightarrow \frac{qr}{b^2+c^2-a^2} = \frac{rp}{c^2+a^2-b^2} = \frac{pq}{a^2+b^2-c^2}$$

$$\Leftrightarrow \frac{rp+pq}{2a^2} = \frac{pq+qr}{2b^2} = \frac{qr+rp}{2c^2}$$

$\Leftrightarrow a^2 : b^2 : c^2 = p(q+r) : q(r+p) : r(p+q)$

$\Leftrightarrow a : b : c = \sqrt{p(q+r)} : \sqrt{q(r+p)} : \sqrt{r(p+q)}$

Deduction : a^2, b^2, c^2 are in AP

$\Leftrightarrow p(q+r)$, $q(r+p)$, $r(p+q)$ are in AP.

$\Leftrightarrow q(r+p) - p(q+r) = r(p+q) - q(r+p)$

$\Leftrightarrow qr - pr = pr - pq$

$$\Leftrightarrow 2pr = q(p+r) \Leftrightarrow q = \frac{2pr}{p+r}$$

$\Leftrightarrow p, q, r$ are in H.P. $\Leftrightarrow \tan A, \tan B, \tan C$ are in HP.

3. Note that $\tan^{-1}x : R \to (-\pi/2, \pi/2)$ and $\cot^{-1}x : R \to (0, \pi)$

If x is positive, then $\cot^{-1}x = \tan^{-1}\frac{1}{x}$

If x is negative, then $\cot^{-1}x = \pi - \cot^{-1}(-x)$

Also, if x and y are both positive or both negative, we have

$$\tan^{-1}x - \tan^{-1}y = \tan^{-1}\frac{x-y}{1+xy}$$

Since p, q, r are of the same sign

$pq + 1$, $qr + 1$, $rp + 1$ are all positive.

We observe that $(p-q) + (q-r) + (r-p) = 0$.

$\therefore$ All of $p-q$, $q-r$, $r-p$ can not be positive or can not be negative.

Two cases arise

(i) two are positive, one is negative

Let $p-q$, $q-r$, be positive and $r-p$ be negative.

$$\text{Then } \cot^{-1}\frac{pq+1}{p-q} = \tan^{-1}\frac{p-q}{pq+1} = \tan^{-1}p - \tan^{-1}q \quad \ldots(1)$$

$$\cot^{-1}\frac{qr+1}{q-r} = \tan^{-1}\frac{q-r}{qr+1} = \tan^{-1}q - \tan^{-1}r \quad \ldots(2)$$

$$\cot^{-1}\frac{rp+1}{r-p} = \pi - \cot^{-1}\frac{rp+1}{p-r} = \pi - \tan^{-1}\frac{p-r}{rp+1}$$

$$= \pi - (\tan^{-1}p - \tan^{-1}r) \quad \ldots(3)$$

Adding (1), (2), (3) we get

$$\cot^{-1}\frac{pq+1}{p-q} + \cot^{-1}\frac{qr+1}{q-r} + \cot^{-1}\frac{rp+1}{r-p} = \pi.$$

(ii) One is positive and two are negative

Let $p-q$ be positive and $q-r$, $r-p$ are negative.

As before, we have $\cot^{-1}\frac{pq+1}{p-q} = \tan^{-1}p - \tan^{-1}q$

$$\cot^{-1}\frac{qr+1}{q-r} = \pi - (\tan^{-1}r - \tan^{-1}q)$$

$$\text{and } \cot^{-1}\frac{rp+1}{r-p} = \pi - (\tan^{-1}p - \tan^{-1}r)$$

Adding, we get $\sum\cot^{-1}\frac{pq+1}{p-q} = 2\pi.$

Hence the result.

4. Let $c = (x_3, y_3)$

Equation of the line AB is

$(x - x_1)(y_2 - y_1) - (y - y_1)(x_2 - x_1) = 0$ ...(1)

The equation of the circle on AB as diameter is

$(x - x_1)(x - x_2) + (y - y_1)(y - y_2) = 0$ ...(2)

From (1) and (2)

The equation of any circle through A, B is given by

$S + \lambda L = 0$

$\Rightarrow (x - x_1)(x - x_2) + (y - y_1)(y - y_2)$

$+ \lambda\{(x - x_1)(y_2 - y_1) - (y - y_1)(x_2 - x_1) = 0$ ...(3)

This passes through $c(x_3, y_3)$ if

$$\lambda = \frac{-\{(x_3-x_1)(x_3-x_2)+(y_3-y_1)(y_3-y_2)\}}{(x_3-x_1)(y_2-y_1)-(y_3-y_1)(x_2-x_1)}$$

$$= \frac{(x_2-x_3)(x_1-x_3)+(y_2-y_3)(y_1-y_3)}{(x_2-x_1)(y_3-y_1)-(y_2-y_1)(x_3-x_1)} = \frac{l}{k}, \text{ say}$$

We observe that $l = \overline{CB}\cdot\overline{CA} = ab\cos C$

and $k = \pm\, ab\sin C$

according as the orientation of A, B, C is anticlockwise or clockwise.

$$\therefore\ \lambda = \frac{ab\cos C}{\pm ab\sin C} = \pm\cot C.$$

Now, circumcentre of ΔABC = centre of the circle (3)

$$= \left(\frac{x_1+x_2+\lambda(y_1-y_2)}{2}, \frac{y_1+y_2-\lambda(x_1-x_2)}{2}\right)$$

$$= \left(\frac{x_1+x_2\pm\cot C(y_1-y_2)}{2}, \frac{y_1+y_2\mp\cot C(x_1-x_2)}{2}\right) \quad \ldots(4)$$

(i) Take $C = 30°$.

Then $\angle ASB = 60°$ where

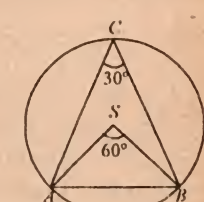
S is the circumcentre of ΔABC

Also, $SA = SB$

$\therefore\ \angle SAB = \angle SBA = 60°$

$\therefore\ ABS$ is an equilateral triangle

and



$$S = \left(\frac{x_1+x_2\pm\sqrt{3}(y_1-y_2)}{2}, \frac{y_1+y_2\mp\sqrt{3}(x_1-x_2)}{2}\right), \text{ from (4)}$$

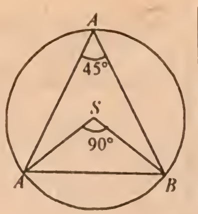
(ii) Take $C = 45°$.

Then $\angle ASB = 90°$,

Also SA = SB.

$\therefore ABS$ is right angled isosceles triangle

and S =



$\left(\frac{x_1 + x_2 \pm (y_1 - y_2)}{2}, \frac{y_1 + y_2 \mp (x_1 - x_2)}{2}\right)$ from (4).

5.(a) Take the circle $x^2 + y^2 + 2gx + 2fy + c = 0$ ...(1) and a point $P(x_1, y_1)$ on it.

Then we have $x_1^2 + y_1^2 + 2gx_1 + 2fy_1 + c = 0$ ...(2)

The equation of $P(x_1, y_1)$ as a point circle is

$(x - x_1)^2 + (y - y_1)^2 = 0^2$

$\Rightarrow x^2 + y^2 - 2x_1x + 2y_1y + x_1^2 + y_1^2 = 0$ ...(3)

The condition of orthogonality for (1) and (3) is

$2g(-x_1) + 2f(-y_1) = c + x_1^2 + y_1^2 = 0$

which is same as (2). So (1) and (3) are orthogonal.

(b) The given circle is $x^2 + y^2 + 4x - 6y - \frac{3}{2} = 0$ ...(1)

Any circle passing the $A(3, 1)$ is orthogonal to the point circle A. If it is orthogonal to (1) as well, then it is orthogonal to the coaxal system determined by (1) and $A(3, 1)$. Hence it must pass through the limiting points of the system. The other limiting point B is the inverse point of $A(3, 1)$ w.r.t. the member (1) of the system. It is enough to find this B.

The centre of (1) is $C(-2, 3)$

The equation of the line AC is $2x + 5y = 11$ ...(2)

The polar of A w.r.t. (1) is

$3x + y + 2(x + 3) - 3(y + 1) - \frac{3}{2} = 0$

$\Rightarrow 5x - 2y = -\frac{3}{2}$ ...(3)

Now B is the point of intersection of (2) and (3) solving, we get $B = (\frac{1}{2}, 2)$.

This is the required point.

6. Consider the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

The normal at any point (x_1, y_1) of the curve is

$\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2 = a^2e^2$

$\Rightarrow a^2y_1x - b^2x_1y = a^2e^2x_1y_1$

Taking $(x_1, y_1) = (a\cos\theta, b\sin\theta)$, the equation of the normal is

$a^2b\sin\theta x - b^2a\cos\theta y = a^2e^2ab\cos\theta\sin\theta$

$\Rightarrow a\sin\theta x - b\cos\theta\ y = a^2e^2\sin\theta\cos\theta$ ...(1)

Take the focus $S(ae, 0)$. If the normal (1) passes through S then we have

$a\sin\theta\ ae - 0 = a^2e^2\sin\theta\cos\theta$

$\Rightarrow \sin\theta(1 - e\cos\theta) = 0$

But $\cos\theta = \frac{1}{e}$ is impossible since $0 < e < 1$

So we have $\sin\theta = 0 \Rightarrow \theta = 0$ or π

The corresponding points on the ellipse are the vertices $(a, 0)$ and $(-a, 0)$.

Instead, if we take the other focus $S'(-ae, 0)$ then also we get the same points $(a, 0)$ and $(-a, 0)$.

Hence there are exactly two points on the ellipse, the normals at which pass through S or S'.

7. Let $f(x) = a^2\sin x + b^2\operatorname{cosec}x$

We may assume that a and b are positive.

$f'(x) = a^2\cos x - b^2\operatorname{cosec}x\cot x$

and $f''(x) = -a^2\sin x + b^2(\operatorname{cosec}^3x + \operatorname{cosec}x\cot^2x)$ ...(1)

$$= -a^2\sin x + \frac{b^2(1+\cos^2 x)}{\sin^3 x}$$

$f'(x) = 0$ when $a^2\cos x = \frac{b^2\cos x}{\sin^2 x}$.

i.e. $\cos x = 0$ or $\sin x = \pm\frac{b}{a}$.

Case (i) : Let $a < b$

The $\sin x = \pm\frac{b}{a}$ is impossible

So $\cos x = 0$, which gives $x = \frac{\pi}{2}$ (given that $x \in (0, \pi)$)

At $x = \frac{\pi}{2}$, $f''(x) = b^2 - a^2$ which is positive

$\therefore f(x)$ has local minimum at $x = \frac{\pi}{2}$

and has no maximum. Also the minimum value

$= a^2 + b^2$.

Case (ii) Let $a > b$.

Then $f'(x) = 0$ if $x = \frac{\pi}{2}$ or $\sin^{-1}\frac{b}{a}$ or $\pi - \sin^{-1}\frac{b}{a}$

At $x = \frac{\pi}{2}$, $f''(x) = b^2 - a^2$ = negative

So $f(x)$ has local maxima at $x = \frac{\pi}{2}$

and the maximum value $f\left(\frac{\pi}{2}\right) = a^2 + b^2$.

At $x = \sin^{-1}\frac{b}{a}$ or $\pi - \sin^{-1}\frac{b}{a}$, we have $\sin x = \frac{b}{a}$ and so

$$f''(x) = -a^2\left(\frac{b}{a}\right) + \frac{b^2\left(1 + 1 - \frac{b^2}{a^2}\right)}{\frac{b^3}{a^3}}$$

$$= -ab + \frac{2a^3 - ab^2}{b} = \frac{2a^3 - 2ab^2}{b}$$

10 Best Problems

Binomial Theorem & Mathematical Induction

1. If S_n denote the sum of the first n natural numbers, prove that

(i) $(1-x)^{-3} = s_1 + s_2x + s_3x^3 + ... + s_nx^{n-1} + ...$

(ii) $2(s_1s_{2n} + s_2s_{2n-1} + ... + s_ns_{n+1}) = \dfrac{(2n+4)!}{5!\,(2n-1)!}$.

Soln.: (i) $s_n = 1 + 2 + ... + n = \dfrac{1}{2}n(n+1)$.

In the expansion of $(1-x)^{-3}$,

$$T_{r+1} = \frac{(-3)(-4)(-5)....(-3-r+1)}{r!}(-x)^r$$

$$= (-1)^{2r}\frac{3\cdot 4\cdot 5...r(r+1)(r+2)}{r!}x^r = \frac{1}{2}(r+1)(r+2)x^r$$

$\therefore$ Coefficient of $x^r = \dfrac{1}{2}(r+1)(r+2) = s_{r+1}$

Hence, $(1-x)^{-3} = s_1 + s_2x + s_3x^2 + ... + s_nx^{n-1} + ...$

(ii) Squaring both sides of the expression proved in part (i), we have

$$(1-x)^{-6} = [s_1 + s_2x + ... + s_nx^{n-1} + s_{n+1}x^n + ...+ s_{2n-1}x^{2n-2} + s_{2n}x^{2n-1} + ...]^2 \quad ....(1)$$

Now coeff. of x^{2n-1} in R.H.S.

$$= 2(s_1s_{2n} + s_2s_{2n-1} + ... + s_ns_{n+1}).$$

Also coeff. of x^{2n-1} in L.H.S.

$$= \frac{(-6)(-7)(-8)....\{-6-(2n-1)+1\}}{(2n-1)!}(-1)^{2n-1}$$

$$= (-1)^{2n-2}\frac{5!\,6\cdot 7\cdot 8....(2n+4)}{5!\,(2n-1)} = \frac{(2n+4)!}{5!(2n-1)!}$$

Hence equating the coefficient of x^{2n-1} on both sides of (1), we get

$$2(s_1s_{2n} + s_2s_{2n-1} + ... + s_ns_{n+1}) = \frac{(2n+4)!}{5!\,(2n-1)!}.$$

2. Find the term independent of x in the expansion of

$$(1+x+2x^3)\left\{\left(\frac{3x^2}{2}\right)-\frac{1}{3x}\right\}^9$$

Soln.: We have

$$(1+x+2x^3)\left(\frac{3}{2}x^2-\frac{1}{3x}\right)^9 = (1+x+2x^3)$$

$$\left[\left(\frac{3}{2}x^2\right)^9 - {}^9C_1\left(\frac{3}{2}x^2\right)^8\left(\frac{1}{3x}\right) + ... + (-1)^9\left(\frac{1}{3x}\right)^9\right] \quad ...(1)$$

Term independent of x in the expansion

$$= 1\cdot a_0 + 1\cdot a_{-1} + 2\cdot a_{-3}.$$

where a_m is the coefficient of x^m in the second bracket [ ] of (1).

Now $(r+1)$th term in [ ] of (1)

$$= {}^9C_r\left[\left(\frac{3}{2}x^2\right)^{9-r}\left(\frac{-1}{3x}\right)^r\right] = (-1)^r\,{}^9C_r\left(\frac{3}{2}\right)^{9-r}\cdot\frac{1}{3^r}\cdot x^{18-3r}$$

$\therefore$ a_{18-3r} = coefficient of x^{18-3r}.

$$= (-1)^r\,{}^9C_r\left(\frac{3}{2}\right)^{9-r}\cdot\frac{1}{3^r}$$

Now for a_0, $18-3r=0 \Rightarrow r=6$

$$\therefore \quad a_0 = (-1)^6\,{}^9C_6\left(\frac{3}{2}\right)^{9-6}\cdot\frac{1}{3^6} = \frac{7}{18}.$$

for a_{-1}, $18-3r=-1 \Rightarrow r = 19/3$ which is fractional,

$\therefore$ $a_{-1} = 0$ and for a_{-3}, $18-3r=-3 \Rightarrow r=7$

$$\therefore \quad a_{-3} = (-1)^7\,{}^9C_7\left(\frac{3}{2}\right)^{9-7}\cdot\frac{1}{3^7} = -\frac{1}{27}.$$

Hence from (2), the required term

$$= 1\cdot\frac{7}{18} + 0 + 2\cdot\left(-\frac{1}{27}\right) = \frac{17}{54}.$$

3. Let $u_1 = 1$, $u_2 = 1$, $u_{n+2} = u_{n+1} + u_n$ for $n \ge 1$. Prove that $u_n = \dfrac{1}{\sqrt{5}}\left[\left(\dfrac{1+\sqrt{5}}{2}\right)^n - \left(\dfrac{1-\sqrt{5}}{2}\right)^n\right]$

Soln.: $u_1 = \dfrac{1}{\sqrt{5}}\left[\dfrac{1+\sqrt{5}}{2} - \dfrac{1-\sqrt{5}}{2}\right] = \dfrac{1}{\sqrt{5}}\cdot\sqrt{5} = 1,$

$$u_2 = \frac{1}{\sqrt{5}}\left[\left(\frac{1+\sqrt{5}}{2}\right)^2 - \left(\frac{1-\sqrt{5}}{2}\right)^2\right] = \frac{1}{\sqrt{5}}\left[\frac{1}{4}\cdot 4\sqrt{5}\right] = 1.$$

Thus $u_1 = 1$ and $u_2 = 1$ are verified ....(1)

Assume the result for $2 \le n \le k$ ....(2)

Now $u_{k+1} = u_k + u_{k-1}$. (given)

$$= \frac{1}{\sqrt{5}}\left\{\left[\left(\frac{1+\sqrt{5}}{2}\right)^k - \left(\frac{1-\sqrt{5}}{2}\right)^k\right]\left[\left(\frac{1+\sqrt{5}}{2}\right)^{k-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\right]\right\}$$

$$=\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{3+\sqrt{5}}{2}\right)^{k}-\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{3-\sqrt{5}}{2}\right)\right\}$$

$$=\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}\left(\frac{\sqrt{5}+1}{2}\right)^{2}-\left(\frac{1-\sqrt{5}}{2}\right)^{k-1}\left(\frac{1-\sqrt{5}}{2}\right)^{2}\right\}$$

$$=\frac{1}{\sqrt{5}}\left\{\left(\frac{1+\sqrt{5}}{2}\right)^{k-1}-\left(\frac{1-\sqrt{5}}{2}\right)^{k+1}\right\}=u_{k+1}$$

From (1), (2) and (3) we have the result by mathematical induction.

4. Prove that

$$\frac{d^n}{dx^n}\left(\frac{x}{x^2+a^2}\right)=\frac{(-1)^n\cdot n!}{a^{n+1}}\cos(n+1)\theta\sin^{n+1}\theta,$$

where $a = x\tan\theta$.

Soln.: Since $x = a\cot\theta$, $dx/d\theta = -a\operatorname{cosec}^2\theta$

$\therefore\quad d\theta/dx = -(1/a)\sin^2\theta$ ....(1)

Also $\dfrac{x}{x^2+a^2}=\dfrac{a\cot\theta}{a^2\operatorname{cosec}^2\theta}=\dfrac{1}{2a}\sin 2\theta.$ ....(2)

For $n = 1$,

$$\text{L.H.S.} = \frac{d}{dx}\left(\frac{x}{x^2+a^2}\right)=\frac{1}{2a}\frac{d}{d\theta}(\sin 2\theta)\cdot\frac{d\theta}{dx}$$

$$=\frac{1}{2a}\cdot 2\cdot\cos 2\theta\cdot\left(-\frac{1}{a}\right)\sin^2\theta$$

$$=\frac{(-1)^1\,1!}{a^2}\cos 2\theta\sin^2\theta=\text{R.H.S.}\qquad ...(2)$$

$\therefore$ The result is true for $n = 1$.

Now assume the result for $n = k$,

i.e. $$\frac{d^k}{dx^k}\left(\frac{x}{x^2+a^2}\right)=\frac{(-1)^k k!}{a^{k+1}}\cos(k+1)\theta\ \sin^{k+1}\theta\quad ...\ (3)$$

For $n = k + 1$, $\text{L.H.S.} =\dfrac{d^{k+1}}{dx^{k+1}}\left(\dfrac{x}{x^2+a^2}\right)$

$$=\frac{d}{dx}\left[\frac{(-1)^k k!}{a^{k+1}}\cos(k+1)\theta\cdot\sin^{k+1}\theta\right]\qquad\text{[using (3)]}$$

$$=\frac{(-1)^k k!}{a^{k+1}}[(k+1)\sin^k\theta\cos\theta\cos(k+1)\theta$$
$$-(k+1)\sin(k+1)\theta\sin^{k+1}\theta]\frac{d\theta}{dx}$$

$$=\frac{(-1)^{k+1}(k+1)!}{a^{k+2}}\sin^{k+2}\theta[\cos(k+1)\theta\cos\theta-\sin(k+1)\theta\sin\theta]$$

$$=\frac{(-1)^{k+1}(k+1)!}{a^{k+2}}\cdot\cos(k+2)\theta\sin^{k+2}\theta\qquad ....(4)$$

$\therefore$ The result is true for $n = k + 1$.

Hence by induction the result follows $\forall n\in N$.

5. Use the principle of mathematical induction to prove that $\displaystyle\int_0^{\pi}\frac{\sin(2nx)}{\sin x}dx=0\quad\forall n\in N.$

Soln.: Let $\displaystyle f(n)=\int_0^{\pi}\frac{\sin(2nx)}{\sin x}\,dx$ ...(1)

$$f(1)=\int_0^{\pi}\frac{\sin 2x}{\sin x}\cdot dx=2\int_0^{\pi}\cos x\,dx=0$$

as $\cos(\pi - x) = -\cos x$ ....(2)

$\therefore$ The result is true for $n = 1$.

Now assume the result for $n = k$

i.e. $\displaystyle f(k)=\int_0^{\pi}\frac{\sin 2kx}{\sin x}dx=0$ ....(3)

Now $$f(k+1)=\int_0^{\pi}\frac{\sin(2k+2)}{\sin x}dx$$

$$=\int_0^{\pi}\frac{\sin 2kx\cos 2x+\cos 2kx\sin 2x}{\sin x}dx$$

$$=\int_0^{\pi}\frac{\sin 2kx(1-2\sin^2 x)+2\cos 2kx\sin x\cos x}{\sin x}dx$$

$$=\int_0^{\pi}\frac{\sin 2kx}{\sin x}dx+2\int_0^{\pi}(\cos 2kx\cos x-\sin 2kx\sin x)\,dx$$

$$=0+2\int_0^{\pi}\cos(2k+1)x\,dx\qquad\text{[using (3)]}$$

$$=2\left|\frac{\sin(2k+1)x}{(2k+1)}\right|_0^{\pi}=0.\qquad ....(4)$$

Hence the result is true for $n = k + 1$.

$\therefore$ By the principle of mathematical induction $f(n) = 0\quad\forall n\in N$.

6. Prove that

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+}}}}\ .....\text{to } n \text{ radicals}=2\cos\frac{\pi}{2^{n+1}}.$$

Soln.:

$P(n)$: $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+}}}}\ .....\text{to } n \text{ radicals}=2\cos\dfrac{\pi}{2^{n+1}}$

$P(1)$: L.H.S. = $\sqrt{2}$,

$$\text{R.H.S.} = 2\cos\frac{\pi}{2^{1+1}}=2\cos\frac{\pi}{4}=2\times\frac{1}{\sqrt{2}}=\sqrt{2}$$

$\therefore$ The result is true for $n = 1$. Let the result be true for $n = m$.

$\therefore$ $P(m)$:

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+}}}}\ .....\text{to } m \text{ radicals}=2\cos\frac{\pi}{2^{m+1}}\ ...\ (1)$$

Now, to prove the result for $n = m + 1$

$P(m+1)$:

$$\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+}}}}\ .....\text{to }(m+1)\text{ radicals}=2\cos\frac{\pi}{2^{(m+1)+1}}$$

$$\Rightarrow\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+}}}}\ .....\text{to }(m+1)\text{ radicals}=2\cos\frac{\pi}{2^{m+2}}\quad .....(2)$$

L.H.S. = $\sqrt{2+\sqrt{2+\sqrt{2+\sqrt{2+}}}}\ .....\text{to }(m+1)\text{ radicals}$

$$=\sqrt{2+2\cos\frac{\pi}{2^{m+1}}}\qquad \text{(Using (1))}$$

$$=\sqrt{2\left(1+\cos\frac{\pi}{2^{m+1}}\right)}=\sqrt{2\left\{2\cos^2\left(\frac{\pi}{2^{m+1}\cdot 2}\right)\right\}}$$

$$=\sqrt{4\cdot\cos^2\frac{\pi}{2^{m+1+1}}}=2\cos\frac{\pi}{2^{m+2}}$$

since $\{1+\cos\theta=2\cos^2(\theta/2)\}$

which is equal to the R.H.S. $\therefore$ the result is true universally for n.

7. Given that $f(a+n)=f(a)\cdot f(n)$; prove that $f(a)=f(a-n)\cdot[f(1)]^n$ (a is a constant) $\forall n\in N$.

Soln.: $P(n)$: $f(a)=f(a-n)[f(1)]^n$

$P(1)$: L.H.S. $=f(a)$

R.H.S. $=f(a-1)[f(1)]^1=f(a-1)\cdot f(1)$

$f(a-1+1)=f(a)$ (Using given condition)

$\therefore$ The result stands true for $n=1$.

Let the result stands true for $n=m$.

$\therefore$ $P(m): f(a)=f(a-m)[f(1)]^m$ .....(1)

Now, to prove the result for $n=m+1$.

$P(m+1)$: $f(a)=f(a-(m+1)]f(1)]^{m+1}$ ....(2)

R.H.S. $=f(a-m-1)\cdot[f(1)]^m\cdot f(1)$

$=f(a-m-1+1)\cdot[f(1)]^m$

(using given condition)

$=f(a-m)\cdot[f(1)]^m=f(a)=$ R.H.S.

(using (1))

$\therefore$ The result is true universally for n.

8. $$\sum_{k=1}^{n}\left(x^k+\frac{1}{x^k}\right)^2=\frac{x^{2n}-1}{x^2-1}\left(\frac{x^{2n+2}+1}{x^{2n}}\right)+2n\quad\forall n\in N.$$

Soln.: $P(n)$: $\displaystyle\sum_{k=1}^{n}\left(x^k+\frac{1}{x^k}\right)^2=\frac{x^{2n}-1}{x^2-1}\left(\frac{x^{2n+2}+1}{x^{2n}}\right)+2n$

$P(1)$: L.H.S. =

$$\left(x^1+\frac{1}{x^1}\right)^2=x^2+\frac{1}{x^2}+2\cdot\frac{1}{x}\cdot x=x^2+\frac{1}{x^2}+2$$

R.H.S. $\displaystyle =\frac{x^{2\cdot 1}-1}{x^2-1}\left(\frac{x^{2\cdot 1+2}+1}{x^{2\cdot 1}}\right)+2(1)=\frac{x^4+1}{x^2}+2$

$$=\frac{x^4}{x^2}+\frac{1}{x^2}+2=x^2+\frac{1}{x^2}+2.$$

$\therefore$ The result stands true for $n=1$. Let the result be true for $n=m$.

$$P(m):\ \sum_{k=1}^{m}\left(x^k+\frac{1}{x^k}\right)^2=\frac{2^{2m}-1}{x^2-1}\left(\frac{x^{2m+2}+1}{x^{2m}}\right)+2m\quad ...(1)$$

Now to prove the result for $n=m+1$

$P(m+1)$:

$$\sum_{k=1}^{m+1}\left(x^k+\frac{1}{x^k}\right)^2=\frac{x^{2(m+1)}-1}{x^2-1}\left(\frac{x^{2(m+1)+2}+1}{x^{2(m+1)}}\right)+2(m+1)$$

$$\Rightarrow\sum_{k=1}^{m+1}\left(x^k+\frac{1}{x^k}\right)^2=\frac{x^{2m+2}-1}{x^2-1}\left(\frac{x^{2m+4}+1}{x^{2m+2}}\right)+2(m+1)\quad ...(2)$$

L.H.S. =

$$\sum_{k=1}^{m+1}\left(x^k+\frac{1}{x^k}\right)^2=\sum_{k=1}^{m}\left(x^k+\frac{1}{x^k}\right)^2+\left(x^{m+1}+\frac{1}{x^{m+1}}\right)^2$$

$$=\frac{x^{2m}-1}{x^2-1}\left(\frac{x^{2m+2}+1}{x^{2m}}\right)+2m+\left(x^{m+1}+\frac{1}{x^{m+1}}\right)^2\quad\text{using (1)}$$

$$=\frac{x^{2m}-1}{x^2-1}\left(\frac{x^{2m+2}+1}{x^{2m}}\right)+2m+x^{2(m+1)}+\frac{1}{x^{2(m+1)}}+2$$

$$=\frac{x^{2m}-1}{x^2-1}\left(\frac{x^{2m+2}+1}{x^{2m}}\right)+x^{2(m+1)}+\frac{1}{x^{2(m+1)}}+2(m+1)$$

$$=\frac{x^{2m}-1}{x^2-1}\left(\frac{x^{2m+2}+1}{x^{2m}}\right)+\frac{x^{(2m+2)+(2m+2)}}{x^{2m+2}}+2(m+1)$$

$$=\frac{x^{2m}-1}{x^2-1}\left(\frac{x^{2m+2}+1}{x^{2m}}\right)+\frac{x^{4m+4}}{x^{2m}x^2}+2(m+1)$$

$$=\frac{(x^{2m}-1)(x^{2m+2}+1)x^2+(x^{4m+4}+1)(x^2-1)}{(x^2-1)(x^{2m}\cdot x^2)}+2(m+1)$$

$$=\frac{(x^{2m+2}-x^2)(x^{2m+2}+1)+x^{4m+6}-x^{4m+4}+x^2-1}{(x^2-1)(x^{2m}\cdot x^2)}+2(m+1)$$

$$=\frac{x^{4m+4}+x^{2m+2}-x^{2m+4}-x^2+x^{4m+6}-x^{4m+4}+x^2-1}{(x^2-1)(x^{2m+2})}+2(m+1)$$

$$=\frac{x^{2m+2}-x^{2m+4}+x^{4m+6}-1}{(x^2-1)(x^{2m+2})}+2(m+1)$$

$$=\frac{x^{2m+2}+x^{4m+6}-x^{2m+4}-1}{(x^2-1)(x^{2m+2})}+2(m+1)$$

$$=\frac{x^{2m+2}(1+x^{2m+4})-1(1+x^{2m+4})}{(x^2-1)(x^{2m+2})}+2(m+1)$$

$$=\frac{(x^{2m+2}-1)(1+x^{2m+4})}{(x^2-1)(x^{2m+2})}+2(m+1)$$

$$=\frac{(x^{2m+2}-1)}{(x^2-1)}\left[\frac{1+x^{2m+4}}{x^{2m+2}}\right]+2(m+1)=\text{R.H.S.}$$

$\therefore$ The result is true universally for n.

9.

$$\sum_{k=1}^{n}\frac{8}{5}(-1)^k+\frac{2^{2k-3}}{5}=\frac{4}{5}\{(-1)^n-1\}+\frac{1}{30}(2^{2n}-1)\quad \forall n\in N$$

Soln.: $P(n)$:

$$\sum_{k=1}^{n}\frac{8}{5}(-1)^k+\frac{2^{2k-3}}{5}=\frac{4}{5}\{(-1)^n-1\}+\frac{1}{30}(2^{2n}-1)$$

$P(1)$: L.H.S.

$$=\frac{8}{5}(-1)^1+\frac{2^{2-3}}{5}=-\frac{8}{5}+\frac{1}{25}=\frac{-16+1}{10}=-\frac{3}{2}.$$

R.H.S. =

$$\frac{4}{5}\{(-1)^1-1\}+\frac{1}{30}(2^2-1)=\frac{-2\cdot 4}{5}+\frac{3}{30}=\frac{-8}{5}+\frac{1}{10}=-\frac{3}{2}.$$

$\therefore$ The result stands true for n = 1. Let the result be true for $n = m$.

$\therefore P(m)$:

$$\sum_{k=1}^{m}\frac{8}{5}(-1)^k+\frac{2^{2k-3}}{5}=\frac{4}{5}\{(-1)^m-1\}+\frac{1}{30}(2^{2m}-1)\quad \ldots (1)$$

Now prove the result for $n = m + 1$.

$P(m+1)$:

$$\sum_{k=1}^{m+1}\frac{8}{5}(-1)^k+\frac{2^{2k-3}}{5}=\frac{4}{5}\{(-1)^{m+1}-1\}+\frac{1}{30}(2^{2m+2}-1)\quad \ldots(2)$$

$$\text{L.H.S.}=\sum_{k=1}^{m+1}\frac{8}{5}(-1)^k+\frac{2^{2k-3}}{5}$$

$$=\sum_{k=1}^{m}\frac{8}{5}(-1)^k+\frac{2^{2k-3}}{5}+\frac{8}{5}(-1)^{m+1}+\frac{2^{2m-1}}{5}$$

$$=\frac{4}{5}\{(-1)^m-1\}+\frac{1}{30}(2^{2m}-1)+\frac{8}{5}(-1)^{m+1}+\frac{2^{2m-1}}{5}\quad \text{(using (1))}$$

$$=\frac{4}{5}\{(-1)^m-1\}+\frac{1}{30}(2^{2m}-1)+\left\{\frac{-8}{5}(-1)^m+\frac{2^{2m}}{25}\right\}$$

$$=\frac{4}{5}\{(-1)^m-1\}+\frac{1}{30}(2^{2m}-1)-\frac{8}{5}(-1)^m+\frac{2^{2m}}{10}$$

$$=\left\{\frac{4}{5}(-1)^m-\frac{4}{5}-\frac{8}{5}(-1)^m\right\}+\left\{\frac{1}{30}(2^{2m}-1)+\frac{3\cdot 2^{2m}}{30}\right\}$$

$$=\left\{\frac{-4}{5}(-1)^m-\frac{4}{5}\right\}+\left\{\frac{2^{2m}-1+3\cdot 2^{2m}}{30}\right\}$$

$$=\frac{-4}{5}\{(-1)^m(-1)-1\}+\left\{\frac{4\cdot 2^{2m}-1}{30}\right\}$$

$$=\frac{4}{5}\{(-1)^{m+1}-1\}+\left\{\frac{2^{2m+2}-1}{30}\right\}=\text{R.H.S.}$$

$\therefore$ The result is true universally for n.

10. $$\sum_{k=1}^{n}(1+3^{k-1}-2^{k-1})x^{k-1}=\frac{1-x^n}{1-x}+\frac{1-3^n\cdot x^n}{1-3x}-\frac{1-2^n\cdot x^n}{1-2x}\quad \forall n\in N.$$

Soln.: $P(n)$:

$$\sum_{k=1}^{n}(1+3^{k-1}-2^{k-1})x^{k-1}=\frac{1-x^n}{1-x}+\frac{1-3^n\cdot x^n}{1-3x}-\frac{1-2^n\cdot x^n}{1-2x}$$

$P(1)$: L.H.S. = $(1 + 3^{1-1} - 2^{1-1})x^{1-1} = (1 + 1 - 1)1 = 1$

$$\text{R.H.S.}=\frac{1-x^1}{1-x}+\frac{1-3^1x^1}{1-3x}-\frac{1-2^1x^1}{1-2x}=1+1-1=1.$$

$\therefore$ The result stands true for n = 1. Let the result be true for $n = m$.

$\therefore$ $P(m)$:

$$\sum_{k=1}^{n}(1+3^{k-1}-2^{k-1})x^{k-1}=\frac{1-x^m}{1-x}+\frac{1-3^m\cdot x^m}{1-3x}-\frac{1-2^m\cdot x^m}{1-2x}\quad \ldots(1)$$

Now, to prove the result for $n = m + 1$.

$$P(m+1):\ \sum_{k=1}^{m+1}(1+3^{k-1}-2^{k-1})x^{k-1}$$

$$=\frac{1-x^{m+1}}{1-x}+\frac{1-3^{m+1}\cdot x^{m+1}}{1-3x}-\frac{1-2^{m+1}\cdot x^{m+1}}{1-2x}\quad \ldots(2)$$

$$\text{L.H.S.}=\sum_{k=1}^{m+1}(1+3^{k-1}-2^{k-1})x^{k-1}$$

$$=\sum_{k=1}^{m}(1+3^{k-1}-2^{k-1})x^{k-1}+(1+3^m-2^m)x^m\quad \text{[using (i)]}$$

$$=\frac{1-x^m}{1-x}+\frac{1-3^mx^m}{1-3x}-\frac{1-2^mx^m}{1-2x}+(1+3^m-2^m)x^m$$

$$=\frac{1-x^m}{1-x}+\frac{1-3^mx^m}{1-3x}-\frac{1-2^mx^m}{1-2x}+x^m+3^mx^m-2^mx^m$$

$$=\left\{\frac{1-x^m}{1-x}+x^m\right\}+\left\{\frac{1-3^mx^m}{1-3x}+3^mx^m\right\}-\left\{\frac{1-2^mx^m}{1-2x}+2^mx^m\right\}$$

$$=\left\{\frac{1-x^m+x^m(1-x)}{1-x}\right\}+\left\{\frac{1-3^mx^m+(1-3x)3^mx^m}{1-3x}\right\}-\left\{\frac{1-2^mx^m+2^mx^m-2^{m+1}x^{m+1}}{1-2x}\right\}$$

$$=\left\{\frac{1-x^m+x^m-x^{m+1})}{1-x}\right\}+\left\{\frac{1-3^mx^m+3^mx^m-3^{m+1}x^{m+1}}{1-3x}\right\}-\left\{\frac{1-2^mx^m+2^mx^m-2^{m+1}x^{m+1}}{1-2x}\right\}$$

$$=\frac{1-x^{m+1}}{1-x}+\frac{1-3^{m+1}x^{m+1}}{1-3x}-\frac{1-2^{m+1}x^{m+1}}{1-2x}=\text{R.H.S.}$$

$\therefore$ The result is true universally for n.

35. The length of latus rectum of the ellipse $\frac{x^2}{36}+\frac{y^2}{49}=1$ is

(a) $\frac{98}{6}$ (b) $\frac{72}{7}$

(c) $\frac{72}{14}$ (d) $\frac{98}{12}$.

36. $\lim_{x\to\infty}\left[1+\frac{4}{x-1}\right]^{x+3}$ is

(a) 1 (b) e^2

(c) e^3 (d) e^4.

37. $a = \log_{24}12$, $b = \log_{36}24$, $c = \log_{48}36$, then $1 + abc$ is

(a) $2ab$ (b) $2bc$

(c) $2ac$ (d) 0.

38. If $x = \log_a bc$, $y = \log_b ca$, $z = \log_c ab$, then $\frac{1}{1+x}+\frac{1}{1+y}+\frac{1}{1+z}$ is

(a) 0 (b) 1

(c) abc (d) $ab + bc + ca$.

39. Integral part of $(\sqrt{2} + 1)^6$ is

(a) 198 (b) 197

(c) 196 (d) 163.

40. The real part of $\frac{1}{1-\cos\theta+i\sin\theta}$ is

(a) $\frac{1}{1-\cos\theta}$ (b) $\frac{1}{2}$

(c) $\frac{\tan\theta}{2}$ (d) none of the above.

41. If $\tan^{-1}x + \tan^{-1}y + \tan^{-1}z = \frac{\pi}{2}$, then $xy + yz + zx =$

(a) 1 (b) xyz

(c) $x + y + z$ (d) 0.

42. If $y = \log_e[\log_e(\log_e x)]$, then y' is

(a) $\frac{1}{\log_e(\log_e x)}$ (b) $\frac{1}{x\log_e x\,\log_e(\log_e x)}$

(c) $\frac{1}{x\log_e x\,\log_e x}$ (d) none of the above.

43. The maximum value of

$$f(x) = \frac{1}{x}\log x,\ 0 < x < 3$$

(a) e (b) $e^{1/2}$

(c) $1/e$ (d) $2/e$.

44. $\int_0^\infty \operatorname{sech} x\, dx =$

(a) $\frac{\pi}{2}+1$ (b) $\frac{\pi}{2}$

(c) π (d) 1.

45. If $x = \log t$ and $y = t^2 - 1$, then $y''(1)$ at $t = 1$ is

(a) 2 (b) 3

(c) 5 (d) 4.

46. The principal value of $\sin^{-1}\left(\sin\frac{2\pi}{3}\right)$ is

(a) $-\frac{2\pi}{3}$ (b) $\frac{2\pi}{3}$

(c) $\frac{4\pi}{3}$ (d) $\frac{\pi}{3}$.

47. $\int\frac{\cos 2x}{\cos^2 x\sin^2 x}dx$ is equal to

(a) $\cot x - \tan x + C$ (b) $\cot x + \tan x + C$

(c) $-\cot x + \tan x + C$ (d) $-\cot x - \tan x + C$.

48. $\sqrt{2+\sqrt{2+2\cos 4x}} =$

(a) $\cos x$ (b) $\cos 2x$

(c) $2\cos x$ (d) $2\cos 2x$.

49. If $y=\sqrt{\cos x+\sqrt{\cos x+\sqrt{\cos x\ldots\ldots\infty}}}$, then y' is

(a) $\frac{\cos x}{1-2y}$ (b) $\frac{\sin x}{1-2y}$

(c) $-\frac{\sin x}{1-2y}$ (d) $\frac{\cos x}{2y-1}$.

50. If x is real and $k=\frac{x^2-x+1}{x^2+x+1}$, then

(a) $\frac{1}{3}\le k\le 3$ (b) $k\ge 5$

(c) $k\le 0$ (d) none of the above.

ANSWERS

1. (b)	2. (c)	3. (c)	4. (c)	5. (b)
6. (a)	7. (a)	8. (c)	9. (b)	10. (c)
11. (c)	12. (c)	13. (d)	14. (b)	15. (b)
16. (b)	17. (b)	18. (a)	19. (c)	20. (b)
21. (c)	22. (a)	23. (d)	24. (a)	25. (a)
26. (a)	27. (a)	28. (a)	29. (c)	30. (b)
31. (b)	32. (b)	33. (b)	34. (a)	35. (a)
36. (d)	37. (b)	38. (b)	39. (b)	40. (b)
41. (a)	42. (b)	43. (c)	44. (c)	45. (d)
46. (d)	47. (d)	48. (c)	49. (b)	50. (a)

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10 Best Problems for Competitive Exams

1. Determine all values for c so that

$F(x) = \dfrac{x-1}{c-x^2+1}$ does not take any value in interval $[-1, -1/3]$.

2. **(a)** $q + r\omega = (1+\omega)p$ and p, q, r are real then show that $p = r = q$.

(b) a, b, c are cube roots of p ($p < 0$), then for any permissible value of x, y, z which is given that

$$\left|\frac{xa+yb+zc}{xb+yc+za}\right| + (a_1^2 - 2b_1^2)\omega + \omega^2([x]+[y]+[z]) = 0.$$

where ω is the cube root of unity and a_1 and b_1 are positive numbers, then find the value of

$[x + a_1] + [y + b_1] + [z]$.

3. Solve the following system of equation:

$$\frac{1}{1!\,9!} + \frac{1}{3!\,7!} + \frac{1}{5!\,5!} + \frac{1}{7!\,3!} + \frac{1}{9!\,1!} = \frac{2^a}{b!}$$

where a and b are positive integers.

(a) Value of x from given expression $(\cos ax)^{100n} - (\sin bx)^{100m} = 1$; $m, n \in I$.

(b) Area enclosed by maxi.$(|x|\,|y|) = b - a$ and $y = 2^{(b-a)x}$.

4. $$2f(x) = (-2)^n\left(\frac{1}{2-\cos x}\right)\left(\frac{1}{2-\cos 2x}\right)\left(\frac{1}{2-\cos 4x}\right)\ .....\ \left(\frac{1}{2-\cos 2^{n-1}x}\right)$$

where $n > 1$, then prove that

$$f^{-1}(1/2) = \frac{\pi k}{2^{n-1} \pm (1/2)}\ \forall k \in I.$$

5. **(a)** $\int \operatorname{cosec}^2 x \log(\cos x + \cos 2x)\,dx$.

(b) $\displaystyle\int \frac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}}$.

6. Let $\Delta_h f(x) = f(x+h) - f(x)$ using mathematical induction show that

$$\Delta_h^n f(x) = \sum_{k=0}^{n} (-1)^k\, {}^nC_k f[x-(n-k)h]$$

$[Hint:\ \Delta_h^n = \Delta_h(\Delta_h^{n-1}) = \Delta_h^{n-1}(\Delta_h)]$

7. Find all possible integral values of x and show that $x^2 + 19x + 88$ is a perfect square.

8. If the perpendiculars from two vertices B and C to the opposite faces of a tetrahedran $ABCD$ intersect, then BC is perpendicular to AC and perpendiculars from A and D to opposite faces also intersect. Prove this.

9. **(a)** Given $|\omega| = 1$, then show that

$|\omega - 1| \le |\arg(\omega)|$.

(b) Prove $|z - i| \le ||z| - 1| + |z|\,|(\pi/2) - \arg(z)|$.

10. If SY and $S'Y'$ be the perpendicular from the foci upon the tangent at any point P of the ellipse, then Y and Y' lie on the auxiliary circle, and $SY \cdot S'Y' = b^2$, prove CY and $S'P$ are parallel, where C is the centre of auxiliary circle.

SOLUTIONS

1. $F(x) = \dfrac{x-1}{c-x^2+1}$; *i.e.* $y = \dfrac{x-1}{c-x^2+1}$

Take $y = -t$, where $t \in [1/3, 1]$.

and given function assume the values $(-t)$ at some $x = x_0$ then $[1/3, 1] = -t$.

$$x^2 - c - 1 = \frac{x-1}{t}$$

Now, we have to just make sure that this quadratic in x does not give any real value of x for the discriminant of the quadratic must be repeat.

$$\frac{1}{t^2} - 4\left(\frac{1}{t} - c - 1\right) < 0\ ;\ \frac{1}{t^2} - \frac{4}{t} + 4 < -4c$$

$$c < -\frac{1}{4}\left(\frac{1}{t} - 2\right)^2.$$

Also useful for IIT-JEE, Roorkee REE, Bihar CEE ....

Now $t \in [1/3, 1]$; $-1 \le \frac{1}{t} - 2 \le 1$

$$-\frac{1}{4} \le -\frac{1}{4}\left(\frac{1}{t}-2\right)^2 \le 0$$

Hence $c \in (-\infty, -1/4]$.

2. **(a)** $q + r\omega = (1 + \omega)p$

$q + r\omega = -\omega^2 p$

$q + r\omega + \omega^2 p = 0$ or $p + q\omega + r\omega^2 = 0$

$$p + q\left(\frac{-1+i\sqrt{3}}{2}\right) + r\left(\frac{-1-i\sqrt{3}}{2}\right) = 0$$

Equating real and imaginary part

$$p - \frac{q}{2} - \frac{r}{2} = 0 \Rightarrow p = \frac{r}{2} + \frac{q}{2}$$

$$\frac{\sqrt{3}}{2}(q - r) = 0 \Rightarrow q = r.$$

or $p = q = r$.

(b) $$\underbrace{\left|\frac{xa+yb+zc}{xb+yc+za}\right|}_{\text{real}} + \underbrace{(a_1^2 - 2b_1^2)}_{\text{real}}\omega + \omega^2(\underbrace{[x]+[y]+[z]}_{\text{real}}) = 0$$

So from above given condition,

$$\left|\frac{xa+yb+zc}{xb+yc+za}\right| = a_1^2 - 2b_1^2 = [x]+[y]+[z]$$

and we know a, b, c are roots of $(p)^{1/3}$.

Let roots $a = t$, $b = t\omega$, $c = t\omega^2$

Now $$\left|\frac{xa+yb+zc}{xb+yc+za}\right| = \left|\frac{xt + y\cdot t\omega + t\omega^2 \cdot z}{x\cdot t\omega + y\cdot t\omega^2 + zt}\right|$$

$$= \left|\frac{x + y\omega + z\omega^2}{x\omega + y\omega^2 + z}\right| = \left|\frac{1}{\omega}\right| = 1.$$

So $a_1^2 - 2b_1^2 = 1.$; $a_1^2 = 1 + 2b_1^2$ (odd) ... (i)

a_1 can be write in $(2n + 1)$ form.

$\Rightarrow$ $(2n + 1)^2 = 1 + 2b_1^2$

$4n^2 + 4n = 2b_1^2$; $b_1 = 2n(n + 1)$.

$b_1^2 = 2n(n + 1)$ - even and given b_1 is prime.

So b_1^2 is also prime.

So $b_1 = 2$ [because 2 is the only even prime number].

$a_1^2 = 9 \Rightarrow a_1 = 3$ from (i)

$b_1 = 2$, $a_1 = 3$

$[x] + [y] + [z] = 1$ from above.

So $[x + a_1] + [y + b_1] + [z] = [x + 3] + [y + 2] + [z]$

$a = 5 + [x] + [y] + [z] = 6.$

3. $$\frac{10!}{10!}\left(\frac{1}{1!\,9!} + \frac{1}{3!\,7!} + \frac{1}{5!\,5!} + \frac{1}{7!\,3!} + \frac{1}{9!\,1!}\right) = \frac{2^a}{b!}$$

$$\frac{1}{10!}\left(\frac{10!}{1!\,9!} + \frac{10!}{3!\,7!} + \frac{10!}{5!\,5!} + \frac{10!}{7!\,3!} + \frac{10!}{9!\,1!}\right) = \frac{2^a}{b!}$$

$$\frac{2^{(10-1)}}{10!} = \frac{2^a}{b!} \Rightarrow a = 9,\ b = 10.$$

$[\because {}^{10}C_1 + {}^{10}C_3 + {}^{10}C_5 + {}^{10}C_7 + {}^{10}C_9 = 2^{(10-1)} = 2^9.]$

$$\frac{(1+x)^n - (1-x)^n}{2} \Rightarrow \text{odd term}$$

Put $x = 1$, we get $2^{n-1} = 2^9$.

Now,

(a) $(\cos ax)^{100n} - (\sin bx)^{100m} = 1$

Let $f_1 = (\cos ax)^{100n}$ and $f_2 = (\sin bx)^{100m}$

and the maximum value of f_1 and f_2 is 1.

$f_1 - f_2 = 1$, $f_2 = 0$.

So, $\sin bx = 0$; $bx = k\pi$, k is an integer.

$\therefore$ $x = \frac{k}{b}\pi$.

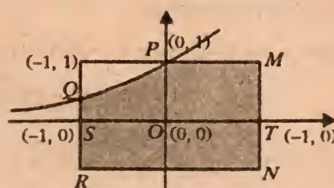
(b) $\max\{|x|, |y|\} = b - a = 1$ and $y = 2^x$.

If $x = -1$, then

$y = 2^{-1}$ or $y = 1/2$.

$y = 1 \Rightarrow 1 = 2^x$;

$x = 0$.



Now area of $QRNMPQ$

$= 3 \times$ area of $OTMP + OPQS$

$$= 3\,(1 \times 1) + \int_{-1}^{0} 2^x\,dx$$

$$= 3 + \left.\frac{2^x}{\log_e 2}\right|_{-1}^{0} = 3 + \frac{1}{2\log_e 2}.$$

4. $$2f(x) = (-2)^n\left(\frac{1}{2} - \cos x\right)\left(\frac{1}{2} - \cos 2x\right) \ldots \left(\frac{1}{2} - \cos 2^{n-1}x\right)$$

$2f(x) = (2\cos x - 1)(2\cos 2x - 1)(2\cos 2^2x - 1)$ ....

$2(2\cos x + 1)\,f(x) = (4\cos^2 x - 1)(2\cos 2x - 1)$ .....

$2(2\cos x + 1)\,f(x) = (2\cos 2x + 1)(2\cos 2x - 1)$ .....

Like this $f(x) = (2\cos 2x + 1)(2\cos 2x - 1)$ ... $(2\cos 2^{n-1}x - 1)$

$$f(x) = \frac{1}{2}\frac{(2\cos 2^n x + 1)}{(2\cos x + 1)}$$

$$\left[\ f^{-1}\left(\frac{1}{2}\right) = \frac{\pi k}{2^{n-1} \pm (1/2)} \Rightarrow f\left(\frac{2\pi k}{2^n \pm 1}\right) = \frac{1}{2}\right.$$

If we take $f(x) = 1/2$, then after solving, x will get $\left.\frac{2\pi k}{2^n \pm 1}.\ \right]$

Now let $f(x) = 1/2$, then $2\cos 2^n x + 1 = 2\cos x + 1$

or, $(2n \pm 1)x = 2k\pi$; $x = \dfrac{k\pi}{2^{n-1} \pm (1/2)}$.

So $f\left(\dfrac{k\pi}{2^{n-1} \pm (1/2)}\right) = \dfrac{1}{2}$

$\Rightarrow \quad f^{-1}\left(\dfrac{1}{2}\right) = \dfrac{k\pi}{2^{n-1} \pm (1/2)}$.

5. $\int \operatorname{cosec}^2 \log(\cos x + \cos 2x)dx$

$= -\cot x \log(\cos x + \sqrt{\cos 2x})$

$+\int \cot x \dfrac{1}{\cos x + \sqrt{\cos 2x}}\left(-\sin x + \dfrac{1}{2\sqrt{\cos 2x}}(-2\sin 2x)\right)dx$

$= -\cot x \log(\cos x + \sqrt{\cos 2x}$

$\quad - \int \cot x \dfrac{\sin x\sqrt{\cos 2x} + \sin 2x}{\sqrt{\cos 2x}\,\{\cos x + \sqrt{\cos 2x}\}} dx$

Multiply $\sqrt{\cos 2x}\,\{\cos x + \sqrt{\cos 2x}\}$ to numerator and denominator,

$I = -\cot x \log(\cos x + \sqrt{\cos 2x})$

$\quad -\int \dfrac{\cos x\sqrt{\cos 2x} - \cos^2 x \cos 2x}{\cos 2x \sin^2 x}$

$= -\cot x \log(\cos x + \sqrt{\cos 2x})$

$\quad -\underbrace{\int \dfrac{\cos x}{\sin^2 x\sqrt{\cos 2x}} dx}_{I_1} + \underbrace{\int \cot^2 x\, dx}_{I_2}$

$I_1 = \int \dfrac{\cos x}{\sin^2 x\sqrt{\cos 2x}} dx = \int \dfrac{\cos x}{\sin^2 x\sqrt{1-2\sin^2 x}} dx$ (say)

Put $\sin x = t$; $dt = \cos x\, dx$

$I_1 = \int \dfrac{dt}{t^2\sqrt{1-2t^2}}$ and now $t = 1/v$, $dt = -dv/v^2$

$I_1 = \int \dfrac{vdv}{\sqrt{v^2-1}} = -(v^2-2)^{1/2} = -\sqrt{\operatorname{cosec}^2 x - 2}$.

$I_2 = \int \cot^2 x dx = \int \operatorname{cosec}^2 x\, dx - \int dx = -\cot x - x.$

So $I = -\cot x \log(\cos x + \sqrt{\cos 2x}) + \sqrt{\operatorname{cosec}^2 x - 2}$
$\quad - \cot x - x + C.$

(b) $\int \dfrac{dx}{\sqrt[4]{(x-1)^3(x+2)^5}} = \int \dfrac{dx}{(x-1)^{3/4}(x+2)^{5/4}}$

$= \int \dfrac{dx}{\left(\dfrac{x-1}{x+2}\right)^{3/4}(x+2)^2}$.

$\dfrac{x \cdot 1}{x+2} = t^4$; $x = \dfrac{-2t^4-1}{t^4-1}$

$dx = \dfrac{8(1-t^4)t^3 + 4t^3(2t^4+1)}{(1-t^4)^2} = \dfrac{12t^3}{(1-t^4)^2} dt$

$x + 2 = \dfrac{3}{1-t^4}$

$I = 12\int \dfrac{t^3}{(1-t^4)^2} \cdot \dfrac{(1-t^4)^2}{t^3 \cdot 9} dt = \dfrac{4}{3}\int dt = \dfrac{4}{3}t$

$= \dfrac{4}{3}\left(\dfrac{x-1}{x+2}\right)^{1/4} + C$.

6. $\Delta_b f(x) = f(x+b) - f(x)$

Let $\Delta_b^n f(x) = \sum\limits_{k=0}^{n} (-1)^k\, {}^nC_k f[x-(n-k)b]$

Let for $n = 1$, $\Delta_b f(x) = \sum\limits_{k=0}^{1} (-1)^k\, {}^1C_k f[x+(1-k)b]$

L.H.S $= \Delta_b f(x) = f(x+b) - f(x)$

R.H.S. $= \sum\limits_{k=0}^{1} (-1)^k\, {}^1C_k f[x+(1-k)b]$

$= f(x+b) - f(x)$

So for $n = 1$, this is true.

Let this is true for $n = n$ and we want to show for $n = n + 1$.

$$\Delta_b^{n+1} f(x) = \sum_{k=0}^{n+1} (-1)^k\, {}^{n+1}C_k\, f[x+(n+1-k)b]$$

L.H.S. $= \Delta_b^{n+1} f(x) = \Delta\left(\Delta_b^n f(x)\right)$

$= \Delta\left[\sum\limits_{k=0}^{n} (-1)^k\, {}^nC_k\, f[x+(n-k)b]\right]$

$= \Delta\, [{}^nC_0\, f(x+nb) - {}^nC_1\, f(x+(n-1)b)$
$\quad + {}^nC_2\, f(x+(n-2)b) \ldots (-1)^n f(x)]$

$= {}^nC_0\, \Delta f(x+nb) - {}^nC_1\, \Delta f(x+(n-1)b)$
$\quad + {}^nC_2\, \Delta f(x+(n-2)b) + \ldots$

$= {}^nC_0\, \{f(x+(n+1)b) - f(x+nb)\}$
$\quad - {}^nC_1\, \{f(x+nb) - f(x+(n-1)b\} + \ldots.$

$= {}^{n+1}C_0\, f(x+(n+1)b - ({}^nC_0 + {}^nC_1)\, f(x+nb)$
$\quad + ({}^nC_1 + {}^nC_2)\, f(x+(n-1)b) + \ldots$
$\quad + (-1)^{n+1} f(x)$

$= {}^{n+1}C_0\, f(x+(n-1)b) - {}^{n+1}C_1\, f(x+nb)$
$\quad + {}^{n+1}C_2\, f(x+(n-1)b) + \ldots$
$\quad + (-1)^{n+1}\, {}^{n+1}C_{n+1}\, f(x)$

$= \sum\limits_{k=0}^{n+1} (-1)^k\, {}^{n+1}C_k\, f(x+(n+1-k)b) =$ R.H.S.

This is also true for $n = n + 1$.

7. $x^2 + 19x + 88 = m^2$

$x^2 + 19x + 88 - m^2 = 0$

Equation must have integral roots.

$(19)^2 - 4(88 - m^2) = k^2$

$(k - 2m)(k + 2m) = 9$

$k - 2m = 3$ and $k + 2m = 3 \Rightarrow k = 3, m = 0$

$k - 2m = -3$ and $k + 2m = -3 \Rightarrow k = -3, m = 0$

$k - 2m = 9$ and $k + 2m = 1 \Rightarrow k = 5, m = -2$

$k - 2m = 1$ and $k + 2m = 9 \Rightarrow k = 5, m = 2$

$k - 2m = -9$ and $k + 2m = -1 \Rightarrow k = -5,\ m = 2$

$k - 2m = -1$ and $k + 2m = -9 \Rightarrow k = -5,\ m = -2$

if $m^2 = 0$, $x^2 + 19x + 88 = 0$

$(x + 11)(x + 8) = 0$; $x = -8, -11$.

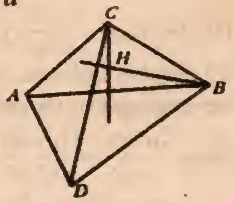
if $m^2 = 4$, $x^2 + 19x + 88 - 4 = 0$

$(x + 12)(x + 7) = 0$; $x = -12, -7$.

$\therefore$ Possible values of x are $-7, -8, -11, -12$.

8. Take A as the origin of reference.

Let $\vec{AB} = \vec{b}$, $\vec{AC} = \vec{c}$, $\vec{AD} = \vec{d}$



Let the perpendicular from B and C to opposite faces ACD and ABD meet at point H whose position vector is $\vec{b}$.

Thus we have

$$\vec{BH} \perp \vec{AC} \Rightarrow (\vec{b} - \vec{b}) \cdot \vec{c} = 0$$

$$\vec{BH} \perp \vec{AD} \Rightarrow (\vec{b} - \vec{b}) \cdot \vec{d} = 0$$

$$\vec{CH} \perp \vec{AB} \Rightarrow (\vec{b} - \vec{c}) \cdot \vec{b} = 0$$

$$\vec{CH} \perp \vec{AD} \Rightarrow (\vec{b} - \vec{c}) \cdot \vec{d} = 0$$

These gives

$\vec{b} \cdot \vec{c} = \vec{b} \cdot \vec{c}$, $\vec{b} \cdot \vec{d} = \vec{b} \cdot \vec{d}$, $\vec{b} \cdot \vec{b} = \vec{b} \cdot \vec{c}$, $\vec{b} \cdot \vec{d} = \vec{c} \cdot \vec{d}$

$\Rightarrow \quad \vec{b} \cdot \vec{d} = \vec{c} \cdot \vec{d} \Rightarrow (\vec{b} \cdot \vec{c}) \cdot \vec{d} = 0 \Rightarrow BC \perp AD.$

Let K be the foot of the perpendicular from A to opposite face BCD and let $\vec{k}$ be its position vector. As AK is perpendicular to the plane BCD, we have

$$\vec{k} \cdot (\vec{c} - \vec{b}) = 0 \ ; \ \vec{k} \cdot (\vec{b} - \vec{d}) = 0$$

$$\vec{k} \cdot \vec{c} = \vec{k} \cdot \vec{b} = \vec{k} \cdot \vec{d}.$$

We shall show that there is a point L on AK such that DL is perpendicular to the face ABC, if λk be the position vector of this point, the two equations $(\lambda\vec{k} - \vec{d}) \cdot \vec{b} = 0$, $(\lambda\vec{k} - \vec{d}) \cdot \vec{c} = 0$ must be consistent in relation to λ.

These equations are really the same for

$$\vec{k} \cdot \vec{b} = \vec{k} \cdot \vec{c} \text{ and } \vec{b} \cdot \vec{d} = \vec{c} \cdot \vec{d}$$

so that λ is determined. Hence the result.

9. **(a)** $|\omega| = 1$, then $|\omega - 1| \le |\arg \omega|$

Take $\omega = r(\cos\theta + i\sin\theta)$ $\qquad |\omega| = 1,\ r = 1$

$|r\cos\theta + ir\sin\theta - 1|$

$= \sqrt{(r\cos\theta - 1)^2 + r^2\sin^2\theta} = \sqrt{r^2 + 1 - 2r\cos\theta}$

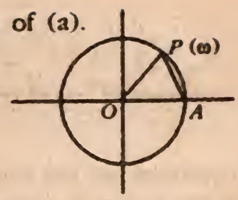
$= \sqrt{2 - 2\cos\theta}$ $\qquad (\because r = 1)$

$= |2\sin(\theta/2)|$

$\le |2(\theta/2)| \le |\theta|$ $\qquad (\because \sin\theta \le \theta)$

$|\omega - 1| \le |\arg \omega|$.

Geometrical representation of (a).



$|\omega| = 1$ is a unit circle.

Centre at origin.

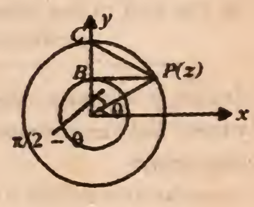
Now $|\omega - 1|$

$= AP \le \text{Arc } AP \le \alpha/1$

$|\omega - 1| \le |\arg \omega|$.

(b) $|OB| = |i|$

$|PB| = |z - i|$



$|PB| \le |BC| + |PC|$ ... (i)

As $PC <$ arc PC

y-axis refers imaginary values. So i on y-axis.

$\Rightarrow \quad |z - i|$

$\le ||z| - |i|| + |\text{arc } PC|$ from (i)

$|z - i| \le ||z| - 1| + |z|\,(\pi/2 - \arg z|$

$$\left(\frac{\pi}{2} - \arg z = \frac{|PC|}{|z|}\right)$$

$|z - i| \le ||z| - 1| + |z||\pi/2 - \arg z|$.

10. The equation to any tangent is

$x\cos\alpha + y\sin\alpha = p$ ... (i)

where $p = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$

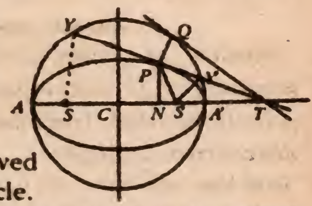
The perpendicular SY to (i) passes through the point $(-ae, 0)$ and its equation

$(x + ae)\sin\alpha - y\cos\alpha = 0$ ... (ii)

If Y be the point (b, k) then since Y lies on both (i) and (ii), we have

$$b\cos\alpha + k\sin\alpha = \sqrt{a^2\cos^2\alpha + b^2\sin^2\alpha}$$

$$b\sin\alpha - k\cos\alpha = -ae\sin\alpha = -\sqrt{a^2 - b^2}\sin\alpha$$



Squaring and adding these equations, we have $b^2 + k^2 = a^2$ so that Y is on the auxilliary circle $x^2 + y^2 = a^2$.

Similarly it may be proved that Y' lies on that circle.

Again S is the point $(-ae, 0)$ and S' $(ae, 0)$. Hence from (i),

$SY = p + ae\cos\alpha$ and $S'Y' = p - ae\cos\alpha$

$SY \cdot S'Y' = p^2 - a^2e^2\cos^2\alpha$

$= a^2\cos^2\alpha + b^2\sin^2\alpha - (a^2 - b^2)\cos^2\alpha = b^2$

$CT = a^2/CN$

and therefore $S'T = \dfrac{a^2}{CN} - ae = \dfrac{a(a - eCN)}{CN}$

$$\therefore \quad \frac{CT}{S'T} = \frac{a}{a - eCN} = \frac{CY}{S'P}$$

Hence CY and $S'P$ are parallel. Similarly CY' and SP are parallel.

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Complex Numbers

Learnfast will aid students in quick understanding of concepts on the above topic. Exercises are given at the end of the topic so that application of concepts can be made. *Remember it is not the number of problems that you have solved that counts, but your level of understanding.*

Definition : A number which can be written in the form $a + ib$, where a and b are real numbers and $i = \sqrt{-1}$, is called a complex number.

A number of the form $a + ib$, where a and b are reals and $b \neq 0$ is called an imaginary number.

e.g. $\sqrt{-3} = \sqrt{3}\sqrt{-1} = \sqrt{3}i$, $3+\sqrt{-4} = 3+\sqrt{4}\sqrt{-1} = 3 + 2i$ are imaginary numbers.

We use the following : $i = \sqrt{-1}$

By definition, $i^2 = -1$

Note that $i^2 = i \times i = \sqrt{-1} \times \sqrt{-1} = \sqrt{(-1)(-1)} = \sqrt{1} = 1$ is wrong.

Because $\sqrt{a} \times \sqrt{b} = \sqrt{ab}$, if and only if, at least one of $\sqrt{a}$ or $\sqrt{b}$ is real.

$i^3 = i^2 \times i = (-1) \times (i) = -i$; $i^4 = i^2 \times i^2 = (-1)(-1) = 1$ etc.

In general, $i^{4n} = +1$, where n is an integer.

Now $i^{11} = i^8 \times i^3 = (1)(-i) = -i$;

$i^{25} = i^{24} \times i = (1)(i) = i$;

$i^{1978} = i^{1976} \times i^2 = (1)(-1) = -1$;

$(-i)^7 = (i^3)^7 = i^{21} = i^{20} \times i = (1)(i) = i$.

Conjugate complex numbers

Definition : $a + bi$ and $a - bi$, where a and b are real numbers, $i = \sqrt{-1}$ and $b \neq 0$ are said to be *complex conjugate* of each other. (Here the sign of i is changed to obtain complex conjugate).

Note that, sum $= (a + bi) + (a - bi) = 2a$ which is real.

and product $= (a + bi)(a - bi) = (a^2) - (bi)^2$

$= a^2 - b^2 i^2 = a^2 - b^2(-1)$

$= a^2 + b^2$ which is real.

The set of complex numbers *does not possess the property of order.*

$\therefore$ the statement $5 + 3i > 2 + i$ is false.

i.e. $a + ib <$ or $> c + id$ are false where b and $d \neq 0$.

Algebraic operations with complex numbers

I. Addition : $(a + ib) + (c + id) = (a + c) + i(b + d)$

II. Subtraction : $(a + ib) - (c + id) = (a - c) + i(b - d)$

III. Multiplication : $(a + ib)(c + id)$

$= ac + iad + ibc + i^2bd = ac + i(ad + bc) + (-1)bd$

$= (ac - bd) + i(ad + bc)$.

IV. Division : $\dfrac{a+ib}{c+id}$ $(c \neq 0, d \neq 0)$

To simplify this case, we always multiply and divide by the conjugate of the denominator.

$$\text{Thus } \frac{a+ib}{c+id} = \frac{(a+ib)}{(c+id)} \times \frac{(c-id)}{(c-id)} = \frac{ac+ibc-iad-i^2bd}{c^2-i^2d^2}$$

$$= \frac{ac+i(bc-ad)-(-1)bd}{c^2-(-1)d^2}$$

$$\frac{a+ib}{c+id} = \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}$$

$$\therefore \frac{a+ib}{c+id} = \frac{(ac+bd)-i(bc-ad)}{c^2+d^2}$$

$$\therefore \frac{a+ib}{c+id} = \frac{(ac+bd)}{c^2+d^2} + i\frac{(bc-ad)}{c^2+d^2}$$

How to represent a complex number

Method (I):

A complex number $Z = x + iy$ can be represented by a point on the plane (known as Argand plane) by the ordered pair (x, y).

Consider a line segment joinong O and P.

Its length $= \sqrt{x^2+y^2}$ and it makes an angle $\theta = \tan^{-1}\dfrac{y}{x}$ with the axis of x.

Definition : The absolute length $OP = \sqrt{x^2+y^2}$ is called the modulus of the complex number $x + iy$ which is denoted by z. i.e $Z = x + iy$ and $|Z| = \sqrt{x^2+y^2}$ the length of OP.

Similarly the amplitude (or argument) of the complex number is defined as the angle made by OP with the axis of x.

θ = amplitude Z = amplitude $(x + iy) = \tan^{-1}\dfrac{y}{x}$

Note: $Z_1 = Z_2 \Leftrightarrow |Z_1| = |Z_2|$ and amplitude Z_1 = amplitude Z_2. There exists a one - one correspondence between the points of the plane and the members of the set of complex numbers, *i.e.* for every complex number Z there ex-

= 533715 crore = Rs. 53371500

$\therefore$ Investment in State-issued bonds
= 26% of Rs. 53371500 = Rs. 13876590.

67. **(b)** : In (a), Rs. 2.988804 crore.
In (b), Rs. 1.387659 crore
In (c), Rs. 5.33715 crore.

68. **(a)** : Investment in Municipal Bonds = 56% of 53371500 = 29888040 (from Q. 66)

(a) Investment in Municipal Bonds = (7% - 9%) = 65% of 29888040 = 19427226.

(b) Investment in State-issued Bonds = 13876590. (from Q. 66)

(c) Investment in High Risk Stock = 9834500. (from Q. 65)

(d) Investment in Municipal Bonds (above 9%) = 35% of 29888040 = 10460814.

69. **(c)** : 35% of 91.9 = 32.165 ≈ 32 lakh.

70. **(c)** : 21% of 25.5 – 10% of 29.2
= 5.355 – 2.920 = 2.435 lakh

71. **(b)** : Slum population in
Kolkota = 32.165 lakh; Mumbai = 31.312 lakh
Delhi = 17.190 lakh; Chennai = 13.728 lakh
Ahmedabad = 6.630 lakh; Hyderabad = 5.355 lakh
Bangalore = 2.920 lakh.

72. **(d)** **73.** **(a)**

74. **(d)** : Slum population (total) = 109.3 lakh.
Mumbai + Ahmedabad = 107.9 lakh.

75. **(b)** : Let $\dfrac{32.165}{91.9} = k \times \dfrac{2.92}{29.2} \Rightarrow k = 3.5.$

76. **(d)**

77. **(c)** : Percent increase in investment

For SAIL = $\dfrac{372}{5933} \times 100 = 6.27\%$

For Coal India = $\dfrac{811}{4730} \times 100 = 17.15\%.$

For NTPC = $\dfrac{1401}{3119} \times 100 = 44.92\%.$

For ONGC = $\dfrac{428}{2432} \times 100 = 17.60\%.$

For REC = $\dfrac{308}{1522} \times 100 = 20.24\%.$

For NTC = $\dfrac{117}{933} \times 100 = 12.54\%.$

78. **(a)** **79.** **(c)**

80. **(c)** : Investment in 1995–96 increased from 1994-95 by 22108 – 18669

= Rs. 3439 crore = $\dfrac{3439}{18669} \times 100 = 18.42\% \approx 18\%.$

Challenging Problems from World Mathematics Championship

1. Show that the polynomial
$x^4 - 1994x^3 + (1993 + m)x^2 - 11x + m,\ m \in Z.$
has at most one integral root.

Soln.: Consider

$$x^4 - 1994x^3 + (1993 + m)x^2 - 11x + m \quad \ldots (1)$$

Suppose the given polynomial has two integral roots. Then neither can be odd for otherwise

$$(x^4 + 1993x^2) - (1994x^3) + m(x^2 + 1) - 11x$$

will be odd (as each of the terms in brackets is even) and hence non-zero.

Suppose $x_1 = 2^{r_1}a$, $r_1 > 0$ and a odd is a solution. Considering the polynomial (mod 2^{2r_1}), we then have that $m \equiv 11x$ (mod 2^{2r_1}).

Hence $m \equiv 2^{r_1}(11a)$ (mod 2^{2r_1}), and since a is odd, m must be of the form $2^{r_1}l_1$, l_1 odd.

If $x_2 = 2^{r_2}b$, b odd, is also a solution, then $m = 2^{r_2}l_2$, l_2 odd, so we must have $r_1 = r_2$.

Thus, if there are two integral roots, both must be of the form $2^r k$, $r > 1$, k odd. The product of the two roots must be a multiple of 2^{2r}. The quadratic which has these two roots as zeros is

$x^2 + px + 2^{2r}q$, where p, q are integers.

Now the given polynomial (1) can be factorized into two quadratics.

$$(x^2 + px + 2^{2r}q)(x^2 + sx + t).$$

If s were not integral, then the coefficient of x^3 would not be integral in the quartic, and if t were not integral, the coefficient of x^2 would not be integral in the quartic. Thus s and t must be integers and $m = 2^{2r}qt$. But the highest power of 2 dividing m is 2^r, so $r \geq 2r$ giving $r = 0$, a contradiction.

Hence the given quartic cannot have more than one integral root.

2. Find the smallest number $n > 4$ such that there is a set of n people with the following properties:

(i) any two people who know each other have no common acquaintances;

(ii) any two people who do not know each other have exactly two commom acquaintances.

Note : Acquaintance is a symmetric relation.

Soln.: Choose one of the people A, and suppose A knows $x_1, x_2, \ldots, x_r$. Then by (i) no x_i knows an x_j for $i \neq j$. Therefore, by (ii), for each pair (x_i, x_j) there must exist an X_{ij} who knows both x_i and x_j in order that x_i and x_j have two common acquaintances A and X_{ij}. Now A cannot know any of the X_{ij}. Thus by (ii) each X_{ij} can have only two acquaintances among $x_1, x_2, \ldots, x_r$, namely x_i and x_j so all the X_{ij} are distinct.

Any person who is not A, nor an acquaintance of A must by (ii) be an X_{ij}. Thus the total number of people must be $\binom{r}{2} + r + 1$.

Now $r > 2$ and $n > 4$.

If $r = 3$, then $n = 7$ ... (A)

If $r = 4$, then $n = 11$ ... (B)

If $r = 5$, then $n = 16$ ... (C)

Let us label the people 1, 2, ..., n.

Case (A) : Without loss of generality suppose 1 knows 2, 3 and 4 and that 5 knows 2 and 3, 6 knows 2 and 4 and 7 knows 3 and 4.

Now 5 must have three acquaintances, so he must know one of 6 and 7. But he has common acquaintances with both 6 and 7, contradicting (i).

Case (B) : Without loss of generality, suppose 1 knows 2, 3, 4 and 5 and 6, 7, 8, 9, 10, 11 know pairs {2, 3}, {2, 4}, {2, 5}, {3, 4}, {3, 5} and {4, 5} respectively.

Then 6 cannot know 7, 8, 9 or 10 as he has common acquaintances with each of them. So 6 can only know 2, 3 and 11, while he must know $r = 4$ people, a contradiction.

Case (C) : We claim that $n = 16$ is the smallest number of people required in such a set, by noting that cases (A) and (B) fail and that the following acquaintance table satisfies (i) and (ii).

Person	*Acquaintances*					*Person*	*Acquaintances*				
1	2	3	4	5	6	9	2	5	11	13	15
2	1	7	8	9	10	10	2	6	11	12	14
3	1	7	11	12	13	11	3	4	9	10	16
4	1	8	11	14	15	12	3	5	8	10	15
5	1	9	12	14	16	13	3	6	8	9	14
6	1	10	13	15	16	14	4	5	7	10	13
7	2	3	14	15	16	15	4	6	7	9	12
8	2	4	12	13	16	16	5	6	7	8	11

3. Prove that there does not exist a function $f: Z \to Z$, for which $f(f(x)) = x + 1$ for every $x \in Z$.

Soln.: Suppose that there is such a function.
Then $f(f(f(x))) = f(x) + 1$.
Since $f(f(x)) = x + 1$, we get $f(x + 1) = f(x) + 1$.
By induction $f(x + n) = f(x) + n$ for every $n \in N$.
Also $f(x) = f(x - n + n) = f(x - n) + n$.
So, $f(x - n) = f(x) - n$ for every $n \in N$.
Finally $f(x + y) = f(x) + y$ for $x, y \in Z$.
For $x = 0$, $f(y) = f(0) + y$.
For $y = f(0)$, $f(f(0)) = f(0) + f(0)$.
But $f(f(0)) = 1$; thus $2f(0) = 1$, a contradiction.

4. Put a natural number in every empty field of the table so that you get an arithmetic sequence in every row and every column.

	74			
				186
		103		
0				

Soln.: Let us say the numbers adjacent to 0 are a_0 and a_1, with a_0 in the row and a_1 in the column. We know that in an arithmetic sequence every term is the arithmetic mean of the term before and after. Therefore, we can put numbers in the chart as follows:

$3a_1$	74			
$2a_1$	$a_1 - a_0 + 103$			186
a_1	$\frac{1}{2}(a_1 + 103)$	103	$\frac{1}{2}(309 - a_1)$	$206 - a_1$
0	a_0	$2a_0$	$3a_0$	$4a_0$

Now, $\frac{1}{2}(186 + 4a_0) = 206 - a_1$
$\Rightarrow 93 + 2a_0 = 206 - a_1$
$\Rightarrow 2a_0 + a_1 = 113$ ... (i)
and $74 + \frac{1}{2}(a_1 + 103) = 2(103 + a_1 - a_0)$
$\Rightarrow 3a_1 - 4a_0 = -161$ ... (ii)
Solving (i) and (ii) gives $a_0 = 50$ and $a_1 = 13$.
So we can easily put numbers in every field as shown.

52	82	112	142	172
39	74	109	144	179
26	66	106	146	186
13	58	103	148	193
0	50	100	150	200

5. Prove that every number of the sequence 49, 4489, 444889, 44448889, ... is a perfect square (in every number there are n fours, $n - 1$ eights and a nine).

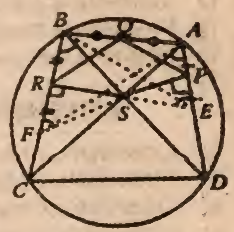
Soln.: Let $4_3 8_2 9$ denote the number 444889 and $6_2 7$ denote the number 667.
We shall show that $(6_{n-1}7)^2 = 4_n 8_{n-1} 9$ for each positive integer n.

Because $(6_{n-1}7)^2 = \left(\frac{6(10^n - 1)}{9} + 1\right)^2$, it suffices to establish that $\left(\frac{6(10^n - 1)}{9} + 1\right)^2 = 4_n 8_{n-1} 9$ for each positive integer n.

Let n be an arbitrary positive integer. Then

$$\left(\frac{6(10^n - 1)}{9} + 1\right)^2 = \left(\frac{6 \cdot 10^n + 3}{9}\right)^2$$
$$= \frac{(2 \cdot 10^n + 1)^2}{9} = \frac{4 \cdot 10^{2n} + 4 \cdot 10^n + 1}{9}$$
$$= \frac{40_{n-1}40_{n-1}1}{9} = 4_n 8_{n-1} 9.$$

6. Let Q be the mid-point of the side AB of an inscribed quadrilateral $ABCD$ and S the intersection of its diagonals. Denote by P and R the orthogonal projections of S on AD and BC respectively. Prove that $|PQ| = |QR|$.



Soln.: Let E be the point symmetric to A with respect to P, and let F be the point symmetric to B with respect to R. Then we have

$SE = SA$, $\angle SEA = \angle SAE$
and $SF = SB$, $\angle SFB$, $\angle SBF$.
As A, B, C, D are concyclic we get $\angle CAD = \angle CBD$.
Thus, $\angle SEA = \angle SAE = \angle SBF = \angle SFB$.
Consequently we have $\angle ASE = \angle BSF$.
Thus we get, $\angle BSE = \angle BSA + \angle ASE$
$= \angle BSA + \angle BSF = \angle FSA$.
Since $SB = SF$ and $SE = SA$, we have
$\Delta SEB \cong \Delta SAF$. (SAS)
Thus we get $EB = AF$. Since P, Q, R are mid-points of AE, AB, BF respectively, we have

$$PQ = \frac{1}{2} EB \text{ and } QR = \frac{1}{2} AF.$$

Therefore we have $PQ = QR$.

10 Selected Problems

Series, Sequence & Progression

1. Let $a_1, a_2, a_3, \ldots, a_n$ be in AP where $a_1 = 0$ and common difference $\neq 0$. Show that

$$\frac{a_3}{a_2}+\frac{a_4}{a_3}+\frac{a_5}{a_4}+\ldots\ldots+\frac{a_n}{a_{n-1}}-a_2\left(\frac{1}{a_2}+\frac{1}{a_3}+\ldots.+\frac{1}{a_{n-2}}\right)$$
$$=\frac{a_{n-1}}{a_2}+\frac{a_2}{a_{n-1}}$$

Soln.: Here $t_1 = a_1 = 0$;
common difference $= a_2 - a_1 = a_2 - 0 = a_2$
L.H.S.

$$=\frac{a_3-a_2}{a_2}+\frac{a_4-a_2}{a_3}+\frac{a_5-a_2}{a_4}+\ldots..+\frac{a_{n-1}-a_2}{a_{n-2}}+\frac{a_n}{a_{n-1}}$$

$$=\frac{a_2}{a_2}+\frac{a_3}{a_3}+\frac{a_4}{a_4}+\ldots\ldots+\frac{a_{n-2}}{a_{n-2}}+\frac{a_n}{a_{n-1}}$$

$\{\because\ t_n - \text{common difference} = t_{n-1}\}$

$$=(n-3)+\frac{a_n}{a_{n-1}}=(n-3)+\frac{a_1+(n-1)a_2}{a_1+(n-2)a_2}$$

$\{\because\ a_2 = \text{common difference}\}$

$$=(n-3)+\frac{n-1}{n-2} \quad (\because\ a_1 = 0)$$

$$\text{RHS} = \frac{a_1+(n-2)a_2}{a_2}+\frac{a_2}{a_1+(n-2)a_2}$$

$$=(n-2)+\frac{1}{n-2}=(n-3)+1+\frac{1}{n-2}=(n-3)+\frac{n-1}{n-2}$$

$\therefore$ LHS = RHS.

2. The sum of squares of three distinct real numbers which are in GP is S^2. If their sum is $\alpha \cdot S$, show that $\alpha^2 \in (\frac{1}{3}, 1) \cup (1, 3)$.

Soln.: Three numbers in GP are $\frac{a}{r}$, a, ar. As they are distinct, $r \neq 1, -1$.

Let $r+\frac{1}{r}=x$. So $x \neq 2, -2$ as $r \neq 1, -1$

$$a^2\left(r^2+\frac{1}{r^2}\right)+a^2=S^2 \qquad \text{(given)}$$

$$\text{or} \quad a^2\left\{\left(r+\frac{1}{r}\right)^2-2\right\}+a^2=S^2$$

$$\therefore \quad x^2-2+1=\frac{S^2}{a^2} \quad \text{or} \quad x^2-1=\frac{S^2}{a^2} \qquad \ldots (1)$$

$$a\left(r+\frac{1}{r}\right)+a=\alpha\cdot S \qquad \text{(given)}$$

$$\text{or} \quad ax+a=\alpha\cdot S \quad \text{or} \quad x+1=\alpha\cdot\frac{S}{a} \qquad \ldots (2)$$

$$\frac{(x+1)^2}{x^2-1}=\alpha^2 \qquad \text{from (1) and (2)}$$

$$\Rightarrow \quad x=\frac{\alpha^2+1}{\alpha^2-1} \quad \text{or} \quad r+\frac{1}{r}=\frac{\alpha^2+1}{\alpha^2-1}$$

or $(\alpha^2 - 1)r^2 - (\alpha^2 + 1)r + \alpha^2 - 1 = 0$
As r is real, $D \geq 0$
or $(\alpha^2 + 1)^2 - 4(\alpha^2 - 1)^2 \geq 0$
or $(3\alpha^2 - 1)(3 - \alpha^2) \geq 0$

$\therefore$ either $3\alpha^2 - 1 \geq 0$	or $3\alpha^2 - 1 \leq 0$
and $3 - \alpha^2 \geq 0$	and $3 - \alpha^2 \leq 0$
$\therefore$ either $\alpha^2 \geq \frac{1}{3}$	$\therefore$ $\alpha^2 \leq \frac{1}{3}$
and $\alpha^2 \leq 3$	and $\alpha^2 \geq 3$
$\therefore$ $\alpha^2 \geq \frac{1}{3}$	and $\alpha^2 \leq 3$

From (5), $\alpha^2 \neq 3, \frac{1}{3}$ because $x \neq 2, -2$

Also $\alpha^2 \neq 1$ for otherwise $\frac{x+1}{x-1}=1$, which implies $1 = -1$. $\therefore$ $\frac{1}{3}<\alpha^2<1$ and $1 < \alpha^2 < 3$

So $\alpha^2 \in \left(\frac{1}{3}, 1\right) \cup (1, 3)$.

3. Given a GP and an AP with positive terms $a, a_1, a_2, a_3, \ldots\ldots, a_n, \ldots.$ and $b, b_1, b_2, b_3, \ldots.., b_n, \ldots.$ respectively where the common ratio of the GP $\neq 1$. Prove that there exists a positive real number x such that $\log_x a_n - b_n = \log_x a - b$ for all $n \in N$. Also find that x

Soln.: Let the common ratio of the GP be r and the common difference of the AP be d. As the progressions have positive terms.

$a > 0,\ r > 0,\ a_n > 0,\ b > 0,\ b_n > 0,\ d > 0$ for all n.
Now, $a_n = ar^n$ and $b_n = b + rd$

$\therefore\quad \frac{a_n}{a} = r^n$ and $b_n - b = rd$

$\therefore\quad \log_x \frac{a_n}{a} = \log_x r^n$

$\therefore\quad \log_x \frac{a_n}{a} = b_n - b$ if $\log_x r^n = rd$

$\Rightarrow\quad n \log_x r = rd \Rightarrow \log_x r = \frac{rd}{n}$

$\Rightarrow\quad x^{\frac{rd}{n}} = r \Rightarrow x = r^{\frac{n}{rd}} > 0$

$\therefore\quad \log_x (a_n - b_n) = \log_x (a - b)$ where $x = r^{\frac{n}{rd}} > 0$

4. Find the coefficient of x^{98} in the continued product $(x+1)(x+2)(x+3) \ldots (x+100)$.

Soln.: We know that $(x + a_1)(x + a_2)(x + a_3)\ldots(x + a_n)$
$= x^n + (\Sigma a_1)x^{n-1} + (\Sigma a_1 a_2)x^{n-2} + (\Sigma a_1 a_2 a_3)x^{n-3} + \ldots + (a_1 a_2 a_3 \ldots a_n)$

$\therefore$ the coefficient of x^{98} in the product of 100 linear factors will be sum of the products of 1, 2, 3 ...., 100 taking two at a time.

Now $(1 + 2 + 3 + \ldots + 100)^2$
$= 1^2 + 2^2 + 3^2 + \ldots + 100^2 + 2 \times$ (required sum of products)

$\therefore$ the required coefficient

$$= \frac{1}{2}[(1+2+3+\ldots+100)^2 - (1^2+2^2+3^2+\ldots+100^2)]$$

$$= \frac{1}{2}\left[\left(\frac{100\times 101}{2}\right)^2 - \frac{100\times 101\times 201}{6}\right] = 12582075.$$

5. If there be m quantities in a GP whose common ratio is r and S_n denotes the sum of the first n terms of the GP, prove that the sum of their products (two by two) is $\frac{r}{r+1}\ S_m \cdot S_{m-1}$.

Soln.: Let $S_n = a + ar + ar^2 + \ldots + ar^{n-1} = \frac{a(1-r)^n}{1-r}$

$$\therefore\quad S_m = \frac{a(1-r^m)}{1-r},\quad S_{m-1} = \frac{a(1-r^{m-1})}{1-r}.$$

Let the sum of the products of $a, ar, ar^2, \ldots, ar^{m-1}$ taken two at a time be S.

Now $(a_1 + a_2 + a_3 + \ldots + a_m)^2$
$= a_1^2 + a_2^2 + a_3^2 + \ldots + a_m^2 + 2S$

$\therefore\quad (a + ar + ar^2 + \ldots + ar^{m-1})^2$
$= a^2 + a^2r^2 + a^2r^4 + \ldots + a^2r^{2(m-1)} + 2S$

or $$\left\{\frac{a(1-r^m)}{1-r}\right\}^2 = \frac{a^2(1-r^{2m})}{1-r^2} + 2S$$

$$\therefore\quad 2S = \frac{a^2(1-r^m)^2}{(1-r)^2} - \frac{a^2(1-r^{2m})}{1-r^2}$$

$$= a^2 \cdot \frac{2(r^{2m} - r^{m+1} - r^m + r)}{(1-r)^2\cdot(1+r)}$$

$$\therefore S = a^2 \cdot \frac{r^m(r^m-1) - r(r^m-1)}{(1-r)^2(1+r)} = a^2 \cdot \frac{(r^m-1)(r^m-r)}{(1-r)^2(1+r)}$$

$$= \frac{a(r^m-1)}{(1-r)} \cdot \frac{a(r^m-r)}{1-r} \cdot \frac{1}{1+r}$$

$$= \frac{a(1-r^m)}{1-r} \cdot \frac{a(1-r^{m-1})}{1-r} \cdot \frac{r}{1+r} = \frac{r}{1+r} S_m \cdot S_{m-1}.$$

6. If $\log_e \frac{1}{1+x+x^2+x^3}$ be expanded in ascending powers of x, show that the coefficient of x^n is $-1/n$ if n is odd or of the form $4m+2$ and $3/n$ if n is of the form $4m$.

Soln.: $\log_e \frac{1}{1+x+x^2+x^3} = \log_e \frac{1}{(1+x)(1+x^2)}$
$= -\log_e[(1+x)(1+x^2)] = -\log(1+x) - \log(1+x^2)$

$$= -\left\{x - \frac{1}{2}x^2 + \frac{1}{3}x^3 - \ldots\right\} - \left\{x^2 - \frac{1}{2}x^4 + \frac{1}{3}x^6 - \ldots\right\}$$

Let n be odd. Then there is no term containing x^n in the second series.

$\therefore$ coefficient of $x^n = -1/n$.

Let $n = 4m + 2$. Then
coefficient of x^n = coefficient of x^{4m+2}

$$= \frac{1}{4m+2} - \frac{1}{2m+1} = -\frac{1}{4m+2} = -\frac{1}{n}.$$

Let $n = 4m$. Then
coefficient of x^n = coefficient of x^{4m}

$$= \frac{1}{4m} + \frac{1}{2m} = 3\cdot\frac{1}{4m} = \frac{3}{n}.$$

7. If $a_1, a_2, \ldots, a_n$ are in A.P., with common difference d, then

$$\sum_{r<s} a_r a_s = \frac{1}{2}n(n-1)[a_1^2 + (n-1)a_1 d + \frac{1}{12}(3n^2 - 7n + 2)d^2]$$

Soln.: $(a_1 + a_2 + \ldots + a_n)^2 = \sum_{k=1}^{n} a_k^2 + 2\sum_{r<s=1}^{n} a_r\, a_s$ ....(1)

$$\sum_1^n a_k = \frac{n}{2}[2a_1 + (n-1)d_1]$$

$$\sum_1^n a_k^2 = \sum_1^n a_1 + (k-1)d]^2$$

$$= \sum_1^n [a_1^2 + 2a_1(k-1)d + (k-1)^2 d^2]$$

$$= na_1^2 + 2a_1 d \cdot \frac{(n-1)n}{2} + \frac{1}{6}d^2(n-1)n(2n-1)$$

Substituting in (1), we get the required result.

8. If $x_1, x_2, \ldots, x_n$ are n non-zero real numbers such that $(x_1^2 + x_2^2 + \ldots + x_{n-1}^2)(x_2^2 + x_3^2 + \ldots + x_n^2) \le (x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n)^2$, then prove that $x_1, x_2, \ldots, x_n$ are in G.P.

Soln: From the question,

$$(x_1^2 + x_2^2 + \ldots + x^2_{n-1})(x_2^2 + x_3^2 + \ldots + x_n^2) - (x_1x_2 + x_2x_3 + \ldots + x_{n-1}x_n)^2 \le 0$$

Using Lagranges Identity, this gives,

$$(x_1x_3 - x_2x_2)^2 + (x_2x_4 - x_3x_3)^2 + \ldots + (x_{n-2}x_n - x_{n-1}x_{n-1})^2 + (x_{n-1}x_2 - x_nx_1)^2 \le 0 \quad \ldots(1)$$

Since $x_1, x_2 \ldots$ are real numbers (1) gives

$$x_1x_3 = x_2^2,\ x_2x_4 = x_3^2,\ \ldots,\ x_{n-2}x_n = x^2_{n-1},\ x_{n-1}x_2 = x_nx_1$$

or $\dfrac{x_1}{x_2} = \dfrac{x_2}{x_3} = \dfrac{x_3}{x_4} = \ldots\ldots = \dfrac{x_{n-1}}{x_n}$,

so, $x_1, x_2, x_3, \ldots, x_n$ are in G.P.

9. If $S_1, S_2, S_3 \ldots S_n$ are the sums of infinite geometric series whose first terms are 1, 2, 3, ... n and whose common ratios are $\dfrac{1}{2}, \dfrac{1}{3}, \dfrac{1}{4}, \ldots\ldots \dfrac{1}{n+1}$ respectively, then find the value of $S_1^2 + S_2^2 + S_3^2 + \ldots S^2_{2n-1}$.

Soln: Here $S_n = \dfrac{n}{1 - \dfrac{1}{n+1}} = \dfrac{n(n+1)}{n+1-1} = n+1$

$\therefore \quad S_n^2 = (n+1)^2$

So $S_1^2 + S_2^2 + \ldots\ldots S^2_{2n-1} = 2^2 + 3^2 + \ldots\ldots + (2n)^2$

$= (1/6)(2n)(2n+1)(4n+2-1) - 1$

$= \dfrac{1}{3}[n(2n+1)(4n+1) - 3]$.

10. The sum of an infinite geometric series is 2 and the sum of the geometric series made from the cubes of this infinite series is 24. Then find the series.

Soln: Let first term = a, common ratio = r where $-1 < r < 1$. Then $\dfrac{a}{1-r} = 2$ and $\dfrac{a^3}{1-r^3} = 24$

$\therefore \quad \dfrac{1-r^3}{(1-r)^3} = \dfrac{1}{3}$ *i.e.* $1 - 2r + r^2 = 3(1 + r + r^2)$

or $2r^2 + 5r + 2 = 0$

$\therefore \quad r = -2$ or $-1/2$. As $-1 < r < 1$,

$\therefore$ we have $r = -1/2$.

Putting this value of r, we get $a = 3$.

$\therefore$ The series is $3 - \dfrac{3}{2} + \dfrac{3}{4} - \dfrac{3}{8} + \ldots$

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Mahaviracharya - A Great Jain Mathematician

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Mahaviracharya, briefly known as Mahavira was the most celebrated Jain mathematician of the 9th century. He glorified the court of Amoghavarsha Nriptunga, a famous king of Rashtrakoota dynasty which flourished in a part of the present state of Karnataka. His book "*Ganita Sara Sangraba*" (*G.S.S.*) was written in 850 A.D. and was discovered some 50 years ago. The book has been translated to English and edited by M. Rangacharya of the Madras University, and published by the Government of Madras.

Mahavira was not an astronomer and his work was confined to pure mathematics only. The G.S.S. is the first arithmetic text book in the present-day form, except for the fact that there is no chapter on decimals. Mahavira's work does not include any profoundly fundamental discoveries. His main contributions are essentially improving and extending the results of his predecessors. His contributions of arithmetic, algebra and geometry are the following:

Arithmetic and Algebra

(i) Garland product : The following products resemble a garland *i.e.* they give the same numbers read from left to right or right to left.

139 × 109 = 15151

27994681 × 441 = 123456 54321

12345679 × 9 = 111 111 111

333333 666667 × 33 = 110000 110000 11 etc.

(ii) Problem on Quadratic Equation

चरांत कमल षंडे सारसानां चतुर्थो
नवमश्चरण भागौ सप्तमूलानि चाद्रौ
तिलकबकुलमध्ये संप्तनिःप्राष्टमाना:
कतिकथय सखेत्वं प्रशिगणोदगा सारसात् ?

(Out of certain number of Sarasa birds, 1/4th the number are moving about in lotus plants; 1/9th coupled with; 1/4th as well as 7 times the square root of the number move on a hill; 56 birds remain in vakul trees. What is the total number of birds?) If x is the total number of birds, this gives the equation

$$x = \frac{x}{4} + \frac{x}{9} + \frac{x}{4} + 7\sqrt{x} + 56 \Rightarrow x = 576.$$

(iii) Value of a^3 : The following formulae of cubing was given by Mahavira:

$$a^3 = a(a+b)(a-b) + b^2(a-b) + b^3$$
$$= a + 3a + 5a + \ldots \text{ upto } a \text{ terms}$$
$$= a^2 + (a-1)(1 + 3 + 5 + \ldots \text{ upto } a \text{ terms})$$
$$= 3[1 \cdot 2 + 2 \cdot 3 + 3 \cdot 4 + \ldots + (a-1)a] + a$$

(iv) Four different sums of money equal to 40, 30, 20 and 50 respectively are lent out at the same rate of interest of 5, 4, 3 and 6 units of time respectively. The total interest is 34. What is the interest that each amount fetches?

Let the rate of interest be r and x_1, x_2, x_3, x_4 be the interests earned by the given amounts.

$$r = \frac{x_1}{40 \times 5} = \frac{x_2}{30 \times 4} = \frac{x_1}{20 \times 3} = \frac{x_4}{50 \times 6}$$
$$= \frac{x_1 + x_2 + x_3 + x_4}{200 + 120 + 60 + 300} = \frac{34}{680} = \frac{1}{20}$$
$$\Rightarrow x_1 = 10,\ x_2 = 6,\ x_3 = 3,\ x_4 = 15,\ r = \frac{1}{20}.$$

(v) Mahavira is the world's first mathematician to give the general formula

$$^nC_r = \frac{n(n-1)(n-2)\ldots(n-r+1)}{1 \cdot 2 \cdot 3 \cdot \ldots \cdot r}$$

(vi) *Unit fractions* : A unit fraction is one whose numerator is unity. The most interesting results in Mahavira's work are his methods of obtaining unit fractions for any given fraction. He gives the following

(a) $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{3^2} + \frac{1}{3^3} + \ldots + \frac{1}{3^{n-2}} + \frac{1}{2 \cdot 3^{n-2}}$.

Thus for $n = 5$, $1 = \frac{1}{2} + \frac{1}{3} + \frac{1}{9} + \frac{1}{27} + \frac{1}{54}$.

(b)

$$1 = \frac{1}{2 \cdot 3 \cdot \frac{1}{2}} + \frac{1}{3 \cdot 4 \cdot \frac{1}{2}} + \ldots + \frac{1}{(2n-1) \cdot 2n \cdot \frac{1}{2}} + \frac{1}{2n \cdot \frac{1}{2}}$$

A few exceptional Problems

Problem 1 : Determine all possible integral solutions of the following equation:

$$x_1^6 + x_2^6 + x_3^6 + x_4^6 + x_5^6 + x_6^6 + x_7^6 = 2699999$$

Soln.: First we prove two Lemmas.

Lemma-1

Prove that

$$\left.\begin{aligned} x^{k(k-1)} &\equiv 1 \pmod{k^2} \text{ if } (x, k) = 1 \\ &\equiv 0 \pmod{k^2} \text{ if } (x, k) = k \end{aligned}\right\}$$

Proof: We know that in terms of the Fermat Theorem:

$$\left.\begin{aligned} x^{k-1} &\equiv 1 \pmod{k} \text{ for } (x, k) = 1 \\ &\equiv 0 \pmod{k} \text{ for } (x, k) = 0 \end{aligned}\right\} \quad \text{...(i)}$$

Then, substituting $y = x^k$, we have

$(x, k) = 1 \Rightarrow (x^k, k) = 1 \Rightarrow (y, k) = 1$

& $(x, k) = k \Rightarrow (y, k) = k$.

$$\therefore \quad \left.\begin{aligned} y &\equiv 1 \pmod{k} \text{ for } (y, k) = 1 \\ &\equiv 0 \pmod{k} \text{ for } (y, k) = 0 \end{aligned}\right\},$$

Case - I : If $y \equiv 1 \pmod{k}$

Then, $\dfrac{y^k - 1}{y-1} = y^{k-1} + y^{k-2} + \ldots\ldots + y^2 + 1$

$$= \sum_{i=0}^{k-1} y^i = S_k \text{ (say)}$$

But, $y \equiv 1(\text{mod } k) \Rightarrow y^i \equiv 1(\text{mod } k)$.

$\Rightarrow \quad S_k = \sum\limits_{i=0}^{k-1} y^i \equiv 1 \times k \equiv k(\text{mod } k) \equiv 0(\text{mod } k)$

i.e. there exists an integer $r \in I^+$

$S_k = k \cdot r \Rightarrow \dfrac{y^{k-1}}{y-1} = kr$

$\Rightarrow y^{k-1} = kr(y - 1) + 1$

But, $y - 1 \equiv 0(\text{mod } k)$

$\Rightarrow y - 1 = tk$ for some $t \in I^+$

$\therefore \quad y^{k-1} = (kr)(tk) + 1 = rtk^2 + 1$

$\Rightarrow y^{k-1} \equiv 1(\text{mod } k^2)$

But, $y = x^k$ (by assumption)

$\Rightarrow x^k(k - 1) \equiv 1 \pmod{k^2}$.

Case II : If $y \equiv 0(\text{mod } k)$, where k is prime

$\Rightarrow y = mk$ for some $m \in I^+$

$\Rightarrow y^k = (mk)^k = m^k \cdot k^k$

But since, k is a prime number.

$\Rightarrow \quad k \geq 2 \Rightarrow k^k$ is divisible by k^2

$\Rightarrow y^k \equiv 0(\text{mod } k^2)$

$\Rightarrow x^{k(k-1)} \equiv 0(\text{mod } k^2)$

$$\therefore \quad \left.\begin{aligned} x^{k(k-1)} &\equiv 1(\text{mod } k^2), \text{ if } (x, k) = 1 \\ &\equiv 0(\text{mod } k^2), \text{ if } (x, k) = k \end{aligned}\right\}$$

Lemma 2 : Prove that for $x \in I^+$:

$$\left.\begin{aligned} x^6 &\equiv 0(\text{mod } 9) \text{ if } (x, 3) = 3 \\ &\equiv 1(\text{mod } 9) \text{ if } (x, 3) = 1 \end{aligned}\right\}$$

Proof : Method 1.

Putting $k = 3$ in *Lemma 1*.

We have $k(k - 1) = 3 \times 2 = 6$ & $k^2 = 3^2 = 9$.

Hence the result.

Method 2 : Case I :

If $(x, 3) = 3, \Rightarrow x \equiv 0(\text{mod } 3)$

$\Rightarrow x = 3t$ (say) for some $t \in I^+$

$\Rightarrow x^6 = 729t^6 = 9 \times (81t^6) \Rightarrow x^6 \equiv 0(\text{mod } 9)$.

Case II : If $(x, 3) = 1$

Then clearly, $x \equiv 1$ or $2 \pmod{3}$

$\Rightarrow \quad x \equiv 1$ or $-1 \pmod{3}$

$\Rightarrow x = 3t \pm 1$ for some $t \in I^+$

Now, $(3t \pm 1)^2 = 9t^2 \pm 6t + 1 = 3(3t^2 \pm 6t) + 1$

$= 3m + 1$ where $m = 3t^2 \pm 6t$.

Hence, $(3t \pm 1)^6 = (3m + 1)^3$

$= 27m^3 + 27m^2 + 9m + 1$

$= 9(3m^3 + 3m^2 + m) + 1$

$= 9N + 1$, where $N = 3m^3 + 3m^2 + m$.

$\Rightarrow \quad (3t \pm 1)^6 \equiv 1(\text{mod } 9) \Rightarrow x^6 \equiv 1(\text{mod } 9)$

Hence, by combining cases (I) and (II)

$$\left.\begin{aligned} x^6 &\equiv 0(\text{mod } 9) \text{ if } (x, 3) = 3 \\ &\equiv 1(\text{mod } 9) \text{ if } (x, 3) = 1 \end{aligned}\right\}.$$

Hence, by Lemmas 1 and 2,

we must have

$x_i^6 \equiv 0$ or $1 \pmod{9}$ where $x_i \in I^+ \ \forall\, i = 1\ (1)\ 7$ ... (iii)

Now, taking $u = \sum_{i=1}^{7} x_i^6$ ...(iv)

We have if

$x_i^6 \equiv p \pmod 9$ $\forall\, i = 1\ (1)\ 7$ ...(v)

Then, by (iii), (iv), (v)

$0 \le p \le 1 \times 7 \Rightarrow 0 \le p \le 7$ ...(vi)

But, we know that for any number

$$N = \sum_{j=0}^{t} 10^j x_j \Rightarrow N \equiv \sum_{j=0}^{t} x_j \pmod 9$$

In the given problem,

Digit sum of 2699999 = 53 & $53 \equiv 8 \pmod 9$

$\Rightarrow$ $2699999 \equiv 8 \pmod 9$

$\therefore$ By the problem, we must have

$$\sum_{i=1}^{7} x_i^6 \equiv 8 \pmod 9$$

But, clearly by (iii) – (vi)

$$\sum_{i=1}^{7} x_i^6 \equiv p \pmod 9 \text{ where } 1 \le p \le 7.$$

But as 8 > 7. This is a contradiction.

Hence, no integral solution exists for the problem under reference.

Problem 2 : Determine all possible positive integral solutions to the following equation

$$a + b + c = abc.$$

Soln.: *Case A :* If $a = b = c$ (= k, say)

Then $a + b + c = abc$

$\Rightarrow k^3 = 3k \Rightarrow k^2 = 3$ since $k = 0$ is inadmissible as 0 is not a positive integer.

$\Rightarrow k = \sqrt{3}$.

But $\sqrt{3}$ is an irrational number and hence is not an integer. Hence, no solution is possible.

Case B : If exactly two numbers among a, b, c are equal without any loss of generality. Let us take $a = b$ (= l, say)

Then, $a + b + c = abc \Rightarrow 2l + c = l^2c$

$$\Rightarrow c = \frac{2l}{l^2 - 1}.$$

Now $(l, l^2 - 1) = 1$

and $l^2 - 1$ divides 2 only if $l^2 - 1 = 1$ or 2

$\Rightarrow l^2 = 2$ or $3 \Rightarrow l = \sqrt{2}$ or $\sqrt{3}$.

$\Rightarrow a = b = \sqrt{2}$ or $a = b = \sqrt{3}$, but $\sqrt{2}$ and $\sqrt{3} \notin I^+$ which is a contradiction. So, no solution is possible.

Case C : If $a \ne b \ne c$.

Then, let (a_0, b_0, c_0) be such that $a_0 < b_0 < c_0$ & (a_0, b_0, c_0) satisfies $a + b + c = abc$ ...(*)

Then clearly all possible solutions for the equation (*) is satisfied by permuting the triplet (a_0, b_0, c_0). Hence, we can take, without any loss of generality: $a < b < c$.

$\Rightarrow$ for three non-negative integers a_1, b_1, c_1 we can assume that

$a = 1 + a_1$, $b = 2 + b_1$ and $c = 3 + c_1$.

Now, $abc - (a + b + c)$

$= (1 + a_1)(2 + b_1)(3 + c_1) - [(1 + a_1) + (2 + b_1) + (3 + c_1)]$

$= (6 + 6a_1 + 3b_1 + 2c_1 + 3a_1b_1 + 2a_1c_1 + b_1c_1) - (6 + a_1 + b_1 + c_1)$

$= 5a_1 + 2b_1 + c_1 + 3a_1b_1 + 2a_1c_1 + b_1c_1$

$= a_1(5 + 3b_1 + 2c_1) + b_1(2 + c_1) + c_1$

$\therefore$ $a \cdot b \cdot c = a + b + c$

$\Rightarrow a_1 \cdot (5 + 3b_1 + 2c_1) + b_1(2 + c_1) + c_1 = 0$

which is possible only if $a_1 = b_1 = c_1 = 0$

where a_1, b_1, c_1 are three non-negative integers.

$\Rightarrow$ $a = 1 + a_1 = 1 + 0 = 1$

$b = 2 + b_1 = 2 + 0 = 2$

$c = 3 + c_1 = 3 + 0 = 3$.

Hence there is only one basic solution for the equation $a + b + c = abc$ given by $(a, b, c) \equiv (1, 2, 3)$ and the other solutions are permutations of (1, 2, 3). Hence the required solutions to the problem under reference are given by

$(a, b, c) \equiv (1, 2, 3); (1, 3, 2); (2, 1, 3); (2, 3, 1); (3, 1, 2)$ and $(3, 2, 1)$.

Problem 3 : Show that no positive integral solution is possible for the equation

$$a^3 + 2b^3 = ab^2.$$

Soln.: *Case I :* When $a = b$.

We take, $a = b = k$ (say)

Then, $a^3 + 2b^3 = 3k^3$ and $a^2b = k^2 \cdot k = k^3$

Clearly, $3k^3 > k^3$

$\Rightarrow a^3 + 2b^3 > a$, which is a contradiction.

Hence, no solution exists for case - I.

Case II : When $a \ne b$

$a^3 + 2b^3 = ab^2$ ...(i)

For, $a < b$

$\Rightarrow$ R.H.S. $< b \cdot b^2 = b^3$ and L.H.S. $= 2b^3 + a^3 > b^3$ which is a contradiction.

In terms of Case-I, no solution exists for $a = b$.

For, $a > b$

$\Rightarrow$ there exists $t_1 > 0$, t_1 is a non-negative integer, $a = b + t_1$

Then from (i), $2b^3 + (b + t_1)^3 = b^2(b + t_1)$

$\Rightarrow \; b^3 + (b + t_1)^3 = b^2 t_1$ ...(ii)

Again, $t_1 \le b$

$\Rightarrow$ R.H.S. $= b^2\, t_1 \le b^2 \cdot b = b^3$

and L.H.S. $= b^3 + (b + t_1^3) > b^3$

which is a contradiction. $\therefore \; t_1 > b$.

$\Rightarrow$ there exists a non-negative integer t_2 such that $t_1 = k + t_2$

$\therefore$ from (ii), $b^3 + (2b + t_2)^3 = b^3 + b^2 \cdot t_2$

$\Rightarrow \; (2b + t_2)^3 = b^2 \cdot t_2$ ...(iii)

Clearly as L.H.S. of (iii) is a perfect cube.

$\Rightarrow$ there exists a non-negative integer t_3 such that

$$b^2 t_2 = t_3^3 \quad \text{...(iv)}$$

$\Rightarrow b$ divides $t_3 \Rightarrow t_3 = b \cdot t_4$ ...(v)

Hence, from (iv) and (v)

$$b^2 \cdot t_2 = b^3\, t_4^3$$

$\Rightarrow \; t_2 = b \cdot t_4^3$ ...(vi)

$\therefore$ By (iii) & (vi),

$\Rightarrow (2b + b \cdot t_4^3)^3 = b^3 t_4^3 \;\Rightarrow\; t_4^3 + 2 = t_4$

which is not possible for any non-negative integer t_4 as $t_4^3 \ge t_4$ with equality $t_4^3 = t_4$ occuring at $t_4 = 1$ implying interaction, $t_4^3 + 2 > t_4$, which is a contradiction.

Hence no positive integral solution is possible for the problem under reference.

Prepared by : K. Sengupta, Kolkata.

Trigonometry

Q1. α and β are two acute angles. Prove that if $\sin^2(\alpha) + \sin^2(\beta) = \sin(\alpha + \beta)$, then $\alpha + \beta = \pi/2$.

Q2. It is known that for any x,

$$a\cos(x) + b\cos(3x) \le 1$$

Prove that $|b| \le 1$.

Q3. Prove that for every natural n

$$|\sin(1)| + |\sin(2)| + \dots + |\sin(3n-1)| + |\sin(3n)| > \frac{8n}{5}.$$

Q4. Find all the values of a for which the series $\cos(a)$; $\cos(2a)$; $\cos(4a)$; $\cos(8a)$; ... $\cos(2^n a)$, ... consists of negative numbers only.

Q5. Prove that

$$\cos\left(\frac{2\pi}{5}\right) + \cos\left(\frac{4\pi}{5}\right) = -\frac{1}{2}.$$

How Boolean Logic Works?

Have you ever thought about how a computer can do something like balance a check book or spell-check and grammar-check a document? These are things that, just a few decades ago, only humans could do. Now computers do them with apparent ease. How can a "chip" made up of silicon and wires do something that seems like it requires human thought?

If you want to understand the answer to this question down at the very core, the first thing you need to understand is something called **Boolean Logic**. Boolean logic, originally developed by George Boole in the mid 1800's, allows quite a few unexpected things to be mapped into *bits and bytes*. The great thing about Boolean logic is that, once you get the hang of things, Boolean logic (or at least the parts you need in order to understand the operations of computers) is outrageously simple. We will first discuss simple logic "gates", and then see how to combine them into something useful.

Simple Adders : We assure that you have knowledge of binary addition. Now you will learn how you can create a circuit capable of binary addition using the gates.

Let's start with a single bit adder. Let's say that you have a project where you need to add single bits together and get the answer. The way you would start designing a circuit for that is to first look at all of the logical combinations. You might do that by looking at the following four sums:

$$\begin{array}{r} 0 \\ +0 \\ \hline 0 \end{array} \quad \begin{array}{r} 0 \\ +1 \\ \hline 1 \end{array} \quad \begin{array}{r} 1 \\ +0 \\ \hline 1 \end{array} \quad \begin{array}{r} 1 \\ +1 \\ \hline 10 \end{array}$$

That looks fine until you get to 1 + 1. In that case you have carry bit to worry about. If you don't care about carrying (because this is, after all, a 1 bit addition problem), then you can see that you can solve this problem with an XOR gate. But if you do care then you might rewrite your equations to always include 2 bits of output, like this

$$\begin{array}{r} 0 \\ +0 \\ \hline 00 \end{array} \quad \begin{array}{r} 0 \\ +1 \\ \hline 01 \end{array} \quad \begin{array}{r} 1 \\ +0 \\ \hline 01 \end{array} \quad \begin{array}{r} 1 \\ +1 \\ \hline 10 \end{array}$$

From these equations you can form the logic table:

1-bit Adder with Carry-Out

A	*B*	*Q*	*CO*
0	0	0	0
0	1	1	0
1	0	1	0
1	1	0	1

Simply by looking at this table you can see that you can implement *Q* with an XOR gate and *CO* (carry-out) with an AND gate. Nothing could be simpler. What if you want to add two 8-bit bytes together? This becomes slightly harder. The easiest solution is to modularize the problem into reusable components and then replicate components. In this case we need to create only one component : a full binary adder. The difference between a full adder and the previous adder we looked at is that a full adder accepts an *A* and a *B* input plus a carry-in (*CI*) input. Once we have a full adder then we can string 8 of them together to create a byte-wide adder and cascade the carry bit from one adder to the next.

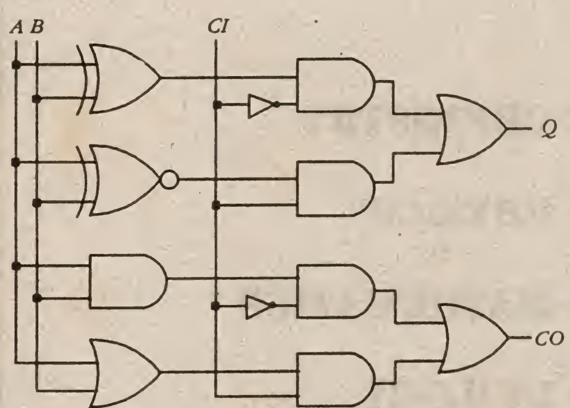
The logic table for a full adder is slightly more complicated than the tables we have used before because now we have three input bits. It looks like this:

1-bit 8 Strings Full Adder with Carry-In and Carry-Out

CI	*A*	*B*	*Q*	*CO*
0	0	0	0	0
0	0	1	1	0
0	1	0	1	0
0	1	1	0	1
1	0	0	1	0
1	0	1	0	1
1	1	0	0	1
1	1	1	1	1

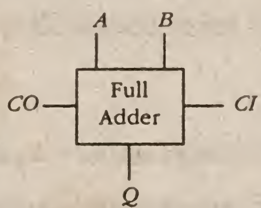
There are many different ways that you might

implements this table. I am going to present one method here that has the benefit of being easy to understand. If you look at the Q bit, you can see that the top four bits are behaving like an XOR gate with respect to A and B, while the bottom four bits are behaving like an XNOR gate with respect to A and B. Similarly, the top four bits of CO are behaving like an AND gate with respect to A and B and the bottom four behave like an OR gate. Taking those facts, the following circuit implements a full adder:

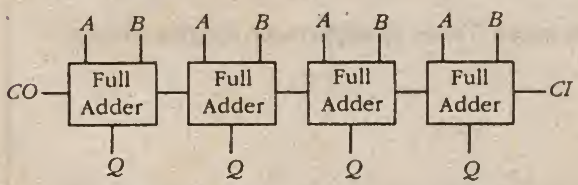


This definitely is not the most efficient way to implement a full adder, but it is extremely easy to understand and trace through the logic. If you are so inclined, see what you can do to implement this logic with fewer gates.

Now we have a piece of functionality called a "full adder". What a computer engineer then does is "black-box" it so that he/she can stop worrying about the details of the component. A black box for a full adder would look like this:



With that black box it is now easy to draw a 4-bit full adder:



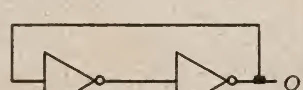
In this diagram the carry-out from each bit feeds directly into the carry-in of the next bit over. A zero is hard-wired into the initial carry-in bit. If you input two 4-bit numbers on the A and B lines, you will get the 4-bit sum out on the Q lines, plus one additional bit for the final carry-out. You can see that this chain can extend as far as you like, through 8, 16 or 32 bits if desired.

The 4-bit adder we just created is called a "ripple-carry" adder. It gets that name because the carry bits "ripple" from one adder to the next. This implementation has the advantage of simplicity but the disadvantage of speed problems. In a real circuit, gates take time to switch states (the time is on the order of nanoseconds, but in high speed computers nanoseconds matter). 32-bit or 64-bit ripple carry adders might take 100 to 200 nanoseconds to settle into their final sum because of carry ripple. For this reason, engineers have created more advanced adders called "carry-lookahead" adders. The number of gates required to implement carry-lookahead is large, but the settling time for the adder is much better.

Flip Flops

One of the more interesting things that you can do with Boolean gates is to create memory with them. If you arrange the gates correctly, they will remember an input value. This simple concept is the basis of RAM (Random Access Memory) in computers, and also makes it possible to create a wide variety of other useful circuits.

Memory relies on a concept called *feedback*. That is, the output of a gate is fed back into the input. The simplest possible feedback circuit using 2 inverters is shown in the figure:

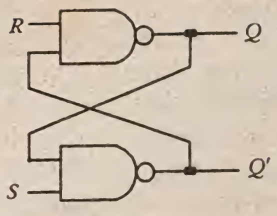


If you follow the feedback path, you can see that if Q happens to be 1, it will always be 1. If it happens to be 0, it will always be 0. Since it's nice to be able to control the circuits we create this one doesn't have much use, but it does let you see how feedback works. It turns out that in "real" circuits you can actually use this sort of simple inverter feedback approach.

A more useful feedback circuit using 2 NAND gates is shown in the next page.

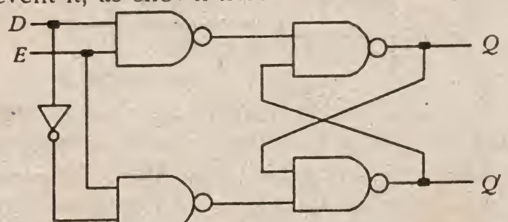
This circuit has 2 inputs (R and S) and 2 outputs (Q and Q'). Because of the feedback, it's logic

table is a little unusual compared to the ones we have seen previously:



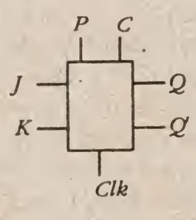
R	S	Q	Q'
0	0	Illegal	
0	1	1	0
1	0	0	1
1	1	Remembers	

What the logic table shows is that, if *R* and *S* are opposites of one another, *Q* follows *S* and *Q'* is the inverse of *Q*. If both *R* and *S* are switched to 1 simultaneously, then the circuit remembers what was previously presented on *R* and *S*. There is also the funny "illegal" state. In this state *R* and *S* both go to 1, which has no value in the memory sense. Because of the illegal state, you normally add a little conditioning logic on the input side to prevent it, as shown here:



In this circuit there are 2 inputs (*D* and *E*). You can think of *D* and "Data" and *E* as "Enable". If *E* is 1 then *Q* will follow *D*. If *E* changes to 0, however, *Q* will remember whatever was last seen on *D*. A circuit that behaves in this way is generally referred to as a "flip-flop".

A very common form of flip-flop is the "*J-K* flip-flop". It is unclear, historically, where the name "*J-K*" came from, but it is generally represented in a black box like this:

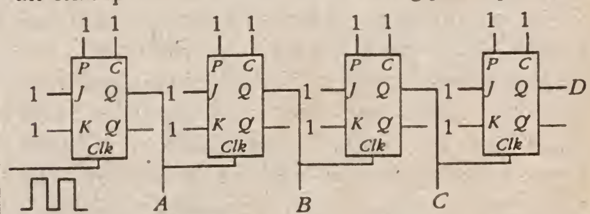


In this diagram *P* stands for "Preset", *C* stands for "Clear" and "*Clk*" stands for "Clock". The logic table looks like this:

P	C	Clk	J	K	Q	Q'
1	1	1-to-0	1	0	1	0
1	1	1-to-0	0	1	0	1
1	1	1-to-0	1	1	Toggles	
1	0	X	X	X	0	1
0	1	X	X	X	1	0

Here is what the table is saying. First, Preset and Clear override *J*, *K* and *Clk* completely. So if Preset goes to 0 then *Q* goes to 1, and if Clear goes to 0 then *Q* goes to 0 no matter what *J*, *K* and *Clk* are doing. However, if both Preset and Clear are 1 then *J*, *K* and *Clk* can operate. The "1-to-0" notation means that, when the clock changes from *a* 1 to *a* 0, the value of *J* and *K* are remembered if they are opposites. At the "low-going edge" of the clock (the transition from 1 to 0), *J* and *K* are stored. However, if both *J* and *K* happen to be 1 at the low-going edge, then *Q* simply "toggles". That is, *Q* changes from its current state to the opposite state.

You might be asking yourself right now, "What in the world is that good for?" It turns out that the concept of "edge triggering" is very useful. The fact that *J-K* flip-flop only "latches" the *J-K* inputs on a transition from 1 to 0 makes it much more useful as a memory device. *J-K* flip-flops are also extremely useful in counters (which we will use extensively when creating a digital clock). Here is an example of a 4-bit counter using *J-K* flip-flops:

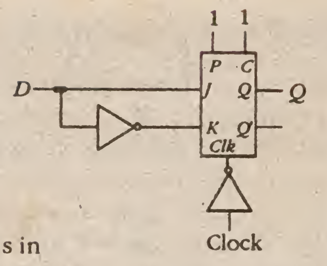


The outputs for this circuit are *A*, *B*, *C* and *D*, and they represent a 4-bit binary number. Into the clock input of the left-most flip-flop comes a signal changing from 1 to 0 and back to 1 repeatedly (an oscillating signal). The counter will count the low-going edges it sees in this signal. That is, every time the incoming signal changes from 1 to 0, the 4-bit number represented by *A*, *B*, *C* and *D* will increment by 1. So the count will go from 0 to 15 and then cycle back to 0. You can add as many bits as you like to this counter and count anything you like. For example, if you put a magnetic switch on a door, the counter will count the number of times the door is opened and closed. If you put an optical sensor on a road, the counter could count the number of cars that drive by. And so on.

Another use of a *J-K* flip-flop is to create an "edge-triggered latch", as shown here:

In this arrangement, the value on *D* is "latched"

when the clock edge goes from low to high. Latches are extremely important in the design of things like Central Processing Units (CPUs) and peripherals in computers.



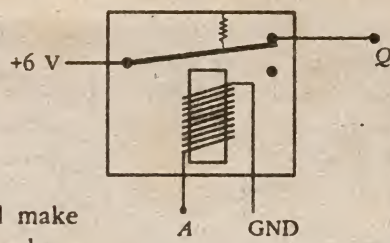
Implementing Gates

In the previous sections we saw that, by using very simple Boolean gates, we can implement adders, counter, latches and so on. That is a big achievement, because not so long ago human beings were the only ones who could do things like add two numbers together. With a little work it is not hard to design Boolean circuits that implement subtraction, multiplication, division... You can see that we are not that far away from a pocket calculator. From there it is not too far a jump to full-blown CPUs used in computers.

So how might we implement these gates in real life? Mr. Boole came up with them on paper, and on paper they look great. To use them, however, we need to implement them in physical reality so that the gates can perform their logic actively. Once we make that leap, then we have started down the road toward creating real computation devices.

The easiest way to understand the physical implementation of Boolean logic is to use relays. This is, in fact, how the very first computers were implemented. No one implements computers with relays anymore - today people use sub-microscopic transistors etched onto silicon chips. These transistors are incredibly small and fast, and they consume very little power compared to a relay. However, relays are incredibly easy to understand, and they can implement Boolean logic very simply. Because of that simplicity, you will be able to see that mapping from "gates on paper" to "active gates implemented in physical reality" is possible and straightforward. Performing the same mapping with transistors is just as easy.

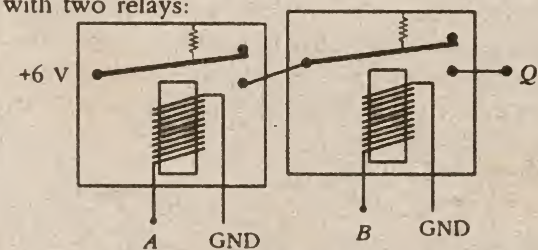
Let's start with an inverter. Implementing a NOT gate with a relay is trivial. What we are going to do is use voltages to represent bit states. We will define a binary 1 to be 6 volts and a binary 0 to be zero volts (ground). Then we will use a 6-volt battery to power our circuits. Our NOT gate would therefore look like this:



This figure will make sense to you if you have knowledge of relays.

You can see in this circuit that if you apply zero volts to *A* then you get 6 volts out on *Q*, and if you apply 6 volts to *A* you get zero volts out on *Q*. It is very easy to implement an inverter with a relay!

It is similarly easy to implement an AND gate with two relays:



Here you can see that if you were to apply 6 volts to *A* and *B*, *Q* will have 6 volts. Otherwise, *Q* will have zero volts. That is exactly the behaviour we want from an AND gate. An OR gate is even simpler - just hook two wires for *A* and *B* together to create an OR. You can get fancier than that if you like and use two relays in parallel.

You can see from this discussion that you can create the three basic gates - NOT, AND and OR - from relays. You can then hook those physical gates together using the logic diagrams shown above to create a physical 8-bit ripple-carry adder. If you use simple switches to apply *A* and *B* inputs to the adder and hook all 8 *Q* lines to light bulbs, you will be able to add any two numbers together and read the results on the lights (light-on = 1, light-off = 0).

In this article you have seen that Boolean logic in the form of simple gates is very straightforward. You have seen that from simple gates you can create more complicated functions, like addition. You have also seen that physically implementing the gates is possible and easy. From these three facts you have the heart of the digital revolution, and you understand, at the core, how computers work.

By Smriti Arora, Ludhiana, Punjab.

Mathematics in China

A brief outline of the history of Chinese mathematics

Primary sources are Mikami's *The Development of Mathematics in China and Japan* and Li Yan and Du Shiran's *Chinese Mathematics, a Concise History.*

1. Numerical notation, arithmetical computations, counting rods

- Traditional decimal notation — one symbol for each of 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 100, 1000, and 10000. Ex. 2034 would be written with symbols for 2,1000,3,10,4, meaning 2 times 1000 plus 3 times 10 plus 4. Goes back to origins of Chinese writing.
- Calculations performed using small bamboo counting rods. The positions of the rods gave a decimal place-value system, also written for long-term records. 0 digit was a space. Arranged left to right like Arabic numerals. Back to 400 B.C.E. or earlier.
- Addition: the counting rods for the two numbers placed down, one number above the other. The digits added (merged) left to right with carries where needed. Subtraction similar.
- Multiplication: multiplication table to 9 times 9 memorized. Long multiplication similar to ours with advantages due to physical rods. Long division analogous to current algorithms, but closer to "galley method."

2. *Zhoubi suanjing (The Arithmetical Classic of the Gnomon and the Circular Paths of Heaven)* (c. 100 B.C.E.-c. 100 C.E.)

- Describes one of the theories of the heavens. Early Han dynasty (206 B.C.E -220 C.E.) or earlier. Book burning of 213 B.C.E..
- States and uses the Pythagorean theorem for surveying, astronomy, etc. Proof of the Pythagorean theorem.
- Calculations including with common fractions.

3. *The Nine Chapters on the Mathematical Art (Jiuzhang Suanshu)* (c. 100 B.C.E.-50 C.E.) Collects mathematics to beginning of Han dynasty. 246 problems in 9 chapters. Longest surviving and most influential Chinese math book. Many commentaries

- Chapter 1, Field measurement: systemati discussion of algorithms using counting rods fo common fractions including alg. for GCD, LCM; area of plane figures, square, rectangle, triangle, trapezoid circle, circle segment, sphere segment, annulus — some accurate, some approximations.
- Chapter 2,3,6 on proportions, Cereals Proportional distribution, Fair taxes.
- Chapter 4, What width?: given area or volum find sides. Describes usual algorithms for squar and cube roots but takes advantage of computation with counting rods
- Chapter 5, Construction consultations: volume of cube, rectangular parallelepiped, prism frustums pyramid, triangular pyramid, tetrahedron, cylinde cone, and conic frustum, sphere — som approximations, some use $pi = 3$
- Chapter 7, Excess and deficients: false positio and double false position
- Chapter 8, Rectangular arrays: Gives eliminatio algorithm for solving systems of three or mor simultaneous linear equations. Involves use o negative numbers (red reds for pos numbers, blac for neg numbers). Rules for signed numbers.
- Chapter 9, Right triangles: applications o Pythagorean theorem and similar triangles, solve quadratic equations with modification of square roc algorithm, only equations of the form $x^2 + a x = b$, with a and b positive.

4. Sun Zi (c. 250? C.E.) Wrote his mathematica manual. Includes "Chinese remainder problem" c "problem of the Master Sun": find n so that upo division by 3 you get a remainder of 2, upon divisio by 5 you get a remainder of 3, and upon divisio by 7 you get a remainder of 2. His solution: Tak 140, 63, 30, add to get 233, subtract 210 to get 2

5. Liu Hui (c. 263 C.E.)

- Commentary on the *Nine Chapters*

Approximates *pi* by approximating circles polygon doubling the number of sides to get bette approximations. From 96 and 192 sided polygon

e approximates *pi* as 3.141014 and suggested 3.14 s a practical approx. States principle of exhaustion or circles Suggests Calvalieri's principle to find ccurate volume of cylinder

Haidao suanjing (Sea Island Mathematical Manual). Originally appendix to commentary on Chapter 9 of the *Nine Chapters.* Includes nine urveying problems involving indirect observations.

6. Zhang Qiujian (c. 450?) Wrote his mathematical nanual. Includes formula for summing an arithmetic equence. Also an undetermined system of two linear equations in three unknowns, the "hundred fowls problem"

7. Zu Chongzhi (429-500) Astronomer, nathematician, engineer.

- Collected together earlier astronomical writings. Made own astronomical observations. Recommended new calendar.
- Determined *pi* to 7 digits: 3.1415926. Recommended use 355/113 for close approx. and 22/7 for rough approx.
- With father carried out Liu Hui's suggestion for volume of sphere to get accurate formula for volume of a sphere.

8. Liu Zhuo (544-610) Astronomer Introduced quadratic interpolation (second order difference method).

9. Wang Xiaotong (fl. 625) Mathematician and astronomer. Wrote *Xugu suanjing (Continuation of Ancient Mathematics)* of 22 problems. Solved cubic equations by generalization of algorithm for cube root.

10. Translations of Indian mathematical works. By 600 C.E., 3 works, since lost. Levensita, Indian astronomer working at State Observatory, translated two more texts, one of which described angle measurement (360 degrees) and a table of sines for angles from 0 to 90 degrees in 24 steps ($3\frac{3}{4}$ degree) increments. Hindu decimal numerals also introduced, but not adopted.

11. Yi Xing (683-727) tangent table.

12. Jia Xian (c. 1050)
Written work lost. Streamlined extraction of square and cube roots, extended method to higher-degree roots using binomial coefficients.

13. Qin Jiushao (c. 1202 - c. 1261)
Shiushu jiuzhang (Mathemtaical Treatise in Nine Sections), 81 problems of applied math similar to the *Nine Chapters.* Solution of some higher-degree (up to 10th) equations. Systematic treatment of indeterminate simultaneous linear congruences (Chinese remainder theorem). Euclidean algorithm for GCD.

14. Li Chih (a.k.a. Li Yeh) (1192-1279)
Ceyuan haijing (Sea Mirror of Circle Measurements), 12 chapters, 170 problems on right triangles and circles inscribed within or circumscribed about them. *Yigu yanduan (New Steps in Computation)*, geometric problems solved by algebra.

15. Yang Hui (fl. c. 1261-1275)
Wrote sevral books. Explains Jiu Xian's methods for solving higher-degree root extractions. Magic squares of order up through 10.

16. Guo Shoujing (1231-1316).
Shou shi li (Works and Days Calendar). Higher-order differences (i.e., higher-order interpolation).

17. Zhu Shijie (fl. 1280-1303)
Suan xue qi meng (Introduction to Mathematical Studies), and *Siyuan yujian (Precious Mirror of the Four Elements).* Solves some higher degree polynomial equations in several unknowns. Sums some finite series including (1) the sum of n^2 and (2) the sum of $\frac{n(n+1)(n+2)}{6}$. Discusses binomial coefficients. Uses zero digit. ■

INTERNATIONAL MATH CONTEST PROBLEMS

1. Find all ordered triples of real numbers (x, y, z) such that $x + y + z > 2$, and $x^2 + y^2 = 4 - 2xy$, $x^2 + z^2 = 9 - 2xz$, and $y^2 + z^2 = 16 - 2yz$.

Soln.: $(x + y)^2 = 4 \Rightarrow x + y = \pm 2$;
Similarly, $x + z = \pm 3$ and $y + z = \pm 4$. Then

$$x + y + z = \frac{\pm 2 \pm 3 \pm 4}{2}.$$

Since $x + y + z > 2$, we can only use

$$\frac{2+3+4}{2} \text{ and } \frac{-2+3+4}{2}. \text{ Thus}$$

$x + y + z = 9/2$ or $x + y + z = 5/2$

Successively subtracting	Successively subtracting
$x + y = 2$	$x + y = -2$
$x + z = 3$	$x + z = 3$
$y + z = 4$ from	$y + z = 4$ from
$x + y + z = 9/2$ gives	$x + y + z = 5/2$ gives
$\left(\frac{1}{2}, \frac{3}{2}, \frac{5}{2}\right)$ and	$\left(-\frac{3}{2}, -\frac{1}{2}, \frac{9}{2}\right)$.

2. The graphs of $x^2 + y^2 + 6x - 24y + 72 = 0$ and $x^2 - y^2 + 6x + 16y - 46 = 0$ intersect at four points. Compute the sum of the distances from these four points to the point $(-3, 2)$.

Soln.: The intersections must lie on [add the equations] $2x^2 + 12x - 8y + 26 = 0$, which is the parabola $(x + 3)^2 = 4(y-1)$. This parabola has vertex $(-3, 1)$, focus $(-3, 2)$, and directrix $y = 0$. Thus the sum of the distances to $(-3, 2)$ equals the sum of the distances to the X-axis, which is the sum of the ordinates of the intersection points. Those points must also satisfy [subtract the original equations] $y^2 - 20y + 59 = 0$, so the sum of two of the ordinates is 20, and the sum of all four (this is a hyperbola crossing a circle, producing pairs of points with the same ordinates) is 40.

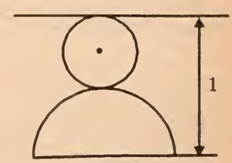
3. Let x and y be real numbers satisfying the equation $x^2 - 4x + y^2 + 3 = 0$. If the maximum and minimum values of $x^2 + y^2$ are M and m respectively, compute the numerical value of $M - m$.

Soln.: Graphically, the equation $(x - 2)^2 + y^2 = 1$ represents a circle of radius 1 and center $(2, 0)$. The quantity $x^2 + y^2$ represents the square of the distance from any point to the origin. For points on the circle, the maximum distance to the origin is from the point $(3, 0)$ and the minimum distance is from $(1, 0)$. Thus $M - m = 9 - 1 = 8$.

4. The function $f(x)$ is defined for all real x. If $f(a + b) = f(ab)$ for all a, b, and $f(-\frac{1}{2}) = -\frac{1}{2}$, compute $f(1984)$.

Soln.: Let $f(0) = k$. Letting $a = 0$ gives $f(b) = f(0) = k$; letting $b = 0$ gives $f(a) = f(0) = k$. Thus $f(a) = f(b)$ for all real a, b. Then $f(x)$ must be all the constant function, so $f(1984)$ also equals $-\frac{1}{2}$.

5. A circle and a semicircle are tangent to one another and to two parallel lines that are 1 unit apart, as shown (the line of centres of the two figures is perpendicular to the parallel lines). Compute the maximum possible product for the areas of the circle and semicircle.



Soln.: Let the radii of the semicircle be a and b respectively. Then the product called for is

$$P = \frac{\pi a^2}{2} \cdot \pi b^2 = \frac{\pi a^2}{2} \cdot \pi\left(\frac{1-a}{2}\right)^2 = \frac{\pi^2 a^2}{8}(1-a)^2$$

Then $y = \sqrt{P} = \dfrac{\pi a}{2\sqrt{2}}(1-a) = \dfrac{\pi}{2\sqrt{2}}(-a^2 + a)$.

Getting maximum $P \Rightarrow$ getting maximum y. This parabola has its maximum at $a = ½$, so

$P = \dfrac{\pi^2(½)^2}{8}\left(\dfrac{1}{2}\right)^2 = \dfrac{\pi^2}{128}$. Note how each figure occupies ½ the distance between the parallel lines.

The same occurs for an equilateral triangle atop a square, or a circle atop a square.

6. Prove that, for all non-negative integers n, $\frac{1}{2}n \le \bar{n}$ £ $2n$.

Soln.: Let I be the set of i such that $a_i \neq 0$. Then

$n+\bar{n} = \sum_{i\in I} a_i 3^i = 3\sum_{i\in I} \bar{a}_i 3^i \le 3\sum_I 3^i \le 3n$, with equality holding if and only if n has all 0's and 1's as digits. Subtracting n from both sides of the inequality $n+\bar{n} \le 3n$ yields $\bar{n} \le 2n$.

Similarly, $n+\bar{n} = \frac{3}{2}\sum_I 2\cdot 3^i \ge \frac{3}{2}n$, so $\bar{n} \ge \frac{1}{2}n$.

7. For how many positive integral values of $x \le 100$ is $3^x - x^2$ divisible by 5?

Soln.: Set up a chart of units digits.

x	3^x	x^2	
1	3	1	
2	9	4	←
3	7	9	
4	1	6	←
5	3	5	
6	9	6	
7	7	9	
8	1	4	
9	3	1	
10	9	0	
11	7	1	
12	1	4	
13	3	9	
14	9	6	
15	7	5	
16	1	6	←
17	3	9	
18	9	4	←
19	7	1	
20	1	0	

(cycles now repeat)

The arrows indicate where the units digits differ by 0 or 5. Thus there are 4 values of x that work in each set of 20. Thus the answers is $4 \times 5 = 20$.

8. (a) If $0 \le r < 1$ and $0 \le s < 1$, then $[2r] + [2s] \ge [r] + [s] + [r + s]$.

(b) If x and y are real numbers, then $[2x] + [2y] \ge [x] + [y] + [x + y]$.

Soln.: (a) Note that $[r] = [s] = 0$.

Method I : If both r and s are < ½, then both sides are 0 and the relationship holds.

It either is ≥ ½, then the left side is ≥ 1, whereas the right side is at most 1, and the relationship holds.

Method II : Suppose $r \ge s$. Then $2r \ge r + s$ and $[2r] \le [r + s]$. Then the relationship clearly holds.

(b) Let $x = a + r$ and $y = b + s$, where a and b are integers and $0 \le r < 1$ and $0 \le s < 1$. Then $[2x] + [2y] = [2a + 2r] + [2b + 2s] =$ (by 1a) $2a + 2b + [2r] + [2s] \ge$ (by 2a) $2a + 2b + [r] + [s] + [r + s] =$ (by 1a) $[a + r] + [b + s] + [a + b + r + s] = [x] + [y] + [x + y]$.

9. If $\log_{\sin x}\cos x = \frac{1}{2}$, and $0 < x < \pi/2$, compute $\sin x$.

Soln.:

$$\sqrt{\sin x} = \cos x \Rightarrow \sin^2 x + \sin x - 1 = 0$$
$$\Rightarrow \sin x = \frac{\sqrt{5}-1}{2}.$$

10. If x and y are real, and $x^2 + y^2 = 1$, compute the maximum value of $(x + y)^2$.

Soln.: Let $x = \sin\theta$ and $y = \cos\theta$ (since clearly neither x nor y can exceed 1).

Then $(x + y)^2 = 2\sin\theta\cos\theta + 1 = \sin 2\theta + 1$. The maximum value of this is 2. Note that the greatest height of the graph of $z = \sin\theta + \cos\theta$ is $\sqrt{2}$.

■

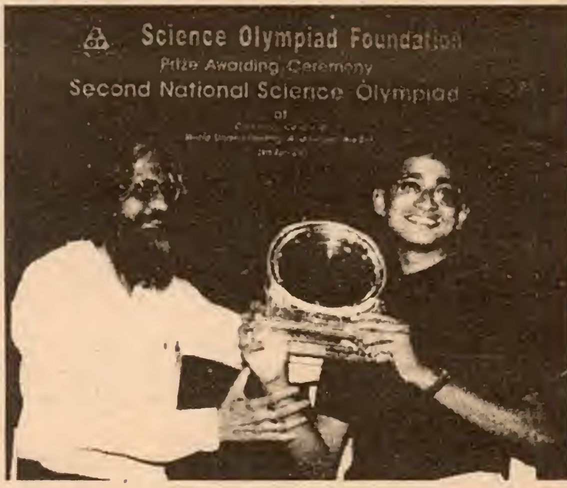
24.08.05

National Science Olympiad Awards

The Prize Awarding Ceremony of the 2nd National Science Olympiad was held at Chinmaya Centre of World Understanding Auditorium, New Delhi on 28th April, 2001. Five Hundred science students from classes 8 to 12 from all over the country were awarded with various medals, Cash Prizes, Discount Coupons and Certificates for their achievements in the 2nd National Olympiad conducted in January/July 2000. The National Science Olympiads are conducted annually by the Science Olympiad Foundation(SOF), a registered non-profit Organization. It is the brainchild of leading academicians, scientists and media persons, was set in 1998 to identify and nurture science talents among school boys & girls.

Speaking at the occasion, the Chief Guest Shri Y.S. Rajan, Scientific Secretary to the Principal Scientific Advisor to the Government of India, said that there were several areas in which India's economic, business or technological strengths are not world class. The country has made tremendous achievements in the past five decades but it was not enough and we have to do a lot more, he said. While the older generation had a tremendous role to play in the realization of the Vision, the real task belonged to the young people. If young people became bold and determined they could make India the most desired place to live in the world, he said.

Shri Bachi Singh Rawat, Minister of State for Science and Technology in his message commended the Science Olympiad Foundation for its efforts to improve the quality of science education and promoting scientific temper among children. He hosted a dinner on 29th April to honour the meritorious students.



Shri. Y.S. Rajan, Scientific Secretary to the Principal Scientific Advisor to the Govenment of India, presenting the award.

Mr. V.S. Ramamurthy, Secretary, Department of Science & Technology and Shri Y.S. Rajan were also present at the occasion.

The Foundation will conduct the 4th NSO in January, 2002. The detailed information will be sent to the schools all over the country in July/August.

The Foundation is also introducing the National Cyber Olympiad which will be a National Level exam to test students aptitude for taking up careers in computer. The first Cyber Olympiad is scheduled for 3rd October, 2001.

Besides the annual Olympiads, the Foundation conducts seminars, workshops and teacher training programmes in the various State capitals.

International

MATH OLYMPIAD PROBLEMS

1. Let $p(x) = x^5 + x$ and $q(x) = x^5 + x^2$. Find all pairs (ω, z) of complex numbers with $\omega \neq z$ for which $p(\omega) = p(z)$ and $q(\omega) = q(z)$.

Soln.: Let $P(x, y) = \dfrac{p(x)-p(y)}{x-y}$

$= x^4 + x^3y + x^2y^2 + xy^3 + y^4 + 1$

and $Q(x, y) = \dfrac{q(x)-q(y)}{x-y}$

$= x^4 + x^3y + x^2y^2 + xy^3 + y^4 + x + y$

We need those pairs which satisfy

$P(\omega, z) = Q(\omega, z) = 0.$

From $P - Q = 0$, we have $\omega + z = 1$. Let $c = \omega z$. A short calculation shows that $c^2 - 3c + 2 = 0$, which has solutions $c = 1$ and $c = 2$.

From the system $\omega + z = 1$, $\omega z = c$, we obtain the following pairs:

$$\left(\frac{1 \pm \sqrt{3}i}{2}, \frac{1 \mp \sqrt{3}i}{2}\right) \text{ and } \left(\frac{1 \pm \sqrt{7}i}{2}, \frac{1 \mp \sqrt{7}i}{2}\right).$$

2. Let $f: R \to (0, \infty)$ be an increasing differentiable function for which $\lim\limits_{x\to\infty} f(x) = \infty$ and f' is bounded.

Let $F(x) = \int\limits_0^x f(t)dt$. Define the sequence $\{a_n\}$ inductively by $a_0 = 1$, $a_{n+1} = a_n + \dfrac{1}{f(a_n)}$ ... (i)

and the sequence $\{b_n\}$ simply by $b_n = F^{-1}(n)$.

Prove that $\lim\limits_{n\to\infty}(a_n - b_n) = 0$. [Ed. The original said ∞, but the solution proves 0].

Soln.: From the conditions, it is clear that F is increasing and that $\lim\limits_{n\to\infty} b_n = \infty$.

By Lagrange's theorem and the recursion in (i), for all non-negative integers k, there exists a real number $\xi \in (a_k, a_{k+1})$ such that

$$F(a_{k+1}) - F(a_k) = f(\xi)(a_{k+1} - a_k) = \frac{f(\xi)}{f(a_k)} \quad \text{... (ii)}$$

By the monotonicity, $f(a_k) \le f(\xi) \le f(a_{k+1})$.

Thus, $1 \le F(a_{k+1}) - F(a_k) \le \dfrac{f(a_{k+1})}{f(a_k)}$

$$= 1 + \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} \quad \text{... (iii)}$$

Summing (iii) over k and subtracting $F(b_n) = n$, we have

$F(b_n) < n + F(a_0) \le F(a_n)$

$$\le F(b_n) + F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)} \quad \text{... (iv)}$$

From the first two inequalities, we already have $a_n > b_n$ and $\lim\limits_{n\to\infty} a_n = \infty$.

Let $e > 0$. Choose an integer K_e such that $f(a_{K_e}) > 2/e$. If n is sufficiently large, we have

$$F(a_0) + \sum_{k=0}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}$$

$$= \left(F(a_0) + \sum_{k=0}^{K_e-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}\right) + \sum_{k=K_e}^{n-1} \frac{f(a_{k+1}) - f(a_k)}{f(a_k)}$$

$$< O_e(1) + \frac{1}{f(a_{K_e})} \sum_{k=K_e}^{n-1} (f(a_{k+1}) - f(a_k))$$

$$< O_e(1) + \frac{e}{2}(f(a_n) - f(a_{K_e})) < ef(a_n). \quad \text{... (v)}$$

Inequalities (iv) and (v) together imply that for any positive e, if n is sufficiently large, we have

$F(a_n) - F(b_n) < ef(a_n)$.

Again, by Lagrange's theorem, there is a real number $\xi \in (b_n, a_n)$ such that

$$F(a_n) - F(b_n) = f(\xi)(a_n - b_n) > f(b_n)(a_n - b_n) \quad \text{... (vi)}$$

Thus, $f(b_n)(a_n - b_n) < ef(a_n)$ ... (vii)

Let B be an upperbound for f'.

Apply $f(a_n) < f(b_n) + B(a_n - b_n)$ in (vii).

$f(b_n)(a_n - b_n) < e(f(b_n) + B(a_n - b_n))$

$(f(b_n) - eB)(a_n - b_n) < ef(b_n)$.

Because of $\lim\limits_{n\to\infty} f(b_n) = \infty$, we see that the first factor is positive and we have,

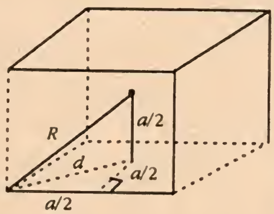
$$a_n - b_n < e\frac{f(b_n)}{f(b_n) - eB} < 2e.$$

for sufficiently large n.

Thus, for arbitrary positive e, we have proved that $0 < a_n - b_n < 2e$ if n is sufficiently large, and the proof is complete.

3. There are a series of 9 coloured plastic figures of decreasing sizes, alternating cube, sphere, cube, sphere etc. Each figure may be opened and the succeeding one may be placed inside, fitting exactly. The largest and the smallest figures are both cubes. Determine the ratio between their side lengths.

Soln.: If a sphere with radius R is circumscribed to a cube with edge a then the sphere and the cube have the same centre, and the vertices of the cube are points of the sphere.



From Pythagoras' theorem,

$$R^2 = d^2 + \frac{a^2}{4} = \frac{3a^2}{4}.$$

Thus, $R = \frac{\sqrt{3}}{2}a$ .... (i)

If a sphere with radius R is inscribed in a cube with edge b, then the sphere and cube have the same centre, and the centres of the sides of the cube are points of the sphere. Then

$R = b/2$ .... (ii)

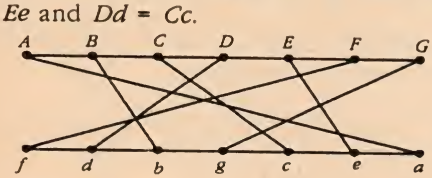
From (i) and (ii), it follows that the ratio between the side-lengths of the "outside cube" and the "inside cube" is

$$b/a = \sqrt{3}.$$

Since there are 5 cubes, the ratios between the side-lengths of the largest and the smallest figures is $(\sqrt{3})^4 = 9$.

4. In a ballroom 7 gentlemen A, B, C, D, E, F and G are sitting opposite 7 ladies a, b, c, d, e, f and g in arbitrary order. When the gentlemen walk across the dance floor to ask each of their ladies for a dance, one observes that at least two gentlemen walk distances of equal length. Is that always the case?

The figure shows an example. In this example $Bb = Ee$ and $Dd = Cc$.



Soln.: Note first that the observation is only correct if we assume that the 7 gentleman are "evenly" spaced.

We show that in general, if there are n gentleman and n ladies, then the same conclusion holds when $n \equiv 2, 3 \pmod 4$.

Suppose the n ladies are situated at $(k, 0)$ and the n gentlemen, at $(k, 1)$; $k = 1, 2, ..., n$. Suppose that the gentlemen at $(k, 1)$ walks a distance of d_k to the lady at $(a_k, 0)$, $a_k \in \{1, 2, ..., n\}$. Then $(a_1, a_2, ..., a_n)$ is a permutation of $(1, 2, ..., n)$ and thus, $\sum_{k=1}^{n}(a_k - k) = 0$.

Since $d_k = (1 + (a_k - k)^2)^{1/2}$, we have

$d_k^2 = 1 + (a_k - k)^2$.

We show that the values of the d_k's cannot all be distinct.

Note that $a_k - k \in \{0, \pm 1, \pm 2, ..., \pm(n-1)\}$. Suppose to the contrary that $(a_k - k)^2 \neq (a_j - j)^2$ for all $j \neq k$, j, $k = 1, 2, ..., n$. Then we have

$\{|a_k - k| : k = 1, 2, ..., n\} = \{0, 1, 2, ..., n-1\}$

and thus

$$\sum_{k=1}^{n}|a_k - k| = \sum_{k=0}^{n-1} t = \frac{n(n-1)}{2} \quad ... \text{(i)}$$

On the other hand, since $|t| - t$ must be even for all integers t, we have

$$\sum_{k=1}^{n}|a_k - k| = 2d + \sum_{k=1}^{n}(a_k - k) = 2d$$

for some ineger d ... (ii)

Comparing (i) and (ii), we get $n(n-1) = 4d$ which implies that $n \equiv 0, 1 \pmod 4$.

Therefore, if $n \equiv 2, 3 \pmod 4$ then we must have $d_j = d_k$ for some $j \neq k$.

5. Let $P(x) = x^4 + x^3 + x^2 + x + 1$. Show that there exist polynomials $Q(y)$ and $R(y)$ of positive degrees, with integer coefficients, such that $Q(y) \cdot R(y) = P(5y^2)$ for all y.

Soln.: Since $P(5y^2) = 5^4y^8 + 5^3y^6 + 5^2y^4 + 5y^2 + 1$, we try factors of the form

$(25y^4 + ay^3 + by^2 + cy + 1)$

$(25y^4 - ay^3 + by^2 - cy + 1)$.

On expanding out, these are factors :

$a = 25$, $b = 15$ and $c = 5$.

6. Let a, b, c be distinct real numbers and $P(x)$ a polynomial with real coefficients, if

- the remainder on division of $P(x)$ by $x - a$ equals a,
- the remainder on division of $P(x)$ by $x - b$ equals b,
- and the remainder on division of $P(x)$ by $x - c$ equals c;

determine the remainder on division of $P(x)$ by $(x - a)(x - b)(x - c)$.

Soln.: As is well known, the remainder on division of $P(x)$ by $x - a$ is $P(a)$. So, the hypothesis imply:

$$P(a) = a,\ P(b) = b,\ P(c) = c.$$

Let $R(x)$ be the remainder on division of $P(x)$ by $(x - a)(x - b)(x - c)$, so that the degree of $R(x)$ is ≤ 2 and $P(x) = (x - a)(x - b)(x - c)\,Q(x) + R(x)$ for a polynomial $Q(x)$.

We remark that $R(a) = P(a) = a$ and similarly $R(b) = b$ and $R(c) = c$. From this observation, we may conclude through one of the three following ways:

(i) The polynomial $R(x) - x$ has degree ≤ 2 and three distinct zeros a, b, c. Hence $R(x) - x$ is the zero polynomial and $R(x) = x$.

(ii) $R(x)$ has the form $ux^2 + vx + w$ where (u, v, w) is the solution of the system

$$\begin{cases} ua^2 + va + w = a \\ ub^2 + vb + w = b \\ uc^2 + vc + w = c \end{cases} \qquad \ldots \text{(S)}$$

The determinant of (S) is a Vandermonde determinant and is not zero (since a, b, c are distinct), so (S) has a unique solution, which clearly is $u = 0$, $v = 1$, $w = 0$. Thus $R(x) = x$ again.

(iii) $R(x)$ is the Lagrange's interpolation polynomial.

$$R(x) = a\frac{(x-b)(x-c)}{(a-b)(a-c)} + b\frac{(x-a)(x-c)}{(b-a)(b-c)} + c\frac{(x-a)(x-b)}{(c-a)(c-b)}$$

Multiplying out and grouping similar terms, a lengthy but easy calculation provides $R(x) = x$ again. ■

Attention XI, XII & XII F

What a tutor teaches in t

INTERACTIVE PH

teaches in three w

Dr. Indu kumar
M.Sc.,Ph.
Department of Physic
S. N. S. College, Hajip

Dear sirs,

"Interactive Physics (Vol-1) is a dawn after long years of darkness" – that's what I've concluded after studying the boo In fact, I would like to share a secret with you. This boo has taught me how to teach Physics!

What makes this book different from (the unwanted) crowd of books is its easy approach, super-simple language, in spite of a 'desciplined' approach to basic co On the whole, the book lets the students know their pitfalls where they were applying wrong concepts and were gettin carried away by confusions.

I had seen the book (1st part) for the first time in one of m students' hand, and the rest is a revolution – as it has occupied an enevitable place in our institution.

Sincerely Your

Indu Kumari
2/10/09

10 Mathematical Challenges

1. The length of the sides of a right triangle are three consecutive terms of an arithmetic progression. Prove that the lengths are in the ratio 3 : 4 : 5.

2. Alice, Betty and Carol took the same series of examinations. For each examination there was one mark of x, one mark of y and one mark of z, where x, y, z are distinct positive integers. The total of the marks obtained by each of the girls was : Alice-20, Betty - 10, Carol - 9. If Betty placed first in the algebra examination, who placed second in the geometry examination?

3. A father, mother and son decide to hold a family tournament, playing a particular two-person board game which must end with one of the players winning (i.e. no tie is possible). After each game the winner then plays the person who did not play in the game just completed. The first player to win two games (not necessarily consecutive) wins the tournament. It is agreed that, because he is the oldest, the father may choose to play in the first game or to sit out the first game. Advice the father what to do: play or not to play in the first game.

4. During an election campaign n different kinds of promises are made by the various political parties, $n > 0$. No two parties have exactly the same set of promises. While several parties may make the same promise, every pair of parties have at least one promise in common. Prove that there can be as many as 2^{n-1} parties, but no more.

5. Let $a_1, a_2, \ldots, a_n$ be n positive integers. Show that for some i and k $(1 \le i \le i + k \le n)$,

$a_i + \ldots + a_{i+k}$ is divisible by n.

6. A boy lives in each of n houses on a straight line. At what point should the n boys meet so that the sum of the distances that they walk from their houses is as small as possible?

7. If $2\log(x - 2y) = \log x + \log y$, find x/y.

8. Two flag poles of heights h and k are situated $2a$ units apart on a level surface. Find the set of all points on the surface which are so situated that the angles of elevation, at each point of the tops of the poles are equal.

9. Let $S_n = 1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \ldots + \frac{1}{\sqrt{n}}$.

Show that $2\sqrt{n+1} - 2 < S_n < 2\sqrt{n} - 1$.

10. A triangle has sides of lengths a, b, c and respective altitudes of lengths h_a, h_b, h_c. If $a \ge b \ge c$ show that $a + h_a \ge b + h_b \ge c + h_c$.

SOLUTIONS

1. The three lengths are of the form $a - d$, a, $a + d$ with $(a - d)^2 + a^2 = (a + d)^2$. This reduces to $a(a - 4d) = 0$, or $a = 4d$. Thus the sides are $3d$, $4d$, $5d$.

2. We may assume that $x > y > z \ge 1$. Letting N denote the number of examinations, we have $N > 1$ (implied by the word series in the statement of the problem) and

$(x + y + z)N = 20 + 10 + 9 = 39.$

Since $x + y + z \ge 3 + 2 + 1 = 6$, we know that $N \le 6$, and since N divides 39, we deduce $N = 3$, so that $x + y + z = 13$. The solutions (x, y, z) of the equation $x + y + z = 13$ with $x, y, z \ge 1$, and x, y, z distinct integers, are:

$(x, y, z) = (10, 2, 1), (9, 3, 1), (8, 4, 1), (8, 3, 2), (7, 5, 1), (7, 4, 2), (6, 5, 2), (6, 4, 3)$.

Except for (8, 4, 1), all these possibilities are eliminated by the fact that Alice's marks sum to 20. Now we know that Betty's algebra mark is 8 (the largest of 8, 4, 1) and the problem is to fill in the table so that each row is a permutation of 8, 4, 1 and the column sums are 20, 10, 9.

	Alice	Betty	Carol
algebra		8	
geometry			
other subject			
	20	10	9

The only solution is easily seen to be

	Alice	Betty	Carol
algebra	4	8	1
geometry	8	1	4
other subject	8	1	4

Thus, Carol placed second in geometry.

3. Advise him to play in the first game. We show that father wins the tournament with greater probability when he plays in the first game than he does by sitting out of the first game.

Let F, M and S denote father, mother and son respectively and let $X > Y$ denote that X wins the game playing against Y.

If F and M plays the first game, then F wins the tournament in the following three sequences.

(i) $F > M,\ F > S$

(ii) $F > M,\ S > F,\ M > S,\ F > M$

(iii) $M > F,\ S > M,\ F > S,\ F > M.$

If F and S play the first game, the winning sequences for F are the same as those above with M and S interchanged.

If M and S play the first game, then F wins the tournament in the following two sequences.

(iv) $S > M,\ F > S,\ F > M;$

(v) $M > S,\ F > M,\ F > S.$

Let $\overline{AB}$ denote the probability that $A > B$. Note that $\overline{AB} + \overline{BA} = 1$.

If F and M play the first game, the probability P_{FM} that F wins the tournament is

$$P_{FM} = \overline{FM} \cdot \overline{FS} + \overline{FM} \cdot \overline{SF} \cdot \overline{MS} \cdot \overline{FM} + \overline{MF} \cdot \overline{SM} \cdot \overline{FS} \cdot \overline{FM}$$

If F and S play the first game, the probability P_{FS} and F wins the tournament is

$$P_{FS} = \overline{FS} \cdot \overline{FM} + \overline{FS} \cdot \overline{MF} \cdot \overline{SM} \cdot \overline{FS} + \overline{SF} \cdot \overline{MS} \cdot \overline{FM} \cdot \overline{FS}$$

If M and S play the first game, the probability P_{MS} that F wins the tournament is

$$P_{MS} = \overline{SM} \cdot \overline{FS} \cdot \overline{FM} + \overline{MS} \cdot \overline{FM} \cdot \overline{FS}$$

$$= (\overline{SM} + \overline{MS}) \cdot (\overline{FS} \cdot \overline{FM}) = \overline{FS} \cdot \overline{FM}\ .$$

It is clear that $P_{FM} > P_{MS}$ and $P_{FS} > P_{MS}$.

4. Suppose there are N parties, and their promises are the sets $S_1, S_2, \ldots, S_N$. We know that no two of these sets are equal (because no two parties make exactly the same promises). Furthermore $S_i \cap S_j \neq 0$ for $i \neq j$ (because every pair of parties make a promise in common). Thus, no S_i in the complement of an S_j, and hence there are at most $\frac{1}{2}2^n = 2^{n-1}$ subsets (of the set of all promises) in the list $S_1, S_2, \ldots, S_N$, *i.e.* $N \leq 2^{n-1}$.

Let $p_1, p_2, \ldots, p_n$ be the n promises, and let A_i, $i = 1, 2, \ldots, 2^{n-1}$, be the subsets of $\{p_2, p_3, \ldots, p_n\}$. The 2^{n-1} sets $\{p_1\} \cup A_i$ show that there can be as many as 2^{n-1} parties.

5. Consider the n sums.

$s_1 = a_1,\ s_2 = a_1 + a_2,\ s_3 = a_1 + a_2 + a_3, \ldots, s_n = a_1 + a_2 + \ldots + a_n$.

Let s_i leave a remainder of r_i on division by n, *i.e.* $s_i = q_i n + r_i,\ 0 \leq r_i \leq n-1,\ i = 1, 2, \ldots, n$.

If for some i, $r_i = 0$, then s_i has the desired properties. If $r_j \neq 0$ for $j = 1, 2, \ldots, n$ then we have n integers $r_1, \ldots, r_n$ all in the set $\{1, 2, \ldots, n-1\}$, which contains only $n-1$ integers. By the Pigeonhole principle two of the r_i's must be equal, say $r_l = r_m$ with $m > l$; in this case

$$a_{l+1} + \ldots + a_m = s_m - s_l = (q_m - q_l)n.$$

6. Observe that the meeting place should be between the first and last houses. For, given any point beyond either end house, the sum of the distances would exceed the sum of the distances to the closer end house. Now note that for all points chosen between the first and last houses, the sum of the two distances from the meeting place is constant, so we can remove these houses from further consideration and minimize the sum of the distances to the remaining houses. Repeating the above argument, we deduce that the meeting place should be between the second and second-last house, between the third and third-last house, and so on. Thus, if n is even, the boys should meet anywhere between the two middle houses; if n is odd, at the middle house.

An analytic version goes as follows. On a coordinate line, let the houses have coordinates $a_1, a_2, \ldots, a_n$ with $0 \leq a_1 < a_2 < \ldots < a_n$,

and let the point P where they meet have coordinate x. Then the total distance walked is

$$D(x) = |x - a_1| + |x - a_2| + \ldots + |x - a_n|.$$

If n is even, say $n = 2m$, then

$$D = \sum_{i=1}^{m}(|x - a_i| + |a_{2m+1-i} - x|)$$

(and by the triangle inequality)

$$D \geq \sum_{i=1}^{m}|x - a_i + a_{2m+1-i} - x| = \sum_{i=1}^{m}(a_{2m+1-i} - a_i)$$

$$= a_{m+1} + \ldots + a_{2m} - (a_1 + \ldots + a_m),$$

with equality if $a_m \leq x \leq a_{m+1}$.

If n is odd, say $m = 2m + 1$, then

$$D = \sum_{i=1}^{m}(|x - a_i| + |a_{2m+2-i} - x|) + |x - a_{m+1}|$$

$$\geq \sum_{i=1}^{m}|x - a_i + a_{2m+2-i} - x| + |x - a_{m+1}|$$

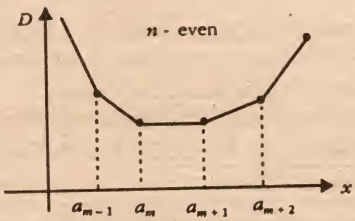
$$= \sum_{i=1}^{m}|a_{2m+2-i} - a_i| + |x - a_{m+1}|$$

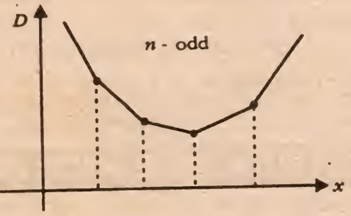
$$= \sum_{i=1}^{m}(a_{2m+2-i} - a_i) + |x - a_{m+1}|$$

$$\geq a_{m+2} + \ldots + a_{2m+1} - (a_1 + \ldots + a_n)$$

with equality if $x = a_{m+1}$.

If one plots the graph of $D(x)$ against x, $0 \leq x < a_n$, one gets one of the two unbounded polygonal figures in the figure.





7. We have $x > 0$, $y > 0$, $x - 2y > 0$, and

$$2\log(x - 2y) = \log x + \log y,$$

so, $\log(x - 2y)^2 = \log xy$,

$$(x - 2y)^2 = xy, \quad \left(\frac{x}{y} - 2\right)^2 = \frac{x}{y}$$

$$\left(\frac{x}{y}\right)^2 - 5\left(\frac{x}{y}\right) + 4 = 0 \; ; \; \left(\frac{x}{y} - 4\right)\left(\frac{x}{y} - 1\right) = 0$$

$$x = 4y \text{ or } x = y.$$

The latter situation is impossible, for then $x - 2y < 0$. Hence $x/y = 4$.

8. Set up a coordinate system so that the pole of height h is based at $(-a, 0)$ and the pole of height k at $(a, 0)$. Then a point (x, y) subtends equal elevations if

$$\frac{h}{\sqrt{(x+a)^2 + y^2}} = \frac{h}{\sqrt{(x-a)^2 + y^2}}$$

or, $h^2(x^2 - 2ax + a^2 + y^2) = k^2(x^2 + 2ax + a^2 + y^2)$,

or, $x^2(k^2 - h^2) + 2ax(k^2 + h^2) + y^2(k^2 - h^2) + a^2(k^2 - h^2) = 0$

If $k = h$, the equation becomes $x = 0$, so the locus is the y-axis.

If $k \neq h$, say $k > h$, then

$$x^2 + 2ax\left(\frac{k^2 + h^2}{k^2 - h^2}\right) + y^2 + a^2 = 0$$

and the locus is a circle with a diameter joining the two obvious points on the x-axis with equal elevations.

9. For $k = 1, 2, 3, \ldots$ we have $\sqrt{k+1} - \sqrt{k}$

$$= \frac{(\sqrt{k+1} - \sqrt{k})(\sqrt{k+1} + \sqrt{k})}{\sqrt{k+1} + \sqrt{k}} = \frac{1}{\sqrt{k+1} + \sqrt{k}}$$

so that $\dfrac{1}{2\sqrt{k+1}} < \sqrt{k+1} - \sqrt{k} < \dfrac{1}{2\sqrt{k}}$.

Adding the left inequalities for $k = 1, 2, \ldots, n - 1$ and the right inequalities for $k = 1, 2, \ldots, n$ yields the desired inequalities.

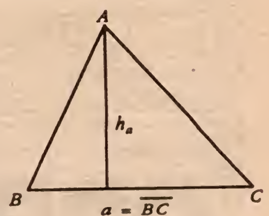
10. Let Δ denote the area of the triangle ABC. Then

$$\Delta = \frac{1}{2}a h_a = \frac{1}{2}b h_b = \frac{1}{2}c h_c$$

Thus, $a + h_a \geq b + h_b$ is equivalent to

$$a - b \geq h_b - h_a$$

$$= 2\Delta\left(\frac{1}{b} - \frac{1}{a}\right) = \frac{2\Delta(a - b)}{ab}$$



or, $(a - b)(ab - 2\Delta) \geq 0$. Since $2\Delta = ab \sin C \leq ab$ it follows that $ab - 2\Delta \geq 0$; this with $a - b \geq 0$ yields the desired inequality.

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International

MATH OLYMPIAD PROBLEMS

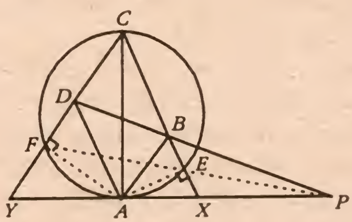
1. Let $\prod_{n=1}^{1996}(1+nx^{3^n}) = 1+a_1x^{k_1}+a_2x^{k_2}+...+a_mx^{k_m}$ where a_1, a_2, ..., a_m are non-zero and $k_1 < k_2 < ... < k_m$. Find a_{1996}.

Soln.: In general, if $k = 2^{\alpha_1} + 2^{\alpha_2} + ... + 2^{\alpha_p}$, then

$a_k = \alpha_1 + \alpha_2 + ... + \alpha_p$.

In particular, $a_{1996} = 10 + 9 + 8 + 7 + 6 + 3 + 2 = 45$.

2. In a parallelogram $ABCD$ with $m(\hat{A}) < 90°$, the circle with diameter $[AC]$ intersects the lines CB and CD at E and F besides C, and the tangent to this circle at A intersects the line BD at P. Show that the points P, F, E are collinear.



Soln.: Since AC is a diameter we get

$$\angle AEC = \angle AFC = 90°.$$

Since AP is tangent to the circle with diameter AC we have $AP \perp AC$.

Let X and Y be the intersections of AP with BC and CD, respectively.

Since $\angle XAC = 90°$ and $AE \perp XC$, we have

$AX^2 = XE \cdot XC$ and $AC^2 = EC \cdot XC$.

So, $AX^2 : AC^2 = XE : EC$ ... (i)

Similarly, we have $AY^2 : AC^2 = YF : FC$ ... (ii)

It follows that, from (i) and (ii),

$$\frac{XE}{EC}\cdot\frac{CF}{FY} = \frac{AX^2}{AC^2}\cdot\frac{AC^2}{AY^2} = \frac{AX^2}{AY^2} \quad ...\text{(iii)}$$

Since $AB \parallel CY$ and $AD \parallel XC$, we get

$$\frac{AX}{AY} = \frac{XB}{BC} \text{ and } \frac{AX}{AY} = \frac{CD}{DY}.$$

Hence, we have

$$\frac{AX^2}{AY^2} = \frac{XB}{BC}\cdot\frac{CD}{DY} \quad ...\text{(iv)}$$

By Menelau's theorem for triangle CXY, we have

$$\frac{YP}{PX}\cdot\frac{XB}{BC}\cdot\frac{CD}{Dy} = 1 \quad ...\text{(v)}$$

Thus, we obtain from (iii), (iv) and (v),

$$\frac{YP}{PX}\cdot\frac{XE}{EC}\cdot\frac{CF}{FY} = 1$$

Therefore, P, F, E are collinear, by the converse of Menelau's theorem.

3. Given real numbers

$0 = x_1 < x_2 < ... < x_{2n} < x_{2n+1} = 1$

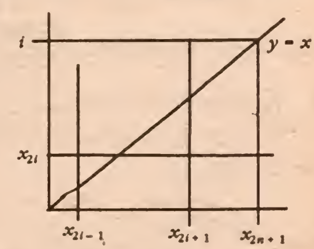
with $x_{i+1} - x_i \le h$ for $1 \le i \le 2n$, show that

$$\frac{1-h}{2} < \sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) \le \frac{1+h}{2}.$$

Soln.: This is equivalent to showing that

$$\left|\sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) - \frac{1}{2}\right| < \frac{h}{2}$$

Now $\sum_{i=1}^{n} x_{2i}(x_{2i+1} - x_{2i-1}) - \frac{1}{2}$ is the difference between the area of the rectangles formed by the four lines $x = x_{2i-1}$, $x = x_{2i+1}$, $y = 0$ and $y = x_{2i}$ and the triangle formed by the three lines $x = 0$, $y = 1$, $x = y$. The area contained in the rectangles but not in the triangle (respectively contained in the rectangle but not in the rectangles) is a union of triangles of total base less than 1 and height $\le h$. Hence, we have the required inequality.



4. In a convex quadrilateral $ABCD$, area (ABC) = area (ADC) and $[AC] \cap [BD] = \{E\}$. The parallels from E to the line segments $[AD]$, $[DC]$, $[CB]$, $[BA]$ intersect $[AB]$, $[BC]$, $[CD]$, $[DA]$ at the points K, L, M, N respectively. Compute the ratio $\frac{\text{Area }(KLMN)}{\text{Area }(ABCD)}$.

Soln.: We denote the area of polygon $A_1A_2 ... A_n$ by $[A_1A_2 ... A_n]$. Let B', D' be the feet of perpendiculars from B, D to AC, respectively.

Since $[ABC] = [ADC]$, we get $BB' = DD'$, so that

$BE : ED = BB' : DD' = 1 : 1$.

Thus, we have

$BE = ED$.

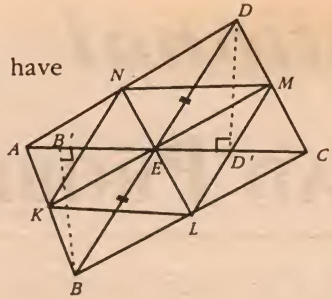
Since $EK \| DA$, we have

$BK : KA = BE : ED$

$= 1 : 1$.

Hence, $BK = KA$.

Similarly, we have $BL = LC$, $CM = MD$ and $DN = NA$.



In triangle ABD, note that K, E and N are the mid-points of AB, BD and DA respectively. Thus

$$[ENK] = \frac{1}{4}[ABD].$$

Similarly, we have $[EKL] = \frac{1}{4}[ABC]$,

$$[ELM] = \frac{1}{4}[BCD] \text{ and } [EMN] = \frac{1}{4}[CDA].$$

Hence,

$$[KLMN] = [ENK] + [EKL] + [ELM] + [EMN]$$

$$= \frac{1}{4}[ABD] + \frac{1}{4}[ABC] + \frac{1}{4}[BCD] + \frac{1}{4}[CDA]$$

$$= \frac{1}{4}([ABD] + [BCD]) + \frac{1}{4}([ABC] + [CDA])$$

$$= \frac{1}{4}[ABCD] + \frac{1}{4}[ABCD] = \frac{1}{2}[ABCD].$$

Therefore, we obtain $\frac{[KLMN]}{[ABCD]} = \frac{1}{2}$.

5. For which ordered pairs of positive real numbers (a, b) is the limit of every sequence (x_n) satisfying the condition

$$\lim_{n\to\infty}(ax_{n+1} - bx_n) = 0 \qquad \dots \text{(i)}$$

Soln.: If $b > a$, then, for $x_n = (b/a)^n$, we have $ax_n + 1 - bx_n = 0$, but $\lim_{n\to\infty}(b/a)^n = \infty$.

If $b = a$, we have the well-known example:

$$x_n = 1 + \frac{1}{2} + \dots + \frac{1}{n},$$

for which we have $\lim_{n\to\infty}(x_{n+1} - x_n) = 0$, but $\{x_n\}$ does not converge to a finite limit.

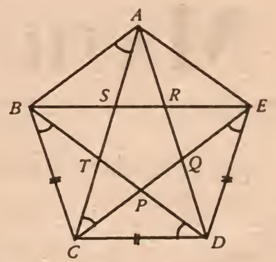
Let us assume that $b < a$. We shall prove that

$$\underline{\lim}\, x_n = \overline{\lim}\, x_n = 0.$$

Denote by m (resp. M), the $\underline{\lim}$ (resp. $\overline{\lim}$) of $\{x_n\}$.

By (i), we have $M \le \frac{b}{a}m$ and since $m \le M$, we deduce that $M \le \frac{b}{a}M$ and consequently that $m, M \le 0$. Similarly, by (i), we have $M\frac{b}{a} \le m$ and since $M \ge m$, we deduce that $m \ge \frac{b}{a}m$ and consequently that $m, M \ge 0$; whence, $m = M = 0$ and $\lim_{n\to\infty} x_n = 0$.

6. Let $ABCDE$ be a convex pentagon such that $BC = CD = DE$ and each diagonal of the pentagon is parallel to one of its sides. Prove that all the angles in the pentagon are equal, and that all sides are equal.



Soln.: As shown in the figure, we label the intersections of diagonals.

Since $BE \| CD$ and $AC \| ED$, $SCDE$ is a parallelogram, so that $CS = DE = CB$.

Hence, $\angle CBE = \angle CBS = \angle CSB = \angle DEB$.

Since $\angle CBE = \angle DEB$ and $BC = ED$, it follows that $BCDE$ is an isosceles quadrilateral, so that B, C, D, E are concyclic. Since $AB \| CE$, $AC \| DE$, we have

$$\angle BAC = \angle ACE = \angle CED = \angle CBD = \angle BDC.$$

Thus, A, B, C, D are concyclic, giving that A, B, C, D, E are concyclic. Since $BC \| AD$, we get $AB = CD$, and since $AC \| ED$, we have $AE = CD$. Therefore, $AB = BC = CD = DE = EA$.

Consequently, corresponding minor arcs AB, BC, CD, DE and EA are equal, and also corresponding inscribed angles are equal.

We put $\angle BAC = \alpha$. Then we have

$\angle EAB = \angle ABC = \angle BCD = \angle CDE = \angle DEA = 3\alpha$.

7. Let $p(x)$ be a cubic polynomial with roots r_1, r_2, r_3. Suppose that $\frac{p(1/2) + p(-1/2)}{p(0)} = 1000$.

Find the value of $\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1}$.

Soln.: $p(0)$ is supposed non-zero, so that r_1, r_2, r_3 are non-zero.

Let $p(x) = ax^3 + bx^2 + cx + d$.

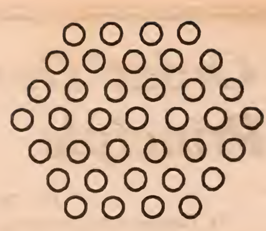
The hypothesis is :

$$\left(\frac{a}{8} + \frac{b}{4} + \frac{c}{2} + d\right) + \left(-\frac{a}{8} + \frac{b}{4} - \frac{c}{2} + d\right) = 1000d$$

that is, $b = 1996\,d$. Now

$$\frac{1}{r_1r_2} + \frac{1}{r_2r_3} + \frac{1}{r_3r_1} = \frac{r_3 + r_1 + r_2}{r_1r_2r_3} = \frac{-b/a}{-d/a} = \frac{b}{d} = 1996.$$

8. A number of tubes are bundled together into a hexagonal form. The number of tubes in the bundle can be 1, 7, 19, 37 (as shown), 61, 91, ... If this sequence is continued, it will be noticed that the total number of tubes is often a number ending in 69. What is the 69th number in the sequence which ends in 69?



Soln.: The number is 1417969. Note first that the sequence $\{a_n\}$ is given by the formula

$$a_n = 1 + 6(1 + 2 + 3 + \dots + (n-1))$$
$$= 1 + 3n(n-1) \quad \text{where } n \geq 1.$$

Clearly, a_n ends in 69 if and only if $100 \mid a_n - 69$, and $a_n \geq 69$; that is,

$$100 \mid 3n(n-1) - 68 \quad \dots \text{(i) where } n \geq 6.$$

In particular, $5 \mid 3n(n-1) - 68$ and so

$$3n(n-1) \equiv 68 \equiv 3 \pmod 5.$$

Since $(3, 5) = 1$, we have $n(n-1) \equiv 1 \pmod 5$, which holds if and only if $n \equiv 3 \pmod 5$.

Hence, $n = 5k + 3$ for some integer $k \geq 1$. Then

$$n(n-1) = (5k+3)(5k+2) = 25k^2 + 25k + 6.$$

and (i) becomes $100 \mid 75k^2 + 75k - 50$,

or $4 \mid 3k^2 + 3k - 2$.

Thus, we have $3k(k+1) \equiv 2 \equiv 6 \pmod 4$. Since $(3, 4) = 1$, we have $k(k+1) \equiv 2 \pmod 4$, which holds if and only if $k \equiv 1, 2 \pmod 4$. Thus, $k = 4t + 1$ or $4t + 2$, and $n = 20t + 8$ or $20t + 13$ for some non-negative integer t.

Conversely, if $n = 20t + 8$,

then $3n(n-1) - 68 = 1200t^2 + 900t + 100$,

and if $n = 20t + 13$,

then $3n(n-1) - 68 = 1200t^2 + 1500t + 400$.

In both cases, (i) holds. Therefore, we conclude that a_n ends in 69 if and only if $n = 20t + 8$ or $20t + 13$ for $t = 0, 1, 2, \dots$. To find the 69^{th} such number, we put $t = 34$ into $n = 20t + 8$ to obtain $n = 688$ and $a_{688} = 1 + 3 \times 688 \times 687 = 1417969$.

9. For which positive integers n can we rearrange the sequence $1, 2, \dots, n$ to $a_1, a_2, \dots, a_n$ in such a way that $|a_k - k| = |a_1 - 1| \neq 0$ for $k = 2, 3, \dots, n$?

Soln.: The required permutations exist if and only if n is even. First of all, since $a_1 - 1 \neq 0$, we have $a_1 > 1$ and so $|a_1 - 1| = a_1 - 1$.

Partition $S = \{1, 2, \dots, n\}$ has $S = A \cup B$ where $A = \{i \in S \mid a_i - i \geq 0\}$ and $B = \{j \in S \mid a_j - j < 0\}$. Since $a_i - i = a_1 - 1$ for all $i \in A$ and $a_j - j = 1 - a_1$ for all $j \in B$, we have

$$\sum_{i \in A}(a_i - i) + \sum_{i \in B}(a_j - j) = |A|(a_1 - 1) + |B|(1 - a_1)$$
$$= (|A| - |B|)(a_i - 1)$$

Since $\sum_{i \in A}(a_i - i) + \sum_{i \in B}(a_j - j) = \sum_{k \in S} a_k - \sum_{k \in S} k = 0$, we conclude that $(|A| - |B|)(a_1 - 1) = 0$, and so, that $|A| = |B|$.

Hence, $n = |S| = |A| + |B| = 2|A|$, showing that n must be even.

Conversely, if $n = 2k$ is even, then the permutation σ below clearly has the described property:

$$\sigma = \begin{pmatrix} 1 & 2 & 3 & 4 & \dots & 2k-1 & 2k \\ 2 & 1 & 4 & 3 & \dots & 2k & 2k-1 \end{pmatrix}$$

that is, $\sigma = (1, 2)(3, 4) \dots (2k-1, 2k)$ is the product of k disjoint transpositions.

10. Let $a_1, a_2, \dots a_n$ be real numbers and s, a non-negative real number, such that

(i) $a_1 \leq a_2 \leq \dots \leq a_n$; (ii) $a_1 + a_2 + \dots + a_n = 0$;
(iii) $|a_1| + |a_2| + \dots + |a_n| = s$.

Prove that $a_n - a_1 \geq \dfrac{2s}{n}$.

Soln.: The result is clear when $s = 0$, so we will suppose $s > 0$. This implies that at least one of the a_i's is non-zero. Since $a_1 + a_2 + \dots + a_n = 0$, the a_i's cannot all be non-negative, or all be non-positive. Thus: $a_1 = \min(a_i) < 0$ and $a_n = \max(a_i) > 0$.

There exists $k \in \{1, 2, \dots, n-1\}$ such that

$$a_1 \leq a_2 \leq \dots \leq a_k \leq 0 < a_{k+1} \leq \dots \leq a_n.$$

Then $a_{k+1} + \dots + a_n = -(a_1 + a_2 + \dots + a_k)$
$= |a_1| + |a_2| + \dots + |a_k| = s - (|a_{k+1}| + \dots + |a_n|)$
$= s - (a_{k+1} + \dots + a_n)$

Hence, $a_{k+1} + \dots + a_n = s/2 = -(a_1 + a_2 + \dots + a_k)$

For $i \in \{1, 2, \dots k\}$, $j \in \{k+1, \dots, n\}$, we have

$$a_j - a_i \leq a_n - a_1 \ (= \delta, \text{ say}) \quad \text{so that}$$
$$a_n - a_1 \leq \delta,\ a_n - a_2 \leq \delta,\ \dots,\ a_n - a_k \leq \delta.$$

Adding up, we get $ka_n + \dfrac{s}{2} \leq k\delta$.

Substracting successively $a_{n-1}, \dots, a_{k+1}$ for a_n, we get similarly

$$ka_{n-1} + \frac{s}{2} \leq k\delta, \dots, ka_{k+1} + \frac{s}{2} \leq k\delta.$$

Adding up again, we obtain

$$k(a_n + \dots a_{k+1}) + (n-k)\frac{s}{2} \leq (n-k)k\delta$$

or, $k\dfrac{s}{2} + (n-k)\dfrac{s}{2} \leq (n-k)k\delta$,

which leads to $\delta \geq \dfrac{ns}{2k(n-k)}$. But $(n-k) + k = n$;

hence, $k(n-k) \leq \dfrac{n^2}{4}$ and $\dfrac{ns}{2k(n-k)} \geq \dfrac{2s}{n}$.

Thus, $\delta \geq \dfrac{2s}{n}$. ■

Second and Higher Order Derivatives from First Principle

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The evaluation of second and higher order derivatives from first principle i.e. by limit, can hardly be seen in text-books of calculus. Here, in this article, the present author intends to advance the said principle for the second and higher order derivatives.

Let $f(x)$ be a bounded and continuous function defined for every value of x in the closed interval $[a, b]$ and let $x = \alpha$ be an interior point in the interval, then the second and higher order derivatives of $f(x)$ at $x = \alpha$ are defined as :

$$\left(\frac{d^2y}{dx^2}\right)_{x=\alpha} \quad \text{or} \quad f''(\alpha) = \operatorname*{Lt}_{h\to 0} \frac{f(\alpha+2h) - 2f(\alpha+h) + f(\alpha)}{h^2}$$

provided $\alpha + 2h \in [a, b]$ and the limit exists,

$$\left(\frac{d^3y}{dx^3}\right)_{x=\alpha} \quad \text{or} \quad f'''(\alpha) = \operatorname*{Lt}_{h\to 0} \frac{f(\alpha+3h) - 3f(\alpha+2h) + 3f(\alpha+h) - f(\alpha)}{h^3}$$

provided $\alpha + 3h \in [a, b]$ and the limit exists, and finally $\left(\frac{d^ny}{dx^n}\right)_{x=\alpha}$

$$\text{or} \quad f^n(\alpha) = \operatorname*{Lt}_{h\to 0} \frac{f(\alpha+nh) - {}^nC_1 f(\alpha+\overline{n-1}\,h) + {}^nC_2 f(\alpha+\overline{n-2}\,h) - \ldots (-1)^{n-1}\,{}^nC_{p-1} f(\alpha+\overline{n-r+1}\,h) + \ldots + (-1)^n f(\alpha)}{h^n}$$

provided $\alpha + nh \in [a, b]$ and the limit exists.

Moreover, we may recall the following definition of first order derivative:

$$\left(\frac{dy}{dx}\right)_{x=\alpha} \quad \text{or} \quad f'(\alpha) = \operatorname*{Lt}_{h\to 0} \frac{f(\alpha+h) - f(\alpha)}{h}$$ provided $\alpha + h \in [a, b]$ and the limit exists.

Example : Obtain $\frac{d^2y}{dx^2}$, when $y = x^3$.

Soln.: $$\frac{d^2y}{dx^2} = \operatorname*{Lt}_{h\to 0} \frac{(x+2h)^3 - 2(x+h)^3 + x^3}{h^2} = \operatorname*{Lt}_{h\to 0} \frac{6h^2(x+h)}{h^2} = \operatorname*{Lt}_{h\to 0} 6(x+h) = 6x \quad h \neq 0$$

Above formulas are of theoretical importance no doubt, because, in practice, it is wise to apply the results of first order derivatives successively and to get higher order derivatives without going through the troublesome and manipulative steps involved in the said formulas.

Like first order derivative, be it noted that second and higher order derivatives have also some geometrical significance. Apart from maxima and minima, these are closely related to concavity, convexity, inflexibility and even curvature of a curve at a point on it.

International

Math Olympiad Problems

1. Let $k \geq 1$ be an integer. Show that there are exactly 3^{k-1} positive integers n with the following properties.

(a) The decimal representation of n consists of exactly k digits.

(b) All digits of n are odd.

(c) The number n is divisible by 5.

(d) The number $m = n/5$ has k odd (decimal) digits.

Soln.: Call a natural number *ideal* if it satisfies conditions (a) – (d).

We show that $n = a_k a_{k-1} \ldots a_1 \in N$ is ideal if and only if $a_i \in \{5, 7, 9\}$ for all $i = 1, 2, \ldots k$ and $a_1 = 5$. This is clearly true when $k = 1$, and thus, we assume that $k \geq 2$.

If n is ideal, then (c) implies that $a_1 = 5$. We show that none of the a_i's can equal 1 or 3.

Let $n/5 = b_k b_{k-1} \ldots b_1$ and consider the process of long division of n by 5.

Clearly, $a_k \neq 1, 3$ for otherwise $n/5$ would have only $k - 1$ digits. If $a_i = 1$ or 3 for any i, $2 \leq i \leq k - 1$, then clearly $b_i = 0, 2, 4, 6$, or 8, depending on whether the remainder *carried over* when dividing a_{i+1} by 5 is 0, 1, 2, 3 or 4, respectively. This is a contradiction to (d).

Conversely, suppose $n = a_k a_{k-1} \ldots a_1$, where $a_i \in \{5, 7, 9\}$ for all $i = 1, 2, \ldots, k$ and $a_1 = 5$. To show that n is ideal, it clearly suffices to verify (d). Since $a_k \geq 5$, $m = n/5$ has k digits, and we can write $m = b_k b_{k-1} \ldots b_1$. Clearly, b_1 is odd. If b_i is even for some i, $2 \leq i \leq k$, then from $n = 5m$, we see that $a_i = 0, 1, 2, 3$ or 4 depending on whether they *carry* (when b_{i-1} is multiplied by 5) is 0, 1, 2, 3 or 4 respectively. This is a contradiction. Hence, all the b_i's are odd. This completes the proof of our claim.

Finally, since $a_1 = 5$ and each of the a_i's ($i = 2, 3, \ldots, k$) can take on any one of the 3 values 5, 7, or 9, the total number of ideal integers is 3^{k-1}.

2. A convex hexagon $ABCDEF$ satisfies the following conditions.

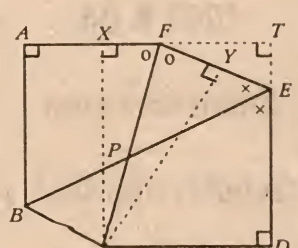
(a) The opposite sides are parallel (that is, $AB \parallel DE$, $BC \parallel EF$, $CD \parallel FA$).

(b) The distances between the opposite sides are equal [that is, $d(AB, DE) = d(BC, EF) = d(CD, FA)$, where $d(g, h)$ denotes the distance between lines g and h]

(c) $\angle FAB$ and $\angle CDE$ are right angles.

Show that diagonals BE and CF intersect at an angle of 45°.

Soln.: Let X, Y be the feet of the perpendiculars from C to AF, EF respectively.



Since $AF \parallel CD$, it follows that

$$CX = d(AF, CD)$$

Since $BC \parallel EF$, it follows that $CY = d(BC, EF)$.

Since $d(AF, CD) = d(BC, EF)$, we have $CX = CY$. Thus CF bisects $\angle AFE$. Similarly BE bisects $\angle DEF$.

Let T be the intersection of AF and DE. Since $AB \parallel DE$ and $\angle A = 90°$, we get $\angle FTE = 90°$.

Then, $\angle AFE + \angle DEF = (\angle T + \angle TEF) + (\angle T + \angle TFE)$
$= 2\angle T + (\angle TEF + \angle TFE) = 90° \times 2 + 90° = 270°$.

Thus, we have $\angle PFE + \angle PEF$
$$= \tfrac{1}{2}(\angle AFE + \angle DEF) = 135°.$$

Hence, we get $\angle FPE = 180° - (\angle PFE + \angle PEF)$
$$= 180° - 135° = 45°.$$

3. The polynomials $P_n(x)$ are defined recursively by $P_0(x) = 0$, $P_1(x) = x$ and

$P_n(x) = xP_{n-1}(x) + (1 - x)P_{n-2}(x)$ for $n \geq 2$.

For every natural number $n \geq 1$, find all real numbers x satisfying the equation $P_n(x) = 0$.

Soln.: We will prove that for $n \geq 1$, the only real solution of $P_n(x) = 0$ is $x = 0$.

For $n \geq 2$,

$$P_n(x) - P_{n-1}(x) = (x-1)[(P_{n-1})(x) - P_{n-2}(x)]$$

Then an easy induction leads to

$$P_n(x) - P_{n-1}(x) = (x-1)^{n-1}(P_1(x) - P_0(x)) = x(x-1)^{n-1}$$

That is, $P_n(x) = P_{n-1}(x) + x(x-1)^{n-1}$ for $n \geq 2$, and we note that it remains true for $n = 1$.

We deduce that, for $n \geq 1$,

$$P_n(x) = x(x-1)^{n-1} + x(x-1)^{n-2} + \ldots + x + P_0(x)$$
$$= x[(x-1)^{n-1} + (x-1)^{n-2} + \ldots + 1)$$

Then, if $x = 2$, $P_n(2) = 2n \neq 0$ and if $x \neq 2$,

$$P_n(x) = x \cdot \frac{(x-1)^n - 1}{x-2}.$$

Thus, $P_n(x) = 0$ if and only if $x = 0$ or $(x-1)^n = 1$ for $x \neq 2$.

If n is even, $(x-1)^n = 1$ if and only if $x = 0$ or $x = 2$. Then $P_n(x) = 0$ if and only if $x = 0$.

If n is odd, $(x-1)^n = 1$ if and only if $x = 2$. Then $P_n(x) = 0$ if and only if $x = 0$.

Thus, for $n \geq 1$, $P_n(x) = 0$ if and only if $x = 0$.

4. The real numbers x, y, z, t satisfy the equalities $x + y + z + t = 0$ and $x^2 + y^2 + z^2 + t^2 = 1$. Prove that $-1 \leq xy + yz + zt + tx \leq 0$.

Soln.: First we have

$xy + yz + zt + tx = (x+z)(y+t) = -(x+z)^2 \leq 0$ (since $y + t = -(x+z)$). Then

$$|xy + yz + zt + tx| \leq (x^2 + y^2 + z^2 + t^2)^{1/2} (y^2 + z^2 + t^2 + x^2)^{1/2} = 1$$

(by the Cauchy-Schwarz inequality). The conclusion follows.

Remark : Equality $xy + yz + zt + tx = 0$ holds if and only if $x + z = y + t = 0$. Therefore, inequality $xy + yz + zt + tx \leq 0$ becomes an equality for the quadruplets $(x, y, z, t) = (a, b, -a, -b)$ where a, b are real numbers such that $a^2 + b^2 = 1/2$.

If equality $xy + yz + zt + tx = -1$ holds, then we have equality in the Cauchy-Schwarz inequality so that (x, y, z, t) and (y, z, t, x) are proportional. Since at least one of the numbers x, y, z, t must be non-zero, this leads to $x = y = z = t$ or $y = -x$, $z = x$, $t = -x$. The first case is incompatible with the hypothesis, and the second case provides only the quadruplets.

$$\left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right) \text{ and } \left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$$

Conversely, these two quadruplets satisfy $xy + yz + zt + tx = -1$ and we may conclude : $xy + yz + zt + tx = -1$ holds if and only if $(x, y, z, t) = \left(\frac{1}{2}, -\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}\right)$ or $\left(-\frac{1}{2}, \frac{1}{2}, -\frac{1}{2}, \frac{1}{2}\right)$.

5. Natural numbers k, n are given such that $1 < k < n$. Solve the system of n equations $x_i^3(x_i^2 + x_{i+1}^2 + \ldots + x_{i+k-1}^2) = x_{i-1}^2$ for $1 \leq i \leq n$, with n real unknowns $x_1, x_2, \ldots, x_n$.

Note : $x_0 = x_n$, $x_{n+1} = x_1$, $x_{n+2} = x_2$ and so on.

Soln.: We prove that there are two solutions.

$$x_1 = x_2 = \ldots = x_n = 0 \quad \text{and}$$
$$x_1 = x_2 = \ldots = x_n = \frac{1}{\sqrt[3]{k}}.$$

First, suppose that $(x_1, \ldots, x_n)$ is a solution.

First case : There is $i \in \{1, \ldots n\}$ such that $x_i = 0$. From the cyclic symmetry, we suppose, without loss of generality, that $x_1 = 0$. Then

$$x_1^3(x_1^2 + \ldots + x_k^2) = 0 = x_n^2.$$

Thus, $x_n = 0$.

Continuing, we deduce that $x_i = 0$ for all i.

Second case : $x_i \neq 0$ for all i.

Since $x_i^3(x_i^2 + \ldots + x_{i+k-1}^2) = x_{i-1}^2$, we deduce that $x_i > 0$ for all i.

From cyclic symmetry, without loss of generality, we have

$$x_1 = \min\{x_i : i = 1, 2, \ldots, n\}.$$

Then $x_1^3 \leq x_2^3$ and $x_1^2 \leq x_{k+1}^2$. Thus

$$x_1^2 + x_2^2 + \ldots + x_k^2 \leq x_2^2 + x_3^2 + \ldots + x_{k+1}^2.$$

We deduce that

$$x_n^2 = x_1^3(x_1^2 + \ldots + x_k^2) \leq x_2^3(x_2^2 + \ldots + x_{k+1}^2) = x_1^2.$$

Thus, $x_n \leq x_1$.

What we have actually shown is that if $x_i = \min\{x_1, \ldots, x_n\}$, then $x_{i-1} = x_i (= \min\{x_1, \ldots, x_n\})$. An easy induction leads to $x_1 = x_2 = \ldots = x_n$.

Let a denote the common value. Then

$$a^3(ka^2) = a^2 \text{; that is } a = \frac{1}{\sqrt[3]{k}}$$

Conversely, it is easy to verify that $(0, 0, \ldots, 0)$ and $\left(\frac{1}{\sqrt[3]{k}}, \ldots, \frac{1}{\sqrt[3]{k}}\right)$ are solutions.

6. Show that there do not exist non-negative integers k and m such that $k! + 48 = 48(k+1)^m$.

Soln.: Suppose the given equation holds for some non-negative integers k and m. Then $48 \mid k!$. Since $48 = 2^4 \times 3$, we must have $k \geq 6$. If $k = 6$ or 7, the equation becomes $16 = 7^m$ or $106 = 8^m$, respectively. Clearly, neither is possible. Hence, $k \geq 8$ and the given equation can be rewritten as

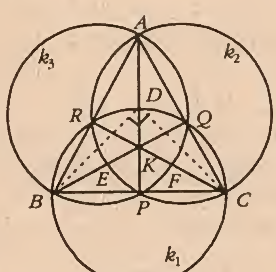
$3 \times 5 \times 7 \times 8 \times \ldots \times (k-1) \times k + 1 = (k+1)^m$ ... (i)
Suppose $k + 1$ is a composite. Then it has a prime divisor q. Since $q \le k$, we have $q \mid k!$, which implies $q \mid 48$. Since $k \ge 8$, the left side of (i) is odd and thus, q must be odd. Hence, $q = 3$, which is clearly impossible in view of (i), since $3 \nmid 1$.
Therefore, $k + 1 = p$ is a prime. Then

$$k! = (p-1)! \equiv -1 \pmod p$$

by *Wilson's theorem*; that is $p \mid k! + 1$.
Rewriting the given equation as $k! + 1 + 47 = 48p^m$, we infer that $p \mid 47$ and so $p = 47$. Then we have $46! + 48 = 48 \times 47^m$ or $46! - 48(47^m - 1)$. Since the prime divisors of 46! include 5, 7, 11 which are all coprime with 48, we have $47^m \equiv 1$ (mod 5, 7, 11). Now, straightforward checking reveals that $\mathrm{ord}_5(47) = 4$ (that is, the least positive integer n such that $47^n \equiv 1 \pmod 5$ is $n = 4$), $\mathrm{ord}_7(47) = 6$ and $\mathrm{ord}_{11}(47) = 5$. Hence, m is divisible by lcm {4, 6, 5}; that is $60 \mid m$. So $m \ge 60$ and we have $48 \times 47^m \ge 48 \times 47^{60}$. Clearly, $48 \times 47^{60} > 46! + 48$ and we have a contradiction.

7. For an acute triangle ABC, k_1, k_2, k_3 are the circles with diameters $[BC]$, $[CA]$, $[AB]$ respectively. If K is the radical centre of these circles, $[AK] \cap k_1 = \{D\}$, $[BK] \cap k_2 = \{E\}$, $[CK] \cap k_3 = \{F\}$ and area$(ABC) = u$, area$(DBC) = x$, area$(ECA) = y$ and area $(FAB) = z$, show that $u^2 = x^2 + y^2 + z^2$.

Soln.: Let P, Q and R be the feet of the perpendiculars from A, B and C to BC, CA and AB respectively. AP, BQ and CR are common chords of k_2, k_3 ; k_3, k_1 and k_1, k_2 respectively. Thus, K is the orthocentre of ΔABC. Since $\angle BDC = 90°$ and $DP \perp BC$, we get



$$DP^2 = BP \cdot CP \quad \ldots \text{(i)}$$

Since $BK \perp AC$ and $AP \perp BC$, we have $\angle BKP = \angle ACP$. Further, we have $\Delta BKP \sim \Delta ACP$. It follows that $BP : AP = KP : CP$, that is

$$AP \cdot KP = BP \cdot CP \quad \ldots \text{(ii)}$$

From (i) and (ii), we have

$$DP^2 = AP \cdot KP$$

Hence, we have $DP^2 \cdot BC^2 = (AP \cdot BC) \times (KP \cdot BC)$
From this we get $x^2 = u \times$ area (KBC) ... (iii)
Similarly, we have $y^2 = u \times$ area (KCA) ... (iv)
and $z^2 = u \times$ area(KAB) ... (v)
Therefore, we obtain from (iii), (iv) and (v),

$$x^2 + y^2 + z^2 = u \times \{\text{area}(KBC) + \text{area}(KCA) + \text{area}(KAB)\}$$
$$= u \times \text{area}(ABC) = u^2.$$

8. Let N denote the set of positive integers. Let A be a real number and $(a_n)_{n=1}^{\infty}$ be a sequence of real numbers such that $a_1 = 1$ and

$$1 < \frac{a_{n+1}}{a_n} \le A \text{ for all } n \in N.$$

(a) Show that there is a unique non-decreasing surjective function $k: N \to N$ such that $1 < \dfrac{A^{k(n)}}{a_n} \le A$ for all $n \in N$.
(b) If k takes every value at most m times, show that there exists a real number $C > 1$ such that $C^n \le Aa_n$ for all $n \in N$.

Soln.: **(a)** The condition $1 < \dfrac{A^{k(n)}}{a_n} \le A$ is equivalent to $A^{k(n)-1} \le a_n < A^{k(n)}$ or $k(n) - 1 \le \dfrac{\ln(a_n)}{\ln(A)} < k(n)$.
Hence, k is necessarily the function given by

$$k(n) = 1 + \left\lfloor \frac{\ln(a_n)}{\ln(A)} \right\rfloor = 1 + \lfloor \log A(a_n) \rfloor \text{ for all } n \in N.$$

This shows the unicity of k. Since $\{a_n\}$ is increasing and $a_1 = 1$, we have $a_n \ge 1$. We also note $A > 1$.
Hence, $\dfrac{\ln(a_n)}{\ln(A)}$ is a non-negative real number and $1 + \lfloor \log A(a_n) \rfloor$ is a positive integer. Thus, we can define a function $k : N \to N$ by the formula $k(n) = 1 + \lfloor \log A(a_n) \rfloor$. For all $n \in N : a_n < a_{n+1}$, so that $\log_A(A_n) < \log_A(a_n + 1)$ and $k(n) \le k(n+1)$. Thus, k is non-decreasing.
Now, let us remark that, for $s \in N$, $k(n) = s$ is equivalent to $A^{s-1} \le a_n < A^s$. Therefore, if $\{a_n\}$ is bounded above, say $a_n \le M$ for all n, as s satisfying $A^{s-1} > M$ (such an s exists since $A > 1$) cannot be an image under k. Thus, if $\{a_n\}$ satisfies the hypothesis and is convergent (for instance $a_n = \dfrac{1}{2}\left(1 + \dfrac{1}{n}\right)^n$), the surjectivity of k cannot be obtained. Consequently, we will henceforth assume that $\{a_n\}$ is not bounded above.
Given $s \in N$, we prove that the equation $k(n) = s$ has at least one solution. For $s = 1$, $n = 1$ is obviously a solution, so now we suppose that $s \ge 2$. From the supplementary hypothesis, there exists $n \in N$ such that $a_n \ge A^{s-1}$. Let r be the least of these n's, so that $a_{r-1} < A^{s-1} \le a_r$. Then $a_r \le Aa_{r-1} < A^s$ so that $A^{s-1} \le a_r < A^s$ and r is a solution. The function k

fulfills all the demands and (a) follows.

(b) For each $s \in N$, denote by $N(s)$ the number of n such that $k(n) = s$ that is, such that $A^{s-1} \le A_n < A^s$. By hypothesis, we have $N(s) \le m$ for each s.

We will show that (b) holds with $C = A^{1/m} > 1$.

Let n be an arbitrary integer. If $k(n) = s$, then

$$n = N(1) + N(2) + \dots + N(s-1) + j$$

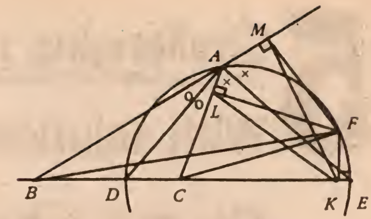
where $j \in \{1, 2, \dots N(s)\}$

(because there are $N(1) + N(2) + \dots + N(s-1)$ terms of the sequence $\{a_n\}$ which are $< A^{s-1}$). On the one hand, $a_n A \ge A^{s-1} A = A^s = C^{ms}$ and on the other hand,

$$n = N(1) + N(2) + \dots + N(s-1) + j$$
$$\le N(1) + N(2) + \dots + N(s-1) + N(s) \le sm.$$

Hence, $C^n \le C^{ms}$ and we obtain $C^n \le a_n A$.

9. In a triangle ABC with $|AB| \ne |AC|$, the internal and external bisectors of the angle A intersect the line BC at D and E, respectively. If the feet of the perpendiculars from a point F on the circle with diameter $[DE]$ to the lines BC, CA, AB are K, L, M respectively, show that $|KL| = |KM|$.



Soln.: Since the circle with diameter DE is the Apollonius circle, we have

$$\frac{FB}{FC} = \frac{AB}{AC} \quad \dots \text{(i)}$$

By the law of Sines, we have $\dfrac{AB}{AC} = \dfrac{\sin C}{\sin B}$ ... (ii)

From (i) and (ii), we get

$$\frac{FB}{FC} = \frac{\sin C}{\sin B}$$

that is $FB \sin B = FC \sin C$ ... (iii)

Since the circle with diameter BF passes through K and M, we have

$$KM = FB \sin B.$$

Similarly, we have $KL = FC \sin C$.

Therefore, we obtain from (iii),

$$KL = KM.$$

■

5 Challenging problems With solutions

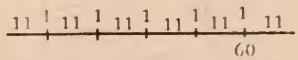
PROBABILITY

1. (a) : A railroad numbers its locomotives in order, 1, 2, ..., N. One day you see a locomotive and its number is 60. Guess how many locomotives the company has.

(b) You have looked at 5 locomotives and the largest number observed is 60. Again guess how many locomotives the company has.

Soln.: While the questions as stated provide no "right" answers, still there are some reasonable things to do. According to symmetry principle when one point is dropped, on the average the two segments will be of equal size, and so you might estimate in part (a) that the number is 119, because the segment to the left of 60 has 59, 2(59) = 118, and 118 + 1 = 119.

Similarly in part (b), you might estimate that the 5 observed numbers separate the complete series into 6 pieces. Since 60 – 5 = 55, the average length of the first 5 pieces is 11, and so you might estimate the total number as 60 + 11 or 71. Of course, you cannot expect your estimate to be exactly right very often.



The method just described makes sure that in many such estimates you average close to the correct value. That is, imagine many problems in which the unknown number N is to be guessed. Follow the estimation program described above each time (draw a sample, make an estimate). Then the set of estimates will average close to the true value in the long run.

On the other hand, you might not be interested in being close in the long run, or in averaging up well. You might want to try to be exactly right this time, however faint the hope. Then a reasonable strategy is just to guess the largest number you have seen. If you have seen 2 locomotives, then the chance that a sample of 2 contains the largest is $(N-1)/\binom{N}{2}$ or $\frac{2}{N}$.

The method of confidence limits is often used to make an interval estimate. If the company has N locomotives and we draw a random one, then the probabilities of the numbers 1, 2, ..., N are each $1/N$. Therefore we can be sure that the chance that our locomotive is in some set is the size of the set divided by N. For example, let n be the random number to be drawn, then for even values of N, $P(n > N/2) = \frac{1}{2}$, and for odd values of N the probability is slightly more. Then we can read the statement $n > N/2$ and say that the probability is at least $\frac{1}{2}$ that it is true when n is a random variable. If we have observed the value of n and do not know N, but wanted to say something about it, we could say $2n > N$, and that would put an upper bound on N. The statement itself is either right or wrong in any particular instance, and it is right in more than half of experiments and statements made in this manner. If one wanted to be surer, then one could change the limits. For example, $P\left(n \geq \frac{1}{3}N\right) \geq \frac{2}{3}$. The confidence statement would be $3n \geq N$, and we would be at least $\frac{2}{3}$ sure it was correct. In our problem, if we wanted to be at least 2/3 sure of making a statement that contains the correct value of N, we say N is between 60 and 180.

Another method of estimation that is much in vogue is maximum likelihood. One would choose the value of N that makes our sample most likely. For example, if $N = 100$, our sample value of 60 would have probability $\frac{1}{100}$; but if $N = 60$, its probability is $\frac{1}{60}$. We can't go lower than 60 because if $N = 59$ or less we can't observe 60, and our sample would have probability 0. Consequently, if n is the observed value, the maximum likelihood estimate of N is n.

2. Labour laws in Erewhon require factory owners to give every worker a holiday whenever one of them has a birthday and to hire without discrimination on grounds of birthdays. Except for

these holidays they work a 365-day year. The owners want to maximize the expected total number of man-days worked per year in a factory. How many workers do factories have in Erewhon ?

Soln.: With 1 worker in the factory, the owner gets 364 man-days, with 2 he usually gets 2(363) = 726, and so we anticipate more than 2 workers to maximize working days in a factory. On the other hand, if the factory population is enormous, every day of the year is practically certain to be someone's birthday, and the factory never works. Consequently, there must be a finite maximum.

If we can get the expected total number of days worked, we are a long step forward. Each day is either a working day or it isn't. Let's replace 365 by N so that we solve the problem generally, and let n be the number of workers. Then the probability that the first day is a working day is $(1 - 1/N)^n$, because then every worker has to have a birthday on one of the other $N - 1$ days. The expected number of man-days contributed by the first working day is

$$n\left(1-\frac{1}{N}\right)^n . 1 + n\left[1-\left(1-\frac{1}{N}\right)^n\right] . 0 = n\left(1-\frac{1}{N}\right)^n .$$

Every day contributes this same number, and so the expected number of man-days worked by n workers is $nN(1 - 1/N)n$. To maximize this function of n, we must find n so that increasing or decreasing n reduces the total, or in symbols :

$$(n+1)N\left(1-\frac{1}{N}\right)^{n+1} \le nN\left(1-\frac{1}{N}\right)^n$$

and $$(n+1)N\left(1-\frac{1}{N}\right)^{n-1} \le nN\left(1-\frac{1}{N}\right)^n .$$

The first inequality reduces to :

$$(n+1)\left(1-\frac{1}{N}\right) \le n , \; N \le n + 1.$$

The second inequality reduces to :

$$n - 1 \le n\left(1-\frac{1}{N}\right), \; n \le N.$$

Combining these results gives us $n \le N \le n + 1$, and so either $n = N$ or $n = N - 1$. When these values are substituted for n in the formula for the expected man-days, we get $N^2(1 - 1/N)^N$ and $(N - 1)N(1 - 1/N)^{N-1}$, which are equal. Since the Nth man adds nothing, $N - 1$ must be the factory size. Since $(1 - 1/N)^N \approx e^{-1}$, we get at last N^2e^{-1} as the approximate expected number of days worked. If all N men worked every day, they would work N^2 days, and so e^{-1} is the expected fraction that the actual man-days worked is of the potential N^2 man-days. Thus the fraction is about 0.37. The factory size is 364, and the man-days worked are roughly 49,000, assuming no other absenteeism. The 364th worker adds only 0.37 days to the total expectation!

3. (a) : If a stick is broken in two at random, what is the average length of the smaller piece?
(b) : What is the average ratio of the smaller length to the larger ?

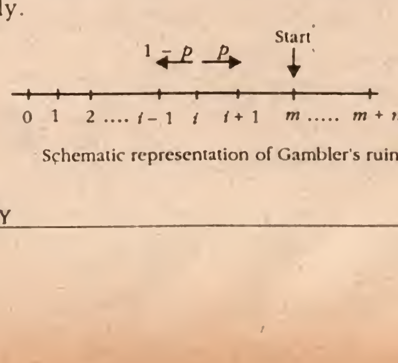
Soln.: **(a) :** Breaking "at random" means that all points of the stick are equally likely as a breaking point (uniform distribution). The breaking point is just as likely to be in the left half as the right half. If it is in the left half, the smaller piece is on the left; and its average size is half of that half, or one-fourth the length of the stick. The same sort of argument applies when the break is in the right half of the stick, and so the answer is one-fourth of the length.

(b) : We might suppose that the point fell in the right-hand half. Then $(1 - x)/x$ is the fraction if the stick is of unit length. Since x is evenly distributed from $\frac{1}{2}$ to 1, the average value, instead of the intuitive $\frac{1}{3}$, is

$$2\int_{\frac{1}{2}}^{1}\frac{1-x}{x}dx = 2\int_{\frac{1}{2}}^{1}\left(\frac{1}{x}-1\right)dx = 2\log_e 2 - 1 \approx 0.386.$$

4. Player M has \$1, and Player N has \$2. Each play gives one of the players \$1 from the other. Player M is enough better than Player N that he wins $\frac{2}{3}$ of the plays. They play until one is bankrupt. What is the chance that Player M wins ?

Soln.: This problem is a special case of the general random walk problem with two absorbing barriers. Historically, the problem arose as a gambling problem, called gambler'e ruin. Let us restate the problem generally.



Schematic representation of Gambler's ruin

Player M has m units; player N has n units. On each play of a game one player wins and the other loses 1 unit. On each play, the probability that player M wins is p, that N wins is $q = 1 - p$. Play continues until one player is bankrupt. The figure represents the amount of money Player M has at any time. He starts at $x = m$. When $x = 0$, he is bankrupt; when $x = m + n$, Player N is bankrupt. We know that, had Player M played against a bank with unlimited resources, he would have become bankrupt with probability $(q/p)^m$. In the course of a trip of bankruptcy, either he attains an amount of money $m + n$ (n is now finite), or he is never that well off. Let the probability that he loses to Player N be Q (that is equivalent to the infinite bank winning without Player M ever reaching $m + n$). Then

$$(q/p)^m = Q + (1 - Q)(q/p)^{m+n} \quad \text{... (i)}$$

because Q is the fraction of the sequences that are absorbed before reaching $m + n$, and of the fraction $1 - Q$ that do reach $m + n$, the portion $(q/p)^{m+n}$ is also absorbed at 0 if the game is allowed to proceed indefinitely. Then $P = 1 - Q$ is the probability that player M wins. Making substitution into equation (1) and solving for P gives

$$P = \frac{1-(q/p)^m}{1-(q/p)^{m+n}} \quad \text{... (ii)}$$

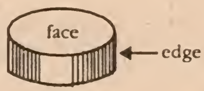
For our players $p = \frac{2}{3}$, $q = \frac{1}{3}$, $m = 1$, $n = 2$, and $P = \frac{4}{7}$. So in this instance it is better to be twice as good a player rather than twice as wealthy.

If $q = p = \frac{1}{2}$, then P in equation (2) takes the indeterminate form 0/0. When L'Hospital's rule is applied, we find

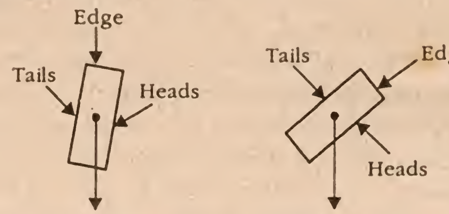
$$P = \frac{m}{m+n}, \quad \text{... (iii)} \quad ; \quad p = q = 1/2.$$

Thus, had the players been evenly matched, Player M's chance would be $\frac{1}{3}$ and his expectation would be $\frac{1}{3}(2) + \frac{2}{3}(-1) = 0$. Thus the game is fair, that is, has 0 expectation of gain for each player.

5. How thick should a coin be to have a 1/3 chance of landing on edge ?



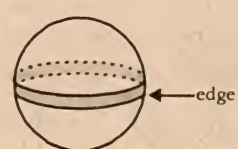
Soln.: This problem has no definite answer without some simplifying conditions. The elasticity of the coin, the intensity with which it is tossed, and the properties of the surface on which it lands combine to make the real life question an empirical one.



Coin falls on edge | Coin falls on face

The simplifying conditions that spring to mind are those that correspond to inscribing the coin in a sphere, where the center of the coin is the center of the sphere. The coin itself is regarded as a right circular cylinder. Then a random point on the surface of the sphere is chosen. If the radius from that point to the center strikes the edge, the coin is said to have fallen on edge.

To simulate this in reality, the coin might be tossed in such a way that it fell on a thick sticky substance that would grip the coin when it touched, and then the coin would slowly settle to its edge or its face.



A key theorem in solid geometry simplifies this problem. When parallel planes cut a sphere, the orange-peel-like band produced between them is called a zone. The surface area of a zone is proportional to the distance between the planes, and so our coin should be 1/3 as thick as the sphere. How should the thickness compare with the diameter of the coin ?

Let R be the radius of the sphere and r that of the coin.

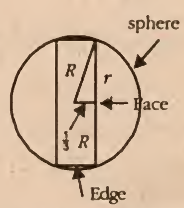
The Pythagorean theorem gives

$$R^2 = r^2 + \frac{1}{9}R^2,$$

or $$\frac{8}{9}R^2 = r^2, \quad \frac{R^2}{9} = \frac{r^2}{8},$$

$$\frac{1}{3}R = \frac{\sqrt{2}}{4}r \approx 0.354r.$$

And so the coin should be about 35% as thick as the diameter of the coin.



Cross section showing relation between radius R of sphere and radius r of coin

■

39. (b) : Since $\overrightarrow{OP}$ has projection $\frac{13}{5}, \frac{19}{5}$ and $\frac{26}{5}$ on the coordinate axes, therefore

$$\overrightarrow{OP} = \frac{13}{5}\hat{i} + \frac{19}{5}\hat{j} + \frac{26}{5}\hat{k}.$$

Suppose P divides the join of $Q(2, 2, 4)$ and $R(3, 5, 6)$ in the ratio $\lambda : 1$. Then the position vector of P is

$$\left(\frac{3\lambda+2}{\lambda+1}\right)\hat{i} + \left(\frac{5\lambda+2}{\lambda+1}\right)\hat{j} + \left(\frac{6\lambda+4}{\lambda+1}\right)\hat{k}.$$

$$\therefore\ \frac{13}{5}\hat{i} + \frac{19}{5}\hat{j} + \frac{26}{5}\hat{k} = \left(\frac{3\lambda+2}{\lambda+1}\right)\hat{i} + \left(\frac{5\lambda+2}{\lambda+1}\right)\hat{j} + \left(\frac{6\lambda+4}{\lambda+1}\right)\hat{k}$$

$$\Rightarrow\ \frac{3\lambda+2}{\lambda+1} = \frac{13}{5},\ \frac{5\lambda+2}{\lambda+1} = \frac{19}{5},\ \frac{6\lambda+4}{\lambda+1} = \frac{26}{5}$$

$\Rightarrow\ 2\lambda = 3 \ \Rightarrow\ \lambda = 3/2.$

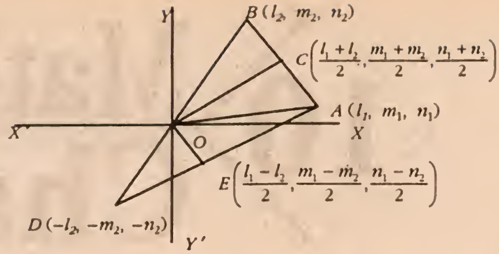
Hence, P divided QR in the ratio $3 : 2$.

40. (b) : Let OA and OB be two lines with DCs l_1, m_1, n_1 and l_2, m_2, n_2.

Let $OA = OB = 1$. Then the coordinates of A and B are (l_1, m_1, n_1) and (l_1, m_2, n_2) respectively.

Let OC be the bisector of $\angle AOB$. Then C is the midpoint of AB and so its coordinates are

$$\left(\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2} + \frac{n_1+n_2}{2}\right).$$



$\therefore$ DRs of OC are $\frac{l_1+l_2}{2}, \frac{m_1+m_2}{2}, \frac{n_1+n_2}{2}$.

we have $OC = \sqrt{\left(\frac{l_1+l_2}{2}\right)^2 + \left(\frac{m_1+m_2}{2}\right)^2 + \left(\frac{n_1+n_2}{2}\right)^2}$

$$= \frac{1}{2}\sqrt{(l_1^2+m_1^2+n_1^2)+(l_2^2+m_2^2+n_2^2)+2(l_1l_2+m_1m_2+n_1n_2)}$$

$= \frac{1}{2}\sqrt{2+2\cos\theta}$ ($\cos\theta = l_1l_2 + m_1m_2 + n_1n_2$ (given))

$= \frac{1}{2}\sqrt{2(1+\cos\theta)} = \cos\left(\frac{\theta}{2}\right)$

$\therefore$ DCs of $\overrightarrow{OC}$ are $\frac{l_1+l_2}{2(OC)}, \frac{m_1+m_2}{2(OC)}, \frac{n_1+n_2}{2(OC)}$

or, $\frac{l_1+l_2}{2\cos\theta/2}, \frac{m_1+m_2}{2\cos\theta/2}, \frac{n_1+n_2}{2\cos\theta/2}$. ■

10 Mathematical Challenges

1. Show that among any seven distinct positive integers not greater than 126, one can find two of them, say x and y, satisfying the inequalities $1 < \frac{y}{x} \leq 2$.

2. Show that if m is a positive rational number then $m + \frac{1}{m}$ is an integer only if $m = 1$.

3. Four distinct lines L_1, L_2, L_3, L_4 are given in the plane, with L_1 and L_2 respectively parallel to L_3 and L_4. Find the locus of a point moving so that the sum of its perpendicular distances from the four lines is constant.

4. Show that if n is a positive integer greater than 1, then

$$1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}$$

is not an integer

5. Prove that if 5 pins are stuck onto a piece of cardboard in the shape of an equilateral triangle of side length 2, then some pair of pins must be within distance 1 of each other.

6. Let f be a function with the following properties:
(1) $f(n)$ is defined for every positive integer n;
(2) $f(n)$ is an integer;
(3) $f(2) = 2$;
(4) $f(mn) = f(m)\,f(n)$ for all m and n;
(5) $f(m) > f(n)$ whenever $m > n$.
Prove that $f(n) = n$ for $n = 1, 2, 3, \ldots$.

7. You are given 6 congruent balls, two each of colours red, white, and blue, and informed that one ball of each colour weighs 15 grams while the other weighs 16 grams. Using an equal arm balance only twice, determine which three are the 16-gram balls.

8. What is the maximum number of terms in a geometric progression with common ratio greater than 1 whose entries all come from the set of integers between 100 and 1000 inclusive.

9. Let $f(x) = a_n x^n + a_{n-1} x^{n-1} + \ldots + a_1 x + a_0$ be a polynomial of degree $n \geq 1$ with integer coefficients. Show that there are infinitely many positive integers m for which
$f(m) = a_n m^n + a_{n-1} m^{n-1} + \ldots + a_1 m + a_0$ is not prime.

10. If a, b, c, d are positive integers for which $ab = cd$, show that $a^2 + b^2 + c^2 + d^2$ is composite.

SOLUTION

1. If the integers 1, 2, 3, .., 126 are split into 6 sets, then by the Pigeonhole Principle one of the sets will contain two (or more) of the seven chosen integers. Thus, the problem is to arrange the splitting so that in each of the 6 sets the largest integer is at most twice the smallest. Clearly, the following splitting does the trick:
{1, 2}, {3, 4, 5, 6}, {7, 8, ..., 13, 14}, {15, 16, ..., 29, 30}, {31, 32, ..., 61, 62}, {63, 64, ..., 125, 126}.

2. *First solution :* Let $m = p/q$, where p and q are positive integers having no common prime factors.

Then $\quad m + \frac{1}{m} = \frac{p^2 + q^2}{pq}$

so that, if this is an integer, then p and q both divide $p^2 + q^2$. But then p divides q^2 and q divides p^2. Since p and q have no common prime factors this means that $P = q = 1$ and $m = 1$.
Second solution : It suffices to show that the only positive integer k for which the equation

$x + \frac{1}{x} = k$ has a positive rational root is $k = 2$. This equation, namely

$x^2 - kx + 1 = 0$ has roots

$$\frac{k+\sqrt{k^2-4}}{2},\ \frac{k-\sqrt{k^2-4}}{2}.$$

If these roots are rational, then $k^2 - 4$ must be a square. But

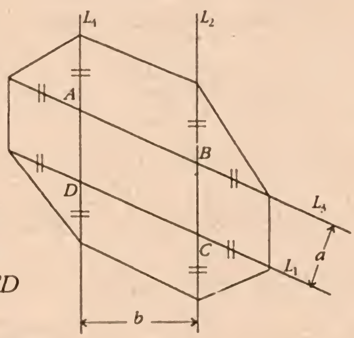
$$(k-1)^2 < k^2 - 4 < k^2 \quad \text{when } k \geq 3,$$

so $k^2 - 4$ cannot be a square when $k \geq 3$, and, when $k = 1$, $k^2 - 4$ is negative and so cannot be a square. Thus the only possible value for k is 2.

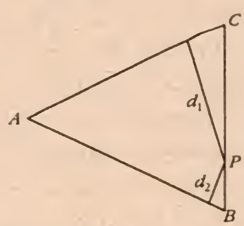
3. Let a be the distance between L_1 and L_3; let b be the distance between L_2 and L_4. Clearly, the sum of the distances from a point to the 4 lines is at least $a + b$. Now let K be positive, and let L denote the locus of points whose sum of distances (from the four lines) equals K.

Case 1 : $K < a + b$. Then L is empty.

Case 2 : $K = a + b$. Then L is the parallelogram $ABCD$ and its interior.



Case 3: $K > a + b$. Then L is a centrally symmetric octagon, as in figure. This is an immediate consequence of the following: If P is any point on the side BC of the isosceles triangle ABC ($a = \overline{AB} = \overline{AC}$) then the sum of the distances from P to AB and AC is a constant. For if these distances are d_1 and d_2, then the area of ΔABC equals

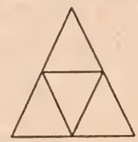


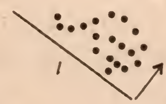
$\frac{1}{2}ad_1 + \frac{1}{2}ad_2 = \frac{1}{2}a(d_1 + d_2)$, as in figure.

4. Express each term of the sum as a fraction with denominator equal to the least common multipleof 2, 3, ..., n. All numerators will be even except for the single term whose original denominator was the highest power of 2 not exceeding n. Thus the sum of the numerators is odd while the common denominator is even.

Remark : It can be shown that, when a, $d > 0$, $\frac{1}{a} + \frac{1}{a+d} + \ldots + \frac{1}{a+(n-1)d}$ (the sum of the reciprocals of an arithmetic progression) is never an integer. A proof depends on Bertrand's Postulate: *There always exists a prime between m and 2m.*

5. Partition the triangle into 4 equilateral triangles of side 1 (figure). By the Pigeon Hole principle, one of these four triangles must contain 2 (or more) of the 5 points, and these 2 points are within distance 1 of each other.





6. First solution: First observe that

$2 = f(2) = f(1 \cdot 2) = f(1)\,f(2) = f(1)^2$,

so $f(1) = 1$. Furthermore

$f(2^2) = f(2 \cdot 2) = f(2)f(2) = 2 \cdot 2 = 2^2$

$f(2^3) = f(2 \cdot 2^2) = f(2)f(2^2) = 2 \cdot 2^2 = 2^3$

and so on, *i.e.*, $f(2^k) = 2^k$ for $k = 0, 1, 2, \ldots$

Now consider successive powers of 2 and the integers between:

$2^k < 2^k + 1 < 2^k + 2 < \ldots < 2^k + 2^k - 1$
$= 2^{k+1} - 1 < 2^{k+1}$.

Their f values satisfy

$2^k < f(2^k + 1) < f(2^k + 2) < \ldots < f(2^{k+1} - 1) < 2^{k+1}$.

Thus, between $2k$ and 2^{k+1} we have $2^k - 1$ distinct integers $f(2^k + j)$, $j = 1, 2, \ldots, 2^k - 1$. Since there are exactly $2^{k+1} - 2^k - 1 = 2^k - 1$ integers between 2^k and 2^{k+1}, it follows that

$$f(2^k + j) = 2^k + j,\ j = 1, 2, \ldots$$

Second solution: As before, we observe that $f(2^k) = 2^k$ for $k = 1, 2, \ldots$ From property (5) we see that $f(m + 1) > f(m)$, i.e., $f(m + 1) \geq f(m) + 1$; hence $f(m) \geq m$ and $f(m + k) \geq f(m) + k$. It remains to show that for no n is $f(n) > n$. Suppose that $f(n) > n$ for some n. Then $2^n > n$ and

$$2^n = f(2^n) = f(n + 2^n - n) \geq f(n) + 2^n - n$$
$$> n + 2^n - n = 2^n, \quad \text{which is absurd.}$$

Third solution (by induction): First we observe that $f(1) = 1$.

Now assume that $f(k) = k$ for $k = 1, 2, .., n$.

We show that $f(n + 1) = n + 1$.

If $n + 1 = 2j$, then $1 \leq j \leq n$ and

$f(n + 1) = f(2j) = 2j = n + 1$.

If $n + 1 = 2j + 1$, then $1 \le j < n$ and

$$2j = f(2j) < f(2j+1) < f(2j+2)$$
$$= f(2(j+1)) = 2f(j+1) = 2(j+1) = 2j+2$$

Thus $2j < f(2j+1) < 2j+2$,
so $f(2j+1) = 2j+1 = n+1$.

7. Let the two red balls R_1 and R_2 weigh r_1 and r_2 respectively, and use analogous notation for the other colours. First, balance R_1 and W_1 against R_2 and B_1. If $r_1 + w_1 = r_2 + b_1$, then either: $r_1 < r_2$ and $w_1 > b_1$, or $r_1 > r_2$ and $w_1 < b_1$. These cases can be distinguished by then balancing W_1 against W_2.
If $r_1 + w_1 > r_2 + b_1$, then certainly $r_1 > r_2$ and one of these possibilities occurs:
(i) $w_1 > w_2$ and $b_1 > b_2$;
(ii) $w_1 > w_2$ and $b_1 < b_2$;
(iii) $w_1 < w_2$ and $b_1 < b_2$.
For the second weighing, balance R_1 and B_1 against W_2 and B_2, and deduce:
if $r_1 + b_1 > w_2 + b_2$, then $b_1 > b_2$ and $w_1 > w_2$;
if $r_1 + b_1 = w_2 + b_2$, then $b_1 < b_2$ and $w_1 > w_2$;
if $r_1 + b_1 < w_2 + b_2$, then $b_1 < b_2$ and $w_1 < w_2$.
The final case $r_1 + w_1 < r_2 + b_1$ can be disposed of similarly.

8. Let us try to construct the longest possible geometric progression. Suppose it has n terms, that is common ratio (necessarily rational) is p/q in its lowest terms and its smallest term is a. The common ratio should be as small as possible p should be $q + 1$. In addition, the last term $\frac{ap^{n-1}}{q^{n-1}}$ is an integer, so q^{n-1} divides a. Thus a is divisible by a large power. Since $3^6 = 729$ we see that a sequence with at least 7 terms must start with 729 or a multiple of 26 = 64. A little thought suggests that the best that can be done is to start with 128, take a common ratio $\frac{3}{2}$ and be content with 6 terms {128, 192, 288, 432, 648, 972}.
A more formal proof follows. The geometric progression
128, 192, 288, 432, 648, 972
with common ratio $\frac{3}{2}$ shows that the longest sequence is at least of length 6. Consider now the geometric progression

$$100 \le a < ar < ar^2 < \dots < ar^{n-1} \le 1000,$$

where the n terms are all integers and $r > 1$. Clearly r must be rational, say $r = \frac{p}{q}$ with $p > q \ge 1$, $(p, q) = 1$. Because $ar^{n-1} = a\left(\frac{p}{q}\right)^{n-1}$ is an integer, q^{n-1} divides a. We may take $p = q + 1$, for the progression.

$$100 \le a < a\left(\frac{q+1}{q}\right) < \dots < a\left(\frac{q+1}{q}\right)^{n-1} \le 1000$$

has length n and the entries are all integers (since q^{n-1} divides a) in the required range. If $q \ge 3$, then

$$1000 \ge a\left(\frac{q+1}{q}\right)^{n-1} \ge (q+1)^{n-1} \ge 4^{n-1},$$

i.e. $n \le 5$. If $q = 1$, then

$$1000 \ge a\left(\frac{q+1}{q}\right)^{n-1} = a2^{n-1} \ge 100 \cdot 2^{n-1},$$

i.e. $n \le 4$. If $q = 2$, then

$$1000 \ge a\left(\frac{q+1}{q}\right)^{n-1} = a\left(\frac{3}{2}\right)^{n-1} \ge 100\left(\frac{3}{2}\right)^{n-1}.$$

i.e., $n \le 6$. Hence the longest sequence has length 6.

9. If $|a_0| \ne 1$, a solution is obvious. For if m is any multiple of a_0 then $f(m)$ is divisible by a_0. In fact, infinitely often $f(m)$ is a proper multiple of a_0, since $f(m)$ can assume the values a_0 and $-a_0$ only a finite number of times. To find a solution when $|a_0| = 1$, proceed as follows. Choose any integer k for which $f(k)$ is not equal to 1 or –1. (This is always possible. Why?) Let p be any prime divisor of $f(k)$. Then $f(k + rp) \equiv f(k) \pmod p$. Furthermore, since a polynomial can assume any value only a finite number of times, $f(k + rp)$ is a multiple of p distinct from $\pm p$ for all but finitely many values of r.

10. Let m be the *gcd* of a and c, and suppose that $a = mu$ and $c = mv$. The $ub = vd$.
Since $gcd(u, v) = 1$, u must divide d.
If $d = nu$, then $b = nv$ and

$$a^2 + b^2 + c^2 + d^2 = (m^2 + n^2)(u^2 + v^2).$$

$$= \lim_{x\to\infty}\left[\left\{1+\frac{4x+1}{x^2+x+2}\right\}^{\frac{x^2+x+2}{4x+1}}\right]^{\frac{4x^2+x}{x^2+x+2}}$$

$$= \lim_{x\to\infty}\left[(1+y)^{1/y}\right]^4 \text{, where } y = \frac{4x+1}{x^2+x+2}$$

$$\left[\because \lim_{x\to\infty}\frac{4x+1}{x^2+x+2} = \lim_{x\to\infty}\frac{\frac{4}{x}+\frac{1}{x^2}}{1+\frac{1}{x}+\frac{2}{x^2}} = 0 \text{ and } \lim_{x\to\infty}\frac{4x^2+x}{x^2+x+2} = \lim_{x\to\infty}\frac{4+\frac{1}{x}}{1+\frac{1}{x}+\frac{2}{x^2}} = 4\right] = e^4.$$

40. (b) : We have $y = \cos^{-1}\left(\frac{2\cos x - 3\sin x}{\sqrt{13}}\right)$

Let $2 = r\cos\theta$, $3 = r\sin\theta$. Then $r = \sqrt{13}$ and $\tan\theta = \frac{3}{2}$

$\therefore\ y = \cos^{-1}[\cos(x+\theta)] = x + \theta = x + \tan^{-1}\left(\frac{3}{2}\right)$

$\Rightarrow\ dy/dx = 1.$

41. (b) : Let $y = \sec^{-1}\left(\frac{1}{2x^2-1}\right), z = \sqrt{1-x^2}$.

Putting $x = \cos\theta$, we get

$y = \sec^{-1}(\sec 2\theta) = 2\theta = 2\cos^{-1} x.\quad \therefore \frac{dy}{dx} = -\frac{2}{\sqrt{1-x^2}}$

Diff. $z = \sqrt{1-x^2}$ w.r.t. x, we get $\frac{dz}{dx} = -\frac{x}{\sqrt{1-x^2}}$

Hence $\frac{dy}{dz} = \frac{-\frac{2}{\sqrt{1-x^2}}}{-\frac{x}{\sqrt{1-x^2}}} = \frac{2}{x}$. Thus, $\left(\frac{dy}{dz}\right)_{x=1/2} = 4.$

42. (b) : We have $ay^2 = x^3$. Differentiating with respect to x, we get $2ay\frac{dy}{dx} = 3x^2 \quad \therefore \frac{dy}{dx} = \frac{3x^2}{2ay}$.

Let (x_1, y_1) be a point on $ay^2 = x^3$.
Then $ay_1^2 = x_1^3$ .... (i)
The equation of the normal at (x_1, y_1) is

$$y - y_1 = -\frac{2ay_1}{3x_1^2}(x - x_1)$$

This meets the coordinate axes at

$$A\left(x_1 + \frac{3x_1^2}{2a}, 0\right) \text{ and } B\left(0, y_1 + \frac{2ay_1}{3x_1}\right)$$

Since the normal cuts off equal intercepts with the coordinate axes, therefore

$$x_1 + \frac{3x_1^2}{2a} = y_1 + \frac{2ay_1}{3x_1} \Rightarrow x_1\frac{(2a+3x_1)}{2a} = y_1\frac{(3x_1+2a)}{3x_1}$$

$\Rightarrow 3x_1^2 = 2ay_1 \qquad \Rightarrow 9x_1^4 = 4a^2y_1^2$ ... (ii)

From (i) and (ii) $9x_1^4 = 4a^2\left(\frac{x_1^3}{a}\right) \Rightarrow x_1 = \frac{4a}{9}$

43. (b) : Since $y = a\log x + bx^2 + c$ has its extremum at $x = -1$ and $x = 2$, therefore $\left(\frac{dy}{dx}\right) = 0$ at $x = -1, 2$

We have $\frac{dy}{dx} = \frac{a}{x} + 2bx + 1$

$\therefore\ \left(\frac{dy}{dx}\right)_{\text{at } x=-1} = 0 \quad \Rightarrow -a - 2b + 1 = 0$ and

$\left(\frac{dy}{dx}\right)_{x=2} = 0 \quad \Rightarrow \frac{a}{2} + 4b + 1 = 0$

Solving these two equations, we get $a = 2, b = -\frac{1}{2}$

44. (c) : $\int\frac{1}{(1+x^2)\sqrt{1-x^2}}dx = \int\frac{-t\,dt}{(t^2+1)\sqrt{t^2-1}}$

Putting $x = \frac{1}{t}, dx = -\frac{1}{t^2}dt$

$\int\frac{-u\,du}{(u^2+2)\sqrt{u^2}} = -\int\frac{1}{u^2+(\sqrt{2})^2}du$

Putting $t^2 - 1 = u^2$, $tdt = udu$

$$= -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{u}{\sqrt{2}}\right) + K = -\frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + K$$

$$= -\frac{1}{\sqrt{2}}\left\{\frac{\pi}{2} - \cot^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right)\right\} + K \quad (\because \tan^{-1}x + \cot^{-1}x = \pi/2)$$

$$= \frac{1}{\sqrt{2}}\cot^{-1}\left(\frac{\sqrt{1-x^2}}{\sqrt{2}x}\right) + \left(K - \frac{\pi}{2\sqrt{2}}\right)$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\sqrt{2}x}{\sqrt{1-x^2}}\right) + C, \text{ where } C = K - \frac{\pi}{2\sqrt{2}}$$

45. (a) : $I = \int\frac{\sqrt{\tan x}}{\sin x\cos x}dx = \int\frac{\sqrt{\tan x}}{\tan x}\sec^2 x\,dx$

$= \int\frac{1}{\sqrt{t}}dt$, where $t = \tan x$

$\therefore\ I = 2t^{1/2} + C = 2\sqrt{\tan x} + C$

46. (d) : Let $I = \int_0^1\frac{1}{x^2+2x\cos\alpha+1}dx$. Then

$$I = \int_0^1\frac{1}{(x+\cos\alpha)^2+\sin^2\alpha}dx = \frac{1}{\sin\alpha}\left(\tan^{-1}\frac{x+\cos\alpha}{\sin\alpha}\right)_0^1$$

$$I = \frac{1}{\sin\alpha}\left[\tan^{-1}\frac{1+\cos\alpha}{\sin\alpha} - \tan^{-1}\frac{\cos\alpha}{\sin\alpha}\right]$$

$$= \frac{1}{\sin\alpha}\left[\tan^{-1}\left(\cot\frac{\alpha}{2}\right) - \tan^{-1}(\cot\alpha)\right]$$

$$= \frac{1}{\sin\alpha}\left[\left(\frac{\pi}{2}-\frac{\alpha}{2}\right) - \left(\frac{\pi}{2}-\alpha\right)\right] = \frac{\alpha}{2\sin\alpha}$$

47. (a) : The given equation when expressed as a polynomial in derivatives is

$$\rho^2\left(\frac{d^2y}{dx^2}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$$

Clearly it is a second order differential equation of degree 2.

48. (b) : We have $\frac{dy}{dx} = 1 - \frac{1}{x^2} \qquad \therefore y = x + \frac{1}{x} + C$

This passes through $\left(2, \frac{7}{2}\right)$, therefore $\frac{7}{2} = 2 + \frac{1}{2} + C$

$\Rightarrow C = 1$

Thus the equation of the curve is

$$y = x + \frac{1}{x} + 1 \text{ or } xy = x^2 + x + 1$$

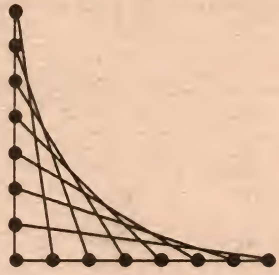
Contd. on page 55

JUNIOR HIGH SCHOOL MATHEMATICS CONTEST (BRITISH) PROBLEMS

PART - A

1. The last (ones) digit of a perfect square cannot be :
(a) 1 (b) 4 (c) 5
(d) 6 (e) 8.

2. Suppose a string art design is constructed by connecting nails on a vertical axis and on a horizontal axis by line segments as follows: The nail furthest from the origin on the vertical axis is connected to the nail nearest the origin on the horizontal axis. Then proceed toward the origin on the vertical axis and away on the horizontal axis as shown in the diagram. If this were done on a project with 10 nails on each axis, the number of points of intersection of line segments would be :



(a) 45 (b) 46 (c) 47
(d) 48 (e) none of these.

3. Assume there is an unlimited supply of pennies, nickels, dimes, and quarters. An amount (in cents) which cannot be made using exactly 6 of these coins is :
(a) 91 (b) 87 (c) 78
(d) 51 (e) 49.

4. Given $x^2 + y^2 = 28$ and $xy = 14$, the value of $x^2 - y^2$ equals :
(a) –14 (b) 0 (c) 14
(d) 28 (e) 42.

5. Given that $0 < x < y < 20$, the number of integer solutions (x, y) to the equation $23x + 3y = 50$ is :
(a) 16 (b) 9 (c) 8
(d) 5 (e) 3.

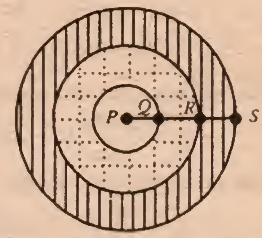
6. The numbers 1, 3, 6, 10, 15... are known as triangular numbers. Each triangular number can be expressed as $\frac{n(n+1)}{2}$ where n is a natural number. The largest triangular number less than 500 is :
(a) 494 (b) 495 (c) 496
(d) 497 (e) 498.

7. An 80 m rope is suspended at its two ends from the tops of two 50 m from flagpoles. If the lowest point to which the mid-point of the rope can be pulled is 36 m from the ground, then the distance, in metres, between the flagpoles is :
(a) $6\sqrt{39}$ (b) $36\sqrt{13}$ (c) $12\sqrt{39}$
(d) $18\sqrt{13}$ (e) $12\sqrt{26}$.

8. At a certain party, the first time the door bell rang 1 guest arrived. On each succeeding ring two more guests arrived than on the previous ring. After 20 rings the number of guests at the party was :
(a) 39 (b) 58 (c) 210
(d) 361 (e) 400.

9. An operation $*$ is defined such that
$$A * B = A^B - B^A$$
The value of $2 * (-1)$ is :
(a) –3 (b) –1 (c) $-\frac{1}{2}$
(d) 0 (e) $\frac{3}{2}$.

10. Three circles with a common centre P are drawn as shown with $PQ = QR = RS$. The ratio of the area of the region between the inner and middle circles (shaded with squares) to the area of the region between the middle and outer circles (shaded with lines) is :



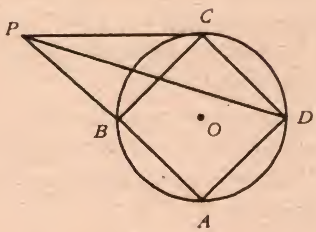
(a) $\frac{1}{3}$ (b) $\frac{4}{9}$ (c) $\frac{1}{2}$
(d) $\frac{3}{5}$ (e) $\frac{2}{3}$.

PART - B

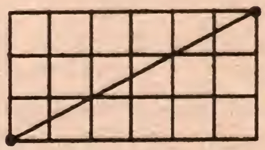
11. (a) How many 3-digit numbers can be formed using only the digits 1, 2 and 3 if both of the following conditions hold :
(i) repetition is allowed;
(ii) no digit can have a larger digit to its left.

(b) Repeat for a 4-digit number using the digits 1, 2, 3 and 4.

12. The square $ABCD$ is inscribed in a circle of radius one unit. ABP is a straight line, PC is tangent to the circle. Find the length of PD. Make sure you explain thoroughly how you got all the things you used to find your solution !



13. If a diagonal is drawn in a 3 × 6 rectangle, it passes through four vertices of smaller squares. How many vertices does the diagonal of a 45 × 30 rectangle pass through ?



14. Let a and b be any real numbers. Then $(a - b)$ is also a real number, and consequently $(a - b)^2 \geq 0$. Expanding gives $a^2 - 2ab + b^2 \geq 0$. If we add $2ab$ to both sides of the inequality, we get $a^2 + b^2 \geq 2ab$. Thus, for any real numbers a and b, we have $a^2 + b^2 \geq 2ab$.

Prove that for any real numbers a, b, c, d:

(i) $2abcd \leq b^2c^2 + a^2d^2$.

(ii) $6abcd \leq a^2b^2 + a^2c^2 + a^2d^2 + b^2c^2 + b^2d^2 + c^2d^2$.

15. A circular coin is placed on a table. Then identical coins are placed around it so that each coin touches the first coin and its other two neighbours.

(a) If the outer coins have the same radius as the inner coin, show that there will be exactly 6 coins around the outside.

(b) If the radius of all 7 coins is 1, find the total area of the spaces between the inner coin and the 6 outer coins.

SOLUTIONS

Part - A

1. (e) : If we square the digits from 0 to 9 and consider the final digit of the square we get only the digits 0, 1, 4, 9, 6 and 5. Since there are no others, we see that 8 is NOT the final digit of any square.

2. (a) : Each of the 10 straight lines intersects each of the others exactly once. This makes for 90 intersections; however, each of the intersections is counted twice in this approach, depending upon which of the two lines we consider first. To get the correct number of intersections we simply divide 90 by 2 to get 45.

3. (e) : Let us try successively to make up each of the given amounts using 6 coins :

$$91 = 1 + 5 + 10 + 25 + 25 + 25,$$
$$87 = 1 + 1 + 10 + 25 + 25 + 25,$$
$$78 = 1 + 1 + 1 + 25 + 25 + 25,$$
$$51 = 1 + 10 + 10 + 10 + 10 + 10.$$

Thus each of the first 4 choices can be made up with 6 coins. In order to make up 49 we would need to use 4 pennies. This would require us to make up the total of 45 cents with only 2 coins, which is clearly impossible.

4. (b) : First observe that

$$(x - y)^2 = x^2 + y^2 - 2xy = 28 - 2(14) = 0.$$

This means that $x - y = 0$; that is, $x = y$. In that event we clearly have $x^2 - y^2 = 0$.

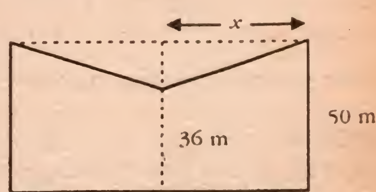
5. (e)

6. (c) : We are seeking the largest integer n such that

$$\frac{n(n+1)}{2} \leq 500 \text{ or } n(n+1) \leq 1000$$

Since $32^2 = 1024$ we see that $n < 32$. Checking $n = 31$ we find $31.32 = 992$. Thus the integer n we seek is 31. The triangular number associated with this value of n is $\frac{1}{2}(992) = 496$.

7. (c) : In order for the rope to be at the lowest possible point, that point must be the middle of the rope. Thus we are faced with solving a right-angled triangle with hypotenuse 40 m and one side of 50 – 36 = 14 m.



By the Theorem of Pythagoras the third side (x in the diagram at the right) is $\sqrt{40^2 - 14^2} = \sqrt{1404} = 6\sqrt{39}$. The distance between the two flagpoles is $2x = 12\sqrt{39}$.

8. (e) : Let a_n be the number of people who arrive at the n^{th} ring of the door bell. Then $a_n = 2n - 1$. Let b_n be the number of people who have arrived after the n^{th} ring of the door bell. Then we have

$$b_1 = 1,$$
$$b_{n+1} = b_n + a_{n+1} \text{ for } n \geq 1,$$
$$= b_n + 2n + 1$$

This can be rewritten as

$$b_1 = 1$$
$$b_{n+1} - b_n = 2n + 1 \text{ for } n \geq 1.$$

If we write out the first 20 of these we get

$$b_1 = 1,$$
$$b_2 - b_1 = 3,$$
$$b_3 - b_2 = 5,$$
$$\vdots$$
$$b_{20} - b_{19} = 39.$$

When we add all 20 of the above equations together we get

$$b_{20} = 1 + 3 + 5 + \cdots + 39 = \frac{1}{2}(20)\cdot(2\cdot 1 + 19\cdot 2)$$
$$= 400,$$

where we have used the well-known formula for the sum of an arithmetic progression with n terms, having first term a and common difference d:

$$\frac{1}{2}n\,(2a + (n-1)\,d).$$

9. (c) : According to the definition of the operation $*$, we have

$$2 * (-1) = 2^{-1} - (-1)^2 = \frac{1}{2} - 1 = -\frac{1}{2}$$

10. (d) : Let a be the length of PQ, QR, and RS. Then the radii of the 3 circles are a, $2a$, and $3a$. The area between the inner and middle circles is then $\pi(2a)^2 - \pi a^2 = 3\pi a^2$, and the area between the middle and outer circles is $\pi(3a)^2 - \pi(2a)^2 = 5\pi a^2$.

Thus the ratio we want is $3\pi a^2 : 5\pi a^2 = \frac{3}{5}$.

Part - B

11. (a) For this part of the question, the simplest method is simply to list all the possible numbers. In increasing order they are :

111, 112, 113, 122, 123, 133, 222, 223, 233 and 333, for a total of 10 numbers.

(b) Again, most junior students will simply try to list all the possible integers. In increasing order they are :

1111, 1112, 1113, 1114, 1122, 1123, 1124, 1133, 1134, 1144, 1222, 1223, 1224, 1233, 1234, 1244, 1333, 1334, 1344, 1444, 2222, 2223, 2224, 2233, 2234, 2244, 2333, 2334, 2344, 2444, 3333, 3334, 3344, 3444 and 4444 for a total of 35 numbers.

A more sophisticated approach (which can be generalized) follows: We first define $n(k, d)$ to be the number of k-digit integers ending with the digit d and satisfying the two conditions (i) and (ii) in the problem statement. Since a k-digit number ending with the digit d consists of appending the digit d to all $(k-1)$ digit numbers ending with a digit less than or equal to d, we have

$n(k, d) = n(k-1,1) + n(k-1,2) + \cdots + n(k-1,d)$ $\quad(*)$

Furthermore, we also have $n(1,d) = 1$ for all digits d and $n(k, 1) = 1$ for all integers k. The relationship $(*)$ allows us to create the following table of values for $n(k, d)$:

k \ d	1	2	3	4
1	1	1	1	1
2	1	2	3	4
3	1	3	6	10
4	1	4	10	20

Each entry in the table is the sum of the entries in the previous row up to and including the column containing the given entry (note the presence of Pascal's Triangle in the table). From that table, the answers to parts (a) and (b) are :

(a) : $n(3,1) + n(3,2) + n(3,3) = 1 + 3 + 6 = 10$,

(b) : $n(4,1) + n(4,2) + n(4,3) + n(4,4) = 1 + 4 + 10 + 20 = 35$.

Clearly this table could have been extended to deal with any number k and with any digit $d \leq 9$.

12. Since $ABCD$ is a square, the lines AC and BD are perpendicular. Since the circle had radius 1 unit, the Theorem of Pythagoras tells us that $AB = BC = CD = DA = \sqrt{2}$. The tangent PC at C is perpendicular to the diameter AC; thus $\angle PCB = 45°$. Since $PA \perp BC$ we also have $\angle CPB = 45°$. This makes ΔPBC isosceles, which means that $PB = BC = \sqrt{2}$. Applying the Theorem of Pythagoras to ΔAPD we have

$$PD^2 = AP^2 + AD^2 = AP^2 + AD^2$$
$$= (2\sqrt{2})^2 + \sqrt{2}^2 = 8 + 2 = 10,$$

from which we see that $PD = \sqrt{10}$.

13. Since the 45×30 rectangle has its sides in the proportion 3 : 2, we will consider first looking at a 3×2 rectangle, in which there are 2 vertices which lie on the diagonal. In the original 45×30 rectangle we need only consider the fifteen 3×2 rectangle which straddle the diagonal in question. The lower left of these has its lower leftmost vertex on the diagonal, and each of these 3×2 rectangles adds a further vertex to the count for its upper rightmost corner. This gives us a total of $1 + 15 = 16$ vertices on the diagonal.

14. (a) We will use the proof in the problem statement as a model. Consider $bc - ad$. Clearly $(bc - ad)^2 \geq 0$. Expanding gives

$$b^2c^2 - 2abcd + a^2d^2 \geq 0.$$

This is easily rearranged to yield $b^2c^2 + a^2d^2 \geq 2abcd$.

(b) We will use part (a) to prove part (b). Since a, b, c, and d are arbitrary real numbers, the inequality in part (a) remains true for any rearrangement of the letters; in particular we have :

$$2abcd \leq b^2c^2 + a^2d^2$$
$$2abdc \leq b^2d^2 + a^2c^2$$
$$2adcb \leq d^2c^2 + a^2b^2.$$

Recognizing that multiplication is commutative for real numbers, we can reorganize the products in each of the above inequalities and sum the three inequalities to get the desired result.

15. (a) Let us place 2 (outer) coins next to the original coin so that they touch each other. Then the centres of the 3 coins form an equilateral triangle with side length equal to twice the radius of a single coin. Therefore the angle between the centres of the 2 (outer) coins measured at the centre of the first coin is 60°. Since 6 such angles make up a full revolution around the inner coin, we can have exactly 6 outer coins each touching the original (inner) coin and also touching its other two neighbours.

(b) There are 6 non-overlapping spaces whose areas we must add; each is found between 3 coins which simultaneously touch other, and whose centres form the equilateral triangle mentioned in part (a) above. This equilateral triangle has side length 2, since we are given the radii of the coins as 1. Our strategy to compute the area of one such space is to find the area of the equilateral triangle and subtract the areas of the 3 circular sectors found within the triangle. The altitude of the equilateral triangle with side length 2 can be easily found (Theorem of Pythagoras) as $\sqrt{3}$. Thus the area of the triangle itself is $\frac{1}{2}\cdot 2\cdot\sqrt{3} = \sqrt{3}$. The area of a single coin is $\pi \cdot 1^2 = \pi$. The circular sectors within the equilateral triangle are each one-sixth of the area of the coin; there are 3 such sectors which gives us a total area of one-half the area of a single coin to be subtracted from the area of the equilateral triangle. Thus the area of a single space is $\sqrt{3} - (\pi/2)$. Since there are 6 such spaces, we have a total area of $6\sqrt{3} - 3\pi$ square units.

■

Contd. from page 51

49. (a) : The composition table is as given below

$\times_{10}$	2	4	6	8
2	4	8	2	6
4	8	6	4	2
6	2	4	6	8
8	6	2	8	4

Since the third row coincides with the top most row, 3rd column coincides with the left most column and their intersection point is 6. Hence 6 is the identify element.

50. (c) : The composition table is as given below :

•	1	2	3	4	5	6
1	1	2	3	4	5	6
2	2	4	6	1	3	5
3	3	6	2	5	1	4
4	4	1	5	2	6	3
5	5	3	1	6	4	2
6	6	5	4	3	2	1

The identity element is 1. In the third row the identity element 1 is obtained from $3 \bullet 5$. Hence the inverse of 3 is 5.

51. (c) : Let e be the identity element. Then

$a \bullet e = a$ for all $a \in G$

$\Rightarrow a + e + 1 = a$ for all $a \in G \quad \Rightarrow e = -1$

Let x be the inverse of a. Then $a \bullet x = e$

$\Rightarrow a + x + 1 = -1 \quad \Rightarrow x = -2 - a.$

52. (c) : Trivial.

53. (c) : $(3 + 5^{-1})^{-1} = (3 + 1)^{-1} = 4^{-1} = 2$ $(\because 4 +_6 2 = 0)$

54. (a) : $a \bullet b = b \bullet a \Rightarrow (a \bullet b)^{-1} = (b \bullet a)^{-1}$

$\Rightarrow b^{-1} \bullet a^{-1} = a^{-1} \bullet b^{-1}.$

55. (d) : $\operatorname{cosec}\theta + 2 = 0 \Rightarrow \sin\theta = -\frac{1}{2} \Rightarrow \theta = 210°$ or $330°$

56. (c) : $\sin A = \sin B$, $\cos A = \cos B \quad \Rightarrow A = 2n\pi + B$, clearly this satisfies both the relations for all $n \in Z$.

57. (b) : $(1 + \tan\theta)(1 + \tan\phi) = 2$

$\Rightarrow \quad 1 + \tan\theta + \tan\phi + \tan\theta + \tan\phi = 2$

$\Rightarrow \tan\theta + \tan\phi = 1 - \tan\theta\tan\phi \quad \Rightarrow \dfrac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} = 1$

$\Rightarrow \tan(\theta + \phi) = 1 \quad \Rightarrow \theta + \phi = \dfrac{\pi}{4}$

58. (c) : $\tan^{-1}\dfrac{x}{y} - \tan^{-1}\left(\dfrac{x-y}{x+y}\right) = \tan^{-1}\dfrac{x}{y} - \tan^{-1}\left(\dfrac{1-y/x}{1+y/x}\right)$

$= \tan^{-1}\dfrac{x}{y} - \left(\tan^{-1}1 - \tan^{-1}\dfrac{y}{x}\right) = \tan^{-1}\dfrac{x}{y} + \tan^{-1}\dfrac{y}{x} - \dfrac{\pi}{4}$

$= \tan^{-1}\dfrac{x}{y} + \cot^{-1}\dfrac{x}{y} - \dfrac{\pi}{4} = \dfrac{\pi}{2} - \dfrac{\pi}{4} = \dfrac{\pi}{4}$

59. (d) : $\cot\left[\cos^{-1}\left(\dfrac{7}{25}\right)\right] = \cot\left[\cot^{-1}\left(\dfrac{7}{24}\right)\right] = \dfrac{7}{24}$

60. (a) : $4\sin^{-1}x + \cos^{-1}x = \pi$

$\Rightarrow 4\sin^{-1}x + \dfrac{\pi}{2} - \sin^{-1}x = \pi$

$\Rightarrow 3\sin^{-1}x = \dfrac{\pi}{2} \qquad \Rightarrow x = \sin\left(\dfrac{\pi}{6}\right) = \dfrac{1}{2}$

■

5 Challenging problems With solutions

PROBABILITY

1. Players A and B match pennies N times. They keep a tally of their gains and losses. After the first toss, what is the chance that at no time during the game will they be even?

Soln.: The probability of not getting a tie is (for N odd and N even)

$$P(\text{no tie}) = \frac{\binom{N-1}{n}}{2^{N-1}},\ N = 2n+1.$$

$$P(\text{no tie}) = \binom{N}{n}\Big/2^{N},\ N = 2n.$$

The formulas show that the probability is the same for an even N and for the following odd number $N+1$. For example, when $N = 4$, the second formula applies. The 16 possible outcomes are

• $A\ A\ A\ A$	$B\ A\ A\ A$	$A\ B\ B\ A$	$B\ A\ B\ B$
• $A\ A\ A\ B$	$A\ A\ B\ B$	$B\ A\ B\ A$	• $B\ B\ A\ B$
• $A\ A\ B\ A$	$A\ B\ A\ B$	$B\ B\ A\ A$	• $B\ B\ B\ A$
$A\ B\ A\ A$	$B\ A\ A\ B$	$A\ B\ B\ B$	• $B\ B\ B\ B$

where the star indicates that no tie occurs. Since the number of combinations of 4 things taken 2 at a time is 6, the formula checks.

For $N = 2n$, the probability of x wins for A is $\dfrac{\binom{N}{x}}{2^N}$.

If $x \le n$, the probability of a tie is $2x/N$, based on the ballot box result, and for $x \ge n$ it is $2(N-x)/N$. To get the unconditional probability of a tie, we weight the probability of the outcome x by the probability of a tie with x wins and sum to get

$$\text{(i)}\ 2(2^{-N})\left[\frac{0}{N}\binom{N}{0}+\frac{1}{N}\binom{N}{1}+\ldots+\frac{n-1}{N}\binom{N}{n-1}\right.$$
$$\left.\frac{n}{N}\binom{N}{n}+\frac{n-1}{N}\binom{N}{n+1}+\ldots+\frac{1}{N}\binom{N}{N-1}+\frac{0}{N}\binom{N}{N}\right]$$

When the binomial coefficients are converted to factorials and their coefficients canceled, we find that, except for a missing term which is $\dfrac{(N-1)!}{n!(n-1)!} = \binom{N-1}{n}$, the sum in brackets would be $\Sigma\binom{N-1}{x}$ over the possible values of x. Consequently, we can rewrite expression (i) as

$$\text{(ii)}\ 2^{-N+1}\left[2^{N-1}-\binom{N-1}{n}\right] = \frac{1-\binom{N-1}{n}}{2^{N-1}}$$

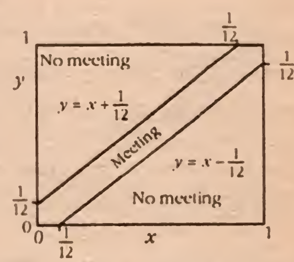
The complement of expression (ii) gives at last the probability of no tie $\binom{N-1}{n}\Big/2^{N-1}$ which a little algebra shows can be written $\binom{N}{n}\Big/2^{N}$ as suggested earlier.

2. Akash gets off work at random times between 3 and 5 P.M. His mother lives uptown, his girl friend downtown. He takes the first subway that comes in either direction and eats dinner with the one he is first delivered to. His mother complains that he never comes to see her, but she says she has a 50-50 chance. He has had dinner with her twice in the last 20 working days. Explain.

Soln.: Downtown trains run past Akash's stop at, say, 3:00, 3:10, 3:20, ..., etc., and uptown trains at 3:01, 3:11, 3:21, ... To go uptown Akash must arrive in the 1-minute interval between a downtown and an uptown train.

3. Duels in the town of Discretion are rarely fatal. There, each contestant comes at a random moment between 5 A.M. and 6 A.M. on the appointed day and leaves exactly 5 minutes later, honour served, unless his opponent arrives within the time interval and then they fight. What fraction of duels lead to violence.

Soln.: Let x and y be the times of arrivals measured in parts of an hour from 5 A.M. The shaded region of the figure shows the arrival times for which the duelists meet.



The probability that they do not meet is $(11/12)^2$ and so the fraction of duels in which they meet is $23/144 \approx 1/6$.

4. (a) The king's minter boxes his coins 100 to a box. In each box he puts 1 false coin. The king suspects the minter and from each of 100 boxes draws a random coin and has it tested. What is the chance the minter's peculations go undetected? **(b)** What if both 100's are replaced by n?

Soln.: **(a)** P (0 false coins) = $\left(1-\frac{1}{100}\right)^{100} \approx 0.366$

(b) Let there be n boxes and n coins per box. For any box the chance that the coin drawn is good is $1 - (1/n)$, and since there are n boxes, P (0 false coins) = $[1 - (1/n)]^n$.

Let us look at this probability for a few values of n.

n	**P (0 false coins)**
1	0
2	0.250
3	0.296
4	0.316
5	0.328
10	0.349
20	0.358
100	0.366
1000	0.3677
∞	0.367879... = $1/e$.

Two things stand out. First, the tabled numbers increase; and second, they may be approaching some number. The number they are approaching is well known, and it is e^{-1} or $1/e$, where e is the base of the natural logarithms, 2.71828...

If we expand $\left(1-\frac{1}{n}\right)^n$ in powers of $1/n$, we get

$$1^n - \binom{n}{1}1^{n-1}\left(\frac{1}{n}\right) + \binom{n}{2}1^{n-2}\left(\frac{1}{n}\right)^2 - \binom{n}{3}1^{n-3}\left(\frac{1}{n}\right)^3 + \ldots$$

or, $$1-\frac{n}{n}+\frac{n(n-1)}{2!n^2}-\frac{n(n-1)(n-2)}{3!n^3}+\ldots \qquad \ldots \text{(i)}$$

If we take one of these terms, say the fourth, and study its behaviour as n becomes very large, we find that it approaches $-1/3!$ because

$$\frac{n(n-1)(n-2)}{n^3} = 1\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right) = 1-\frac{3}{n}+\frac{2}{n^2} \qquad \ldots \text{(ii)}$$

As n grows large, all terms on the right-hand side of equation (ii) except 1 tend to zero. Similarly, for r^{th} term of expansion (i) the factors depending on n tend to 1, and the term itself tends except for sign to $1/(r-1)!$. Therefore, as n grows, the series for $\left(1-\frac{1}{n}\right)^n$ tends to

$$1-1+\frac{1}{2!}-\frac{1}{3!}+\frac{1}{4!}-\frac{1}{5!}+\ldots$$

This series is one way of writing e^{-1}.

Had we investigated the case of 2 false coins in every box, we would have found that $\left(1-\frac{2}{n}\right)^n$ tends to e^{-2} as n grows large, and in general that $\left(1-\frac{m}{n}\right)^n$ tends to e^{-m}. Also $\left(1+\frac{m}{n}\right)^n$ tends to e^m whether m is an integer or not. These facts are important for us.

5. The king's minter boxes his coins n to a box. Each box contains m false coins. The king suspects the minter and randomly draws 1 coin from each of n boxes and has these tested. What is the chance that the sample of n coins contains exactly r false coins?

Soln.: Each of the coins in the king's sample is drawn from a new box and has probability m/n of being counterfeit. The drawings are independent, and so we get the binomial probability for r false (and $n - r$ true) to be

$$P\,(r \text{ false coins}) = \binom{n}{r}\left(\frac{m}{n}\right)^r\left(1-\frac{m}{n}\right)^{n-r}$$

Let us see what happens when n grows large while r and m are fixed. We write P (r false coins) as

$$\frac{1}{r!}\frac{n(n-1)\ldots(n-r+1)}{n^r}\cdot m^r\left(1-\frac{m}{n}\right)^n\left(1-\frac{m}{n}\right)^{-r}$$

As n grows $1/r!$ is unchanged, m^r is unchanged, $n(n-1)\ldots(n-r+1)/n^r$ tends to 1, $\left(1-\frac{m}{n}\right)^n$ tends to e^{-m}, as explained in problem 4 and $\left(1-\frac{m}{n}\right)^{-r}$ tends to 1 (again because m and r are fixed). Therefore for large n

$$P\,(r \text{ false coins}) \approx \frac{e^{-m}m^r}{r!}$$

These terms add up to 1, that is

$$e^{-m}\left(\frac{1}{0!}+\frac{m}{1!}+\frac{m^2}{2!}+\frac{m^3}{3!}+\ldots\right) = e^{-m}e^m = 1.$$

The series in parentheses is an expansion of e^m.

Poisson distribution

The distribution whose probabilities are

$$P(r) = \frac{e^{-m}m^r}{r!}, \quad r = 0, 1, 2, \ldots$$

is called the Poisson distribution, and it approximately represents the probabilistic behaviour of many physical processes.

■

PROBLEMS Of the MONTH

Two problems to test your nerves and your preparation for IIT-JEE and other Engineering Entrance Exams

1. Find all pairs of consecutive integers the difference of whose cubes is a full square.

Soln.: Let a, b be two integers.

$(a + 1)^3 - a^3 = b^2 \Rightarrow 3a^2 + 3a + 1 = b^2$ ... (i)

$\Rightarrow 3(4a^2 + 4a + 1) + 1 = 4b^2$

$\Rightarrow (2b)^2 - 3(2a + 1)^2 = 1$

$\Rightarrow (2a, 2a + 1)$ is a solution of Pell's equation

$X^2 - 3Y^2 = 1$ ... (ii)

The minimal non-trivial solution of (ii) is (2, 1). It is then well known that the solutions of (ii) are the pairs $(\pm x_n, \pm y_n)$ where $x_0 = 1$, $y_0 = 0$ and for all $n \geq 0$.

$$\begin{cases} x_{n+1} = 2x_n + 3y_n \\ y_{n+1} = 2y_n + x_n \end{cases}$$

But we want only those with x_n even and y_n odd. It is easy to see that if x_n is even and y_n is odd then x_{n+1} is odd and y_{n+1} is even, and then x_{n+2} is even and y_{n+2} is odd.

Thus, since $x_1 = 2$ and $y_1 = 1$, we consider only the pairs (x_{2n+1}, y_{2n+1}). Since, for all $n \geq 0$.

$$\begin{cases} x_{n+2} = 7x_n + 12y_n \\ y_{n+2} = 4x_n + 7y_n \end{cases}$$

Then, the solutions of (i) are the pairs (a, b) of the form $\left(\frac{-1 \pm V_n}{2}, \pm\frac{U_n}{2}\right)$, where $U_1 = 2$, $V_1 = 1$.

and for all $n \geq 1$, $\begin{cases} U_{n+1} = 7U_n + 12V_n \\ V_{n+1} = 4U_n + 7V_n \end{cases}$

For example, we first note that $(a + 1)^3 - a^3 = b^2$ if and only if $(-a)^3 - (-a - 1)^3 = b^2$. Then we give only the first positive values of a and b.

n	U_n	V_n	a	b
1	2	1	0	1
2	26	16	7	13
3	362	209	104	181
4	5042	2911	1455	2521
5	70226	40545	20272	35113
⋮				

2. Let $A_1, A_2, ..., A_n$ be a regular n-gon inscribed in the circle of radius 1 with the centre at O. A point M is given on the ray OA_1 outside the n-gon.

Prove that $\sum_{k=1}^{n} \frac{1}{|MA_k|} \geq \frac{n}{|OM|}$.

Soln.: *Ist method :* We may suppose that a system of co-ordinates has been chosen so that the complex numbers associated to $O, A_1, A_2, ..., A_n, M$ are respectively $0, 1, u, ..., u^{n-1}, r$ where $u = \exp(2\pi i/n)$ and r is a real number > 1.

Note that $1, u, ..., u^{n-1}$ are the n^{th} roots of unity. Hence we have the identity

$z^n - 1 = (z - 1)(z - u) ... (z - u^{n-1})$ ... (i)

Now, $\frac{1}{n}\sum_{k=1}^{n}\frac{1}{|MA_k|} = \frac{1}{n}\sum_{k=0}^{n-1}\frac{1}{|r - u^k|}$

$\geq \sqrt[n]{\frac{1}{|r-1|}\cdot\frac{1}{|r-u|}\cdots\frac{1}{|r-u^{n-1}|}}$ (by AM - GM)

$= \sqrt[n]{\frac{1}{|r^n - 1|}}$ (using (i)) $= \sqrt[n]{\frac{1}{r^n - 1}} > \sqrt[n]{\frac{1}{r^n}} = \frac{1}{r} = \frac{1}{|OM|}$

Note that the proof gives a strict inequality.

Second method : By applying the AM-GM inequality it suffices to show that $|OM|^n \geq \prod_{k=1}^{n} |MA_k|$.

This will follow from de Moivre's property that is "if $A_0 A_1 A_2 ... A_{n-1}$ is a regular polygon inscribed in a circle centre O, radius a and P is a point such that $OP = x$, $\angle(OA_0, OP) = \theta$, then

$\angle(OA_r, OP) = \theta + (2r\pi/n)$

and $PA_r^2 = x^2 + a^2 - 2xa\cos[\theta + (2r\pi/n)]$

Also, $PA_0^2 \cdot PA_1^2 .... PA_{n-1}^2 = \prod_{r=0}^{n-1}\left(x^2 - 2xa\cos\left(\theta + \frac{2r\pi}{n}\right) + a^2\right)$

or, $PA_0 \cdot PA_1 .... PA_{n-1} = \sqrt{x^{2n} - 2x^n a^n \cos n\theta + a^{2n}}$

If P lies on OA_0 so that $\theta = 0$, then $PA_0, PA_1, ... PA_{n-1} = |x^n - a^n|$. If OP bisects $\angle A_{n-1}OA_0$, so that $\theta = \pi/n$, then $PA_0 \cdot PA_1 ... PA_{n-1} = x^n + a^n$. These special results are called *Cotes properties* ".

Applying this to (i), we get $|OM|^n \geq |OM|^n -$ from which the result is now obvious.

time, better used in solving other questions. It would puncture the confidence leading to a poor performance.

Say your IIT-JEE has 10 questions/120 minutes. It means democratically speaking, each question can get 12 minutes. But understand that when a question paper of IIT-JEE is made, it is assumed that the examinee has 2 hours of time for thinking as well as writing. If you knew all the questions before hand, would you take 2 hours in writing the paper. Certainly not. May be just 1 hour or so. The rest of the time is the assumed time for thinking.

Now, if you send in the best question to bat, they are as good as questions known before hand. So they certainly would not eat that much time. (12 minutes). They would give more marks in lesser time. That is to say if your choice of the batting order is correct, you would be in 108 for 1 kind of score. You would have scored about 40 marks in the first hour of your writing the paper.

In the IIT mains, you can get through, if you score 40 marks. Score 10 more marks, you would be within 1500 ranks. Add 10 more marks to your score, you quantum leap to the first 500 ranks. Scratch in 10 more, you are in the *la creme da creme* of top 100.

If your opening questions score 40 marks in the first hour, you are already an IITian. The next hour of the exam is being given not by you but you the IITian.

Looking at the above statistics, we can conclude that if your opening questions score 40 marks in the first hour, you are already an IITian. The next hour of the exam is being given not by you but you the IITian.

Now, when an IITian solves the question paper, it would have a class of its own. You are now playing in the positive spiral. The previous batsmen/questions inspire a superlative performance from the subsequent batsmen/questions.

Match temperament

On the day of the exam, you can maximize the mobilisation of your potential. Realise, you don't get marks for what you know, you get marks for what you write.

Irrespective of the score of your past test matches, what will count is the score you make in this one. This is your world cup finals.

The ball that you will face now, has never been bowled before, it has never been faced before by anyone else. It is a fresh ball. Likewise the question that you will face is a question that you have never faced before and neither has anybody else. So what matters, is how well you play the ball that comes to you now.

You are able and capable. You have a solid preparation. Bring in the freshness of your attention on the question that you face.

When waters of the lake are calm, the moon in the sky will be reflected in the lake, in all its pristine glory. Likewise the solutions to the questions already exist within you. They will reflect in the cool and calm mind of yours as the test progresses. Be sure the answers will flow through you. The runs will flow from your bat. Just allow them to do so.

Batting order

It is the day of the match. You are sitting in the examination hall. The audience is expectant. Your parents, well-wishers, friends expect a great performance from you. You take in their encouragement and support but you are not pressured by them. You watch the bowlers and the kinds of balls they would bowl. Based on this you set down your batting order.

Scan the question paper completely, marking against each question, the code of the topic to which it belongs. Categorise them as 1, 2, 3 or 4 as appropriate.

Again scan the questions and discern the kind of ball it is. Mark it as *A*, *B*, *C*, *D* as per your analysis on the left of your numeric codes.

Thus your entire question paper is now divided according to a new sequence, the alphanumeric sequence.

e.g. *A*1, *A*2, *A*3, *A*4, *B*1....*C*1, *C*2 .... *D*3, *D*4. This is your batting order. Your batsmen shall play in the order, you decide. Attempt the questions in this order.

Thus you can maximize your potential on the examination day. Read this letter, atleast 2 times to internalize the idea. Be a cool captain, strategise the way you have been coached, win your match.

All the best.

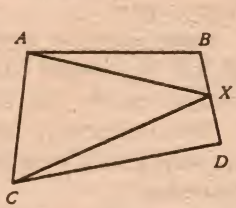
Yours affectionately,
Shyam Sunder

10 Mathematical Challenges

1. Mr. Smith commutes to the city regularly and invariably takes the same train home which arrives at his home station at 5 PM. At this time, his chauffeur always just arrives, promptly picks him up, and drives him home. One fine day, Mr. Smith takes an earlier train and arrives at his home station at 4 PM. Instead of calling or waiting for his chauffeur until 5 PM, he starts walking home. On his way he meets the chauffeur who picks him up promptly and returns home arriving 20 minutes earlier than usual. Some weeks later, on another fine day, Mr. Smith takes an earlier train and arrives at his home station at 4.30 PM. Again instead of waiting for his chauffeur, he starts walking home. On his way he meets the chauffer who picks him up promptly and returns home. How many minutes earlier than usual did he arrive home this time?

2. Given a $(2m + 1) \times (2n + 1)$ checkboard in which the four corners are black squares, show that if one removes any one red square and any two black squares, the remaining board is coverable with dominoes (*i.e.* 1×2 rectangles).

3. Let X be any point between B and C on the side BC of the convex quadrilateral $ABCD$ (as in figure). A line is drawn through B parallel to AX and another line is drawn through C parallel to DX. These two lines intersect at P. Prove that the area of the triangle APD is equal to the area of the qudrilateral $ABCD$.



4. Find all number triplets (x, y, z) such that when any one of these numbers is added to the product of the other two, the result is 2.

5. In a certain town, the blocks are rectangular, with the streets (of zero width) running E - W, the avenues N - S. A man wishes to go from one corner to another m blocks east and n blocks north. The shortest path can be achieved in many ways. How many?

6. $2n + 3$ points $(n \geq 1)$ are given in the plane, no three on a line and no four on a circle. Prove that there exists a circle through three of them such that, of the remaining $2n$ points, n are in the interior and n are in the exterior of the circle.

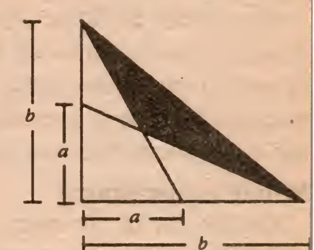
7. If A denotes the number of integers whose logarithms (to base 10) have the characteristic a, and B denotes the number of integers the logarithms of whose reciprocals have characteristic $-b$, determine $(\log A - a) - (\log B - b)$. (The characteristic of $\log x$ is the integer $[\log x]$).

8. Is it possible to color the points (x, y) in the Cartesian plane for which x and y are integers with three colors in such a way that .

(a) each color occurs infinitely often in infinitely many lines parallel to the x-axis, and

(b) no three points, one of each color, are collinear?

9. Find an expression in terms of a and b for the area of the hatched region in the right triangle in the figure.

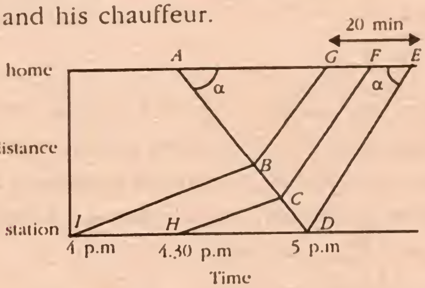


10. Two players play the following game. The first player selectes any integer from 1 to 11 inclusive. The second player adds any positive integer from 1 to 11 inclusive to the number selected by the first player. They continue in this manner alternately. The player who reaches 56 wins the game. Which player has the advantage?

SOLUTIONS

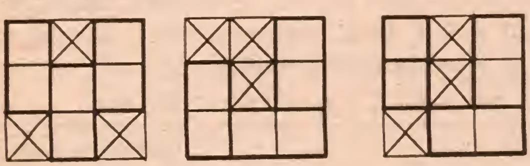
1. *First solution :* In the trip in which Smith got home 20 minutes earlier than usual, the chauffeur saved 10 minutes on each leg of this trip, and consequently picked up Smith at 4.50 P.M. (This part is a well-known problem in which we are assuming that Smith and the chauffeur walk and ride, respectively, at constant rates). Also, it follows now that the chauffeur's speed is five times that of Smith. Suppose Smith meets the chauffeur the second time t minutes after 4.30 P.M. The chauffeur is saved a journey of $t/5$ minutes from the meeting place to the station (which he normally reaches at 5.00 P.M. Hence $t + t/5 = 30$, so $t = 25$ and Smith arrives at home 10 minutes earlier than usual.

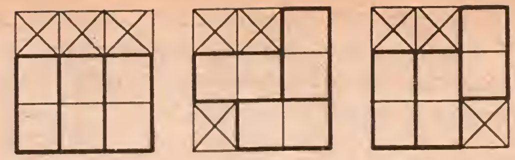
Second method : We plot the distance vs time for Smith and his chauffeur.



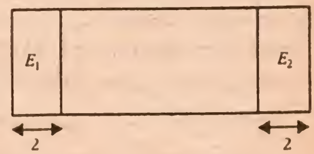
The chauffeur's world line is ADE, where A and E are not known but $\angle DAE = \angle DEA$. Smith's world line after arriving at 4 PM is IB, BG where $BG \| DE$. Then $\overline{GE}$ = 20 minutes. Smith's world line after arriving at 4.30 P.M. is HC, CF where $HC \| IB$ and $CF \| DE$. It then follows that $\overline{BC} = \overline{CD}$ and $\overline{GF} = \overline{FE}$. Thus $\overline{FE} = 10$ minutes.

2. We shall refer to such $(2m + 1) \times (2n + 1)$ checkboard with one red square and two black squares removed as a deleted checkboard. First, we note that the case $m = n = 1$ is easily handled by exhaustion. Owing to the symmetry there are only six cases that need to be considered, and these are shown below.





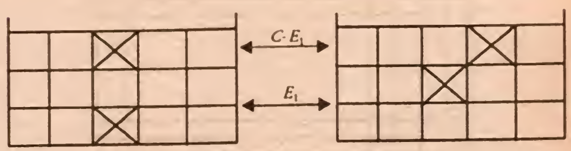
We now proceed by induction. We are given a $(2m + 1) \times (2n + 1)$ deleted checkboard C and we may assume that any smaller $(2k + 1) \times (2l + 1)$ deleted checkboard which is contained in C may be covered with dominoes. Since at least one of the two dimensions of C is the length at least five, C has two oppositely placed, non-overlapping ends E_1 and E_2 of width two.



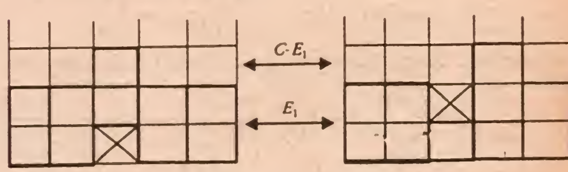
Clearly, we can choose an end containing at most one of the deleted squares of C. Let this end be E_1 and consider the following two cases.

Case I : E_1 contains no deleted square of C. Then $C - E_1$ contains all three of the deleted squares. By the induction assumption, $C - E_1$ can be covered with dominoes. This covering, together with the obvious one for E_1, yields the desired covering of C.

Case II : E_1 contains exactly one deleted square of C. In this case, with the deleted square in E_1, we identify the associated square of the same color in $C - E_1$ as shown in the following figure.



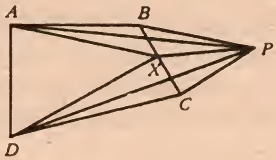
Now delete the associated square in $C - E_1$. By the induction assumption, there is a domino covering of $C - E_1$ with this deletion. Now C with its original deletions, may be covered by making use of the covering just found, together with the scheme shown in the following figure.



This procedure would fail only in the case where the only choice for the associated square in $C - E_1$

was also deleted. This is impossible in the case of a red sauare. In the case of a black square, we infer that the one deleted red square is in E_2 and proceed as before.

3. Because AX is parallel to BP, triangles AXP and AXB have equal areas. Similarly triangles DXP and DXC have equal areas.



Hence area APD = area AXD + area DXP + area AXP
= area AXD + area DXC + area AXB
= area $ABCD$.

4. The system to be solved is :

$$x + yz = 2 \ ; \ y + zx = 2 \ ; \ z + xy = 2.$$

Subtracting the second (third) from the first (second) yields

$$(x - y)(1 - z) = 0 \ ; \ (y - z)(1 - x) = 0$$

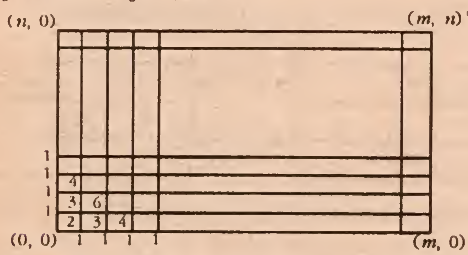
Each of the four cases

$$x - y = 0 = y - z, \ x - y = 0 = 1 - x$$
$$1 - z = 0 = y - z, \ 1 - z = 0 = 1 - x$$

implies $x = y = z = 1$ or $x = y = z = -2$.

5. *First method*: If (m, n) denote the number of paths from (0, 0) to (m, n), it follows that

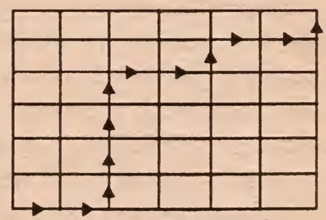
$$f(m, n) = f(m-1, n) + f(m, n-1),$$
$$m \geq 1, \ n \geq 1, \qquad \ldots\ (\text{*})$$
$$f(m, 0) = f(0, n) = 1.$$



Using this recurrence, we can compute $f(m, n)$ for small values of m, n. These are indicated in the diagram. Along the diagonal joining (1, 0) to (0, 1) the numbers are $1 = \binom{1}{1}$, $1 = \binom{1}{0}$.

Along the diagonal joining (2, 0) to (0, 2) they are $1 = \binom{2}{2}$, $2 = \binom{2}{1}$, $1 = \binom{2}{0}$. Along the next diagonal they are : $1 = \binom{3}{3}$, $3 = \binom{3}{2}$, $1 = \binom{3}{0}$. These suggests that $f(m, n) = \binom{m+n}{n}$ and it is easy to verify that $\binom{m+n}{n}$ does indeed satisfy the system (*).

Second method : Every path from (0, 0) to (m, n) is described by a sequence of m symbols R and n symbols U arranged in a straight line



RRUUUURRURRU

where the R corresponding to a step right and the U to a step up. For example, the path in the figure corresponds to RRUUUURRURRU.

The number of sequences (of m R's and n U's) is $\binom{m+n}{n}$.

6. By considering the convex hull of all the points it follows that we can always choose two of the points, say A and B, so that the remaining $2n + 1$ points lie on the same side of line AB. Label these $P_1, P_2, \ldots, P_{2n+1}$ in such a way that

$$\angle AP_1B \leq \angle AP_2B \leq \ldots \leq \angle AP_{2n+1}B.$$

In fact, these angles are all different; for if say $\angle AP_iB = \angle AP_jB$, then the four points A, B, P_i, P_j would lie on a circle. Hence, the circle through AB and P_{n+1} has $P_1, P_2, \ldots, P_n$ in its exterior and $P_{n+2}, P_{n+3}, \ldots P_{2n+1}$ in its interior.

7. Since $A = 10^{a+1} - 10^a = 9 \cdot 10^a$
and $B = 10^b - 10^{b-1} = 9 \cdot 10^{b-1}$,

$$\log A - \log B = a - (b - 1).$$

Hence $(\log A - a) - \log(B - b) = 1$.

8. Such a coloring is possible. Paint (x, y) red if $x + y$ is even, white if x is odd and y is even, and blue if x is even and y is odd. Clearly, condition (a) is satisfied. Now suppose (x_1, y_1) is red, (x_2, y_2) is white and (x_3, y_3) is blue. Then $x_2 - x_1$ and $y_2 - y_1$ have opposite parity ; $x_3 - x_2$ and $y_3 - y_2$ are both odd. Hence

$$(y_2 - y_1)(x_3 - x_2) \neq (y_3 - y_2)(x_2 - x_1)$$

Contd. on page 68

International Math Olympiad

PROBLEMS & SOLUTIONS

1. Let a, b, c be the sides and A, B, C the angles of a triangle. Prove that for any $k \le 1$,

$$\sum\frac{a^k}{A} \ge \frac{3}{\pi}\sum a^k, \text{ where the sums are cyclic.}$$

Soln.: Let $f(x) = \dfrac{\sin^k x}{x}$ for $0 < x < \pi$.

Then, for $0 < k \le 1$, we have

$$f'(x) = \frac{(kx\cos x - \sin x)\sin^{k-1} x}{x^2}$$

For $x \ge \pi/2$, we have $\cos x \le 0$,
so that $kx\cos x - \sin x \le 0$
For $x < \pi/2$, we have $\cos x > 0$ and
$\tan x > x \ge kx$, so that $kx\cos x - \sin x \le 0$.

Therefore $f'(x) = \dfrac{(kx\cos x - \sin x)\sin^{k-1} x}{x^2} \le 0.$

Without loss of generality, we may assume that $A \le B \le C$. For $0 < k \le 1$, we have that $f(x)$ is a non-increasing function, so that $f(A) \ge f(B) \ge f(C)$.

Thus, by Tchebyshev's inequality, we have

$$\left(\frac{\sin^k A}{A} + \frac{\sin^k B}{B} + \frac{\sin^k C}{C}\right)(A+B+C) \ge 3(\sin^k A + \sin^k B + \sin^k C)$$

By the sine rule, we have $a = 2R\sin A$, $b = 2R\sin B$ and $c = 2R\sin C$, where R is the circumradius of ΔABC. Multiply the inequality by $(2R)^k$ and substitute $A + B + C = \pi$ to get

$$\sum\frac{a^k}{A} \ge \frac{3}{\pi}\sum a^k.$$

For $k \le 0$, we have

$$a \le b \le c \Rightarrow a^k \ge b^k \ge c^k \Rightarrow \frac{a^k}{A} \ge \frac{b^k}{B} \ge \frac{c^k}{C}.$$

Thus, by using Tchebyshev's inequality again, (1) holds for $k \le 0$.
In conclusion, (1) holds for any $k \le 1$.

2. ABC is a triangle with incentre I. Let P and Q be the feet of the perpendiculars from A to BI and CI respectively. Prove that

$$\frac{AP}{BI} + \frac{AQ}{CI} = \cot\frac{A}{2}.$$

Soln.: In ΔAPB, we have $\sin\left(\dfrac{B}{2}\right) = \dfrac{AP}{AB}$,

In ΔAQC, we have $\sin\left(\dfrac{C}{2}\right) = \dfrac{AQ}{AC}$,

In ΔABI, we have $\dfrac{BI}{\sin(A/2)} = \dfrac{AB}{\cos(C/2)}$

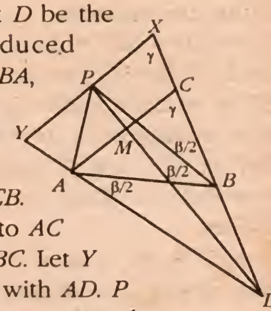
In ΔACI, we have $\dfrac{CI}{\sin(A/2)} = \dfrac{AC}{\cos(B/2)}$

so that

$$\frac{AP}{BI} + \frac{AQ}{CI} = \frac{\sin(B/2)\cos(C/2)}{\sin(A/2)} + \frac{\sin(C/2)\cos(B/2)}{\sin(A/2)}$$

$$= \frac{\sin(B/2 + C/2)}{\sin(A/2)} = \frac{\cos(A/2)}{\sin(A/2)} = \cot(A/2).$$

3. ABC is a triangle. Let D be the point on side BC produced beyond B such that $BD = BA$, and let M be the mid-point of AC. The bisector of $\angle ABC$ meets DM at P. Prove that $\angle BAP = \angle ACB$.



Soln.: Let PX be parallel to AC with X lying on the line BC. Let Y be the intersection of PX with AD. P is the midpoint of XY because M is the mid-point of AC. Then B is the mid-point of DX [PB is parallel to AD since $2\angle DAB = \angle DAB + \angle BDA = \angle ABC = 2\angle PBA$].

Hence $BX = BD = AB$. Triangle BPA is congruent to triangle BPX

[$PB = PB$; $AB = XB$; $\angle ABP = \angle XBP$.]
Therefore, $\angle BAP = \angle BXP = \angle BCA$ [$PX \parallel AC$].

4. In how many ways can 111 be written as a sum of three integers in geometric progression?

Soln.: Suppose $111 = a + ar + ar^2$ where a is an integer and r is a rational number. If $r = 0$, then we get the trivial solution
$111 = 111 + 0 + 0$.

Suppose $r = \frac{n}{m} \neq 0$ where m and n are non zero integers. Without loss of generality, we may also assume that $m > 0$ and $(m, n) = 1$. Since the reverse of the G.P. a, ar, ar^2 is another G.P. ar^2, ar, a, we may also assume that $|r| \geq 1$ and so $0 < m \leq |n|$. From $a(1 + r + r^2) = 111$ we get $a(m^2 + mn + n^2) = 111m^2$. Since clearly $(m^2 + mn + n^2, m^2) = 1$ we have $m^2|a$. Letting $a = km^2$ where k is an integer we then get $k(m^2 + mn + n^2) = 111$ which implies $k|111$. Since $m^2 + mn + n^2 > 0$ and $111 = 3 \times 37$, we have $k = 1, 3, 37$, or 111. Note that $m^2 + mn + n^2 = m^2 + |n|(\pm m + |n|) \geq m^2$.

Case 1 : If $k = 1$, then $m^2 + mn + n^2 = 111 \Rightarrow m^2 \leq 111 \Rightarrow m \leq 10$. When $m = 1$, $a = 1$ and from $n^2 + n = 110$ we get $n = 10, -11$. Thus $r = 10, -11$ and we obtain the solutions:

$111 = 1 + 10 + 100 = 1 - 11 + 121$. ...(2)

For $2 \leq m \leq 9$ it is easily checked that the resulting quadratic equations in n has no integer solutions. When $m = 10$, $a = 100$ and from $n^2 + 10n = 11$ we get $n = 1, -11$. Since $m \leq |n|$, $n = -11$ and $r = -\frac{11}{10}$ yielding the solution:

$111 = 100 - 110 + 121$. ...(3)

Case 2 : If $k = 3$, then $m^2 + mn + n^2 = 37 \Rightarrow m^2 \leq 37 \Rightarrow m \leq 6$. Quick checkings reveal that there are no solutions for $m = 1, 2, 5, 6$.
When $m = 3$, $a = 27$ and from $n^2 + 3n = 28$ we get $n = 4, -7$. Thus $r = 4/3, -7/3$ yielding the solutions:

$111 = 27 + 36 + 48 = 27 - 63 + 147$ ...(4)

When $m = 4$, $a = 48$ and from $n^2 + 4n = 21$ we get $n = 3, -7$. Since $m \leq |n|$, $n = -7$ and $r = -\frac{7}{4}$ yielding the solution:

$111 = 48 - 84 + 147$ ...(5)

Case 3 : If $k = 37$, then $m^2 + mn + n^2 = 3 \Rightarrow m^2 \leq 3 \Rightarrow m = 1 \Rightarrow a = 37$ and from $n^2 + n = 2$ we get $n = 1, -2$.
Thus $r = 1$ or -2 yielding the solutions:

$111 = 37 + 37 + 37 = 37 - 74 + 148$ ...(6)

Case 4 : If $k = 111$, then $m^2 + mn + n^2 = 1 \Rightarrow m^2 \leq 1 \Rightarrow m = 1 \Rightarrow a = 111$, and from $n^2 + n = 0$ we get $n = -1$ as $n \neq 0$. Thus $r = -1$ and we get the solution.

$111 = 111 - 111 + 111$ ...(7)

Reversing the summand in (2) – (7) and noting that two of them are symmetric, we obtain seventeen solutions in all.

$$\begin{aligned}111 &= 111 + 0 + 0 = 1 + 10 + 100 = 100 + 10 + 1\\ &= 1 - 11 + 121 = 121 - 11 + 1 = 100 - 110 + 121\\ &= 121 - 110 + 100 = 27 + 36 + 48 = 48 + 36 + 27\\ &= 27 - 63 + 147 = 147 - 63 + 27 = 48 - 84 + 147\\ &= 147 - 84 + 48 = 37 + 37 + 37 = 37 - 74 + 148\\ &= 148 - 74 + 37 = 111 - 111 + 111.\end{aligned}$$

5. Find all values of λ for which the inequality

$$2(x^3 + y^3 + z^3) + 3(1 + 3\lambda)xyz \geq (1 + \lambda)(x + y + z)(yz + zx + xy)$$

holds for all positive real numbers x, y, z.

Soln.: On setting $x = y = 1$ and $z = 0$, we obtain $4 \geq (1 + \lambda)2$ and thus find that λ must be ≤ 1. We now show that the inequality holds for all $\lambda \leq 1$. First, if $\lambda = 1$, the inequality reduces to

$$x^3 + y^3 + z^3 + 6xyz \geq (x + y + z)(yz + zx + xy),$$

which is equivalent to the special case $n = 1$ of the known Schur inequality

$$x^n(x - y)(x - z) + y^n(y - z)(y - x) + z^n(z - x)(z - y) \geq 0,$$

true for all real n, and which has come up many times in this journal. The rest will follow by showing that for all $\lambda < 1$.

$$\begin{aligned}&(1 + \lambda)(x + y + z)(yz + zx + xy) - 3(1 + 3\lambda)xyz\\ &\leq (1 + 1)(x + y + z)(yz + zx + xy) - 3(1 + 3)xyz \qquad ...(1)\end{aligned}$$

rewrite the original inequality as

$$2(x^3 + y^3 + z^3) \geq (1 + \lambda)(x + y + z)(yz + zx + xy) - 3(1 + 3\lambda)xyz;$$

then (1) says that the right hand side is largest when $\lambda = 1$, so doing the case $\lambda = 1$ would be enough. But (1) can be written

$$(1 - \lambda)[(x + y + z)(yz + zx + xy) - 9xyz] \geq 0$$

which [after cancelling the positive factor $1 - \lambda$] is a known elementary inequality, equivalent to Cauchy's inequality

$$(x+y+z)\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right) \geq 9.$$

6. A quadrilateral has sides a, b, c, d (in that order) and area F. Prove that

$$2a^2 + 5b^2 + 8c^2 - d^2 \geq 4F$$

When does equality hold?

Soln.: Let $ABCD$ be the quadrilateral with $AB = a$, $BC = b$, $CD = c$ and $DA = d$. We can assume, without loss of generality, that $AC = 1$. Therefore, we can locate the quadrilateral in a system of Cartesian coordinates where

$A = (0, 0)$, $B = (p, q)$, $C = (1, 0)$, $D = (r, s)$

We assume that $ABCD$ is simple so that its area is well-defined. If $ABCD$ is not convex we can make it convex and keep the side lengths the same while increasing the area. This means that we will be done if we can show that the result is true for convex quadrilaterals. It's also clear from this that if the result is true for convex quadrilaterals, then equality cannot hold for non-convex quadrilaterals. Therefore, assume $q < 0$ and $s > 0$. Now note that

$$2a^2 + 5b^2 = 2(p^2 + q^2) + 5((p-1)^2 + q^2)$$

$$= 7p^2 - 10p + 5 + 7q^2 = 7\left(p - \frac{5}{7}\right)^2 - \frac{25}{7} + 5 + 7q^2$$

$$2a^2 + 5b^2 \geq 7q^2 + \frac{10}{7}, \qquad \ldots(2)$$

and

$$8c^2 - d^2 = 8((r-1)^2 + s^2) - (r^2 + s^2)$$

$$= 7r^2 - 16r + 8 + 7s^2$$

$$= 7\left(r - \frac{8}{7}\right)^2 - \frac{64}{7} + 8 + 7s^2$$

$$8c^2 - d^2 \geq 7s^2 - \frac{8}{7} \qquad \ldots(3)$$

Combining (2) and (3), we get

$$2a^2 + 5b^2 + 8c^2 - d^2 \geq 7q^2 + 7s^2 + \frac{2}{7}$$

$$= \left(7q^2 + \frac{1}{7}\right) + \left(7s^2 + \frac{1}{7}\right)$$

$$= \left(7\left(|q| - \frac{1}{7}\right)^2 + 2|q|\right) + \left(7\left(|s| - \frac{1}{7}\right)^2 + 2|s|\right)$$

$$\geq 2(|q| + |s|) \qquad \ldots(4)$$

$$= 4(\overline{ABC}) + 4(\overline{CDA})$$

$$2a^2 + 5b^2 + 8c^2 - d^2 \geq 4F,$$

as we wished to prove (where $\overline{ABC}$ and $\overline{CDA}$ refer to the areas of the two triangles ABC and CDA respectively). For equality to hold (when $A = (0, 0)$ and $C = (1, 0)$, it must hold in steps (2), (3) and (4). Therefore $p = 5/7$, $r = 8/7$, $q = -1/7$ and $s = 1/7$. Thus, in general, equality holds if and only if $ABCD$ is directly similar to quadrilateral $A_0B_0C_0D_0$, where

$$A_0 = (0, 0),\ B_0 = \left(\frac{5}{7}, -\frac{1}{7}\right),\ C_0 = (1, 0),\ D_0 = \left(\frac{8}{7}, \frac{1}{7}\right).$$

7. Triangle ABC is not isosceles nor equilateral, and has sides a, b, c. D_1 and E_1 are points of BA and CA or their production so that $BD_1 = CE_1 = a$. D_2 and E_2 are points of CB and AB or their productions so that $CD_2 = AE_2 = b$. Show that $D_1E_1 \parallel D_2E_2$.

Soln.: Let S be the intersection of AB and D_2E_1. Then CS is the bisector of $\angle ACB$, since $CE_1 = CB$ and $CA = CD_2$. Therefore

$$\frac{D_1S}{SE_2} = \frac{BD_1 - BS}{AE_2 - AS} = a = BSb - AS = \frac{a}{b},$$

since $\frac{BS}{AS} = \frac{a}{b}$. It then follows that $D_1E_1 \parallel D_2E_2$, since

$$\frac{E_1S}{SD_2} = \frac{CE_1}{CD_2} = \frac{a}{b} = \frac{D_1S}{SE_2}.$$

8. Prove that

$$\frac{a+b+c}{3} \leq \frac{1}{4}\sqrt[3]{\frac{(b+c)^2(c+a)^2(a+b)^2}{abc}}$$

where $a, b, c > 0$. Equality holds if $a = b = c$.

Soln.: By the arithmetic-geometric mean inequality we have

$$a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 \geq$$

$$6\sqrt[6]{a^6b^6c^6} = 6bac,$$

which implies

$$9(a^2b + ab^2 + b^2c + bc^2 + c^2a + a^2c + 2abc)$$

$$\geq 8(a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2 + 3abc),$$

or

$$9(a+b)(b+c)(c+a) \geq 8(a+b+c)(ab+bc+ca)$$

$$= 4(a+b+c)(a(b+c) + b(c+a) + c(a+b))$$

Using the arithmetic-geometric mean inequality again, we then have

$$\frac{3}{4}(a+b)(b+c)(c+a)$$

$$\geq (a+b+c)\cdot\frac{a(b+c)+b(c+a)+c(a+b)}{3}$$

$$\geq (a+b+c)\sqrt{abc(a+b)(b+c)(c+a)} \qquad \ldots(1)$$

From (1) it follows immediately that

$$\frac{1}{4}\sqrt[3]{\frac{(a+b)^2(b+c)^2(c+a)^2}{abc}} \geq \frac{a+b+c}{3}.$$

Clearly, equality holds if $a = b = c$.

■■

10. If $x+\frac{1}{x}=2\cos\alpha$, $y+\frac{1}{y}=2\cos\beta$, then one of the values of $x^{-1}y-xy^{-1}$ is

(a) $2\cos(\alpha-\beta)$ (b) $2\sin(\alpha-\beta)$

(c) $2i\sin(\alpha-\beta)$ (d) none of these.

SOLUTIONS

1. (c) : $(-1)^{1/3}=[\cos(2k\pi+\pi)+i\sin(2k\pi+\pi)]^{1/3}$

$$=\cos\frac{2k\pi+\pi}{3}+i\sin\frac{2k\pi+\pi}{3},\ k=0,1,2.$$

The values corresponding to $k=0$ and $k=2$ are given.

The value corresponding to $k=1$ is

$\cos\pi+i\sin\pi=-1$.

2. (d) : (To suit the given answer the equation should have been as follows: The general value of θ satisfying the equation

$(\cos\theta+i\sin\theta)(\cos3\theta+i\sin3\theta)\ldots[\cos(2r-1)\theta+i\sin(2r-1)\theta]=1)$ is ......

$(\cos\theta+i\sin\theta)(\cos3\theta+i\sin3\theta)(\cos5\theta+i\sin5\theta)\ldots[\cos(2r-1)\theta+i\sin(2r-1)\theta=$ L.H.S.

LHS $=\cos\{[\theta+3\theta+5\theta\ldots+(2r-1)\theta]\}+i\sin\{[\theta+3\theta+5\theta\ldots+(2r-1)\theta]\}$

$=\cos r^2\theta+i\sin r^2\theta\ (\because \theta+3\theta+5\theta\ldots+(2r-1)\theta=r^2\theta)$

$\therefore$ We have to find the general value of θ satisfying the equation

$\cos r^2\theta+i\sin r^2\theta=1+i\cdot0$

$\therefore$ $\cos r^2\theta=1$ and $\sin r^2\theta=0$

$\therefore$ $r^2\theta=2n\pi,\ n\in I,\ r^2\theta=n\pi,\ n\in I$

$\therefore$ $\theta=\frac{2n\pi}{r^2},\ n\in I$.

3. (a) : $x+\frac{1}{x}=2\cos\frac{\pi}{10}=2\cos\theta$ if $\theta=\frac{\pi}{10}$

$\therefore$ $x^2-2x\cos\theta+1=0$

$\therefore$ $x=\cos\theta+i\sin\theta$; $1/x=\cos\theta-i\sin\theta$

$$x^5+\frac{1}{x^5}=(\cos\theta+i\sin\theta)^5+(\cos\theta-i\sin\theta)^5$$

$=(\cos5\theta+i\sin5\theta)+(\cos5\theta-i\sin5\theta)$

$=2\cos5\theta=2\cos5\cdot\frac{\pi}{10}=2\cos\frac{\pi}{2}=0$.

4. (d) : $z^4=i$. $\therefore$ $z=(i)^{1/4}=\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right)^{1/4}$

$$=\cos\left(\frac{2n\pi}{4}+\frac{\pi}{8}\right)+i\sin\left(\frac{2n\pi}{4}+\frac{\pi}{8}\right)$$

One value is $\cos(\pi/8)+i\sin(\pi/8)$.

5. (d) **6. (a)** **7. (a)** **8. (a)** **9. (b)** **10. (d)**

5 CHALLENGING PROBLEMS — CALCULUS

Contributed by : T. R. Chakravarthy, B.Tech (Mech.), IIT Madras

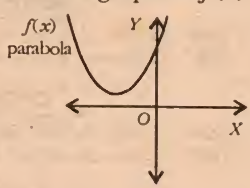
1. $f(x)$ is a polynomial of degree 2 with positive integral coefficients such that for every $\beta > \alpha$, $\int_\alpha^\beta f(x)dx > 0$. Let $g(t) = f'(t)f(t)$, given that $g(0) = 12$. Sketch graph of $f(x)$ roughly. Find maximum value assumed by $f(1)$. If given that $2f(0) > f(1)$ and $f(2) \cdot g(-1)$ is divisible by 10, find range of values taken by $f(x)$. Show that for every natural number $n > 1$, the $(3n - 1)$ degree polynomial $nf(x^n)x^{(n-1)} = f(x)$ has atleast one root in (0, 1).

Soln. **:** Given that $\int_\alpha^\beta f(x)dx > 0$ for all $\beta > \alpha$

Let $f(x) = ax^2 + bx + c$ $f(0) = c > 0$ $(\because c \in Z^+)$.
Also $f(x)$ has no roots. If at all $f(x)$ has roots the condition $\int_\alpha^\beta f(x)dx > 0$ $\forall$ $\beta > \alpha$ will not be fulfilled.

$$f'(x) = 2ax + b = 0 \Rightarrow x = \frac{-b}{2a} < 0$$

So with these inferences, we draw a graph of $f(x)$ as below



$f''(x) = 2a$
$g(0) = f''(0)f(0) = 2a \cdot c$
$= 2ac = 12$ (given)
$\therefore$ $ac = 6$

$\because$ a and c are positive integers, the possible values are $a = 6$, $c = 1$; $a = 1$, $c = 6$; $a = 2$, $c = 3$; $a = 3$, $c = 2$

Also $f(x)$ has no roots
$\therefore$ $b^2 < 4ac$ $\qquad$ $b^2 < 24$
The possible values of b are 1, 2, 3, and 4.
The possible value of $a + c$ are 7 and 5
So maximum value assumed by
$a + b + c$ i.e $f(1)$ is $(a + c)_{max} + b_{max} = 7 + 4 = 11$
If $2f(0) > f(1)$ *i.e.* $2c > a + b + c$ i.e. $a + b < c$
i.e. if $a + b < c$ the possible values are $a = 1$ and $c = 6$. b can assume values 1, 2, 3 and 4.
Given $f(2)g(-1)$ is double by 10.
$f(2) = (4a + 2b + c) = (10 + 2b) = 2(5 + b)$
$g(-1) = f'(-1)f(-1) = 2(7 - b)$
$f(2)g(-1) = 4(7 - b)(5 + b)$ is divisible by 10.
So $b = 2$ is only possible solution out of 1, 2, 3 and 4.
$\therefore$ $f(x) = x^2 + 2x + 6$
$f'(x) = 2x + 2 = 0 \Rightarrow x = -1$
$f(-1) = 1 - 2 + 6 = 5$.
So range of values taken by $f(x) = [5, \infty)$
Consider a function

$$h(t) = \int_t^{t^n} f(x)dx \quad (n > 1,\ n \in N)$$

$h(t)$ is continuous and derivable in R
Note that $h(0) = h(1) = 0$
Thus $h(t)$ satisfies conditions for Rolle's theorem
$\therefore$ $h'(c) = 0$ for some $c \in (0, 1)$
$h'(c) = f(t^n)nt^{n-1} - f(t) = 0$ for some $t \in (0, 1)$
$\therefore$ The $(3n - 1)$ degree polynomial equation $n\, f(x^n)x^{n-1} - f(x) = 0$ has atleast one solution in (0, 1).

2. Evaluate $\displaystyle\int \frac{\cos x\, dx}{(1 - \cos\alpha \sin x)\sqrt{(1 + \cos 2\alpha \sin^2 x - 2\cos\alpha \sin x)}}$

Soln. **:** $\displaystyle = \int \frac{\cos x\, dx}{(1 - \cos\alpha\sin x)\sqrt{(1 - 2\cos\alpha\sin x + (\cos^2\alpha - \sin^2\alpha)\sin^2 x}}$

$$= \int \frac{\cos x\, dx}{(1 - \cos\alpha\sin x)\sqrt{(1 + \cos^2\alpha\sin^2 x - 2\cos\alpha\sin x) - (\sin^2\alpha\sin^2 x)}}$$

$$= \int \frac{\cos x\, dx}{(1 - \cos\alpha\sin x)\sqrt{(1 - \cos\alpha\sin x)^2 - (\sin\alpha\sin x)^2}}$$

$$= \int \frac{(\cos x)\, dx}{(1-\cos\alpha\sin x)^2\sqrt{1-\left(\frac{\sin\alpha\sin x}{1-\cos\alpha\sin x}\right)^2}}$$

Put $\frac{\sin\alpha\sin x}{1-\cos\alpha\sin x} = t$

$$\therefore \frac{\left[\begin{array}{l}(\sin\alpha)[(1-\cos\alpha\sin x)\cos x] - \\ \qquad (\sin x)(-\cos\alpha\cos x)\end{array}\right]}{(1-\cos\alpha\sin x)^2} = \frac{dt}{dx}$$

$$\left(\frac{dt}{dx}\right) = \frac{\sin\alpha}{(1-\cos\alpha\sin x)^2}\left(\begin{array}{l}\cos x - \cos\alpha\sin x\cos x + \\ \qquad \cos\alpha\sin x\cos x\end{array}\right)$$

$$\frac{dt}{dx} = \frac{\sin\alpha\cos x}{(1-\cos\alpha\sin x)^2}$$

$$\therefore \frac{\cos x\, dx}{(1-\cos\alpha\sin x)^2} = dt\,(\operatorname{cosec}\alpha)$$

$$\therefore \int \frac{(\cos x)dx}{(1-\cos\alpha\sin x)^2\sqrt{1-\left(\frac{\sin\alpha\sin x}{1-\cos\alpha\sin x}\right)^2}}$$

after substitution it changes to

$$= \frac{1}{\sin\alpha}\int\frac{dt}{\sqrt{1-t^2}} = \frac{\sin^{-1}(t)}{\sin\alpha} + c$$

where c is integration constant.

$$= \frac{1}{\sin\alpha}\sin^{-1}\left[\frac{\sin\alpha\sin x}{1-\cos\alpha\sin x}\right] + c.$$

3. Find $\int \frac{dx}{\sqrt{\sin(x+\alpha)\cos^3(x-\beta)}}$

Also, show that $\int_0^{\pi/2} \frac{dx}{\sqrt{\sin(x+\alpha)\cos^3(x-\beta)}} =$

$$\frac{2}{(\cos\beta\sqrt{\cos\alpha\sin\beta} + \sin\beta\sqrt{\sin\alpha\cos\beta})}$$

***Soln.* :** $\int \frac{dx}{\sqrt{\sin(x+\alpha)\cos^3(x-\beta)}}$

Put $(x-\beta) = y \Rightarrow dx = dy$

$$\int \frac{dy}{\sqrt{\cos^3 y\sin(y+\beta+\alpha)}}$$ Also let $\alpha+\beta = \theta$

$$= \int \frac{dy}{\sqrt{\cos^3 y\sin(y+\theta)}}$$

$$= \int \frac{dy}{\sqrt{\cos^3 y\,(\sin y\cos\theta + \cos y\sin\theta)}}$$

$$= \int \frac{dy}{\sqrt{\cos^4 y\,(\cos\theta\tan y + \sin\theta)}}$$

$$= \int \frac{\sec^2 y\, dy}{\sqrt{(\sin\theta + \cos\theta\tan y)}}$$

Put $\sin\theta + \cos\theta\tan y = z^2$

$\Rightarrow \cos\theta\sec^2 y\, dy = 2z\, dz$

$$= \int \frac{2\sec\theta\, z\, dz}{z} = 2\sec\theta\, z + c$$

$$= 2\sec\theta\sqrt{\sin\theta + \cos\theta\tan y} + c$$

$$= 2\sec\theta\sqrt{\frac{\sin\theta\cos y + \cos\theta\sin y}{\cos y}} + c$$

$$= 2\sec\theta\sqrt{\frac{\sin(y+\theta)}{\cos y}} + c$$

$$= 2\sec(\alpha+\beta)\sqrt{\frac{\sin(x+\alpha)}{\cos(x-\beta)}} + c$$

$$\int_0^{\pi/2}\frac{dx}{\sqrt{\sin(x+\alpha)\cos^3(x-\beta)}}$$

$$= 2\sec(\alpha+\beta)\sqrt{\frac{\sin(x+\alpha)}{\cos(x-\beta)}}\Bigg|_0^{\pi/2}$$

$$= 2\sec(\alpha+\beta)\left(\sqrt{\frac{\cos\alpha}{\sin\beta}} - \sqrt{\frac{\sin\alpha}{\cos\beta}}\right)$$

$$= \frac{2}{\cos(\alpha+\beta)}\frac{(\sqrt{\cos\alpha\cos\beta} - \sqrt{\sin\alpha\sin\beta})}{\sqrt{\cos\beta\sin\beta}}$$

Multiply numerator and denominator with $(\sqrt{\cos\alpha\cos\beta} + \sqrt{\sin\alpha\sin\beta})$

$$= \frac{2\cos(\alpha+\beta)}{\cos(\alpha+\beta)\sqrt{\cos\beta\sin\beta}\,(\sqrt{\cos\alpha\cos\beta} + \sqrt{\sin\alpha\sin\beta})}$$

$$= \frac{2}{(\cos\beta\sqrt{\cos\alpha\sin\beta} + \sin\beta\sqrt{\sin\alpha\cos\beta})}.$$

4. Evaluate $\int_{\tan\alpha}^{\tan\beta}\frac{(\sin(\alpha+\beta)x - \cos\alpha\cos\beta x^2)}{\sqrt{(x-\tan\alpha)(\tan\beta - x)}}dx$

***Soln.* :** Substitution of $x = (\tan\alpha)\cos^2\theta + (\tan\beta)\sin^2\theta$ may solve the problem but it may be cumbersome. We shall solve in another simple way

$$= \int_{\tan\alpha}^{\tan\beta}\frac{\left(x - \frac{\cos\alpha\cos\beta}{\sin(\alpha+\beta)}x^2\right)\sin(\alpha+\beta)}{\sqrt{(x-\tan\alpha)(\tan\beta-x)}}dx$$

$$= \sin(\alpha+\beta)\int_{\tan\alpha}^{\tan\beta} \frac{\left(x - \frac{x^2}{\tan\alpha+\tan\beta}\right)}{\sqrt{(x-\tan\alpha)(\tan\beta - x)}}dx$$

$$= \frac{\sin(\alpha+\beta)}{\tan\alpha+\tan\beta}\int_{\tan\alpha}^{\tan\beta} \frac{(\tan\alpha+\tan\beta)x - x^2}{\sqrt{(x-\tan\alpha)(\tan\beta - x)}}dx$$

Add and subtract $(1 - \tan\alpha \tan\beta)$

$$= \frac{\sin(\alpha+\beta)}{\tan\alpha+\tan\beta}\int_{\tan\alpha}^{\tan\beta} \frac{1-\tan\alpha\tan\beta - x^2 + (\tan\alpha+\tan\beta)x}{\sqrt{(x-\tan\alpha)(\tan\beta - x)}}\, dx$$

$$+ \frac{\sin(\alpha+\beta)}{\tan\alpha+\tan\beta}(\tan\alpha\tan\beta - 1)\int_{\tan\alpha}^{\tan\beta} \frac{dx}{\sqrt{(x-\tan\alpha)(\tan\beta - x)}}$$

$$= \cos\alpha\cos\beta\int_{\tan\alpha}^{\tan\beta} \frac{1+(x-\tan\alpha)(\tan\beta - x)}{\sqrt{(x-\tan\alpha)(\tan\beta - x)}}dx$$

$$-\cos(\alpha+\beta)\int_{\tan\alpha}^{\tan\beta} \frac{dx}{\sqrt{(x-\tan\alpha)(\tan\beta - x)}},$$

$$= (\cos\alpha\cos\beta - \cos(\alpha+\beta))\int_{\tan\alpha}^{\tan\beta} \frac{dx}{\sqrt{(x-\tan\alpha)(\tan\beta - x)}}$$

$$+ \cos\alpha\cos\beta\int_{\tan\alpha}^{\tan\beta} \sqrt{(x-\tan\alpha)(\tan\beta - x)}\, dx$$

By substitution of $x = a\cos^2\theta + b\sin^2\theta$ we can very easily prove following results.

$$\int_a^b \frac{dx}{\sqrt{(x-a)(b-x)}} = \pi$$

$$\int_a^b \sqrt{(x-a)(b-x)}\, dx = \frac{\pi}{8}(b-a)^2$$

with these results

$$\int_{\tan\alpha}^{\tan\beta} \frac{(\sin(\alpha+\beta)x - \cos\alpha\cos\beta x^2)}{\sqrt{(x-\tan\alpha)(\tan\beta - x)}}dx$$

$$= \pi\sin\alpha\sin\beta + \cos\alpha\cos\beta\frac{\pi}{8}(\tan\beta - \tan\alpha)^2$$

$$= \pi\sin\alpha\sin\beta + \frac{\pi}{8}\frac{\cos\alpha\cos\beta\sin^2(\alpha-\beta)}{\cos^2\alpha\cos^2\beta}$$

$$= \pi\sin\alpha\sin\beta + \frac{\pi\sin^2(\alpha-\beta)}{8\cos\alpha\cos\beta}$$

$$= \frac{2\pi\sin 2\alpha\sin 2\beta + \pi\sin^2(\alpha-\beta)}{8\cos\alpha\cos\beta}.$$

5. After several operations of differentiation and multiplying by $(x + 1)$ performed in an arbitrary order the polynomial $x^8 + x^7$ is changed to $ax + b$. Prove that the difference between the integers a and b is always divisible by 49.

Soln. : Let $f(x) = x^m$

$$f^n(x) = m(m-1)(m-2)....(m+1-n)\, x^{m-n}$$

$f^n(x)$ is n^{th} derivative of $f(x)$

$$f^n(x) = \frac{m!}{(m-n)!}x^{m-n}$$

Let $g(x) = x^8 + x^7$

Let it be differentiated n times

$$g^n(x) = \frac{8!}{(8-n)!}x^{8-n} + \frac{7!}{(7-n)!}x^{7-n}$$

Now multiply by $(x + 1)$

$$(x+1)g^n(x) = \frac{8!}{(8-n)!}x^{8-n} + \frac{7!}{(7-n)!}x^{7-n}$$

$$+ \frac{8!}{(8-n)!}x^{9-n} + \frac{7!}{(7-n)!}x^{8-n}$$

$(x + 1)g^n(x)$ has to be $(8 - n)$ times to get to form of $ax + b$

Let $h(x) = (x + 1)g^n(x)$

Consider $h^{(8-n)}(x)$

$$= \frac{8!}{(8-n)!}\frac{(8-n)!}{0!} + \frac{7!}{(7-n)!}\frac{(8-n)!}{0!}$$

$$+ \frac{8!}{(8-n)!}\frac{(9-n)!}{1!}x$$

$= 8!\,(9 - n)x + 7!(8 - n) + 8 \cdot 7!$

$= 8!\,(9 - n)x + 7!(16 - n)$

Comparing $a = 8!\,(9 - n) = 7!\,(72 - 8n)$

$b = 7!\,(16 - n)$

$a - b = 7!\,(72 - 8n - 16 + n)$

$= 7!(56 - 7n)$

$= 7 \times 7!\,(8 - n) = 49 \cdot 6!\,(8 - n)$

$\Rightarrow a - b$ is divisible by 49.

■■

SOLVED PAPER

Kerala P.E.T. 2002

1. If $A = \{x : x^2 - 5x + 6 = 0\}$, $B = \{2, 4\}$, $C = \{4, 5\}$ then $A \times (B \cap C)$

(a) $\{(2, 4), (3, 4)\}$ (b) $\{(4, 2), (4, 3)\}$

(c) $\{(2, 4), (3, 4), (4, 4)\}$

(d) $\{(2, 2), (3, 3), (4, 4), (5, 5)\}$

(e) null set.

2. In a city 20 percent of the population travels by car, 50 percent by bus and 10 per cent travels by both car and bus. Then persons travelling by car or bus

(a) 80 percent (b) 40 percent

(c) 60 percent (d) 70 percent

(e) 30 percent.

3. If $f(x) = \dfrac{2x+1}{3x-2}$, then $(f \circ f)(2)$ is equal to

(a) 1 (b) 3

(c) 4 (d) 2

(e) none of these.

4. Which one of the following is a bijective function on the set of real numbers?

(a) $2x - 5$ (b) $|x|$

(c) x^2 (d) $x^2 + 1$

(e) $x^4 - x^2 + 1$.

5. If $f(x) = \log \dfrac{1+x}{1-x}$, then $f(x)$ is

(a) even

(b) $f(x_1)f(x_2) = f(x_1 + x_2)$

(c) $\dfrac{f(x_1)}{f(x_2)} = f(x_1 - x_2)$

(d) odd

(e) neither even nor odd.

6. Let the function f be defined by $f(x) = \dfrac{2x+1}{1-3x}$ then $f^{-1}(x)$ is

(a) $\dfrac{x-1}{3x+2}$ (b) $\dfrac{3x+2}{x-1}$

(c) $\dfrac{x+1}{3x-2}$ (d) $\dfrac{2x+1}{1-3x}$

(e) $\dfrac{1-3x}{2x+1}$.

7. If $\sqrt{a+ib} = x + iy$, then possible value of $\sqrt{a-ib}$ is

(a) $x^2 + y^2$ (b) $\sqrt{x^2 + y^2}$

(c) $x + iy$ (d) $x - iy$

(e) $\sqrt{x^2 - y^2}$.

8. If $(1 + i)(1 + 2i)(1 + 3i) \ldots (1 + ni) = a + ib$, then $2 \times 5 \times 10 \ldots \times (1 + n^2)$ is equal to

(a) $\sqrt{a^2 + b^2}$ (b) $\sqrt{a^2 - b^2}$

(c) $a^2 + b^2$ (d) $a^2 - b^2$

(e) $a + b$.

9. If $i^2 = -1$, then sum $i + i^2 + i^3 + \ldots$ to 1000 terms is equal to

(a) 1 (b) -1

(c) i (d) $-i$

(e) 0.

10. $\left(\dfrac{1+\sin\theta + i\cos\theta}{1+\sin\theta - i\cos\theta}\right)^n =$

(a) $\cos\left(\dfrac{n\pi}{2} - n\theta\right) + i\sin\left(\dfrac{n\pi}{2} - n\theta\right)$

(b) $\cos\left(\dfrac{n\pi}{2} + n\theta\right) + i\sin\left(\dfrac{n\pi}{2} + n\theta\right)$

(c) $\sin\left(\dfrac{n\pi}{2} - n\theta\right) + i\cos\left(\dfrac{n\pi}{2} - n\theta\right)$

(d) $\cos n\left(\dfrac{\pi}{2} + 2\theta\right) + i\sin n\left(\dfrac{\pi}{2} + 2\theta\right)$

(e) $\cos n\theta + i\sin n\theta$.

11. If w is a non real cube root of unity, then $(a + b)(a + bw)(a + bw^2)$ is

(a) $a^3 + b^3$ (b) $a^3 - b^3$

(c) $a^2 + b^2$ (d) $a^2 - b^2$

(e) 0.

Maths Clinic

If you have any difficult / unsolved problem or you are unable to understand one, then write to us. Our team of experts will diagnose your problems. The diagnosed problems will be published in the subsequent issues.

1. Evaluate : $\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$

[Chintamani P. Siddheshwar, karwar]

Soln: Let $A = \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x^2}$

$\therefore \quad \log A = \lim_{x\to 0}\frac{1}{x^2}\log\left(\frac{\tan x}{x}\right)$

$= \lim_{x\to 0}\frac{1}{x^2}\log\left[\frac{1}{x}\left\{x+\frac{1}{3}x^3+\frac{2}{15}x^5\ldots\right\}\right]$

$= \lim_{x\to 0}\frac{1}{x^2}\log\left[1+\left(\frac{1}{3}x^2+\frac{2}{15}x^4+\ldots\right)\right]$

$= \lim_{x\to 0}\frac{1}{x^2}\left[\left(\frac{1}{3}x^2+\frac{2}{15}x^4+\ldots\right)\right.$

$\left.-\frac{1}{2}\left(\frac{1}{3}x^2+\frac{2}{15}x^4+\ldots\right)^2+\ldots\right]$

$= \lim_{x\to 0}\left[\frac{1}{3}+\text{term with positive power of } x\right] = 1/3$

$\Rightarrow \log A = 1/3 \Rightarrow A = e^{1/3}$.

2. If $z_1 = a + ib$, and $z_2 = c + id$ are two complex numbers such that $z_1 \neq z_2$ and $|z_1| = |z_2| = 1$ and $\mathrm{Re}(z_1\bar{z}_2) = 0$ then pair for the complex numbers $\omega_1 = a + ic$, $\omega_2 = b + id$. Find $\mathrm{Re}(\omega_1\bar{\omega}_2)$ and $|\omega_2|$.

[Ravish Bhasin, Amritsar]

Soln: Given $|z_1| = 1$, $|z_2| = 1$ and $\mathrm{Re}(z_1\bar{z}_2) = 0$

$\Rightarrow a^2 + b^2 = 1$ ... (i)

$c^2 + d^2 = 1$ ... (ii) ; $\quad ac + bd = 0$ ... (iii)

From (iii) $\frac{a}{d} = \frac{-b}{c} = \lambda$ (say) $\Rightarrow a = \lambda d$, $b = -c\lambda$.

From (i), $\lambda^2(c^2 + d^2) = 1$

$\Rightarrow \lambda^2 = 1$ (using ii) $\Rightarrow \lambda = \pm 1$.

Thus $a = d$, $b = -c$ or $a = -d$, $b = c$.

Putting $a = d$ in (i), we get $|\omega_2| = 1$.

Now $a = d$ and $b = -c$

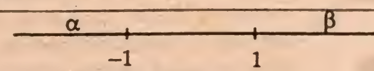
$\Rightarrow ab + cd = 0 \Rightarrow \mathrm{Re}(\omega_1\bar{\omega}_2) = 0$

3. Let a, b, c be real. If $ax^2 + bx + c = 0$ has two real roots α, β where $\alpha < -1$ and $\beta > 1$ then show that $1+\frac{c}{a}+\left|\frac{b}{a}\right| < 0$.

[R. Prasad, Andhra Pradesh]

Soln: As α, β are real and unequal,

$b^2 - 4ac > 0$



Here $\alpha + \beta = -b/a$ and $\alpha\beta = c/a$

Also $\alpha < -1$, $\beta > -1$

$\Rightarrow \alpha + 1 < 0$, $\beta + 1 > 0$ ... (i)

$\alpha < 1$, $\beta > 1 \Rightarrow \alpha - 1 < 0$, $\beta - 1 > 0$ ... (ii)

From (i), $(\alpha + 1)(\beta + 1) < 0$

or $\alpha\beta + \alpha + \beta + 1 < 0 \Rightarrow \frac{c}{a}-\frac{b}{a}+1<0$ ... (iii)

From (ii), $(\alpha - 1)(\beta - 1) < 0$

or $\alpha\beta - (\alpha + \beta) + 1 < 0$ or $\frac{c}{a}+\frac{b}{a}+1<0$ ... (iv)

Combining (iii) and (iv)

$$1+\frac{c}{a}\pm\frac{b}{a}<0 \quad\text{or}\quad 1+\frac{c}{a}+\left|\frac{b}{a}\right|<0.$$

4. The line L has intercepts a and b on the coordinate axes. When keeping the origin fixed, the coordinate axes are rotated through a fixed angle, then the same line has intercepts p and q on the rotated axes. Then prove that $\frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}$.

[Joy Mallik, Kolkata]

Soln: Let the equation of the line with reference to old axes be $\frac{x}{a}+\frac{y}{b}=1$, and let the axes be rotated through an angle α. Let (X, Y) be the new coordinates of any point $P(x, y)$. Then

$x = X\cos\alpha - Y\sin\alpha$, $y = X\sin\alpha - Y\cos\alpha$

The equation of the line with reference to original coordinates is

$$\frac{x}{a}+\frac{y}{b}=1$$

i.e. $\frac{X\cos\alpha - Y\sin\alpha}{a}+\frac{X\sin\alpha + Y\cos\alpha}{b}=1$

and with reference to new coordinates is $\frac{X}{p}+\frac{Y}{q}=1$.

Comparing the two equations, we get

$$\frac{\cos\alpha}{a}+\frac{\sin\alpha}{b}=\frac{1}{p} \quad\text{and}\quad \frac{\cos\alpha}{b}-\frac{\sin\alpha}{a}=\frac{1}{q}$$

or $b\cos\alpha + a\sin\alpha = \frac{ab}{p}$ and $a\cos\alpha - b\sin\alpha = \frac{ab}{q}$

Squaring and adding, we get

$$a^2+b^2=a^2b^2\left(\frac{1}{p^2}+\frac{1}{q^2}\right) \Rightarrow \frac{1}{a^2}+\frac{1}{b^2}=\frac{1}{p^2}+\frac{1}{q^2}.$$

■

My Success Story

Kuntal Loya
IIT-JEE 2002 8th Rank

We heartily congratulate IIT topper (AIR–8) Kuntal Loya for her brilliant success. While interacting with us she tried to assimilate various factors and key points contributing towards her academic excellence. Although this stunning performance itself tells the saga of two years hard work, concentration and serious study, her answering to the pertinent questions asked by us would be beneficial to thousands of aspirants.

MTG : Why did you choose Engineering stream? Why did you sit for these particular examinations?

Kuntal Loya : I have great interest in computer. My parents also wanted me to become an engineer, and IIT is the most premeir Institute for Engineering, so I planned to appear for JEE.

MTG : What other exams you appeared for and your status/ rank in them?

Kuntal Loya : I appeared for Raj PET, IIT–JEE and secured 169th rank in screening.

MTG : Any other achievements ?

Kuntal Loya : (i) I am National Talent Search Examination (NTSE) scholar. Obtained 8th rank in state (Rajasthan).

MTG : How much time does one require for serious preparation for this exam ?

Kuntal Loya : I seriously started preparing for these entrance examination in the beginning of class XI. I strongly believed that with sincerity and dedication I would be able to satisfy my aspiration of topping this most prestigious exam. I feel that devoting time to study may vary according to the grasping power & caliber of the student, however on an average I used to study 8–10 hours per day.

> **"Desire is the key to motivation, but it's the determination and commitment to an unrelenting pursuit of your goal-a commitment to excel-that will enable you to attain the success you seek."**

MTG : In your words what are the components of an ideal preparation plan?

Kuntal Loya : Regular revision with emphasis on the weaker topics, time distribution for the three subjects for competition oriented study. Concentrate more on mathematics giving importance to understanding the concepts as it carries greater weightage and is given priority in case of clashes. Start your preparation earlier (Class 11th beginning) and do not waste even a minute and utilize every second efficiently as there is no use of repenting after one has taken the exams, over one's performance.

MTG : Did you take any extra coaching ?

Kuntal Loya : Yes, I attended Bansal classes, Kota. This coaching class has an excellent academic environment and is one of the best in the country in preparing students for IIT-JEE. For

10 Mathematical Challenges

1. Observe that

$$3 + 5 = 8 = 2^3$$
$$5 + 7 = 12 = 2^2 \cdot 3$$
$$7 + 11 = 18 = 2 \cdot 3^2$$
$$11 + 13 = 24 = 2^3 \cdot 3$$

Show that if p and q are any two consecutive odd primes, then $p + q$ is a product of at least 3 (not necessarily distinct) primes.

Soln.: $p + q$ is even, so $\frac{1}{2}(p+q)$ is an integer between p and q. Since p, q are consecutive primes, the integer $\frac{1}{2}(p+q)$ is not a prime, *i.e.*, it is a product of two or more primes, and now $p + q = 2\left(\frac{p+q}{2}\right)$.

2.(a) Show that $\sqrt{2}+\sqrt{3}$ is not rational

(b) Given the positive integers m and n, under what conditions is $\sqrt{m}+\sqrt{n}$ rational ?

Soln.: (a) Will follow from a solution to **(b)**.

Suppose $\sqrt{m}+\sqrt{n}$ is rational. Then

$$\sqrt{m}-\sqrt{n} = \frac{m-n}{\sqrt{m}+\sqrt{n}} \text{ is rational}$$

Hence

$$\sqrt{m} = \frac{1}{2}(\sqrt{m}+\sqrt{n}) + \frac{1}{2}(\sqrt{m}-\sqrt{n}) \text{ is rational.}$$

Therefore m is a perfect square. Similarly, n is a perfect square. Conversely, if m and n are squares then $\sqrt{m}+\sqrt{n}$ is an integer, hence rational.

3. Let $P_1, P_2, \ldots, P_m$ be m points on a line and $Q_1, Q_2, \ldots, Q_n$ be n points on a distinct and parallel line. All segments P_iQ_j are drawn. What is the maximum number of points of intersection.

Soln.: Each choice of a pair of points P_i and a pair of points Q_j gives rise to one intersection point. By appropriately choosing the points on each line in turn, it can be arranged that the intersection points are all distinct. Hence the number of intersection points are $\binom{m}{2}\binom{n}{2}$.

4. Factor $(x + y + z)^5 - x^5 - y^5 - z^5$.

Soln.: Let $F(x, y, z) = (x + y + z)^5 - x^5 - y^5 - z^5$. Since $F(x, -x, z) = F(x, y, -x) = F(x, y, -y) = 0$,
$F(x, -x, z) = F(x, y, -x) = F(x, y, -y) = 0$,
$F(x, y, z)$ has (by the Factor theorem) the factors $x + y$, $x + z$ and $y + z$. Let

$$F(x, y, z) = (x + y)(x + z)(y + z)G(x, y, z) \quad \ldots(1)$$

Then $G(x, y, z)$ must be a symmetric, homogeneous polynomial of degree 2 and hence has the form $G(x, y, z) = a(x^2 + y^2 + z^2) + b(xy + xz + yz)$. Setting $x = y = z = 1$ in (1) yields $3^5 - 3 = 8(3a + 3b)$, or $a + b = 10$. Setting $x = y = 1$ and $z = 0$ in (1) yields $2^5 - 2 = 2(2a + b)$, or $2a + b = 15$. We conclude that $a = b = 5$ and

$$G(x, y, z) = 5(x^2 + y^2 + z^2 + xy + xz + yz).$$

5. Without using "long" multiplication, a computer or a pocket calculator, verify that

(a) $13! = 112296^2 - 79896^2$

(b) $240^4 + 340^4 + 430^4 + 599^4 = 651^4$.

Soln.: (a) $112296^2 - 79896^2$
$= (112296 - 79896)(112296 + 79896)$
$= (32400)(192192) = 18^2 10^2 (192)(1001)$
$= (2 \cdot 9 \cdot 3 \cdot 6)(2 \cdot 5 \cdot 10)(2 \cdot 8 \cdot 12)(7 \cdot 11 \cdot 13)$
$= 13!$.

(b) The equation suggest that we should try to pull a factor 10^4 out of $651^4 - 591^4$, so we calculate
$651^4 - 599^4 = (651^2 - 599^2)(651^2 + 599^2)$
$= (651 - 599)(651 + 599)(651^2 + 599^2)$

$= 52 \cdot 1250((625 + 26)^2 + (625 - 26)^2)$

$= 2^2 \cdot 13 \cdot 2 \cdot 5^4 \cdot 2(625^2 + 26^2)$

$= 10^4 \cdot 13(25^4 + 26^2)$,

deducing,

$651^4 - 599^4 - 430^4 - 340^4 - 240^4 = 10^4 A$,

where

$A = 13(25^4 + 26^2) - (43^4 + 34^4 + 24^4)$

$= 12.25^4 + 4.13^3 - (34^4 + 24^4) - (43^4 - 25^4)$

$= 12 \cdot 25^4 + 4 \cdot 13^3 - 16(17^4 + 12^4) - 18 \cdot 68 \cdot 2(34^2 + 9^2)$

$= 4B$

with $B = 3 \cdot 25^4 + 13^3 - 4(17^4 + 12^4) - 17 \cdot 36(34^2 + 9^2)$

We have to show that $A = B = 0$. This can be done using the following lemma.

Lemma : Let N be an integer which is divisible by positive integers $n_1, n_2, ..., n_k$. If the least common multiple of $n_1, n_2, ..., n_k$ exceeds N, then $N = 0$.

6. For $n = 1, 2, 3, ...$ find a "closed" expression for the sum

$$\frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + ... + \frac{2n-1}{2^n}.$$

Soln.: $S_n = \frac{1}{2} + \frac{3}{2^2} + \frac{5}{2^3} + ... + \frac{2n-1}{2^n}$

Multiply by 1/2 to obtain

$$\frac{1}{2}S_n = \frac{1}{2^2} + \frac{3}{2^3} + ... + \frac{2n-5}{2^{n-1}} + \frac{2n-3}{2^n} + \frac{2n-1}{2^{n+1}}$$

and now subtract the second equality from the first :

$$S_n - \frac{1}{2}S_n = \frac{1}{2} + \frac{2}{2^2} + \frac{2}{2^3} + ... + \frac{2}{2^{n-1}} + \frac{2}{2^n} - \frac{2n-1}{2^{n+1}}$$

Hence

$$\frac{1}{2}S_n = \frac{1}{2} + \frac{1}{2} + \frac{1}{2^2} + \frac{1}{2^3} + ... + \frac{1}{2^{n-1}} - \frac{2n-1}{2^{n+1}}$$

$$= \frac{1}{2} + \frac{1}{2} \cdot \frac{1-(1/2)^{n-1}}{1-1/2} - \frac{2n-1}{2^{n+1}}$$

and $S_n = 3 - \frac{2n+3}{2^n}$.

7. Let k, m, and n be positive integers with the property: for some number $x \neq 1$, the numbers $\log_k x$, $\log_m x$, $\log_n x$ are consecutive terms of an arithmetic progression. Show that

$$n^2 = (kn)^{\log_k m}.$$

Soln. : By taking logarithms to base n we have to show that

$$2 = (\log_k m)(1 + \log_n k) \qquad ...(1)$$

or $2 = \log_k m + \log_n m$

(using the logarithm property $(\log_c b)(\log_b a) = \log_c a$).

Now from the arithmetic progression condition, we have

$2 \log_m x = \log_n x + \log_k x$

Multiplying both sides by $\log_x m$ and using $(\log_a b)(\log_b a) = 1$ in addition to the previous "log property", we obtain (1).

8. Solve the following system of 100 equations in 100 unknowns:

$$x_1 + x_2 + x_3 = 0$$
$$x_2 + x_3 + x_4 = 0$$
$$\vdots$$
$$x_{98} + x_{99} + x_{100} = 0$$
$$x_{99} + x_{100} + x_1 = 0$$
$$x_{100} + x_1 + x_2 = 0$$

Soln.: Add the 100 equations to obtain

$$3(x_1 + x_2 + ... + x_{100}) = 0.$$

Therefore

$0 = (x_1 + x_2 + x_3) + (x_4 + x_5 + x_6) + ... + (x_{97} + x_{98} + x_{99}) + x_{100}$

$= 0 + 0 + 0 + ... + 0 + x_{100}$,

i.e., $x_{100} = 0$, and similarly $x_1 = x_2 = ... = x_{99} = 0$.

9. Show that for any real numbers x, y and any positive integer n,

(a) $0 \le [nx] - n[x] \le n - 1$,

(b) $[x] + [y] + (n - 1)[x + y] \le [nx] + [ny]$

($[z]$ denotes the greatest integer not exceeding z.)

Soln.: Both (a) and (b) are obviously valid for $n = 1$, so we assume $n > 1$, x can be expressed in the form

$$x = a_0 + \frac{a_1}{n} + \frac{a_2}{n^2} + ... \qquad ...(1)$$

where $a_0, a_1, a_2, ...$ are integers and

$$0 \le a_i \le n - 1 \quad \text{for } i = 1, 2, 3, ...$$

Then $[x] = a_0$

$$nx = na_0 + a_1 + \frac{a_2}{n} + \frac{a_3}{n^2} + ...$$

$[nx] = na_0 + a_1$, and $[nx] - n[x] = a_1$,

establishing (a).

To prove (b), note that (using (1) and a similar expression for y) we can express x and y in the form

$$x = a_0 + \frac{a_1 + \alpha}{n}, \quad y = b_0 + \frac{b_1 + \beta}{n}, \quad 0 \le \alpha, \beta < 1$$

and

$$x + y = a_0 + b_0 + \frac{a_1 + b_1 + \alpha + \beta}{n} \quad 0 \le \alpha, \beta < 1.$$

Either (i) $a_1 + b_1 + \alpha + \beta < n$ or
(ii) $n \le a_1 + b_1 + \alpha + \beta$

In case (i), $[x + y] = a_0 + b_0$, so

$[x] + [y] + (n - 1)[x + y]$

$= a_0 + b_0 + (n - 1)(a_0 + b_0)$

$= na_0 + nb_0 \le na_0 + a_1 + nb_0 + b_1 = [nx] + [ny]$.

In case (ii), since $\alpha + \beta < 2$ we have $a_1 + b_1 > n - 2$, so $a_1 + b_1 \ge n - 1$,
while $n \le a_1 + b_1 + \alpha + \beta < 2n$; hence

$[x + y] = a_0 + b_0 + 1$

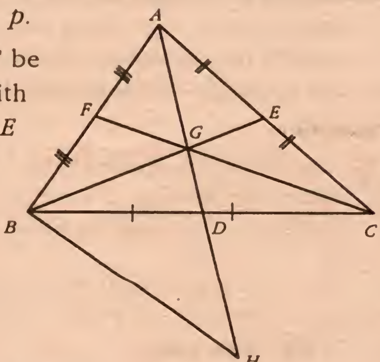
and $[x] + [y] + (n - 1)[x + y]$

$= a_0 + b_0 + (n - 1)(a_0 + b_0 + 1)$

$= na_0 + nb_0 + n - 1$

$\le na_0 + nb_0 + a_1 + b_1 = [nx] + [ny]$.

10. Let p be the perimeter and m the sum of the length of the three medians of any triangle. Prove that $\frac{3}{4}p < m < p$.



Soln.: Let ABC be the triangle, with medians AD, BE and CF. Produce AD to H so that $\overline{AD} = \overline{DH}$.

Then $\Delta BDH \equiv \Delta ADC$, so $\overline{BH} = \overline{AC}$.

By the triangle inequality,

$2\overline{AD} = \overline{AH} < \overline{AB} + \overline{BH} = \overline{AB} + \overline{AC}$.

Similarly, $2\overline{BE} < \overline{AB} + \overline{BC}$ and $2\overline{CF} < \overline{AC} + \overline{BC}$,

and adding these three inequalities, yields

$$\overline{AD} + \overline{BE} + \overline{CF} < \overline{AB} + \overline{BC} + \overline{CA}.$$

On the other hand,

$\overline{BC} < \overline{BG} + \overline{GC} = \frac{2}{3}(\overline{BE} + \overline{CF})$,

$\overline{AC} < \frac{2}{3}(\overline{AD} + \overline{CF})$, $\overline{AB} < \frac{2}{3}(\overline{AD} + \overline{BE})$

and hence

$\overline{AB} + \overline{BC} + \overline{CA} < \frac{4}{3}(\overline{AD} + \overline{BE} + \overline{CF})$. ■

HOW TO BEAT THE CALCULATOR

Calculation Tips & Tricks

Squaring numbers in the 60s

1. Square the last digit (keep the carry) _ _ _ X
2. Multiply the last digit by 12, add the carry _ _ X _
3. The first digits will be 36 plus the carry: X X _ _

Example :

If the number to be squared is 63:

1. Square the last digit (keep the carry): $3 \times 3 = 9$ (keep 3) _ _ _ 9
2. Multiply the last digit by 12, add the carry: $12 \times 3 = 36$ (keep 3) _ _ 6 _
3. The first digits will be 36 plus the carry: 36 (+ carry): $36 + 3 = 39$ 3 9 _ _
4. So **63 × 63 = 3969**.

See the pattern ?

If the number to be squared is 67:

1. Square the last digit (keep the carry): $7 \times 7 = 49$ (keep 4) _ _ _ 9
2. Multiply the last digit by 12, add the carry: $12 \times 7 = 84$, $84 + 4 = 88$ _ _ 8 _
3. The first digits will be 36 plus the carry: 36 (+ carry): $36 + 8 = 44$ 4 4 _ _
4. So **67 × 67 = 4489**.

Squaring numbers in the 70s

1. Square the last digit (keep the carry) : _ _ _ X
2. Multiply the last digit by 14, add the carry : _ _ X _
3. The first digits will be 49 plus the carry: X X _ _

Example :

If the number to be squared is 72:

1. Square the last digit: $2 \times 2 = 4$ _ _ _ 4
2. Multiply the last digit by 14: $14 \times 2 = 28$ (keep the carry) _ _ 8 _
3. The first digits will be 49 plus the carry: 49 (+ carry): $49 + 2 = 51$ 5 1 _ _
4. So **72 × 72 = 5184**.

See the pattern ?

If the number to be squared is 78:

1. Square the last digit (keep the carry): $8 \times 8 = 64$ (keep 6) _ _ _ 4
2. Multiply the last digit by 14, add the carry: $14 \times 8 = 80 + 32 = 112$
   $112 + 6 = 118$ (keep 11) _ _ 8 _
3. The first digits will be 49 plus the carry (11): 49 (+ carry): $49 + 11 = 60$ 6 0 _ _.
4. So **78 × 78 = 6084**.

Squaring numbers in the 80s

1. Square the last digit (keep the carry) _ _ X
2. Multiply the last digit by 16, add the carry _ X _
3. The first digits will be 64 plus the carry: X X _ _

Example :

If the number to be squared is 83:

1. Square the last digit: $3 \times 3 = 9$ _ _ _ 9
2. Multiply the last digit by 16: $16 \times 3 = 30 + 18 = 48$ _ _ 8 _
3. The first digits will be 64 plus the carry: 64 (+ carry): $64 + 4 = 68$ 6 8 _ _
4. So **83 × 83 = 6889**.

See the pattern ?

If the number to be squared is 86:

1. Square the last digit (keep the carry): $6 \times 6 = 36$ (keep 3) _ _ _ 6
2. Multiply the last digit by 16, add the carry: $16 \times 6 = 60 + 36 = 96$ $96 + 3 = 99$ (keep 9) _ _ 9 _
3. The first digits will be 64 plus the carry: 64 (+ carry): $64 + 9 = 73$ 7 3 _ _
4. So **86 × 86 = 7396**.

Squaring numbers in the hundreds

1. Choose a number over 100
2. The last two places will be the square of the last two digits (keep any carry) _ _ _ X X.

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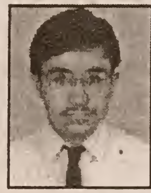
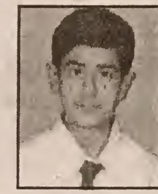


SUCHARIT SARKAR
Kolkata (6th)

ANOSH RAJ
Hyderabad (10th)



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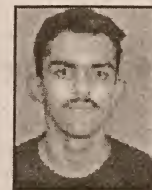

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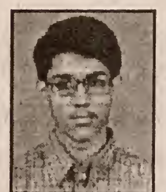

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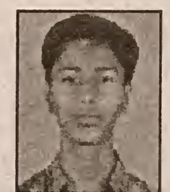

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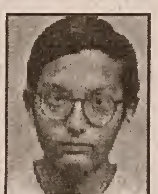

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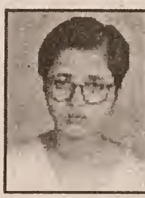

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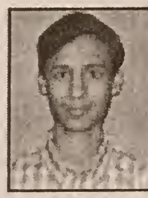

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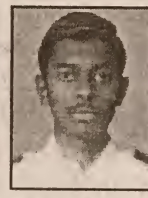

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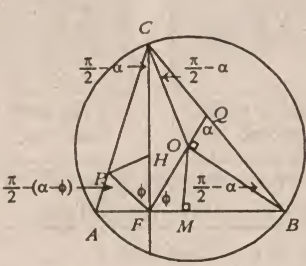
PROBLEMS Of the MONTH

Two problems to test your nerves and your preparation for IIT-JEE Engineering Entrance Exams

1. Let ABC be an acute-angled triangle with $|BC| > |CA|$, and let O be the circumcentre, H its orthocentre, and F the foot of its altitude CH. Let the perpendicular to OF at F meet the side CA at P. Prove that $\angle FHP = \angle BAC$.

Soln. **:** We denote $\angle CFP = \angle OFB = \phi$. M is the mid-point of AB.



Now $OM = R\cos\gamma$, $FM = R\sin(\alpha-\beta)$, so that

$$\tan\varphi = \frac{\cos\gamma}{\sin(\alpha-\beta)} \quad ...(1)$$

From the Law of Sines in ΔCPF,

$CF = 2R\sin\alpha\sin\beta$

$$\angle FCP = \frac{\pi}{2}-\alpha \;;\quad \angle FPC = \frac{\pi}{2}+\alpha-\varphi \qquad (2)$$

so $CP : CF = \sin\varphi : \cos(\alpha-\varphi)$

with (2)

$$CP = \frac{2R\sin\alpha\sin\beta\sin\varphi}{\cos(\alpha-\varphi)} = \frac{2R\sin\alpha\sin\beta}{\cos\alpha\cot\varphi+\sin(\alpha)} \qquad (3)$$

From (1) and (3),

$$CP = \frac{2R\sin\alpha\cos\gamma}{\sin\beta\left(\sin^2\alpha-\cos^2\alpha\right)} = \frac{-2R\sin\alpha\cos\gamma}{\sin\beta\cos2\alpha} \qquad (4)$$

$$OQ \perp OB,\ \angle OBQ = \frac{\pi}{2}-\alpha \Rightarrow \angle OQB = \alpha$$

Now,

$$CQ = a - QB = R\left(2\sin\alpha - \frac{1}{\sin\alpha}\right) = -R\frac{\cos2\alpha}{\sin\alpha}$$

Furthermore, $CH = 2R\cos\gamma$, $CO = R$

It is easy to verify that $CP : CH = CO : CQ$, and

$$-2R\frac{\sin\alpha\cos\gamma}{\sin\beta\cos2\alpha} : 2R\cos\gamma = R : \frac{-R\cos2\alpha}{\sin\alpha} \qquad (5)$$

We also have

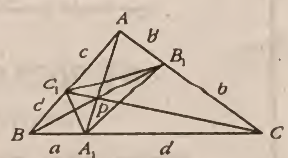
$$\angle PCH = \angle OCQ = \frac{\pi}{2}-\alpha \qquad (6)$$

From (5) and (6), we have that ΔPCH and ΔOCQ are similar. Thus $\angle PHC = \angle OQC = \pi-\alpha$. Thus $\angle FHP = \alpha$.

2. Let ABC be equilateral, and let P be a point in its interior. Let the lines AP, BP, CP meet the sides BC, CA, AB in the points A_1, B_1, C_1 respectively. Prove that

$$A_1B_1 \cdot B_1C_1 \cdot C_1A_1 \geq A_1B \cdot B_1C \cdot C_1A$$

Soln. **:** We put $A_1B = a$, $A_1C = a'$, $B_1C = b$, $B_1A = b'$, $C_1A = c$ and $C_1B = c'$. Then we have by Ceva's Theorem



$$abc = a'b'c' \qquad (1)$$

Since $\angle B_1AC_1 = 60°$ we have

$$B_1C_1^2 = (b')^2 + c^2 - 2b'c\cos60°$$

$$= (b')^2 + c^2 - b'c \geq b'c$$

Similarly we have

$$C_1A_1^2 \geq c'a \text{ and } A_1B_1^2 \geq a'b$$

Multiplying these three inequalities, we get

$$B_1C_1^2 \cdot C_1A_1^2 \cdot A_1B_1^2 \geq b'c \cdot c'a \cdot a'b \qquad (2)$$

From (1) and (2) we have

$$B_1C_1^2 . C_1A_1^2 \cdot A_1B_1^2 \geq a^2b^2c^2$$

Thus we have $\quad B_1C_1 \cdot C_1A_1 \cdot A_1B_1 \geq abc$

That is $\quad B_1C_1 \cdot C_1A_1 \cdot A_1B_1 \geq A_1B \cdot B_1C \cdot C_1A$

■

10 Mathematical Challenges

1. Big fleas have little fleas
Upon their backs that bite 'em,
And little ones have lesser ones,
And so ad infinitum.

If the flea on the bottom weighs $\sqrt{2}$ grams, and every other flea weighs $\sqrt{2-x}$, were x represents the weight of the flea on whose back it rests (while biting, of course), how much does the flea on top weigh ?

Soln.: Let w_n denote the weight of the n^{th} flea (from the bottom), so that

$$w_{n+1} = \sqrt{2-w_n}, \quad n = 0, 1, 2, \ldots, w_1 = \sqrt{2}.$$

If the sequence w_1, w_2, ... converges, to L say, then $L = \sqrt{2-L}$, so L must be 1. We now show that the sequence does indeed converge to 1. It is not difficult to see that w_n is alternately greater than and less than 1, so we examine the relation between every second w_n and 1, as follows:

$$1-w_{n+2}^2 = w_{n+1}-1 = \sqrt{2-w_n}-1 = \frac{1-w_n}{1+\sqrt{2-w_n}}$$

and hence

$$1-w_{n+2} = \frac{1-w_n}{(1+w_{n+2})(1+\sqrt{2-w_n})}.$$

If $w_n < 1$, then $w_{n+2} < 1$ and $1-w_{n+2} < \dfrac{1-w_n}{2}$.

If $w_n > 1$, then

$$w_{n+2} > 1, \ (1+w_{n+2})(1+\sqrt{2-w_n}) > 2$$

and $w_{n+2}-1 < \dfrac{w_n-1}{2}$

Hence $|w_{n+2}-1| < \frac{1}{2}|w_n-1|$. In any case

$|w_n-1| < \dfrac{1}{2^{(n/2)}}$, so $\lim\limits_{n\to\infty}|w_n-1| = 0$.

2. Let r be one of the roots of the quadratic equation $x(1-x) = 1$; the other root is $1-r$. Show that for n = 1, 2, 3, ...

$$r^n+(1-r)^n = \begin{cases} 2(-1)^n & \text{if } 3\mid n \\ (-1)^{n-1} & \text{if } 3\nmid n. \end{cases}$$

Soln.:

Observe that $x^3+1 = (x+1)(x^2-x+1)$,

so $1-r = -r^2$, $r^3 = -1$, $r^4 = -r$, $r^5 = -r^2$, and $r^6 = 1$.

Now $S_n = r^n+(1-r)^n = r^n+(-1)^n r^{2n}$

and we have

$S_{6m} = 1+1 = 2,$

$S_{6m+1} = r-r^2 = 1,$

$S_{6m+2} = r^2+r^4 = -1,$

$S_{6m+3} = r^3-r^6 = -2,$

$S_{6m+4} = r^4+r^8 = -r+r^2 = -1,$

$S_{6m+5} = r^5-r^{10} = -r^2+r = 1$

This agrees with the desired result.

3. The Fibonacci sequence f_1, f_2, f_3, ... is defined by $f_1 = f_2 = 1$, $f_n = f_{n-1}+f_{n-2}$, $n \geq 3$.
Thus the sequence begins

1, 1, 2, 3, 5, 8, 13, 21, 34, 55, ...

Let $Q = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$.

Prove that $Q^n = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}$, $n = 2, 3, 4, \ldots$

Establish the identity :

$$f_{3n} = f_{n+1}^3+f_n^3-f_{n-1}^3, \ n = 1, 2, 3, \ldots$$

Soln.: The proof of the first part is by induction. First note that

$$Q^2 = \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1\cdot1+1\cdot1 & 1\cdot1+1\cdot0 \\ 1\cdot1+1\cdot0 & 1\cdot1+0\cdot0 \end{pmatrix}$$

$$= \begin{pmatrix} 1\cdot1+1\cdot1 & 1\cdot1+1\cdot0 \\ 1\cdot1+1\cdot0 & 1\cdot1+0\cdot0 \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} f_3 & f_2 \\ f_2 & f_1 \end{pmatrix}$$

Now assume that for some $m \geq 2$,

$$Q^m = \begin{pmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{pmatrix}$$

Then

$$Q^{m+1} = Q^m \cdot Q = \begin{pmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{pmatrix}\begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} f_{m+1} + f_m & f_{m+1} \\ f_m + f_{m-1} & f_m \end{pmatrix} = \begin{pmatrix} f_{m+2} & f_{m+1} \\ f_{m+1} & f_m \end{pmatrix}$$

and hence

$$Q^m = \begin{pmatrix} f_{m+1} & f_m \\ f_m & f_{m-1} \end{pmatrix},$$

$m = 2, 3, 4, \ldots$ Now for the second part. Since

$$Q^{3n} = \begin{pmatrix} f_{3n+1} & f_{3n} \\ f_{3n} & f_{3n-1} \end{pmatrix},$$

and $$(Q^n)^3 = \begin{pmatrix} f_{n+1} & f_n \\ f_n & f_{n-1} \end{pmatrix}^3$$

$$= \begin{pmatrix} f_{n+1}^3 + 2f_{n+1}f_n^2 + f_n^2 f_{n-1} & f_{n+1}^2 f_n + f_{n+1}f_n f_{n-1} + f_n^3 + f_n f_{n-1}^2 \\ f_{n+1}^2 f_n + f_{n+1}f_n f_{n-1} + f_n^3 + f_n f_{n-1}^2 & f_{n+1}f_n^2 + 2f_n^2 f_{n-1} + f_{n-1}^3 \end{pmatrix}$$

we equate the entries in row 1, column 2 to find that

$$\begin{aligned} f_{3n} &= f_{n+1}^2 f_n + f_{n+1}f_n f_{n-1} + f_n^3 + f_n f_{n-1}^2 \\ &= f_{n+1}f_n(f_{n+1} + f_{n-1}) + f_n^3 + f_n f_{n-1}^2 \\ &= f_{n+1}(f_{n+1} - f_{n-1})(f_{n+1} + f_{n-1}) + f_n^3 + f_n f_{n-1}^2 \\ &= f_{n+1}(f_{n+1}^2 - f_{n-1}^2) + f_n^3 + f_n f_{n-1}^2 \\ &= f_{n+1}^3 - f_{n+1}f_{n-1}^2 + f_n^3 + f_n f_{n-1}^2 \\ &= f_{n+1}^3 + f_n^3 - f_{n-1}^2(f_{n+1} - f_n) \\ &= f_{n+1}^3 + f_n^3 - f_{n-1}^3. \end{aligned}$$

4. Prove that the function

$$f(x,y) = \frac{(x+y)(x+y+1)}{2} + x$$

is a one-to one map from the set

$$\{(x,y) \mid x, y \text{ integers} \geq 0,\ x^2 + y^2 > 0\}$$

(the lattice points other than (0, 0) in the first quadrant) onto the set

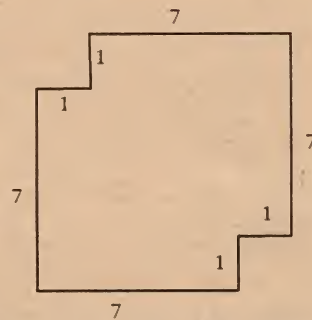
$$\{m \mid m \text{ integral and } > 0\}$$

of positive integers.

Soln.: For $n \geq 1$, the points of the sequence L_n : (0, n), (1, $n - 1$), (2, $n - 2$), ..., (n, 0) (the points on the line $x + y = n$) map onto the integers in the sequence

$$I_n : \frac{n(n+1)}{2}, \frac{n(n+1)}{2} + 1, \frac{n(n+1)}{2} + 2, \ldots, \frac{n(n+1)}{2} + n.$$

It is easy to see that the sets $I_1, I_2, \ldots$ are disjoint and their union is the set of all positive integers.



5. Two equal regular tetrahedra intersect in such a way that each face of either passes through the midpoints of three concurrent edges of the other. The union U of the two tetrahedra is a three-dimensional star. Describe the intersection V of the two tetrahedra. Determine the ratio of the volumes of U and V.

Soln.: Each face of V is an equilateral triangle whose side length is half that of either tetrahedron. There are 8 such triangles, each contained in a face of one of the intersecting tetrahedra. Because of symmetry, V must be an octahedron.

Let v be the volume of each tetrahedral point of star U. The volume of each intersecting tetrahedron is $2^3 v = 8v$, so the volume of V is $8v - 4v = 4v$. Since U is made up of 8 tetrahedral points and the octahedron V, the volume of U is $4v + 8v = 12v$.

Hence $$\frac{\text{volume } U}{\text{volume } V} = \frac{12v}{4v} = 3.$$

6. Prove that for $n = 1, 2, 3, \ldots$

$$n^4 + 2n^3 + 2n^2 + 2n + 1$$

is not the square of an integer.

Soln.: Since

$$\begin{aligned} (n^2 + n)^2 &= n^4 + 2n^3 + n^2 \\ &< n^4 + 2n^3 + 2n^2 + 2n + 1 \\ &< n^4 + 2n^3 + 3n^2 + 2n + 1 = (n^2 + n + 1)^2, \end{aligned}$$

we see that $n^4 + 2n^3 + 2n^2 + 2n + 1$ is between two consecutive squares and so cannot itself be a square.

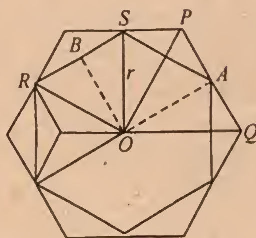
7.(a) If a regular hexagon and an equilateral triangle have the same perimeter, determine the ratio of their areas.

(b) Given a circle, determine the ratio of the area of the circumscribed regular hexagon to the area of the inscribed regular hexagon.

Soln.:

(a) The ratio of the area of the hexagon to that of the triangle is $\frac{6}{4}=\frac{3}{2}$.

(b) The ratio of the areas of the two hexagons is that of the areas of triangles OPQ and ORS, which is equal to $\frac{\overline{OA}^2}{\overline{OB}^2}$. If r is the radius of the circle, then



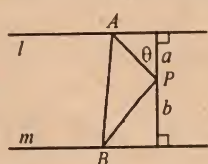
$\overline{OA}=r,\ \overline{OB}=\frac{\sqrt{3}r}{2}$ and

$$\frac{\overline{OA}^2}{\overline{OB}^2}=\frac{4}{3}.$$

An alternative quick solution to (b) can be seen from figure.

8. Let l and m be parallel lines and P a point between them. Find the triangle APB of smallest area, with A on l, B on m, and $\angle APB = 90°$.

Soln.: Let PAB be a right angled triangle, and angle θ as indicated in figure. The area of triangle PAB is



$$\frac{1}{2}(\overline{AP})(\overline{PB}) = \frac{1}{2}\left(\frac{a}{\cos\theta}\right)\left(\frac{b}{\sin\theta}\right) = \frac{ab}{\sin 2\theta}.$$

and this function assumes its minimum value, ab, when $\theta = 45°$.

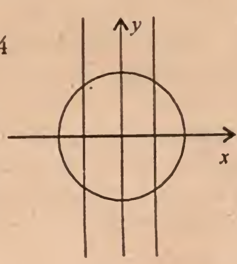
9. Sketch the graph of the curve

$$|3x^2+y^2-12| = |x^2-y^2+4|.$$

Soln.: If $3x^2 + y^2 - 12$ and $x^2 - y^2 + 4$ are both positive or both negative then the equation becomes

$3x^2 + y^2 - 12 = x^2 - y^2 + 4$
or $x^2 + y^2 = 8$

Is every point on the circle $x^2 + y^2 = 8$ also on the given curve? The answer is yes, for if (x, y) satisfies $x^2 + y^2 = 8$, then



$|3x^2 + y^2 - 12| = |2x^2 - 4|$
$= |x^2 - y^2 + 4|.$

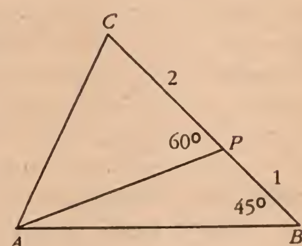
If $3x^2 + y^2 - 12$ and $x^2 - y^2 + 4$ have opposite signs, then the equation becomes

$3x^2 + y^2 - 12$
$= -(x^2 - y^2 + 4)$

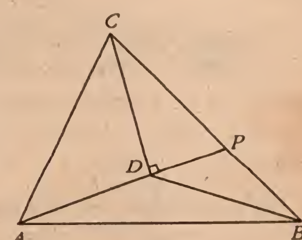
or $|x| = \sqrt{2}$.

Furthermore, it is easy to verify that if (x, y) lies on the curve $|x| = \sqrt{2}$, then it lies on the given curve. Hence the graph we seek is the union of the circle $x^2 + y^2 = 8$ and the lines $x=\sqrt{2}$ and $x=-\sqrt{2}$.

10. The point P divides the BC or triangle ABC into ratio $\frac{\overline{BP}}{\overline{PC}}=\frac{1}{2}$, and $\angle CBA = 45°$ while $\angle APC = 60°$. Find $\angle ACB$ without the use of trigonometry.



Soln.: Let D be the foot of the perpendicular from C to AP, so that $D \neq P$ and $\angle PCD = 30°$. Then $\overline{PD}=\frac{1}{2}\overline{PC}=\overline{PB}$ so triangle BPD is isosceles and so triangle BPD is isosceles and



$\angle PBD = \angle PDB = 30°$. Furthermore,

$\angle DBA = \angle CBA - \angle PBD$
$= 45° - 30° = 15°,$

and

$$30° = \angle PDB = \angle DAB + \angle DBA = \angle DAB + 15°.$$

Then $\angle DAB = 15°$, $\overline{AD}=\overline{BD}=\overline{CD}$,

so $\angle DAC = \angle DCA = 45°$ and $\angle ACB = 75°$.

■■

Irrationality of SQUARE ROOT OF 2

Professor Richard Palais of Brandeis University says.

The (classical) proof of irrationality of the square root of two is clear and elegant enough, but hardly the shortest or most revealing.

In general, proving a negative statement such as, "the square root of two is not rational" can be quite difficult. Often in mathematics, such a statement is proved by contradiction, and that is what we will do here. In other words, we want to assume the contrary statement, namely, we want to begin by assuming that the square root of two is rational, and then see what happens. If we find that this assumption leads to an inescapable contradiction, then we will know that this contrary assumption is false–and we will be able to conclude the statement we are trying to prove.

To begin with, we observe that if $\sqrt{2}$ is rational, then there is some positive integer q such that $q \times \sqrt{2}$ is an integer. Since the positive integers are well ordered, we may suppose that q is the smallest such number.

We next observe that since $1 < \sqrt{2} < 2$, then $\sqrt{2} - 1 < 1$, and consequently

$q \times (\sqrt{2} - 1) = (q \times \sqrt{2} - q)$ is less than q. Let us call this new number r, and observe that it too is a positive integer. But we now have $r \times \sqrt{2}$ is also an integer, since

$r \times \sqrt{2} = (q \times \sqrt{2} - q) \times \sqrt{2} = (2q - q \times \sqrt{2})$. In short, r is a positive integer less than q and $r \times \sqrt{2}$ is an integer. But we said that q was the smallest positive integer with this property, and so we have a contradiction. The nice thing about the proof is how easily it can be generalized. Let us denote $[\sqrt{n}]$ the integer part of $\sqrt{n}$. For example, since the square root of 5 is approximately 2.236, the integer part is 2. For any n. that is not a perfect square, we can prove that $\sqrt{n}$ is irrational exactly as above by considering $9 \times (\sqrt{n} - [\sqrt{n}])$.

For another proof let us assume again that square root of two is rational. By the definition rational of numbers, it means assuming that we can represent the square root of two by a ratio of integers, i.e.,

$\sqrt{2} = \dfrac{p}{q}$ ...where p and q are each integers. If this equation is true, then there is nothing wrong with squaring both sides of the equation, as follows,

$$(\sqrt{2})^2 = \left(\frac{p}{q}\right)^2$$

$$\Downarrow$$

$$2 = \frac{p^2}{p^2}$$

we can multiply both sides by q^2.

$$2 \times q^2 = \frac{p^2}{q^2} \times q^2 \Rightarrow 2q^2 = p^2$$

Now we want to notice something interesting. According to the Fundamental Theorem of Arithmetic, each of the integers p and q factors uniquely into primes. That is, each of p and q is just some unique collection of primes multiplied together. Since in the above equation both p and q are squared, that means that each of these primes must occur in pairs, and moreover there must be exactly the same pairs of each of the primes on each side of the equation, else it wouldn't be a true equation. But this can't be! Thus again our assumption is wrong implies $\sqrt{2}$ is irrational.

This result, among the most elegant in mathematics, was known to the Greeks and is therefore quite ancient. A slightly more general argument along the same lines shows that all square roots of integers are irrational, except when the integers are perfect squares (1, 4, 9, 16, 25 etc.).

■■

5 CHALLENGING PROBLEMS

COMPLEX NUMBERS

1. Solve the system of equations:

$$\left|\frac{z-12}{z-8i}\right| = \frac{5}{3}, \quad \left|\frac{z-4}{z-8}\right| = 1.$$

2. Show that the points z_1, z_2, z_3, z_4 taken in order are concyclic iff $\dfrac{(z_3-z_1)(z_4-z_2)}{(z_3-z_2)(z_4-z_1)}$ is purely real.

Hence show that if $z_1z_2 + z_3z_4 = 0$ and $z_1 + z_2 = 0$, then z_1, z_2, z_3, z_4 are concyclic.

3. Find all the roots of the equation

$$(3z-1)^4 + (z-2)^4 = 0$$

in the simplified form of $a + ib$.

4. Show that the circumcentre of the triangle whose vertices are given by the complex numbers z_1, z_2, z_3 is given by $z = \dfrac{\sum |z_1|^2 (z_2 - z_3)}{\sum \bar{z}_1 (z_2 - z_3)}$.

5. For every real number $c \ge 0$, find all the complex numbers z which satisfy the equation

$$|z|^2 - 2iz + 2c(1+i) = 0.$$

SOLUTION

1. Putting $z = x + iy$, the given equations are equivalent to

$3|x + iy - 12| = 5|x + iy - 8i|$, ...(1)

and $|x + iy - 4| = |x + iy - 8|$ ...(2)

From (1) we get

$9\{(x-12)^2 + y^2\} = 25\{x^2 + (y-8)^2\}$

i.e. $16x^2 + 16y^2 + 216x - 400y + 304 = 0$

or $2x^2 + 2y^2 + 27x - 50y + 38 = 0$, ...(3)

which is a circle

Equation (2) gives

$(x-4)^2 + y^2 = (x-8)^2 + y^2$,

or $-8x + 16 = -16x + 64$ or $x = 6$ ...(4)

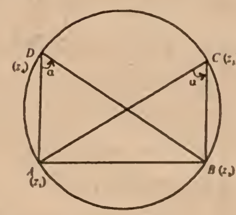
Required points are the points of intersection of (3) and (4), so putting $x = 6$ in (3), we get

$2y^2 - 25y + 136 = 0$ or $(y-8)(y-17) = 0$

$\therefore\ x = 6,\ y = 8, 17.$

Hence $6 + 8i$ and $6 + 17i$ are two solutions.

2. Let A, B, C and D be the points represented by the complex numbers z_1, z_2, z_3 and z_4 respectively.



I. Let A, B, C, D be concyclic

Then $\angle ADB = \angle ACB$

$$\therefore\ \frac{z_2 - z_3}{z_1 - z_3} = \frac{BC}{AC}e^{i\alpha} \quad \text{and} \quad \frac{z_2 - z_4}{z_1 - z_4} = \frac{BD}{AD}e^{i\alpha}$$

Dividing we get

$$\frac{(z_2 - z_4)(z_1 - z_3)}{(z_1 - z_4)(z_2 - z_3)} = \frac{AC \cdot BD}{BC \cdot AD} = \text{real}$$

II. Conversely, let us assume the condition

$$\frac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)} = \text{real} = k \text{ (say)}$$

$$\therefore\ \frac{z_2 - z_3}{z_1 - z_3} = \frac{1}{k}\,\frac{z_2 - z_4}{z_1 - z_4},\ k \text{ is real.}$$

So $\arg\left(\dfrac{z_2 - z_3}{z_1 - z_3}\right) = \arg\left(\dfrac{z_2 - z_4}{z_1 - z_4}\right)$

$\angle ACB = \angle ADB$

Hence, A, B, C, D are concyclic.

Second part. Given that

$z_1 z_2 + z_3 z_4 = 0$, and $z_1 + z_2 = 0$

From the first part, z_1, z_2, z_3, z_4 are concyclic if

$\dfrac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}$ is real

Now $\dfrac{(z_3 - z_1)(z_4 - z_2)}{(z_3 - z_2)(z_4 - z_1)}$

$$= \frac{z_3z_4 + z_1z_2 - z_3z_2 - z_1z_4}{z_3z_4 + z_1z_2 - z_3z_4 - z_3z_1}$$

$$= \frac{z_2z_3 + z_1z_4}{z_1z_3 + z_2z_4} \qquad (\text{using } z_1z_2 + z_3z_4 = 0)$$

$$= \frac{z_2(z_3 - z_4)}{-z_2(z_3 - z_4)} \qquad (\text{using } z_1 + z_2 = 0)$$

$= -1$ (purely real).

Hence z_1, z_2, z_3, z_4 are concyclic.

3. $\left[\dfrac{3z-1}{z-2}\right]^4 = -1 = e^{i\pi} = e^{i(2n+1)\pi}$

$\therefore\ \dfrac{3z-1}{z-2} = e^{i\left\{(2n+1)\frac{\pi}{4}\right\}} \qquad n = 0,\ 1,\ 2,\ 3$

If $\dfrac{3z-1}{z-2} = e^{i\alpha}$ where $\alpha = (2n+1)\dfrac{\pi}{4}$,

then $\dfrac{3(z-2)+5}{z-2} = 3+\dfrac{5}{z-2} = e^{i\alpha}$

$\therefore\ z = \dfrac{5}{e^{i\alpha}-3}+2 = \dfrac{2e^{i\alpha}-1}{e^{i\alpha}-3} = \dfrac{(2e^{i\alpha}-1)(e^{-i\alpha}-3)}{(e^{i\alpha}-3)(e^{-i\alpha}-3)}$

$= \dfrac{5-6e^{i\alpha}-e^{-i\alpha}}{10-3(e^{i\alpha}+e^{-i\alpha})}$

$= \dfrac{5-6\cos\alpha-6i\sin\alpha-\cos\alpha+i\sin\alpha}{10-6\cos\alpha}$

$= \dfrac{5-7\cos\alpha-5i\sin\alpha}{10-6\cos\alpha}$ For $n = 0$, $\alpha = \pi/4$

$\therefore\ z = \dfrac{5-\frac{7}{\sqrt{2}}-\frac{5i}{\sqrt{2}}}{10-\frac{6}{\sqrt{2}}}$

i.e., $z = \dfrac{5\sqrt{2}-7}{10\sqrt{2}-6}-\dfrac{5}{10\sqrt{2}-6}i$

$= \dfrac{(29-20\sqrt{2})}{82} - \dfrac{5(5\sqrt{2}+3)i}{82}$ ...(1)

For $n = 1$, $\alpha = 3\pi/4$

$\therefore\ z = \dfrac{5\sqrt{2}+7}{10\sqrt{2}+6}-\dfrac{5}{10\sqrt{2}+6}i$

$= \dfrac{(29+20\sqrt{2})}{82} - \dfrac{5(5\sqrt{2}-3)i}{82}$ ...(2)

$n=2,\ \alpha = \dfrac{5\pi}{4},\ z = \dfrac{5\sqrt{2}+7}{10\sqrt{2}+6}+\dfrac{5}{10\sqrt{2}+6}i$

$= \dfrac{(29+20\sqrt{2})}{82} + \dfrac{5(5\sqrt{2}-3)i}{82}$ ...(3)

$n=3,\ \alpha = \dfrac{7\pi}{4},\ z = \dfrac{5\sqrt{2}-7}{10\sqrt{2}-6}+\dfrac{5}{10\sqrt{2}-6}i$

$= \dfrac{(29-20\sqrt{2})}{82} + \dfrac{5(5\sqrt{2}+3)i}{82}$ ...(4)

4. Let $O(z)$ be the circumcentre, then

$|z-z_1| = |z-z_2| = |z-z_3|$. .... (1)

Now $|z-z_1| = |z-z_2| \Rightarrow |z-z_1|^2 = |z-z_2|^2$

$\Rightarrow (z-z_1)(\bar{z}-\bar{z}_1) = (z-z_2)(\bar{z}-\bar{z}_2)$

$\Rightarrow |z|^2 - z\bar{z}_1 - z_1\bar{z} + |z_1|^2 = |z|^2 - z\bar{z}_2 - \bar{z}z_2 + |z_2|^2$

$\Rightarrow z(\bar{z}_1-\bar{z}_2)+\bar{z}(z_1-z_2) = |z_1|^2-|z_2|^2$ ....(2)

Similarly, using $|z-z_1| = |z-z_3|$

$z(\bar{z}_1-\bar{z}_3)+\bar{z}(z_1-z_3) = |z_1|^2-|z_3|^2$ ....(3)

Multiplying (2) by $(z_1 - z_3)$ and (3) by $(z_1 - z_2)$ and subtracting, we get

$z\left[(\bar{z}_1-\bar{z}_2)(z_1-z_3)-(\bar{z}_1-\bar{z}_3)(z_1-z_2)\right]$

$= |z_1|^2(z_2-z_3)-|z_2|^2(z_1-z_3)+|z_3|^2(z_1-z_2)$

or $z\left[\bar{z}_1(z_2-z_3)+\bar{z}_2(z_3-z_1)+\bar{z}_3(z_1-z_2)\right]$

$= |z_1|^2(z_2-z_3)+|z_2|^2(z_3-z_1)+|z_3|^2(z_1-z_2)$ ...(4)

Hence $z = \dfrac{\sum|z_1|^2(z_2-z_3)}{\sum\bar{z}_1(z_2-z_3)}$.

5. Let $z = x + iy$, then $|z|^2 = x^2+y^2$ and the equation becomes

$x^2+y^2-2ix+2y+2c+2ci = 0, c \geq 0$ ...(1)

Equating the real and imaginary parts to zero, we obtain the following system of equations:

$x^2 + y^2 + 2y + 2c = 0,$ ...(2)

$-2x + 2c = 0$ ...(3)

From (3) $x = c$, and then from (2) we have

$y^2 + 2y + 2c + c^2 = 0$ ....(4)

since y is real, we should have

$4 - 4(2c + c^2) \geq 0$ or $c^2 + 2c - 1 \leq 0$,

i.e. $(c + 1)^2 \leq (\sqrt{2})^2$. This gives

$-\sqrt{2}-1 \leq c \leq \sqrt{2}-1$; as $c \geq 0$

We have the required condition for y to be real as

$0 \leq c \leq \sqrt{2}-1$ ...(5)

Thus if $0 \leq c \leq \sqrt{2}-1$

$y = \frac{1}{2}\left[-2 \pm \sqrt{\{4-4(2c+c^2)\}}\right]$

i.e. $y = -1 \pm \sqrt{(1-2c-c^2)}$, ...(6)

and for $c > \sqrt{2}-1$, y gives imaginary values which are inadmissible.

Hence for $0 \leq c \leq \sqrt{2}-1$,

$z = c+\left[-1\pm\sqrt{(1-2c-c^2)}\right]i$

and for $c > \sqrt{2}-1$, there are no solutions.

■

Maths Clinic

If you have any difficult / unsolved problem or you are unable to understand one, then write to us. Our team of experts will diagnose your problems. The diagnosed problems will be published in the subsequent issues.

1. Solve the equation

$$\tan^{-1}\left(\frac{x+1}{x-1}\right)+\tan^{-1}\left(\frac{x-1}{x}\right)=\tan^{-1}(-7).$$

[Vidya Bhushan, Nalanda]

Soln.: Taking tan of both sides of the equation, we get

$$\frac{\left(\frac{x+1}{x-1}+\frac{x-1}{x}\right)}{1-\left(\frac{x+1}{x-1}\right)\left(\frac{x-1}{x}\right)}=-7$$

$\Rightarrow \quad \dfrac{2x^2-x+1}{1-x}=-7$

$\Rightarrow \quad 2x^2 - x + 1 = -7(1 - x)$

$\Rightarrow \quad 2x^2 - 8x + 8 = 0$

or $\quad x^2 - 4x + 4 = 0$

or $\quad (x - 2)^2 = 0. \quad \therefore \quad x = 2.$

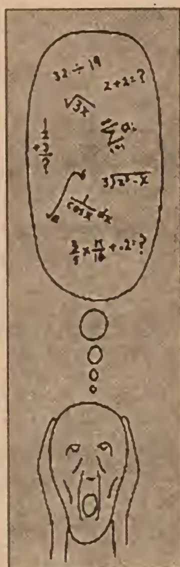
Now let us verify the solution that we have just obtained:

L.H.S. $= \tan^{-1}3 + \tan^{-1}(1/2)$

$= \pi + \tan^{-1}\left(\dfrac{3+(1/2)}{1-(3/2)}\right)$

$= \pi + \tan^{-1}(-7) \neq$ R.H.S.

Since $x = 2$ is not the solution. So the equation has no solution.



2. Solve : $(5+2\sqrt{6})^{x^2-3}+(5-2\sqrt{6})^{x^2-3}=10.$

[Nitin Khadsare, Satara (M.S)]

Soln.: Clearly $(5+2\sqrt{6})(5-2\sqrt{6})=5^2-(2\sqrt{6})^2$
$= 25 - 24 = 1.$

$\therefore \quad 5-2\sqrt{6}=\dfrac{1}{5+2\sqrt{6}}$

$\therefore$ Equation is $(5+2\sqrt{6})^{x^2-3}+\left(\dfrac{1}{5+2\sqrt{6}}\right)^{x^2-3}=10$

Let $(5+2\sqrt{6})=a \quad \Rightarrow \quad a^{x^2-3}+\left(\dfrac{1}{a}\right)^{x^2-3}=10$

$\Rightarrow \quad (a^{x^2-3})^2-10\cdot a^{x^2-3}+1=0$

$\Rightarrow \quad a^{x^2-3}=\dfrac{10\pm\sqrt{100-4}}{2}=5\pm2\sqrt{6}$

$\therefore \quad (5+2\sqrt{6})^{x^2-3}=5+2\sqrt{6},\ 5-2\sqrt{6}$

$\Rightarrow \quad x^2 - 3 = 1, -1 \quad \Rightarrow \quad x^2 = 2$ or $4.$

$\Rightarrow \quad x = \pm 2, \pm\sqrt{2}.$

3. What is the relation between continuity and differentiability?

[Sumit Kumar Panchbhi, Jharkhand]

Soln.: $f(x)$ is differentiable (finitely) at $x = a \quad \Rightarrow f(x)$ is continuous at $x = a$

But $f(x)$ is not continuous at $x = a$

$\Rightarrow \quad f(x)$ is not differentiable at $x = a$.

While examining the continuity and differentiability of a function $f(x)$ at a point $x = a$ if you start with the differentiability and find that $f(x)$ is differentiable than you can conclude that the function is also continuous. But if you find $f(x)$ is not differentiable at $x = a$, you will also have to check the continuity separately.

Instead, if you start with continuity and find that it is non continuous implies the function is also non-differentiable. But if you find $f(x)$ is continuous, you will also have to check the differentiability separately.

4. If $a + b + c = 1$, prove that

$$\frac{8}{27abc}>\left(\frac{1}{a}-1\right)\left(\frac{1}{b}-1\right)\left(\frac{1}{c}-1\right)>8.$$

[Jay Wadhwa, Ujjan]

Soln.: Multiply the inequality by abc, thus now we have to prove

$$\frac{8}{27}>(1-a)(1-b)(1-c)>8abc$$

Now

$$\frac{(1-a)+(1-b)+(1-c)}{3}>[(1-a)(1-b)(1-c)]^{1/3}$$

$\Rightarrow \quad \dfrac{3-(a+b+c)}{3}>[(1-a)(1-b)(1-c)]^{1/3}$

or, $\left(\dfrac{2}{3}\right)^3>(1-a)(1-b)(1-c) \quad \because a + b + c = 1$

$\Rightarrow \quad \dfrac{8}{27}>(1-a)(1-b)(1-c) \quad \ldots$ (i)

Also we have

$$\frac{a+b}{2} > \sqrt{ab},\ \frac{b+c}{2} > \sqrt{bc},\ \frac{c+a}{2} > \sqrt{ac}$$

$\Rightarrow$ By multiplying these we get,

$$\frac{(a+b)(b+c)(c+a)}{8} > abc$$

$\Rightarrow$ $(1-a)(1-b)(1-c) > 8abc$ ... (ii)

From (i) and (ii), we get the required inequality.

5. Prove that $^mC_0\,^nC_k + {}^mC_1\,^nC_{k-1} + \ldots\ {}^mC_k\,^nC_0 = {}^{m+n}C_k$, where m, n, k are positive integers and $^pC_q = 0$ for $p < q$. *[Raj Babadur Patel, M.P]*

Soln.: We have to prove the result.

$$^mC_0\,^nC_k + {}^mC_1\,^nC_{k-1} + {}^mC_2\,^nC_{k-2} + \ldots + {}^mC_{k-1}\,^nC_1 + {}^mC_k\,^nC_0$$

$= {}^{m+n}C_k,\ k \le (m, n),\ \forall\ m, n \in N$ ... (i)

For $m = n = 1$, since $k \le (m, n)$

So $0 < k \le 1$, so $k = 1$.

L.H.S. $= {}^1C_0\,^1C_1 + {}^1C_1\,^1C_0 = 2,$

R.H.S. $= {}^{1+1}C_1 = {}^2C_1 = 2.$

$\therefore$ (i) holds for $m = n = 1$ ... (ii)

Assume the result (i) for $m = p$, $n = q$,

i.e. $^pC_0\,^qC_k + {}^pC_1\,^qC_{k-1} + {}^pC_2\,^qC_{k-2} + \ldots + {}^pC_{k-1}\,^qC_1 + {}^pC_k\,^qC_0$

$= {}^{p+q}C_k,\ k \le (p, q)$ ... (iii)

For $m = p + 1$, $n = q + 1$,

L.H.S. $= {}^{p+1}C_0\,^{q+1}C_k + {}^{p+1}C_1\,^{q+1}C_{k-1} + {}^{p+1}C_2\,^{q+1}C_{k-2} + \ldots + {}^{p+1}C_{k-1}\,^{q+1}C_1 + {}^{p+1}C_k\,^{q+1}C_0$

$= 1\cdot[{}^qC_{k-1} + {}^qC_k] + [{}^pC_0 + {}^pC_1]\,[{}^qC_{k-2} + {}^qC_{k-1}]$
$+ [{}^pC_1 + {}^pC_2]\,[{}^qC_{k-3} + {}^qC_{k-4}] + \ldots + [{}^pC_{k-1} + {}^pC_k]\cdot 1$
$= [{}^pC_0\,^qC_k + {}^pC_1\,^qC_{k-1} + {}^pC_2\,^qC_{k-2} + \ldots]$
$+ [{}^pC_0\,^qC_{k-1} + {}^pC_1\,^qC_{k-2} + {}^pC_2\,^qC_{k-3} + \ldots]$
$+ [{}^pC_0\,^qC_{k-1} + {}^pC_1\,^qC_{k-2} + {}^pC_2\,^qC_{k-2} + \ldots]$
$+ [{}^pC_0\,^qC_{k-2} + {}^pC_1\,^qC_{k-3} + {}^pC_2\,^qC_{k-4} + \ldots]$

Using induction hypothesis (iii), we get

L.H.S. $= {}^{p+q}C_k + {}^{p+q}C_{k-1} + {}^{p+q}C_{k-2}$
$= {}^{p+q+1}C_k + {}^{p+q+1}C_{k-1}$
$= {}^{p+q+2}C_k$
$=$ R.H.S. for $m = p + 1$, $n = q + 1$ ... (iv)

Hence by mathematical induction, we have the result for all $m, n \in N$.

■

HOW TO BEAT THE CALCULATOR

Calculation Tips & Tricks

Squaring numbers in the 600s

1. Choose a number in the 600s (practice with smaller numbers, then progress to larger ones).
2. The first two digits of the square are 36: 3 6 _ _ _ _
3. The next two digits will be 12 times the last 2 digits: _ _ X X _ _
4. The last two places will be the square of the last two digits: _ _ _ _ X X

Example:

If the number to be squared is **607**:

1. The first two digits are 36: 3 6 _ _ _ _
2. The next two digits are 12 times the last 2 digits: 12 × 07 = 84: _ _ 8 4 _ _
3. Square the last 2 digits: 7 × 7 = 49: _ _ _ _ 4 9
4. So **607 × 607 = 368,449**.

For larger numbers reverse the steps:

If the number to be squared is **625**:

1. Square the last two digits (keep carry): 25 × 25 = 625 (keep 6): _ _ _ _ 2 5
2. 12 times the last 2 digits + carry: 12 × 25 = 250 + 50 = 300 + 6 = 306: _ _ 0 6 _ _
3. 36 + carry: 36 + 3 = 39: 3 9 _ _ _ _
4. So **625 × 625 = 390,625**.

Squaring numbers in the 700s

1. Choose a number in the 700s (practice with smaller numbers, then progress to larger ones).
2. Square the last two digits (keep the carry): _ _ _ _ X X
3. Multiply the last two digits by 14 and add the carry: _ _ X X _ _
4. The first two digits will be 49 plus the carry: X X _ _ _ _

Example:

If the number to be squared is **704**:

1. Square the last two digits (keep the carry): 4 × 4 = 16: _ _ _ _ 1 6
2. Multiply the last two digits by 14 and add the carry: 14 × 4 = 56: _ _ 5 6 _ _
3. The first two digits will be 49 plus the carry: 4 9 _ _ _ _
4. So **704 × 704 = 495,616**.

See the pattern?

If the number to be squared is **725**:

1. Square the last two digits (keep the carry): 25 × 25 = 625: _ _ _ _ 2 5
2. Multiply the last two digits by 14 and add the carry: 14 × 25 = 10 × 25 + 4 × 25 = 250 + 100 = 350. 350 + 6 = 356: _ _ 5 6 _
3. The first two digits will be 49 plus the carry: 49 + 3 = 52: 5 2 _ _ _ _
4. So **725 × 725 = 525,625**.

Squaring numbers between 800 and 810

1. Choose a number between 800 and 810.
2. Square the last two digits: _ _ _ _ X X
3. Multiply the last two digits by 16 (keep the carry): _ _ X X _ _
4. Square 8, add the carry: X X _ _ _ _

Example:

If the number to be squared is **802** :

1. Square the last two digits: 2 × 2 = 4: _ _ _ _ 0 4
2. Multiply the last two digits by 16: 16 × 2 = 32: _ _ 3 2 _ _
3. Square 8: 6 4 _ _ _ _
4. So **802 × 802 = 643,204**.

See the pattern?

If the number to be squared is **807**:

1. Square the last two digits: 7 × 7 = 49: _ _ _ _ 4 9
2. Multiply the last two digits by 16 (keep the carry): 16 × 7 = 112: _ _ 1 2 _ _

3. Square 8, add the carry (1): 6 5 _ _ _ _
4. So **807 × 807 = 651, 249**.

Squaring numbers in the 900s

1. Choose a number in the 900s - start out easy with numbers near 1000; then go lower when expert.
2. Subtract the number from 1000 to get the difference.
3. The first three places will be the number minus the difference: X X X _ _ _.
4. The last three places will be the square of the difference: _ _ _ X X X (if 4 digits, add the first digit as carry).

Example:

If the number to be squared is **985**:

1. Subtract 1000 - 985 = 15 (difference)
2. Number - difference: 985 - 15 = 970: 9 7 0 _ _ _
3. Square the difference: 15 × 15 = 225: _ _ _ 2 2 5
4. So **985 × 985 = 970225**.

See the pattern?

If the number to be squared is **920**:

1. Subtract 1000 - 920 = 80 (difference)
2. Number - difference: 920 - 80 = 840: 840 _ _ _
3. Square the difference: 80 × 80 = 6400: _ _ _ 4 0 0
4. Carry first digit when four digits: 8 4 6 _ _ _
5. So **920 × 920 = 846400**.

Squaring numbers between 1000 and 1100

1. Choose a number between 1000 and 1100.
2. The first two digits are: 1,0 _ _, _ _ _
3. Find the difference between your number and 1000.
4. Multiply the difference by 2: 1,0 X X, _ _ _
5. Square the difference: 1,0 _ _, X X X

Example:

If the number to be squared is **1007**:

1. The first two digits are: 1,0 _ _, _ _ _
2. Find the difference: 1007 – 1000 = 7
3. Two times the difference: 2 × 7 = 14: 1,0 1 4, _ _ _
4. Square the difference: 7 × 7 = 49: 1,0 1 4, 0 4 9
5. So **1007 × 1007 = 1,014,049**.

See the pattern?

If the number to be squared is **1012**:

1. The first two digits are: 1,0 _ _, _ _ _
2. Find the difference: 1012 - 1000 = 12
3. Two times the difference: 2 × 12 = 24: 1,0 2 4, _ _ _
4. Square the difference: 12 × 12 = 144: 1,0 2 4, 1 4 4
5. So **1012 × 1012 = 1,024,144**.
6. Start with lower numbers and then extend your expertise to all the numbers between 1000 and 1100. Remember to add the first digit as carry when the square of the difference is four digits.

Squaring numbers between 2000 and 2099

1. Choose a number between 2000 and 2099. (Start with numbers below 2025 to begin with, then graduate to larger numbers.)
2. The first two digits are: 4 0 _ _ _ _ _
3. The next two digits are 4 times the last two digits: 4 0 X X _ _ _
4. For the last three digits, square the last two digits in the number chosen (insert zeros when needed): 4 0 _ _ X X X

Example:

If the number to be squared is **2003**:

1. The first two digits are: 4 0 _ _ _ _ _
2. The next two digits are 4 times the last two: 4 × 3 = 12: _ _ 1 2 _ _ _
3. For the last three digits, square the last two: 3 × 3 = 9: _ _ _ _ 0 0 9
4. So **2003 × 2003 = 4,012,009**.

See the pattern?

For larger numbers, reverse the order:

1. If the number to be squared is **2025**:
2. For the last three digits, square the last two: 25 × 25 = 625: _ _ _ _ 6 2 5
3. The middle two digits are 4 times the last two (keep the carry):
   4 × 25 = 100 (keep carry of 1): _ _ 0 0 _ _ _
4. The first two digits are 40 + the carry: 40 + 1 = 41: 4 1 _ _ _ _ _
5. So **2025 × 2025 = 4,100,625**.

■■

3 IIT Kanpur Scientists solve key math problem

Three Indian computer scientists have solved a longstanding mathematics problem by devising a way for a computer to tell quickly and definitively whether a number is prime – that is, whether it is evenly divisible only by itself and 1.

Prime numbers play a crucial role in cryptography, so devising fast ways to identify them is important. Current computer recipes, or algorithms, are fast, but have a small chance of giving either a wrong answer or no answer at all. The new algorithm – by Manindra Agrawal, Neeraj Kayal and Nitin Saxena of the Indian Institute of Technology in Kanpur – guarantees a correct and timely answer.

Though their paper has not been published, they have distributed it to leading mathematicians who expressed excitement at the finding.

> ***"Methods of determining whether a number is prime have captivated mathematicians since ancient times because understanding prime numbers is the key to solving many important mathematical problems."***

"This was one of the big unsolved problems in theoretical computer science and computational number theory," said Shafi Goldwasser, a professor of computer science at the Massachusetts Institute of Technology and the Weizmann Institute of Science in Israel. "It's the best result I've heard in over 10 years." The new algorithm has no immediate applications, since existing ones are faster and their error probability can be made so small that it is practically zero. Still, for mathematicians and computer scientists, the new algorithm represents a great achievement because, they said, it simply and elegantly solves a problem that has challenged many of the best minds in the field for decades.

Asked why he had the courage to work on a problem that had stymied so many, Agrawal replied in an e-mail message: "Ours was a completely new and unexplored approach. Consequently, it gave us hope that we might succeed."

The paper is now posted on the computer science department webpage at the Indian Institute of Technology (www.cse.iitk.ac.in).

Methods of determining whether a number is prime have captivated mathematicians since ancient times because understanding prime numbers is the key to solving many important mathematical problems. More recently, attention has focussed on tests that run efficiently on a computer, because such tests are part of the underlying mathematics of several widely used systems for encrypting data on computers.

So-called primality testing plays a crucial role in the widely used RSA algorithm, whose security relies on the difficulty of finding a number's prime factors.

RSA is used to secure transactions over the Internet.

On Sunday, the researchers emailed a draft of the paper on the result to dozens of expert mathematicians and computer scientists.

Dr. Carl Pomerance, a mathematician at Bell Labs, said he received the paper on Monday morning and determined it was correct.

After discussing the draft with colleagues in detail over lunch, Dr Pomerance then arranged an impromptu seminar on the result that afternoon attended by various experts.

That he could prepare and give a seminar on the paper so quickly was "a measure of how wonderfully elegant this algorithm is", Dr Pomerance said. "This algorithm is beautiful." ■■

International Math Olympiad

PROBLEMS & SOLUTIONS

1. Let a_1, a_2, a_3, a_4, a_5 be a five-term geometric sequence satisfying the inequality $0 < a_1 < a_2 < a_3 < a_4 < a_5 < 100$, where each term is an integer. How many of these five-term geometric sequence are there?

Soln.: Let $\frac{n}{m}$ be the common ratio of the geometric sequence, where n and m are relatively prime integers, with $n > m$. Now $a_5 = a_1 \times \frac{n^4}{m^4}$, so let $a_1 = km^4$, where k is a positive integer. Thus, our geometric series becomes km^4, km^3n, km^2n^2, kmn^3, kn^4, and $kn^4 < 100$. If $n > 4$, then $kn^4 \geq n^4 > 256 > 100$. So $n \leq 3$. Hence, there are three cases to consider.

If $n = 3$ and $m = 2$, then $81k < 100$, so $k = 1$. The only solution is (16, 24, 36, 54, 81)
If $n = 3$ and $m = 1$, then $81k < 100$, so $k = 1$. The only solution is (1, 3, 9, 27, 81)
If $n = 2$ and $m = 1$, then $16k < 100$, so $k = 1, 2, ..., 6$. There are six solutions, namely (1, 2, 4, 8, 16), (2, 4, 8, 16, 32), (3, 6, 12, 24, 48), (4, 8, 16, 32, 64), (5, 10, 20, 40, 80) and (6, 12, 24, 48, 96). In total, there are eight sequences.

2. Consider the infinite sum

$$S = \frac{a_0}{10^0} + \frac{a_1}{10^2} + \frac{a_2}{10^4} + \frac{a_3}{10^6} + \ldots$$

where the sequence $\{a_n\}$ is defined by $a_0 = a_1 = 1$, and the recurrence relation $a_n = 20a_{n-1} + 12a_{n-2}$ for all positive integers $n \geq 2$. If $\sqrt{S}$ can be expressed in the form $a/\sqrt{b}$, where a and b are relatively prime positive integers, determine the ordered pair (a, b)

Soln.: We have

$$S - \frac{20S}{10^2} - \frac{12S}{10^4} = \left(\frac{a_0}{10^0} + \frac{a_1}{10^2} + \frac{a_2}{10^4} + \frac{a_3}{10^6} + \ldots\right) - \left(\frac{20a_0}{10^2} + \frac{20a_1}{10^4} + \frac{a_2}{10^6} + \frac{a_3}{10^8} + \ldots\right) - \left(\frac{12a_0}{10^4} + \frac{20a_1}{10^6} + \frac{20a_2}{10^8} + \frac{12a_3}{10^{10}} + \ldots\right)$$

$$= \frac{a_0}{10^0} + \frac{a_1}{10^2} - \frac{20a_0}{10^2} + \frac{a_2 - 20a_1 - 12a_0}{10^4} + \frac{a_3 - 20a_2 - 12a_1}{10^6} + \frac{a_4 - 20a_3 - 12a_2}{10^8} + \ldots$$

Since $a_n - 20a_{n-1} - 12a_{n-2} = 0$ for all positive integers $n \geq 2$, we have

$$S - \frac{20S}{10^2} - \frac{12S}{10^4} = \frac{a_0}{10^0} + \frac{a_1}{10^2} - \frac{20a_0}{10^2},$$

and substituting in $a_0 = a_1 = 1$, we have $\frac{7988\,S}{10000} = \frac{81}{100}$, so $S = \frac{2025}{1997}$. Hence $\sqrt{S} = \frac{45}{\sqrt{1997}}$, and so the desired ordered pair is $(a, b) = (45, 1997)$.

3. Let S be the sum of the elements of the set $\{1, 2, 3, ..., (2p)^n - 1\}$. Let T be the sum of the elements of this set whose representation in base $2p$ consists only of digits from 0 to $p - 1$. Prove that $2^n \times \frac{T}{S} = \frac{(p-1)}{(2p-1)}$.

Soln.: We have $S = \frac{[(2p)^n - 1](2p)^n}{2}$. Let R denote the set of numbers that have at most n digits in base $2p$ and contain only the digits from 0 to $p - 1$. Thus T is the sum of the elements of R. For example, when $p = 2$ and $n = 3$, we have $R = \{0, 1, 10, 11, 100, 101, 110, 111\}$ and $T = 1110_4 = 84$. lote R will have p^n elements, because each of the

For more about this exam read MTG's **Math Olympiad Problems and Solutions**

n digits (including leading zeros) can be any one of p different numbers. Since each number in $\{0, 1, ..., p-1\}$ appears the same number of times as a digit in R, there will be exactly p^{n-1} elements in R that have k as their t^{th} digit, for $k = 0, 1, ..., p-1$ and $t = 1, 2, .., n$. Thus,

$$T = \sum_{i=0}^{n-1}(2p)^i \cdot \left[p^{n-1}\times 0 + p^{n-1}\times 1 + ... + p^{n-1}\times(p-1)\right]$$

$$= \sum_{i=0}^{n-1}(2p)^i \cdot p^{n-1} \cdot \frac{p(p-1)}{2} = \frac{p^n(p-1)}{2}\cdot\sum_{i=0}^{n-1}(2p)^i$$

$$= \frac{p^n(p-1)}{2}\cdot\frac{(2p)^n-1}{2p-1}.$$

Hence,

$$= 2^n \times \frac{T}{S} = \frac{2^n \cdot \frac{p^n(p-1)}{2}\cdot\frac{(2p)^n-1}{2p-1}}{\frac{\left[(2p)^n-1\right](2p)^n}{2}} = \frac{p-1}{2p-1},$$

as required.

4. If $f(x) - f(a)$ has order greater than 0 at a, then f is continuous at a. If $f(x) - f(a)$ has order at least 1 at a, then f is differentiable at a.

Soln.: Let r be the order of $f(x) - f(a)$ at a. Assume that $r > 0$. Recall that f is continuous at a if and only if $f(a) = \lim_{x\to a} f(x)$, or equivalently $\lim_{x\to a}(f(x) - f(a)) = 0$. But we see

$$\lim_{x\to a}(f(x)-f(a)) = \lim_{x\to a}\frac{f(x)-f(a)}{(x-a)^r}\cdot\lim_{x\to a}(x-a)^r = 0.$$

In this product, the first limit exists by definition of order, and the second limit exists and is 0 since $r > 0$. Hence, f is continuous at a.

Similarly, recall that f is differentiable at a if and only if $\lim_{x\to a}\frac{f(x)-f(a)}{x-a}$ exists, and the derivative $f'(a)$ is the value of this limit.

Assume that $r \ge 1$. Then

$$\lim_{x\to a}\frac{f(x)-f(a)}{x-a} = \lim_{x\to a}\frac{f(x)-f(a)}{(x-a)^r}\cdot\lim_{x\to a}(x-a)^{r-1},$$

and in this product, both limits exist, so f is differentiable at a.

5. Let $n \ge 2$ be a natural number. Show that there exists a constant $C = C(n)$ such that for all real $x_1, ..., x_n \ge 0$ we have

$$\sum_{k=1}^{n}\sqrt{x_k} \le \sqrt{\prod_{k=1}^{n}(x_k + C)}.$$

Determine the minimum $C(n)$ for some values of n.

Soln.: We show that the inequality is valid for an aggregate of values of C of which the least is

$$C(n) = \frac{n-1}{\sqrt[n-1]{n^{n-2}}},\ n \ge 2.$$

Let us first do the easier task of proving the existence of C's which make the inequality valid. Of course this part will be redundant as soon as we improve the technique to find the least C.

Setting $x_i = y_i^2$ where $y_i \ge 0\ (i = 1, .., n)$, we are to show, equivalently, that for some C we have

$$\left(\sum_{i=1}^{n} y_i\right)^2 \le \prod_{i=1}^{n}(y_i^2 + C) \qquad ...(1)$$

Treating the right hand side of (1) as a polynomial in C, we observe that all coefficients are non-negative and that the coefficient of C^{n-1} is $\sum y_i^2$.

Thus $\prod_{i=1}^{n}(y_i^2 + C) \ge \left(\sum_{i=1}^{n} y_i^2\right)C^{n-1}$

But by the Cauchy-Schwarz inequality we have

$$\left(\sum_{i=1}^{n} y_i\right)^2 \le n\left(\sum_{i=1}^{n} y_i^2\right),$$

so inequality (1) will be valid if we choose $C = n^{\frac{1}{(n-1)}}$ or larger. This completes the easier task.

It turns out that $n^{\frac{1}{(n-1)}}$ is only a slight overestimate of the minimum C, which we now seek. For any C for which (1) is valid, set $w_i = \frac{y_i\sqrt{n-1}}{\sqrt{C}}$, so that (1) becomes

$$\left(\sum_{i=1}^{n} w_i\right)^2 \le \frac{C^{n-1}}{(n-1)^{n-1}}\prod_{i=1}^{n}(w_i^2 + n - 1)$$

or equivalently

$$\left(\sum_{i=1}^{n} w_i\right)^2 \le \frac{C^{n-1}n^n}{(n-1)^{n-1}}\prod_{i=1}^{n}\left(\frac{w_i^2-1}{n}+1\right) \qquad ...(2)$$

To find the minimum C we shall first show that the following inequality is valid:

$$\left(\sum_{i=1}^{n} w_i\right)^2 \le n^2\prod_{i=1}^{n}\left(\frac{w_i^2-1}{n}+1\right). \qquad ...(3)$$

We shall use the Weierstrass inequality

$$\prod_{i=1}^{m}(1 + a_i) \ge 1 + \sum_{i=1}^{m} a_i,$$

which holds if all $a_i \ge 0$ or if $-1 < a_i < 0$ for all i. Without loss of generality let $w_1, ..., w_t \ge 1$ and $0 \le w_{t+1}, ..., w_n < 1$, where $t \in \{0, 1, ..., n\}$. Then

$$\prod_{i=1}^{n}\left(\frac{w_i^2-1}{n}+1\right) = \prod_{i=1}^{t}\left(\frac{w_i^2-1}{n}+1\right)\cdot\prod_{i=t+1}^{n}\left(\frac{w_i^2-1}{n}+1\right)$$

$$\ge \left(1+\sum_{i=1}^{t}\frac{w_i^2-1}{n}\right)\left(1+\sum_{i=t+1}^{n}\frac{w_i^2-1}{n}\right)$$

$$=\frac{1}{n^2}\left(n-t+\sum_{i=1}^{t}w_i^2\right)\left(t+\sum_{i=t+1}^{n}w_i^2\right)$$

$$=\frac{1}{n^2}\left(\sum_{i=1}^{t}w_i^2+\sum_{i=t+1}^{n}1^2\right)\left(\sum_{i=1}^{t}1^2+\sum_{i=t+1}^{n}w_i^2\right)$$

$$\ge\frac{1}{n^2}\left(\sum_{i=1}^{n}w_i\right)^2$$

(the last inequality by the Cauchy-Schwarz inequality), which proves (3). Note that equality occurs for $w_1 = \ldots = w_n = 1$. We conclude that (2) is valid for any C with $\frac{C^{n-1}\,n^n}{(n-1)^{n-1}} \ge n^2$, i.e., with

$$C \ge \frac{n-1}{\sqrt[n-1]{n^{n-2}}},\ n \ge 2.$$

The minimum value $C(n)$ we seek is then as stated at the beginning, since for

$$x_i = y_i^2 = C\left(\frac{w_i}{\sqrt{n-1}}\right)^2 = \frac{C\cdot 1}{n-1}$$

the original inequality reduces to equality.

Quick Note
PYRAMID

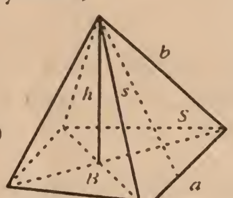
A pyramid is a polyhedron of which one side, the base, is a polygon (not necessarily a regular polygon), and all the rest are triangles sharing a common point, the vertex.

A pyramid is regular if the base is a regular polygen and the other faces are congruent isosceles triangles.

Height : h
Area of base : B
Slant height : s (regular pyramid)
Perimeter of base : P
Lateral surface area : S
Volume : V

$S = \frac{sP}{2}$ (regular pyramid)

$V = \frac{hB}{3}$



5 CHALLENGING PROBLEMS

COORDINATE GEOMETRY

By : T. R. Chakravarthy, B.Tech (Mech), IIT Chennai

1. Two circles with equations $x^2+y^2-8x-2y+13=0$ & $x^2+y^2-16x-18y+109=0$ are excircles of a triangle ABC. Find possible coordinates of vertices and area of the triangle. Also give equation of incircle of triangle ABC (one possible equation).

2. $P(1, -3)$ and $Q(4, 6)$ are two points on the parabola $y^2 = 9x$. Another point R is chosen on parabola such that area of triangle PQR has a fixed area 'C'. If R can assume exactly three different positions for this 'C'. Find 'C' and positions taken by R on the parabola.

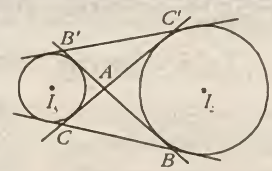
3. Let S and S' be foci of an ellipse. From a variable point P on the ellipse, rays $\overrightarrow{PS}$ and $\overrightarrow{PS'}$ are drawn. An excircle with $\overrightarrow{SS'}$ as tangent is drawn to the triangle PSS'. Prove that normal at P to the ellipse is a normal to the excircle also.

4. From any point P on the curve $b^4x+2a^2y^2=0$ chords are drawn to the hyperbola $\left(\frac{x}{a}\right)^2-\left(\frac{y}{b}\right)^2=1$. Tangents are drawn at the extremites of the chord to the hyperbola which interset at Q. Prove that Q moves along a straight line for a given P. Also show that the straight line always touches a fixed parabola for any P.

5. The circle $x^2+y^2+2\sqrt{2}x-6y+10=0$ is incircle of triangle ABC with $A(-1-\sqrt{2}, 2)$. Find locus of centroid and orthocentre of the triangle.

SOLUTION

1. The excircles and the triangle can be drawn in the following way.



From the figure, it is obvious that two triangles ABC and $AB'C'$ are possible having common vertex A. Also vertex A is internal centre of similitude of the excircles.

Let C_2 be $x^2+y^2-8x-2y+13=0$

C_3 be $x^2+y^2-16x-18y+109=0$

centre of $C_2 = (4, 1)$ radius $r_2 = 2$ units

centre of $C_3 = (8, 9)$ radius $r_3 = 6$ units

$\therefore$ Internal centre of similitude

i.e. $A = \left(\frac{2(8)+6(4)}{2+6}, \frac{2(9)+6(1)}{2+6}\right) = (5, 3)$

$C_2 \equiv (x-4)^2+(y-1)^2=4$

any tangent to C_2 is of form

$$(y-1)=m(x-4)\pm 2\sqrt{1+m^2}$$

$\therefore$ Tangents through A have slopes given by equation

$2 = m \pm 2\sqrt{1+m^2} \Rightarrow (2-m)^2 = 4+4m^2$

$\Rightarrow m^2-4m+4 = 4+4m^2 \Rightarrow 3m^2=-4m$

$m=0$, $m=-4/3$ are roots

$\therefore$ equations of tangents are

$y=3$ and $4x+3y=29$

External centre of similitude

$$= \left(\frac{-2(8)+6(4)}{-2+6}, \frac{-2(9)+6(1)}{-2+8}\right) = (2,-3)$$

Tangents through external centre of similitude give equations of $B'C'$ and BC (see figure for more clarification)

Slopes of $B'C'$ and BC are given by the equation

$-4=-2m\pm 2\sqrt{1+m^2} \Rightarrow 2m-4=\pm 2\sqrt{1+m^2}$

$m-2=\pm\sqrt{1+m^2} \Rightarrow m^2+4-4m = m^2+1$

$4m=3 \Rightarrow m=3/4$.

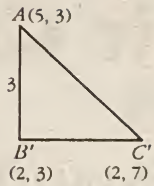
But we have only one root $m=3/4$. $B'C'$ and BC cannot be parallel as $r_2 \neq r_3$. So other root is $m=\infty$. (Students often forget or don't know this important concept)

$\therefore$ Equation of $B'C'$ and BC are $x=2$ and $3x-4y=18$

Coordiantes of B', C', B and C are obtained by careful solving of the linear equations obtained above. Possible coordinates are

$B'(2, 3)$, $C'(2, 7)$, $B(10, 3)$, $C\left(\frac{34}{5}, \frac{3}{5}\right)$

Area of triangle $AB'C'$ which is a right angled one is $\frac{1}{2}(3\times4) = 6$ sq. units



Area of triangle ABC = $A(5, 3)$, $B(10, 3)$, $C\left(\frac{34}{5}, \frac{3}{5}\right)$

$$\Delta = \frac{1}{2}\begin{vmatrix} -5 & \frac{16}{5} \\ 0 & \frac{12}{5} \end{vmatrix} = \frac{1}{2}|-12| = 6 \text{sq.units.}$$

Equation of incircle of triange $AB'C'$

Incircle of triangle OPQ with O as origin $P(4, 0)$, $Q(0, 3)$ is of form (r, r).

$\therefore$ $|7r - 12| = 5r$

$r = 1$ or $r = 6$ But $r = 6$ is not possible. So incentre is (1, 1) for ΔOPQ. So for triangle $AB'C'$, incentre is (3, 4) (by changing origin)

$\therefore$ Equation of incircle of triangle $AB'C'$ is

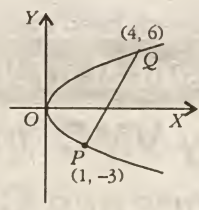
$(x - 3)^2 + (y - 4)^2 = 1$

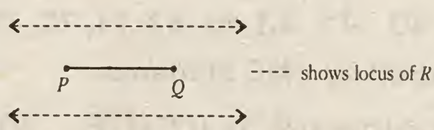
i.e. $x^2 + y^2 - 6x - 8y + 24 = 0$

Similarly, we can obtain equation of incircle of triangle ABC also.

(Try to find area of the triangle using only trigonometry. It is a challenging problem as well!)

2. P and Q are fixed points. The locus of R for a given area of triangle PQR is straight line(s) parallel to PQ as shown below. Each line cuts parabola (generally) in two points. But there are only three positions of R. So one line should be tangent to parabola (parallel to PQ).





Slope of PQ = 3

Equation of PQ : $y = 3x - 6$

Equation of tangent parallel to PQ is $y = 3x + \frac{9}{4\times3}$

i.e. $y = 3x + \frac{3}{4}$

Distance between the parallel lines $= \dfrac{6+\frac{3}{4}}{\sqrt{10}} = \dfrac{27}{4\sqrt{10}}$

lenght of PQ = $3\sqrt{10}$

$\therefore$ Area of $\Delta PQR = \frac{1}{2}\left(3\sqrt{10}\times\frac{27}{4\sqrt{10}}\right)$

$\therefore C = \frac{81}{8}$ sq.units.

Positions assumed by R:

The tangent meets $y = mx + \frac{a}{m}$ parabola at $\left(\frac{a}{m^2}, \frac{2a}{m}\right)$

$\therefore$ One position of R is $\left(\frac{9}{4\times9}, \frac{9}{2\times3}\right) = \left(\frac{1}{4}, \frac{3}{2}\right)$

Other line parallel to PQ at a distance $\frac{27}{4\sqrt{10}}$ units be $y = 3x + C$

$$\frac{|C+6|}{\sqrt{10}} = \frac{27}{4\sqrt{10}} \Rightarrow C + 6 = \pm\frac{27}{4}$$

$$C = \pm\frac{27}{4} - 6 = \frac{3}{4} \text{ or } \frac{-51}{4}$$

$\therefore$ other line $y = 3x - \frac{51}{4}$

Solving with parabola $y^2 = 9x$

$$y^2 = \frac{9}{3}\left(y + \frac{51}{4}\right) \Rightarrow y^2 = 3y + \frac{153}{4}$$

$4y^2 - 12y - 153 = 0$

$$\therefore y = \frac{12 \pm \sqrt{144 + 16\times153}}{8} = \frac{12 \pm 36\sqrt{2}}{8}$$

$$= \frac{3 \pm 9\sqrt{2}}{2} = \frac{3}{2}(1 \pm 3\sqrt{2}).$$

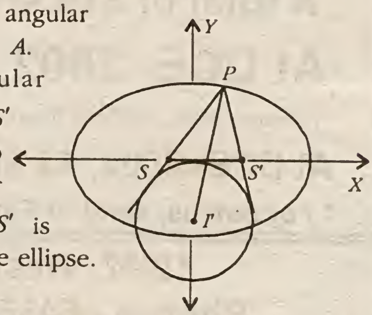
$\therefore$ Other positions of R are given by $\left(\frac{19 \pm 6\sqrt{2}}{4}, \frac{3}{2}(1 \pm 3\sqrt{2})\right)$

3. From trigonometry we have in a triangle ABC; A, I, I_1 are collinear.

i.e. AI_1 is interior angular bisector of angle A.

$\therefore$ PI' is angular bisector of $\angle SPS'$

It is sufficient to prove that angular bisector of $\angle SPS'$ is normal at P to the ellipse.



Let equation of ellipse be $\left(\frac{x}{a}\right)^2+\left(\frac{y}{b}\right)^2=1$

Let P be $(a\cos\theta,\ b\sin\theta)$

Equation of normal at P is given by

$ax\sec\theta - by\operatorname{cosec}\theta = a^2 - b^2$

$\therefore$ slope of normal $= \frac{a}{b}\tan\theta$

It is enough to prove that $\angle SPI' = \angle S'PI'$

Slope of $SP = \dfrac{b\sin\theta}{a\cos\theta+ae}$

$$\tan|\angle SPI'| = \left|\frac{\dfrac{b\sin\theta}{a\cos\theta+ae}-\dfrac{a\sin\theta}{b\cos\theta}}{1+\dfrac{\sin^2\theta}{(\cos\theta+e)\cos\theta}}\right|$$

$$=\left|\frac{b^2\sin\theta\cos\theta-a^2\sin\theta\cos\theta-a^2e\sin\theta}{ab[\cos^2\theta+e\cos\theta+\sin^2\theta]}\right|$$

$$=\left|\frac{(b^2-a^2)\sin\theta\cos\theta-a^2e\sin\theta}{ab(1+e\cos\theta)}\right|$$

$$=\left|\frac{-a^2e^2\sin\theta\cos\theta-a^2e\sin\theta}{ab(1+e\cos\theta)}\right|$$

$$=\left|\frac{a^2e\sin\theta(\cos\theta e+1)}{ab(1+e\cos\theta)}\right| = \frac{ae\sin\theta}{b}.$$

Slope of $S'P = \dfrac{b\sin\theta}{a\cos\theta-ae}$

$$\tan|\angle S'PI'| = \left|\frac{\dfrac{b\sin\theta}{a\cos\theta-ae}-\dfrac{a\sin\theta}{b\cos\theta}}{1+\dfrac{\sin^2\theta}{\cos^2\theta-e\cos\theta}}\right|$$

$$=\left|\frac{b^2\sin\theta\cos\theta-a^2\sin\theta\cos\theta+a^2e\sin\theta}{ab(1-e\cos\theta)}\right|$$

$$=\frac{a^2e\sin\theta|(-e\cos\theta+1)|}{ab(1-e\cos\theta)}=\frac{ae\sin\theta}{b}$$

$\therefore \angle SPI' = \angle S'PI'$

i.e. normal is the angular bisector.

Hence proved.

4. Any point on the curve $b^4x + 2a^2y^2 = 0$

can be represented parametrically as $P\left(\frac{-2a^2t^2}{b^4},\ t\right)$

Clearly Q is polar of P (by definition) which is a straight line.

Equation of polar is given by

$$\frac{x}{a^2}\left(\frac{-2a^2t^2}{b^4}\right)-\frac{yt}{b^2}=1$$

$$\therefore\ \frac{-2xt^2}{b^4}-\frac{yt}{b^2}=1 \text{ or } -2xt^2-ytb^2=b^4$$

or $2xt^2+b^2ty+b^4=0$ is equation of the polar.

Now we have to prove that the polar touches a fixed parabola for all values of t.

$$2xt^2+b^2ty+b^4=0 \quad\text{or}\quad b^2ty=-2xt^2-b^4$$

$$y=\frac{-2xt^2}{b^2t}-\frac{b^4}{b^2t} \quad\text{or}\quad y=\frac{-2t}{b^2}x-\frac{b^2}{t}$$

$$y=\left(-\frac{2t}{b^2}\right)x+\frac{2}{(-2t/b^2)}$$

This is of form $y=mx+\frac{2}{m}$

which always touches the parabola $y^2 = 8x$.

Hence proved.

5. Given circle is

$x^2+y^2+2\sqrt{2}x-6y+10=0$

i.e. $(x+\sqrt{2})^2+(y-3)^2=1$

It is clear that $(-1-\sqrt{2},\ 0)$ and $(0,\ 2)$ lies on circle.

$\therefore x+1+\sqrt{2}=0,\ y-2=0$ are tangents at right angle to the circle.

So put $x+1+\sqrt{2}=X,\ y-2=Y$ so that

$A(-1-\sqrt{2},\ 2)$ becomes origin.

$\therefore$ Equation of circle becomes

$$(X-1)^2+(Y-1)^2=1$$

Any tangent to the circle is of form

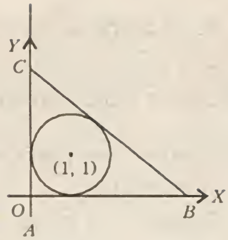
$$Y-1=m(X-1)\pm\sqrt{1+m^2}$$

$$Y=mX-m+1\pm\sqrt{1+m^2}$$

$\therefore$ B and C are given by

$mX = m - 1 - \sqrt{1+m^2}$

$\therefore$ B is $\left(\dfrac{m-1-\sqrt{1+m^2}}{m}, 0\right)$



$y = mx - m + 1 - \sqrt{1+m^2}$ is neglected as perpendicular disance from origin to the line is less than radius (note that m is negative only).

C is given by $(0, \ -m+1-\sqrt{1+m^2}\ )$

Let centroid be (α, β)

$$3\alpha m = m-1-\sqrt{1+m^2}$$

$$\text{or} \quad 3\beta = -m+1+\sqrt{1+m^2}$$

$$\therefore\ \alpha m + \beta = 0 \Rightarrow m = -\frac{\beta}{\alpha}$$

$$\therefore\ \text{locus is } 3\beta = \frac{\beta}{\alpha} + 1 + \frac{\sqrt{\alpha^2+\beta^2}}{\alpha}$$

$$3\alpha\beta = \beta + \alpha + \sqrt{\alpha^2+\beta^2}$$

$$(3\alpha\beta - \beta - \alpha)^2 = \left(\sqrt{\alpha^2+\beta^2}\right)^2$$

$$9\alpha^2\beta^2 + \beta^2 + \alpha^2 - 6\alpha\beta^2 - 6\alpha^2\beta + 2\alpha\beta = \alpha^2 + \beta^2$$

$$\alpha\beta(9\alpha\beta - 6\beta - 6\alpha + 2) = 0$$

On generalisation locus is

$$9xy - 6x - 6y + 2 = 0$$

$$\text{or} \quad (3x-2)(3y-2) = 2$$

So again translating

$$(3(x+1+\sqrt{2})-2)(3(y-2)-2) = 2$$

$$(3x+3\sqrt{2}+1)(3y-8) = 2$$

$$9xy - 24x + 3(3\sqrt{2}+1)y = 24\sqrt{2}+10$$

is required locus of the centroid.

Since triangle is right angled at A, locus is the point $A(-1-\sqrt{2},\ 2)$ itself for the orthocentre.

■

Contd. from page no. 6

DIFFERENTIAL CALCULUS

Real valued functions of a real variable, into, onto and one-to-one functions, sum, difference, product and quotient of two functions, composite functions, absolute value, polynomial, rational, trigonometric, exponential and logarithmic functions.

Limit and continuity of a function, limit and continuity of the sum, difference, product and quotient of two functions.

Even and odd functions, inverse of a function, continuity of composite functions, intermediate value property of continuous functions.

Derivative of a function, derivative of the sum, difference, product and quotient of two functions, chain rule, derivatives of polynomial, rational, trigonometric, inverse trigonometric, exponential and logarithmic functions.

Derivatives of implicit functions, derivatives up to order three, geometrical interpretation of the derivative, tangents and normals, increasing and decreasing functions, maximum and minimum values of a function, applications of Rolle's Theorem and Lagrange's Mean Value Theorem.

INTEGRAL CALCULUS

Integration as the inverse process of differentiation, indefinite integrals of standard functions, definite integrals and their properties, application of the Fundamental Theorem of Integral Calculus.

Integration by parts, integration by the methods of substitution and partial fractions, application of definite integrals to the determination of areas involving simple curves.

Formation of ordinary differential equations, solution of homogeneous differential equations, variables separable method, linear first order differential equations.

VECTORS

Addition of vectors, scalar multiplication, scalar products, dot and cross products, scalar triple products and their geometrical interpretations.

■

International Math Olympiad

PROBLEMS & SOLUTIONS

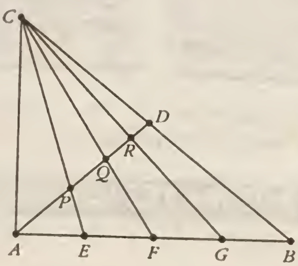
1. Suppose that $V = \{1, 2, 3, ..., 24, 25\}$. Prove that any subset of V with 17 or more elements contains at least two distinct numbers the product of which is the square of an integer.

Soln. **:** The set of numbers $A = \{1, 2, .., 24, 25\}$ contains a total of five perfect squares $\{1, 4, 9, 16, 25\}$. The product of any two of these will also be a perfect square. There is one triplet, the product of any two of its elements will result in a perfect square: $\{2, 8, 18\}$. The only other pairs of numbers from A whose product is a pefect square are $\{3, 12\}$, $\{5, 20\}$, $\{6, 24\}$. The other eleven elements of the set A are $\{7, 10, 11, 13, 14, 15, 17, 19, 21, 22, 23\}$ and they cannot form a perfect square when multiplied with any other element of set A. Group the elements of set A as follows:

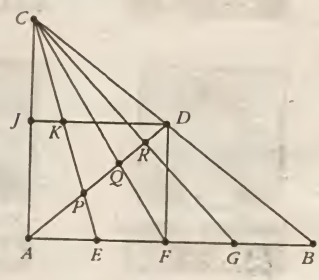
$\{1, 4, 9, 16, 25\}$, $\{2, 8, 18\}$, $\{3, 12\}$, $\{5, 20\}$, $\{6, 24\}$, $\{7\}$, $\{10\}$, $\{11\}$, $\{13\}$, $\{14\}$, $\{15\}$, $\{17\}$, $\{19\}$, $\{21\}$, $\{22\}$, $\{23\}$.

If more than one number is chosen from a given group, a perfect square will result. There is a total of 16 groups, so 16 numbers can be chosen without creating a perfect square product. However, if any 17 numbers are chosen, then two must be contained within the same group, and therefore will form a perfect square product.

2. Given is a triangle ABC, $\angle A = 90°$. D is the midpoint of BC, F is the midpoint of AB, E the midpoint of AF and G the midpoint of FB. AD intersects CE, CF and CG respectively in P, Q and R. Determine the ratio $\dfrac{\overline{PQ}}{\overline{QR}}$.



Soln. **:** We know that two medians in a triangle divide each other in 2 : 1 ratio, or in other words the point of intersection is $\dfrac{2}{3}$ the way from the vertex.



Since CF and AD are both medians in ΔABC, then $\dfrac{\overline{AQ}}{\overline{QD}} = \dfrac{2}{1}$, where Q is the point of intersection.

Also, since D is the midpoint of the hypotenuse in the right triangle ABC, then it is the centre of the circumscribed circle with radius $DA = DC = DB$.

Drop a perpendicular from D onto sides AB and CA. The feet of the perpendiculars will be F and J, respectively, where J is the midpoint of AC, since DF and DJ are altitudes in isosceles triangles ΔADB and ΔADC, respectively. Now consider ΔCFB. The segments CG and FD are medians and therefore intersect at H say in the ratio 2 : 1 so, $\dfrac{\overline{HD}}{\overline{FD}} = \dfrac{1}{3}$.

From here it can be seen that ΔARC and ΔDRH are similar, since their angles are the same. Also since we know that $\overline{FD} = \overline{JA}$, and $2\overline{JA} = \overline{AC}$ then $\overline{HD} = \dfrac{1}{6}\overline{CA}$ and ΔARC is 6 times bigger than ΔDRH.

Now we can see that $\dfrac{\overline{AR}}{\overline{RD}} = \dfrac{6}{1}$ and since $\overline{AR} + \overline{RD} = \overline{AD}$, then $\dfrac{\overline{RD}}{\overline{AD}} = \dfrac{1}{7}$.

Similarly $\Delta APE \sim \Delta KPD$, where medians DJ and CE meet at K. We know that $\overline{AE} = \dfrac{1}{4}\overline{AB}$, so then

$\overline{JK} = \frac{1}{4}\overline{JD}$, since JD is parallel to AB. It now follows that $\frac{\overline{AE}}{\overline{KD}} = \frac{2}{3}$, and from the similarity of the triangles $\frac{\overline{AP}}{\overline{PD}} = \frac{2}{3}$. Also, since $\overline{AP} + \overline{PD} = \overline{AD}$, then $\frac{\overline{AP}}{\overline{AD}} = \frac{2}{5}$.

Combining these results we have $\overline{AP} = \frac{2}{5}\overline{AD}$, $\overline{AQ} = \frac{2}{3}\overline{AD}$, $\overline{QD} = \frac{1}{3}\overline{AD}$ and $\overline{RD} = \frac{1}{7}\overline{AD}$.

Thus $\overline{PQ} = \overline{AQ} - \overline{AP} = \frac{2}{3}\overline{AD} - \frac{2}{5}\overline{AD} = \frac{4}{15}\overline{AD}$

and $\overline{QR} = \overline{QD} - \overline{RD} = \frac{1}{3}\overline{AD} - \frac{1}{7}\overline{AD} = \frac{4}{21}\overline{AD}$.

Form these $\frac{\overline{PQ}}{\overline{QR}} = \frac{7}{5}$.

3. A series of numbers is defined as follows: $u_1 = a$, $u_2 = b$, $u_{n+1} = \frac{1}{2}(u_n + u_{n-1})$ for $n \geq 2$. Prove that $\lim_{n\to\infty} u_n$ exists. Express the value of the limit in terms of a and b.

Soln. **:** For the recurrence $u_{n+1} = \frac{1}{2}(u_n + u_{n-1})$ we obtain the associated equation $2\lambda^2 - \lambda - 1 = 0$, which has roots $\lambda = -\frac{1}{2}, 1$. Thus we seek a solution of the form $u_n = X\left(-\frac{1}{2}\right)^n + Y$. From $u_1 = a$ and $u_2 = b$ we get

$$-\frac{1}{2}X + Y = a \qquad \frac{1}{4}X + Y = b$$

so that $X + \frac{4}{3}(b-a)$ and $Y = \frac{a+2b}{3}$.

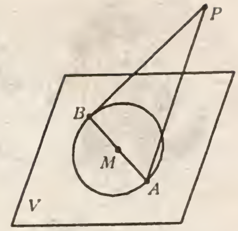
It is now easy to check by induction that

$$u_n = \frac{4}{3}(b-a)\left(-\frac{1}{2}\right)^n + \frac{1+2b}{3}$$

and as

$$n \to \infty,\ u_n \to \frac{1}{3}a + \frac{2}{3}b.$$

4. In a plane V a circle C is given with centre M. P is a point not on the circle C.



(a) Prove that for a fixed point P, $\overline{AP}^2 + \overline{BP}^2$ is a constant for every diameter AB of the circle C.

(b) Let AB be any diameter of C and P a point on a fixed sphere S not intersecting V. Determine the point(s) P on S such that $\overline{AP}^2 + \overline{BP}^2$ is minimal.

Soln. **:** (a) With ΔPAB, we can join P and M to create two new triangles, ΔPMA and ΔPMB. Let $\angle PMA = \theta$. Then $\angle PMB = 180° - \theta$. Because M is the centre of the circle C and A and B both lie on circle C, we have $\overline{MA} = \overline{MB} = r$, the radius of the circle.

By the law of cosines,

$$\overline{BP}^2 = \overline{MP}^2 + r^2 - 2\overline{MP}r\cos(180° - \theta)$$

$$= \overline{MP}^2 + r^2 + 2\overline{MP}r\cos\theta$$

and $$\overline{AP}^2 = \overline{MP}^2 + r^2 - 2\overline{MP}r\cos\theta$$

so $$\overline{AP}^2 + \overline{BP}^2 = 2\overline{MP}^2 + 2r^2.$$

The right hand side is a constant depending only on the radius of the circle and the distance of P from the centre.

(b) From (a), we know that

$$\overline{AP}^2 + \overline{BP}^2 = 2\overline{MP}^2 + 2r^2.$$

For any point P on sphere S, the radius of the circle will remain constant. Therefore the only variable affecting the sum $\overline{AP}^2 + \overline{BP}^2$ is $\overline{MP}$, the distance from the point P to the centre of the circle. $\overline{AP}^2 + \overline{BP}^2$ will be a minimum when $\overline{MP}$ is minimum. Therefore we are looking for the point on the sphere closest to M.

Let T be the centre of the sphere S, D be the point on the segement MT that lies on the sphere, and D' be any other point on S.

We know that $\overline{MD} + \overline{DT} < \overline{MD'} + \overline{D'T}$ becasue the shortest distance between M and T is a straight line.

We know that $\overline{DT} = \overline{D'T}$. Thus $\overline{MD} < \overline{MD'}$.

Thus D is the point on the sphere which minimizes the sum.

■

Examiner's Mind

Quadratic Inequalities

Quadratic Inequalities involve variable expressions that just look like the expressions found in quadratic equations.

The only difference is an inequality symbol ($\le, \ge, <, >$) instead of an equal sign.

For example $x^2 + 6 < 5x$ or $4x^2 - 3x \ge x^2$

☞ Remember while solving quadratic inequality it is best to factoring before go further.

☞ The important key fact is that it is always better to have 0 on one side of the inequality even before the factorization.

Let us being with few illustrations that will make the concept more clear.

Illustration 1 : Solve the inequality $4x^2 - 3x \ge x^2$

Step 1 : Given the quadratic inequality

$$4x^2 - 3x \ge x^2$$

Step 2 : We have to determine the values of x for which the above inequality holds good.

Step 3 : Make sure that you have 0 on one side of the inequality.

$\therefore\ 4x^2 - 3x - x^2 \ge 0$ or $3x^2 - 3x \ge 0$

Factorize : $3x(x-1) \ge 0$

Thus now our aim is to determine the region where the inequality holds true.

Step 4 : $\therefore$ We have either

$$3x(x-1) > 0 \text{ or } 3x(x-1) = 0$$

Now for $3x(x-1) = 0 \Rightarrow x = 0$ or $x = 1$ ...(1)

and $3x(x-1) > 0$

if either $3x > 0$ and $(x-1) > 0$ ...(i)

or $3x < 0$ and $(x-1) < 0$ ...(ii)

From (i) we get $x > 0$ and $x > 1 \Rightarrow x > 1$ ...(2)

and from (ii) we get $x < 0$ and $x < 1 \Rightarrow x < 0$ ...(3)

From (1), (2) and (3) we get $x \in (-\infty, 0]$ or $[1, \infty)$

Thus $x \in (-\infty, 0] \cup [1, \infty)$

Step 5 : $\therefore$ required value of x lies in $(-\infty, 0] \cup [1, \infty)$ which satisfy the given quadratic inequality.

Illustration 2 : Solve $(x+2)(x-1) \ge x^2 + 4x$

Step 1 : Given the quadratic inequality $(x+2)(x-1) \ge x^2 + 4x$

Step 2 : We have to determine the value of x satisfying the given quadratic inequality.

Step 3 : Multiply the factor terms on L.H.S. and get a zero on one side of the inequality.

$\therefore$ We get $x^2 + 2x - x - 2 - x^2 - 4x \ge 0$ or $-3x - 2 \ge 0$

Thus we get the linear inequality

Now we have to determine the region where $-3x - 2 \ge 0$ holds true.

Step 4 : $-3x - 2 \ge 0 \Rightarrow x \le -\dfrac{2}{3}$ $\quad \therefore\ x \in \left(-\infty, -\dfrac{2}{3}\right]$

Step 5 : $\therefore$ Required value of x lies in $\left(-\infty, -\dfrac{2}{3}\right]$ for which x satisfies the given inequality.

Illustration 3 : Solve the inequality and find the value of x for which it satisfy $x(x+7) + 12 > 0$

Step 1 : Given the inequality $x(x+7) + 12 > 0$

Step 2 : We have to determine the value of x for which it satisfies the given quadratic inequality.

Step 3 : Multiply out and make a zero on one side of the inequality

$\therefore\ x^2 + 7x + 12 > 0$

Factorize : $x^2 + 7x + 12 > 0$

$(x+4)(x+3) > 0$

Thus, now our aim is to determine the region or the value of x for which it satisfy the above quadratic inequality.

Step 4 : $(x+4)(x+3) > 0$

$\Rightarrow$ Either $x + 4 > 0$ and $x + 3 > 0$ ...(i)

or $x + 4 < 0$ and $x + 3 < 0$ ...(ii)

From (i) $x + 4 > 0 \Rightarrow x > -4$ and

$x + 3 > 0 \Rightarrow x > -3$

$\Rightarrow x > -3$ ...(A)

From (ii) $x + 4 < 0 \Rightarrow x < -4$ and

$x + 3 < 0 \Rightarrow x < -3 \Rightarrow x < -4$ ...(B)

From (A) and (B) $x > -3$, $x < -4$

$\therefore\ x \in (-\infty, -4)$ or $(-3, \infty)$

Examiner's Mind

Functional Equations

By : **P.K. Roy**

An equation involving an unknown function is called a functional equation. Functional equations occur in many applications of mathematics. We give below some examples to illustrate as to how we can solve functional equations in some simple cases.

Example 1 : If a function f satisfies the functional equation

$f(x+y) = f(x)\cdot f(y)$, then either $f(x) = 0$ or else $f(x) = e^{ax}$, for all rational numbers x, where a is a constant.

Soln. : Taking $x = y = 0$,

we have $f(0) = [f(0)]^2$ i.e. $f(0) = 0$ or 1.

First let us take the case $f(0) = 0$. If x be any rational number then $f(x + 0) = f(x) f(0) = 0$.

$\Rightarrow f(x) = 0 \; \forall$ rational number x.

Now let us take the case when $f(0) = 1$

If x be a positive integer, say $= n$,

then $f(x) = f(n) = f(1 + 1 + 1 + \ldots + 1) = [f(1)]^n$

$= e^{n \log_e f(1)} = e^{an} = e^{ax}$ where $\log_e f(1) = a$ (say)

If $x = \dfrac{p}{q}$, where p and q are both integers, then

$$f(p) = f\left(\frac{p}{q} + \frac{p}{q} + \ldots q \text{ terms}\right) = \left[f\left(\frac{p}{q}\right)\right]^q$$

$$\therefore f\left(\frac{p}{q}\right) = [f(p)]^{1/q} = [e^{ap}]^{1/q}$$

$$\therefore f\left(\frac{p}{q}\right) = [f(p)]^{1/q} = [e^{ap}]^{1/q} = e^{a(p/q)}$$

$\Rightarrow f(x) = e^{ax}$

where x is any positive rational number.

Finally let $x = -y$

where y is a positive rational number

Then $f(x+y) = f(x) f(y) \Rightarrow f(0) = f(x) f(y)$

$$\Rightarrow f(x) = \frac{1}{f(y)} = e^{-ay} = e^{ax}$$

Thus $f(x) = e^{ax} \; \forall$ rational x.

Example 2 : If f is a function of x satisfying the functional equation $f(x+y) = f(x) + f(y)$ for all rational numbers x and y, show that $f(x) = kx$ where k is a constant.

Soln. : Taking $x = y = 0$, we have $f(0) = f(0) + f(0)$ $\Rightarrow f(0) = 0$. Also taking $y = -x$, we get

$\Rightarrow f(0) = f(x) + f(-x) \Rightarrow f(-x) = -f(x)$

If x is a positive integer, say $= n$,

then $f(x) = f(n) = f(1) + f(1) + \ldots + f(1)$, (to n terms) $= nf(1) = nk = kx$ where we have taken $f(1) = k$. If x is negative integer, say $-n$ then we get

$f(x) = f(-x) = -f(n) = -(n) f(1) = kx$

Again let $x = p/q$ where q is a positive integer and p is any integer, then

$$f(p) = f\left(\frac{p}{q}\cdot q\right) = f\left(\frac{p}{q}\right) + f\left(\frac{p}{q}\right) + \ldots q \text{ times}$$

$$= q\, f\left(\frac{p}{q}\right)$$

So that $f\left(\dfrac{p}{q}\right) = \dfrac{1}{q} f(p) = \dfrac{1}{q} kp = k\left(\dfrac{p}{q}\right)$

i.e. $f(x) = kx$ as before

Thus $f(x) = kx \; \forall$ rational x.

Example 3 : If $f : R \to R$ is a function satisfying the properties

(i) $f(-x) = -f(x)$

(ii) $f(x+1) = f(x) + 1$

(iii) $f\left(\dfrac{1}{x}\right) = \dfrac{f(x)}{x^2}$ $x \neq 0$

Prove that $f(x) = x \; \forall \; x \in R$.

Soln.: Let us observe that

(1) If $f(x)$ is known for all $x > 0$, then $f(x)$ can be found for all $x < 0$ by using (1)

(2) By putting $x = 0$ in (i), we find that

$f(-0) = -f(0) \Rightarrow f(0) = 0$

From (1) and (2) we find that it is enough to find $f(x) \; \forall \; x > 0$.

We shall take $x > 0$ and compute $f\left(\dfrac{1}{x+1}\right)$ in terms

of $f(x)$ in two different ways. By equating the two expression for $f\left(\frac{1}{x+1}\right)$ thus obtained we shall find $f(x)$.

Now $f\left(\frac{1}{x+1}\right) = \frac{f(x+1)}{(x+1)^2}$, by (iii)

$$= \frac{f(x)+1}{(x+1)^2}, \text{ by (ii)} \quad \text{...(A)}$$

Again $f\left(\frac{1}{x+1}\right) = f\left(1-\frac{x}{x+1}\right) = f\left(-\frac{x}{x+1}\right)+1$

$$= -f\left(\frac{x}{x+1}\right) + 1 \quad \text{by (i)}$$

$$= -\frac{f\left(\frac{x+1}{x}\right)}{\left(\frac{x+1}{x}\right)^2} + 1 \quad \text{by (iii)}$$

$$= -\frac{x^2}{(x+1)^2} f\left(1+\frac{1}{x}\right) + 1.$$

$$= -\frac{x^2}{(x+1)^2}\left[f\left(\frac{1}{x}\right)+1\right] + 1 \quad \text{by (ii)}$$

$$= -\frac{x^2}{(x+1)^2}\left[\frac{f(x)}{x^2}+1\right]+1 \quad \text{by (iii)} \quad \text{...(B)}$$

from (A) and (B), we have

$$\frac{f(x)+1}{(x+1)^2} = -\frac{x^2}{(x+1)^2}\left[\frac{f(x)}{x^2}+1\right] + 1$$

$$\Rightarrow \frac{2f(x)}{(x+1)^2} = \frac{-x^2}{(x+1)^2}+1-\frac{1}{(x+1)^2} \Rightarrow f(x) = x.$$

Thus $f(x) = x \ \forall \ x > 0$. ...(C)

Now let $x < 0$, Putting $x = -y$, so that $y > 0$

We have $f(x) = f(-y) = -f(y)$ by (i)

$= -y$ by (C)

$= x$.

Since we have already seen that $f(x) = x$ when $x = 0$, it follows that $f(x) = x \ \forall \ x \in R$.

Example 4 : Determine all functions f satisfying the functional relation $f(x)+f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(1-x)}$ where x is a real number $x \neq 0$, $x \neq 1$.

Soln.: We are given that

$$f(x)+f\left(\frac{1}{1-x}\right) = \frac{2(1-2x)}{x(x+1)} = \frac{2}{x} - \frac{2}{1-x} \quad \text{...(1)}$$

for all x other than 0 and 1

Let us introduce another variable y here so that

$$y = \frac{1}{1-x} \left(\text{i.e. } x = 1-\frac{1}{y}\right)$$

From (1) we get $f(x)+f(y) = \frac{2}{x}-\frac{2}{y}$ ...(2)

(2) holds for all values of x and y except 0 and 1. (see what happens for $y = 0$ or 1).

Also we have

$$f(y)+f\left(\frac{1}{1-y}\right) = \frac{2}{y}-\frac{2}{1-y} \quad \text{...(3)}$$

for all real values of y other than 0 and 1 (this is simply (1)).

Introducing another variable

$$z = \frac{1}{1-y} \left(\text{so that } y = 1-\frac{1}{z}\right).$$

In the same manner as above, we have

$$f(y)+f(z) = \frac{2}{y}-2z \quad \text{...(4)}$$

Whenever $y, z \neq 0$ or 1.

The relation $y = \frac{1}{1-x}$ and $z = \frac{1}{1-y}$ give $x = \frac{1}{1-z}$, so that $z = 1-\frac{1}{x}$.

Considering the relation $f(z)+f\left(\frac{1}{1-z}\right)$

$$= \frac{2}{z} - \frac{2}{1-z} \quad \text{...(5)}$$

Substituting $x = \frac{1}{1-z}$, we have

$$f(z)+f(x) = \frac{2}{z}-2x \quad \text{...(6)}$$

whenever $x, z \neq 0$ or 1.

Adding corresponding sides of (2), (4) and (6) and dividing throughout by 2, we have

$$f(x)+f(y)+f(z) = \left[\left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-(x+y+z)\right]$$

By (4) we have

$$f(x) = \left(\frac{1}{x}+\frac{1}{y}+\frac{1}{z}\right)-(x+y+z)$$

$$= \left(\frac{1}{x}-x\right)-\left(y+\frac{1}{y}\right)+\left(z+\frac{1}{z}\right).$$

$$= \left(\frac{1}{x}-x\right)-\left\{\frac{1}{1-x}+1-x\right\}+\left\{1-\frac{1}{x}+\frac{x}{x-1}\right\} = \frac{x+1}{x-1}.$$

Thus $f(x) = \frac{x+1}{x-1} \ \forall \ x \neq 0$ or 1.

Now we place below some good examples with limits enough to enable you to solve it easily try them and enjoy solving them.

Example 5 : A function f is defined for all positive integers and satisfies $f(1) = 1996$; and

$f(1) + f(2) + f(3) + ... + f(n) = n^2 f(n) \ \forall \ n > 1$

Calculate the exact value of $f(1996)$.

Soln.: If you have never seen anything like this before, it seems reasonable to start calculating

$f(1) = 1996$. (given)

$f(2) =$ ...... (from $f(1) + f(2) = 2^2 f(2)$);

$f(3) =$ .... and so on. Use this approach to find $f(2)$, $f(3)$ and $f(4)$. If you preserve, and keep your wits about you, you might just notice some thing interesting. If you are lucky, you may even be able to guess a value of $f(1996)$. But this would not answer the question. In mathematics, the determine means more than just guess; it means that you have to show exactly why your value is correct. For this, it is not the value of $f(2)$, $f(3)$, and so on, that matters, but their form. Thus it is important to express $f(2)$ and $f(3)$ in a form that reveals what is really going on:

$$\text{Thus } f(2) = \frac{1}{2^2 - 1} \cdot f(1)$$

$$f(3) = \frac{1}{3^2 - 1}[f(1) + f(2)]$$

$$= \frac{1}{3^2 - 1}\left[f(1) + \frac{1}{2^2 - 1} f(1)\right] = \frac{1}{2^2 - 1} \cdot \frac{2^2}{3^2 - 1} \cdot f(1)$$

Write out the calculation which shows that

$$f(4) = \frac{1}{2^2 - 1} \cdot \frac{2^2}{3^2 - 1} \cdot \frac{3^2}{4^2 - 1} \cdot f(1).$$

Now guess what you expect to be the corresponding expression for $f(n)$ in terms of $f(1)$, and prove that your guess is correct (by induction on n). Even at this stage it is important to resist the temptation simply to substitute $n = 1996$. Factorize each of the factors $(r^2 - 1)$ in the denominator of your (proven) expression for $f(n)$, and cancel to obtain a greatly simplified formula for $f(n)$ in terms of n and $f(1)$. Finally substitute $n = 1996$.

Example 6 : Let f be a function mapping positive integers into positive integers. Suppose that $f(n + 1) > f(n)$ and $f(f(n)) = 3n$ for all positive integers n. Determine $f(1992)$.

Soln.: Hint of solution is provided in the following six stages.

Stage 1 : Successful problem solving often depends on clinging to the optimistic assumption that although unfamiliar problems often seem impossible at first, they can usually be solved once you understand what is going on. This question looks unfamiliar, and is at first sight very abstract – in that it involves an unknown function f which is not actually given, being described only in terms of some of its properties. Despite all this, you have no option but to take courage and set to. Where should one begin? The obvious place (since f maps positive integers to positive integers) is to start thinking about the value of $f(n)$ when $n \equiv$ ..... Since $f(1)$ has to be a positive integer, you certainly know that $f(1) \geq 1$. You also know that $f(f(1)) =$ ..... It follows that f can not be the identity function. This may not seem to help, until you realise that this, combined with the condition $f(n + 1) > f(n)$ implies that $f(n) > n$. (Why?) use this fact to prove that $f(1) = 2$.

Stage 2 : Use the same ideas to find $f(2)$, $f(3)$ and $f(6)$.

Stage 3 : Once you know $f(3)$ and $f(6)$, the condition $f(n + 1) > f(n)$ should tell you exactly $f(4)$ and $f(5)$ have to be.

Stage 4 : You are now in business. Since the condition $f(f(n)) = 3n$ allows you to fill in quite a few other values (such as $f(7)$, $f(8)$, $f(9)$, and so on). It may still not be clear what is going on; but atleast the horrible feeling that you cannot even begin should have begun to get shorter in size.

Stage 5 : If you construct table of values for the function, you should realize that there are long stretches where $f(n + 1) = f(n) + 1$, followed by stretches where the value of $f(n)$ goes up in jumps. How large are these jumps? Later on you will have to come back and prove the important bits, but the main thing at this stage is to convince yourself that what seemed like a totally inaccessible problem is in fact much more manageable than you thought : the first step is making sense of what is going on is often to risk yourself by making some kind of a guess as to what you expect to find in the next bit of the table..... that is, to formulate simple conjecture and then to test them against the fact to see if they stand up. If your guess turns out to be wrong, stand back and try to improve it. You should decide fairly quickly that you think you then know where the jumps occur. It is not too hard then to decide what you expect the value of $f(1992)$ to be.

Stage 6 : All that remains is to decide what to prove, and how to prove it. This is not nearly as hard as it may appear provided that you sort out what to prove first.

(a) Prove that (by induction on n)
$f(3^n) =$ .... and that $f(2 \cdot 3^n) =$ ....

(b) Then prove that
$f(3^n + k) =$ .... for each k, $0 \leq k \leq 3^n$.

(c) Finally deduce that $f(2 \cdot 3^n + k) = 3^{n+1} + 3k$ for $0 \leq k \leq 3^n$ from a, b, c you can calculate
$f(2 \cdot 3^6) =$ .... and $f(2 \cdot 3^6 + 534) =$ .... ■

LEARN*fast* Understanding Concepts & Problem Solving

Logarithms and their Properties

Learnfast will aid students in quick understanding of concepts on the above topic.
Solved examples are given at the end of the topic so that application of concepts can be understood.
Remember it is not the number of problems that you have solved that counts, but your level of understanding.

- Definition (Natural and Common log)
- Some aspects of logarithms
- Graphing logarithmic function
- Properties of logarithms
- Examples

Definition : Just as the subtraction is the inverse operation of addition, and taking a square root is the inverse operation of squaring, *logarithms* are the inverse operations of exponentiation. More formally, mathematically,

$$y = \log_b x \text{ if and only if } b^y = x$$

where $x > 0$, $b > 0$ and $b \neq 1$. The base 'b' can be any number (except 1). However, two choices are most usual : 10 and $e = 2.7182\ldots$. Logs to the base 10 are often called *common logs* where as log to the base e are often called *natural logs*. It is always assumed, unless stated that log means $\log_{10}$. The natural log i.e. $\log_e$ is also written as ln.

Logarithm Base 10 : The logarithm of base 10 is most often useful when powers of 10 are involved. Logarithm base10 is the logarithmic operation that when carried out on 10 raised to some power ends up giving us the power. It is written as $\log_{10}$. Thus $\log_{10}(10^x) = x$. This is the basic definition of base 10 logs.

Natural logarithm (or base e) : In the cases where the exponential changes are involved we usually use another kind of logarithm called natural logarithm. The natural log can be throught of as logarithm Base-e. It is a logarithmic operation that when carried out on e raised to some power gives us the power itself. This logarithm is labelled with ln (for natural logs) and its definition is $ln(e^x) = x$.

Some Aspects of Logarithms

1. log(1) = 0 : By definition 10 to the power of 0 is equal to 1, so that is why log1 = 0. In fact, since any number except 0 raised to power 0 gives 1, thus logarithm of the value of 1 will always be zero no matter what base we are talking about (as $\log_a 1 = 0 \Rightarrow a^0 = 1$). Thus, the natural logarithm of 1 is also equal to zero. $ln(1) = 0$.

2. Negative answers when taking log : If we take 10^0 we will get 1 and if we take 10^1 we will get 10. Thus, in order to get a values less than 1 we will have to take 10 to the power of some number that is less than zero, thus 10 to the power of a negative number. So when taking log in such condition we will end up with a negative answer.

$$10^{-3} = 0.001 \Rightarrow \log_{10}(0.001) = -3.$$

3. log(0) = undefined : As seen earlier log of a number less than 1 results in the negative answer. Now as we take log of smaller and smaller numbers say log(0.1), log(0.01), log(0.001).... which would give -1, -2, -3, ... respectively. Thus every time we will get a smaller negative number. Well if we will iterate this process until we get very close to zero we will notice that we are getting very large negative numbers finally approaching towards a final huge undefined quantity as approached to zero. Thus log(0) and negative number are undefined.

Graph of logarithmic functions:

Consider function $\log_a(x) = y$

Now by the definition of log functions

$$\log_a(x) = y \Rightarrow a^y = x$$

(1) $\therefore$ as x increases, y also increases

$\therefore$ y is an increasing function.

(2) Since $\log_a x$ is defined for $x > 0$

$\therefore$ the entire graph of y lies in Ist and IVth quadrant. Since no value of y can make $x = 0$ thus the graph never cuts y-axis.

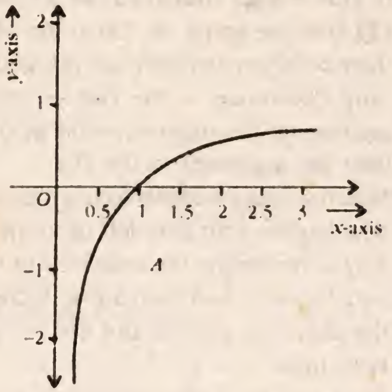
(3) Since $a^0 = 1$ $\therefore$ the graph cuts x axis at $x = 1$.

(4) Notice that the increase in the value of the function is most dramatic between 0 and 1. After $x = 1$, as x gets larger the increasing functions values

begin to slow down (the increase get smaller and smaller as x gets larger).

(5) The function values are positive for x's that are greater than 1 and negative for x less than 1.

Thus from the above discussed facts the graph of $y = \log_a x$ can be plotted as.



Now we will discuss various other cases depending on the change in the value of x, how the graph get transformed.

Transformations

A. $f(-x) = \log(-x)$

(1) The domain of $f(x) = \log x$ is the set of all positive real numbers so domain of $f(-x)$ is the set of all negative numbers.

(2) The graph of $f(x)$ is located in Ist and IVth quadrants. Thus graph of $f(-x)$ is located in II and III quadrants where x values are negative.

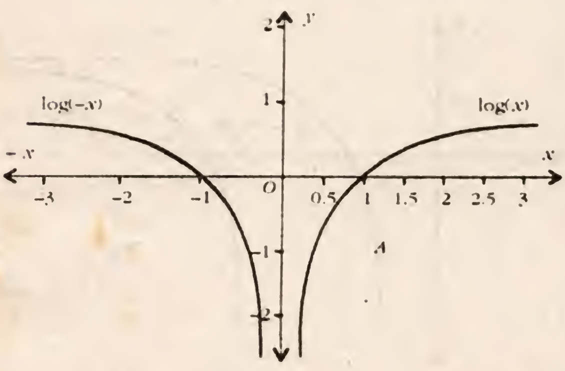
(3) Since $f(x)$ can never cut y-axis so is the case with $f(-x)$. Thus it has no y-intercept.

(4) The graph of $f(x)$ crosses the x-axis at (1, 0). so the graph of $f(-x)$ cuts the x-axis at (–1, 0).

(5) Note that the points of both of the graphs have same y-coordinates and their x-coordinates differ by a minus sign as

$(2, f(2)) = (2, \log 2)$ for $f(x) = \log x$ and

$(-2, f(-2)) = (-2, \log(-(-2))) = (-2, \log(+2))$



Thus $f(x)$ and $f(-x)$ are mirror images of each other over the y-axis i.e. the graphs are symmetric to each other with respect to the y-axis.

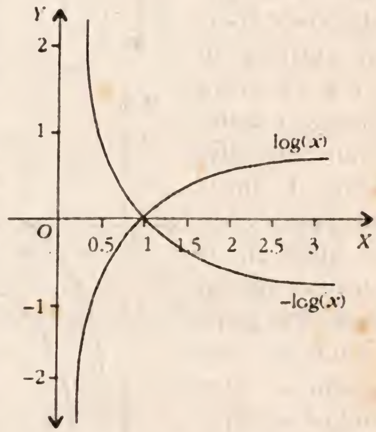
∴ log $(-x)$ is reflection over y-axis of log (x). Finally we can say if there is a minus sign (–) before x in the argument of a function it indicates that there is reflection of the graph of $f(x)$ across y–axis.

B. $f(x) = +\log x \quad g(x) = -f(x) = -\log x$

(1) The domain of both the function is set of +ve real numbers.

(2) Graph of both functions are located in I and IV quadrant i.e. to the right of y-axis.

(3) The graphs of both functions cross the x-axis at $x = 1$. Since neither graph cross the y-axis, thus there is no y-intercept.



(4) Both graphs have coordinates having same x-coordinates but their y-coordinates differ by a minus sign.

(5) Thus both the graph are symmetric to each other with respect to the x-axis.

∴ Whenever the minus sign (–) is in front of the function notation, it indicates a reflection across the x-axis thus the graph of $-f(x)$ is the reflection of $f(x)$ across x-axis.

C. $g(x) = \log x + c; \quad f(x) = \log x$

(1) The only difference between the two equations is the $+c$. Since $f(x) = \log x$, $g(x) = \log x + c$ can be rewritten as $g(x) = f(x) + c$. This means that for every value of x, the function $g(x)$ will always be c units larger than the function $f(x)$.

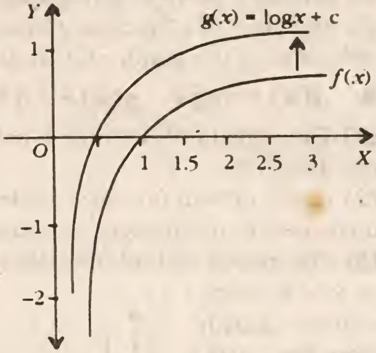
(2) Both the graphs are located in quadrants I and II. Also the domain (values of x) of both functions is the set of positive real numbers.

(3) The graph of the function $f(x)$ has an x-intercept at $x = 1$. The graph of the function $g(x)$ has an x-intercept at the value of x where $\log_a(x) + c = 0$ i.e. $\log_a(x) = -c$ which is some

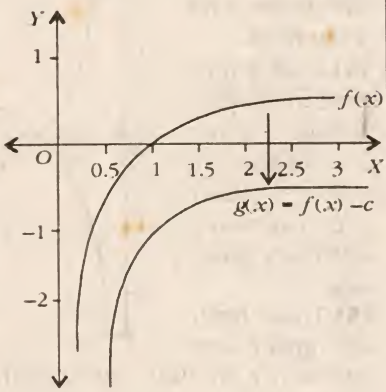
constant values in (0, 1).

(4) For each x-coordinate the y-coordinate of $g(x)$ differs by c units.

(5) Both the graphs have the same shape. The graph of $g(x)$ is nothing more than the graph of $f(x)$ shifted up by c units.



Note that as we have obtained the graph of $g(x) = f(x) + c$ by shifting it up by c units. Similarly we can make the graph of $g(x) = f(x) - c$ by shifting it downwards through c units. Thus finally, vertical shifts takes place when a function is shifted up or down. The graph of $f(x) + c$ is the graph of $f(x)$ shifted up by c units. The graph of $f(x) - c$ is the graph of $f(x)$ shifted down by c units.



D. The graph of $f(x)$ versus the graph of $f(x+c)$ or $f(x-c)$.

Let us consider

$$f(x) = \log x, \quad g(x) = \log(x + c)$$

(1) The domain of the function $f(x)$ is the set of positive real numbers. The domain of $g(x)$ is the set of real numbers where $x + c > 0$ or the set of real numbers greater than $-c$.

(2) The graph of $f(x) = \log x$ is located in I and IV quadrant. This verifies that the domain is the set of +ve real numbers (for $f(x)$). The graph of $g(x)$ is located in quadrants I, II and III to the right of $-c$.

(3) The graph of $f(x)$ crosses x-axis at 1 and never crosses the y-axis. The graph of $g(x)$ crosses x-axis at the point where $x + c = 1 \Rightarrow x = 1 - c$ and also it cuts the y-axis at the point where $x = 0$ i.e. at $\log(c)$.

(4) Both the graphs of $f(x) = \log x$ and $g(x) = \log(x + c)$ have the same shape. The graph of $g(x) = \log(x + c)$ is nothing more than the graph of $f(x) = \log x$ shifted to the left by c units.

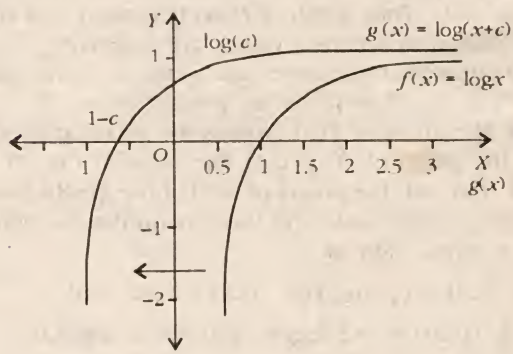
(5) Shift the graph of $f(x)$ to the left c units. It will then be super imposed on the graph of $g(x)$. The only difference in the two equations is in their arguments. The argument in the $g(x)$ is c units greater than the argument in the $f(x)$.

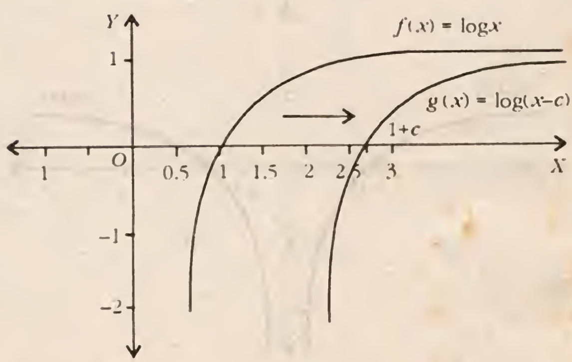
Note that many students have a hard time determining whether the shift is to left or to the right. The easy way to determine the answer is to set the argument equal to zero and then solve. If the answer is –ve, the shift is to the left and if it is +ve then the shift is to right.

For example, if the equation $h(x) = \log(x - c)$ the shift would be to the right

$\because$ $x - c = 0 \Rightarrow x = c$ and in $g(x) = \log(x + c)$, the shift is to left

$\because$ $x + c - 0$ gives $x = -c$.





(E) Let us now discuss the difference between the graph $f(x) = \log x$ and $g(x) = c \log x$ (depending on $c > 1$ or $c < 1$)

Let us first consider

$f(x) = \log x$ and $g(x) = c \log(x)$ where $c > 1$

(1) Both the graphs are located in I and IV quadrants.

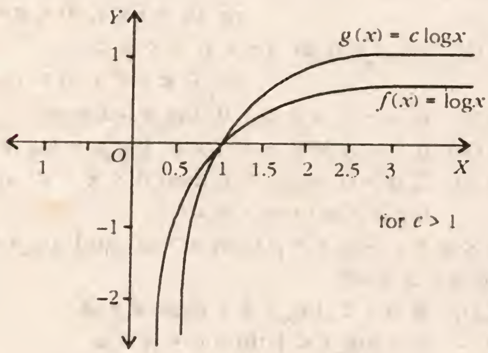
(2) The domain of both the functions is the set of positive real numbers.

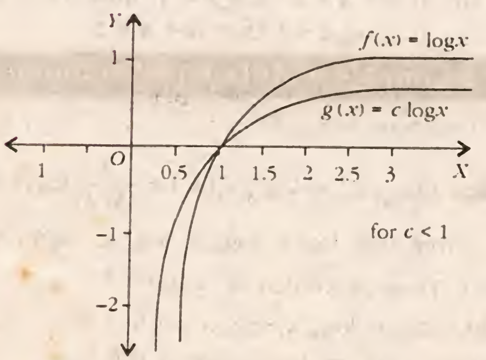
(3) Neither of the graph crosses the y-axis; $\therefore$ the graphs has a y-intercept. Notice that both graphs cross the x-axis at 1 because $\log x = 0$ when $x = 1$ and $c \log x = 0$ when $x = 1$.

The y coordinates of $g(x)$ is 4 times the y-coordinates of $f(x)$.

(4) Both graphs have the basic shape shared by all logarithmic functions. It appears that the graphs of $g(x) = c\,f(x)$ is a result of stretching the graph of $f(x) = \log x$. For every value of x the value of $g(x)$ is c times larger than the value of $f(x)$.

(5) The graph $f(x)$ is stretched up to be superimposed on the graph of $g(x)$.





(F) $f(x) = \log x$, $g(x) = \log(cx)$

Depending on the values of c we have to cases whether $c > 1$ or $c < 1$. let us first discuss for $c > 1$

(1) Both the graphs are located in Ist and IVth quadrant

(2) Domain of both functions is the set of positive real numbers.

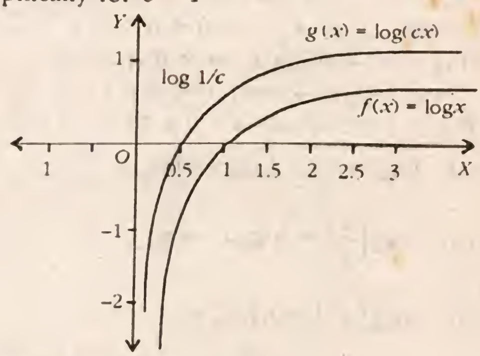
(3) Since neither of the graph crosses the y-axis, $\therefore$ neither of the graphs has a y-intercept

(4) Even though the graph of $g(x)$ seems to be different from the graph of $f(x)$, both graphs have essentially the same shape. The graph of $g(x)$ is located above the graph of $f(x)$ for all positive values of x, and graph of $g(x)$ is located below the graph of $f(x)$ for all negative values of x. The graph of $g(x)$ is thus is a result of stretching and shrinking of graph of $f(x)$. For example for every +ve value of x the value of $g(x)$ is larger than the graph of $f(x)$ by $\log(c)$.

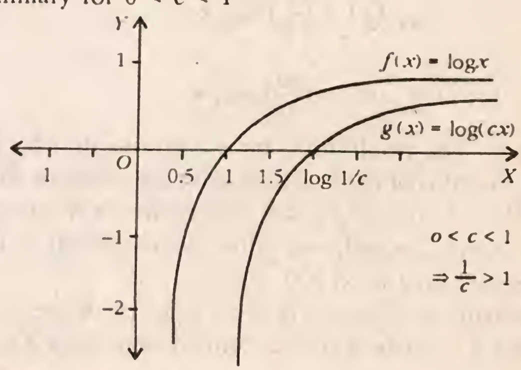
(5) $g(x) = \log(cx) = \log c + \log x = \log c + f(x)$

(6) Graph of $f(x)$ is stretched upward by adding $\log(c)$ to every y-coordinate i.e. if (a, b) is any point on $f(x)$ then $(a, b + \log c)$ is a point on $g(x)$.

Graphically for $c > 1$



Similary for $0 < c < 1$



Properties of Logarithms

Since logarithms are best known as the inverse process of exponentials so most of the properties of logarithms are the consequences of the corresponding properties of exponents. Some of the properties are:

(1) (ii) By definition $a^p = x$ iff $\log_a x = p$

so $a^{\log_a x} = x$ thus $\boldsymbol{a^{\log_a x} = x}$.

(ii) $a^{\log_b x} = x^{\log_b a}$ $a > 0,\ b > 0,\ b \neq 1,\ x > 0$

(2) $\log_a a = 1,\ \log_a 1 = 0$

(3) (i) $\log_a x = \dfrac{1}{\log_x a}$

(ii) $\log_a x = \log_b x \cdot \log_a b = \dfrac{\log_b x}{\log_b a}$

The above two properties are illustrating the cases related to change of base.

(4) For $a > 0,\ a \neq 1$

(i) $\log_a x$ is real if $x > 0$, imaginary if $x < 0$ and undefined if $x = 0$.

(ii) as $x \to 0$, $\log_a x \to -\infty$ if $a > 1$,

and $\log_a x \to \infty$, if $0 < a < 1$

(iii) as $x \to \infty$ $\log_a x \to \infty$ if $a > 1$,

and $\log_a x \to -\infty$, $0 < a < 1$

(5) For $x, y > 0$ (base $a > 0 \neq 1$)

(i) $\log_a(xy) = \log_a x + \log_a y$

(ii) $\log_a\left(\dfrac{x}{y}\right) = \log_a x - \log_a y$

(iii) $\log_a(x^n) = n\log_a x$

(iv) $\log_{a^n}(x) = \left(\dfrac{1}{n}\right)\log_a x$

(v) $\log_{a^n} x^m = \left(\dfrac{m}{n}\right)\log_a x$

Note : The result in (5) have a serious drawbacks as domains of left and right sides are different. The L.H.S. of 5(i) and (ii) are defined for $(xy) > 0$ i.e. both +ve , or both –ve values while the R.H.S. are defined only for $x > 0$, $y > 0$.

Similarly in 5(iii) if n is even, L.H.S. is defined for all $x \neq 0$ while R.H.S. is defined only for $x > 0$.

While using these properties, the risk of using some solution is always there.

The more general properties are

6. (i) $\log_a(xy) = \log_a|x| + \log_a|y|$ $(xy > 0)$

(ii) $\log_a\left(\dfrac{x}{y}\right) = \log_a|x| - \log_a|y|$

$\left(\dfrac{x}{y} > 0,\ y \neq 0\right)$

(iii) $\log_a x^{2k} = 2k\log_a|x| - \log_a|y|$

($x \neq 0$, k is any integer)

(iv) $\log_{a^k} x = \dfrac{1}{2k}\log_{|a|} x$ $\begin{pmatrix} a \neq 0 & |a| \neq 1 \\ k \neq 0 & x > 0 \end{pmatrix}$

These formulae may give some extra solutions which can be discarded by verification.

Properties involving inequalities

(1) (i) $\log_a x > 0$, iff $(x > 1, a > 1)$

or $(0 < x < 1,\ 0 < a < 1)$

(ii) $\log_a x < 0$ iff $(x > 1,\ 0 < a < 1)$

or $(0 < x < 1,\ a > 1)$

(2) (i) If $a > 1$, $x > y > 0$; $\log_a x > \log_a y$

(ii) If $0 < a < 1$, $x > y > 0$, $\log_a x < \log_a y$

(3) (i) If $a > 0$, $\log_a x < p$ then $0 < x < a^p$ and $\log_a x > p$ then $x > a^p$.

If $0 < a < 1$, $\log_a x < p$ then $x > a^p$, and $\log_a x > p$ then $0 < x < a^p$

(4) (i) If $a > 1$, $\log_a x > 1$ then $x > a$,

$0 < \log_a x < 1$ then $1 < x < a$

(ii) If $0 < a < 1$, $\log_a x > 1$ then $0 < x < a$,

$0 < \log_a x < 1$ then $a < x < 1$.

Examples with Detailed Solution

1. Compute $\log_{3\sqrt{3}} 27$

Soln.: $\log_{3\sqrt{3}} 27 = \log_{(3^{3/2})}(3^3) = \dfrac{3}{(3/2)}\log_3 3 = 2.$

2. Given that $\log_l x$, $\log_m x$, $\log_n x$ are in A.P., $x \neq 1$. Then prove that $n^2 = (ln)^{\log_l m}$.

Soln.: $\log_l x$, $\log_m x$, $\log_n x$ are in A.P.

so $\log_x l$, $\log_x m$, $\log_x n$ are in H.P.

$(l,\ m,\ n > 0,\ \neq 1,\ \ x > 0,\ \neq 1)$

$\therefore \log_x m = \dfrac{2\log_x l \cdot \log_x n}{\log_x l + \log_x n} = \dfrac{2\log_x l \cdot \log_x n}{\log_x(ln)}$

so $\log_l m \cdot \log_x(ln) = \log_x n^2$

i.e. $\log_x(ln)^{\log_l m} = \log_x(n^2)$ *i.e.* $n^2 = (ln)^{\log_l m}$.

3. Find $\log_{54}(168)$ if $\log_7 12 = a$ and $\log_{12} 24 = b$.

Soln.: $168 = 24 \times 7$, $\log_{12} 168 = \log_{12} 24 + \log_{12} 7$

$$= b + \frac{1}{a} = \frac{ab+1}{a} \quad \text{and}$$

$$\log_{12} 54 = \frac{\log_2 3^3 \cdot 2}{\log_2 2^2 \cdot 3} = \frac{3\log_2 3 + 1}{2 + \log_2 3},$$

$$b = \frac{\log_2 2^3 \cdot 3}{\log_2 2^2 \cdot 3} = \frac{3 + \log_2 3}{2 + \log_2 3} \qquad \therefore \log_2 3 = \frac{3-2b}{b-1}$$

$$\text{thus } \log_{12} 54 = \frac{3\left(\frac{3-2b}{b-1}\right)+1}{2+\frac{3-2b}{b-1}} = (8-5b).$$

$$\text{Hence } \log_{54} 168 = \frac{\log_{12} 168}{\log_{12} 54} = \frac{ab+1}{a(8-5b)}.$$

4. If n is a natural number > 1, find least value of n for which $\log_n 121 < 3.8$

Soln.: Given $\log_n 121 < 3.8$

$\therefore 121 < n^{3.8}$ (as $n > 1$)

or $3.8 \log_{10} n > \log_{12} 121$, $\log_{10} n > \dfrac{\log_{10} 121}{3.8}$

or $\log_{10} n > .5481$, taking antilog $n > 3.5327$,

$\therefore$ least integral value of $n = 4$.

5. If a, b, c are in G.P., prove that $\log_a n$, $\log_b n$, $\log_c n$ are in H.P.

Soln.: From the problem, it is clear that

$a, b, c > 0, \neq 1$, $n > 0, \neq 1$,

(as no term of H.P. can be zero)

a, b, c are in G.P so $b^2 = ac$

or $2\log_n b = \log_n a + \log_n c$

so $\log_n a$, $\log_n b$, $\log_n c$ are in A.P.

Hence $\log_a n$, $\log_b n$, $\log_c n$ are in H.P.

6. If a, b, c are distinct positive numbers each different from 1 such that

$(\log_b a \log_c a - \log_a a) + (\log_a b \log_c b - \log_b b) + (\log_a c \log_b c - \log_c c) = 0$

then prove that $abc = 1$.

Soln.: As a, b, c are distinct and different from unity, $\log a$, $\log b$, $\log c$ are all different and different from 0.

$$\log_b a \cdot \log_c a - \log_a a = \frac{\log a}{\log b} \cdot \frac{\log a}{\log c} - \frac{\log a}{\log a}$$

$$= \frac{(\log a)^3 - \log a \log b \log c}{\log a \log b \log c} \text{ and similar terms.}$$

Putting in given expression

$$\frac{(\log a)^3 + (\log b)^3 + (\log c)^3 - 3\log a \log b \log c}{\log a \ \log b \ \log c} = 0$$

$$\text{or } (\log a + \log b + \log c)\left[(\log a)^2 + (\log b)^2 + (\log c)^2 - \log a \cdot \log b - \log b \cdot \log c - \log c \cdot \log a\right] = 0$$

i.e. $\log a + \log b + \log c = 0$

as $\log a$, $\log b$, $\log c$ are distinct

or $\log (abc) = 0$ or $a \cdot b \cdot c = 1$.

7. Solve the equation :

$a^{2x} \cdot b^{3y} = m^5$; $a^{3x} \cdot b^{2y} = m^{10}$ $a, b, m > 0, \neq 1$.

Soln.: $a^{2x} \cdot b^{3y} = m^5$ ...(1)

$a^{3x} \cdot b^{2y} = m^{10}$ ...(2)

$a, b, m > 0, \neq 1$

Taking log, the equations (1) and (2) become

$2x \log a + 3y \log b = 5 \log m$

$3x \log a + 2y \log b = 10 \log m$;

Solving we get $x = 4 \log_a m$, $y = -\log_b m$.

8. Solve the equation :

$3\log_{(3x^2)} 27 - 2\log_{(3x)} 9 = 0$.

Soln.: The initial conditions are $x^2 \neq \frac{1}{3}$, $x > 0, \neq \frac{1}{3}$.

Changing all logarithm to base 3, we get

$$\frac{9}{1+2\log_3 x} = \frac{4}{1+\log_3 x} \Rightarrow \log_3 x = -5$$

$$\therefore x = \frac{1}{243}.$$

9. Solve the equation : $\log_{(2x+3)}(6x^2 + 23x + 21) = 4 - \log_{(3x+7)}(4x^2 + 12x + 9)$.

Soln.: In this equation, the domain is defined by

$2x + 3 > 0, \neq 1$, $3x + 7 > 0$, $\neq 1$,

$6x^2 + 23x + 21 = (2x + 3)(3x + 7) > 0$

$4x^2 + 12x + 9 = (2x + 3)^2 > 0$.

The later two are included in first two giving

$x > -3/2, \neq -1$, $x > -\frac{7}{3}, \neq -2$ i.e. $x > -\frac{3}{2}, \neq -1$

is the required domain of the equation (also called initial conditions).

The given equations can be re-written as

$1 + \log_{(2x+3)}(3x + 7) + 2 \log_{(3x+7)}(2x + 3) - 4 = 0$

$$\text{or } z + \frac{2}{z} - 3 = 0 \Rightarrow z^2 - 3z + 2 = 0 \Rightarrow z = 1, 2$$

(putting $\log_{(2x+3)}(3x + 7) = z$)

If $z = \log_{(2x+3)}(3x+7) = 1$,
then $3x + 7 = 2x + 3$ or $x = -4$
If $z = \log_{(2x+3)}(3x+7) = 2$
then $(2x+3)^2 = 3x + 7$, giving $x = -2, -1/4$
Of these only $x = -1/4$ lies in the domain.
Hence $x = -1/4$ is the solution.

10. Solve : $|x-1|^{(\log_3 x^2 - 2\log_x 9)} = (x-1)^7$

Soln.: Domain of the equation is $x > 0, \neq 1$
so $|x - 1| \neq 0, \neq -1$
If $|x - 1| = 1$ i.e. $x - 1 = +1$ i.e. $x = 2$ which satisfies the equation. But $x - 1 = -1$ i.e. $x = 0$ is out of the domain.
If $|x - 1| \neq 1$ ($x - 1 \nless 0$ as then L.H.S. is +ve but R.H.S. is –ve),
So taking $x - 1 > 0$, from given equation, we have

$$2\log_3 x - \frac{4}{\log_3 x} = 7$$

i.e. $2(\log_3 x)^2 - 7\log_3 x - 4 = 0$

$\Rightarrow \log_3 x = 4, -1/2$ giving $x = 81, \frac{1}{\sqrt{3}}$

But $x = \frac{1}{\sqrt{3}}$ being less than unity is not valid. Hence the solutions are $x = 2, 81$.

11. Solve : $x^{\left(\frac{1}{\log_{10} x}\right)}, \log_{10} x < 1$

Soln.: From the equation, it is clear that $x > 0$
Case I : If $x \neq 1$, then inequation can be rewritten as

$$x^{\log_x 10} \cdot \log_{10} x < 1 \text{ i.e. } \log_{10} x < \frac{1}{10}$$

so $0 < x < 10^{1/10}$, $x \neq 1$ ...(1)
(as base is > 1)
Case II. If $x = 1$, we have $1^\infty \cdot 0 = 0 < 1$
so $x = 1$ is also a solution ... (2)
Combining (1) and (2) we have the solution
$0 < x < 10^{1/10}$.

12. If $\log_{12} 18 = a$, $\log_{24} 54 = b$, prove that $ab + 5(a - b) = 1$.

Soln.: Converting all logarithms to any common base, we have

$$a = \frac{2\log 3 + \log 2}{\log 3 + 2\log 2} \quad \text{giving } \frac{\log 2}{\log 3} = \frac{2-a}{2a-1}, \quad ...(1)$$

$$\text{and } b = \frac{3\log 3 + \log 2}{3\log 2 + \log 3}$$

$$\text{giving } \frac{\log 3}{\log 2} = \frac{3b-1}{3-b} \quad ...(2)$$

Multiplying (1) and (2) we get the required result.

13. Prove that if a and b are the lengths of the legs and c the length of the hypotenuse of a right triangle, $c - b \neq 1$, $c + b \neq 1$, then
$\log_{c-b} a + \log_{c+b} a = 2 \log_{c+b} a \cdot \log_{c-b} a$.

Soln.:
By the property of right angled triangle, we have $c^2 = a^2 + b^2$, $a, b, c > 0$

$$\text{L.H.S.} = \frac{\log a}{\log(c-b)} + \frac{\log a}{\log(c+b)}$$

$$= \log a \left\{\frac{\log(c+b) + \log(c-b)}{\log(c+b)\log(c-b)}\right\}$$

$$= \frac{\log a \cdot \log(c^2 - b^2)}{\log(c+b)\log(c-b)} = \frac{\log a \cdot \log a^2}{\log(c+b)\log(c-b)}$$

$$= 2\log_{c+b} a \cdot \log_{c-b} a.$$

14. If $\log(a+b+c) = \log a + \log b + \log c$,

prove that $\log\left(\frac{2a}{1-a^2} + \frac{2b}{1-b^2} + \frac{2c}{1-c^2}\right)$

$$= \log\frac{2a}{1-a^2} + \log\frac{2b}{1-b^2} + \log\frac{2c}{1-c^2}.$$

Soln.: Assuming $a, b, c > 0$, and putting $a = \tan\alpha$, $b = \tan\beta$ and $c = \tan\gamma$, in the given relation, we get
$\log(a + b + c) = \log a + \log b + \log c$,
$a + b + c = abc$
i.e. $\tan\alpha + \tan\beta + \tan\gamma = \tan\alpha \tan\beta \tan\gamma$

$$\text{or } \frac{\tan\alpha + \tan\beta}{1 - \tan\alpha\tan\beta} = -\tan\gamma = \tan(\pi - \gamma)$$

Thus we get $\tan(\alpha + \beta) = \tan(\pi - \gamma)$
Taking α, β, γ in first quadrant, we get
$\alpha + \beta = \pi - \gamma$ i.e. $\alpha + \beta + \gamma = \pi$,
giving $2\alpha + 2\beta + 2\gamma = 2\pi$,
i.e. $\tan 2\alpha + \tan 2\beta + \tan 2\gamma - \tan 2\alpha \tan 2\beta \tan 2\gamma = 0$.
Transposing and taking log, we get the required result.

15. Solve : $x\log_{10}\left(\frac{10}{3}\right) + \log_{10} 3 = \log_{10}(2 + 3^x) + x$.

Soln.: Given equation is (after simplifying)
$x(1 - \log_{10} 3) + \log_{10} 3 = \log_{10}(2 + 3^x) + x$,

$$\Rightarrow \log_{10} 3^{(1-x)} = \log_{10}(2 + 3^x) \Rightarrow \frac{3}{3^x} = 2 + 3^x,$$

$\Rightarrow (3^x + 3)(3^x - 1) = 0$, which gives $3^x = 1 = 3^0$

as $3^x \neq -3$. Hence $x = 0$.

■

mathematical MISCONCEPTIONS

ROUNDING NUMBERS

Question **: What is 14489 to the nearest 1000 ?**

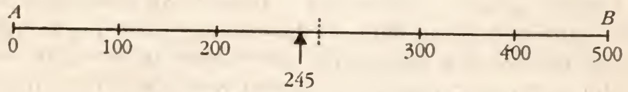
Misconception	**Correct**
To obtain the answer, round to the nearest 10,100 and then 1000; thus: 14489 to the nearest 10 is 14 490, 14490 to the nearest 100 is 14 500, 14500 to the nearest 1000 is 15 000. Hence : the misconception leads to the in correct answer, 15000.	The answer must be either 14000 or 15000, and since 14489 − 14000 = 489, whilst 15000 − 14489 = 511, clearly 14489 is nearer to 14000 than to 15000. Hence : the correct answer is 14000.

Further Explanation :

The correct explanation above, that 14489 is nearer to 14000 than to 15000, is clear-cut; but why is the 'misconceived' method wrong ?

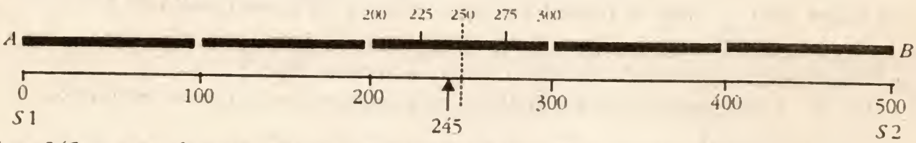
This is best illustrated by taking another example and putting it into context.

Two stations, A and B, are 500 metres apart:



You stand between the stations, 245 m away from A and 255 m from B and you want to walk to the nearest station. Naturally you turn left and walk 245 m.

But now imagine there are five trains parked between the stations, each train 100 m long, and each divided into 4 carriages of equal length.



Your position, 245 m away from A, is now 20 m along the second carriage of the third train, and you decide to find your way to the nearest station by first asking which is the nearest carriage-end (the one on the right, 250 metres from A).

Having moved there, you then seek the nearest end of the train, which (as, by convention, we round mid-points up) is to the right at 300 metres from A. From there finally, you go to the nearest station, and this time arrive at B.

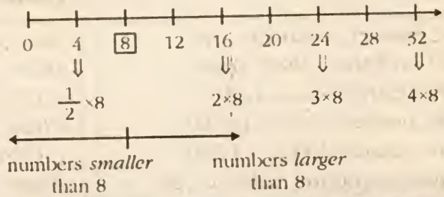
Clearly the result is wrong. Why ?

Because when you want to find the nearest station, the questions about the nearest carriage-end and the nearest train end are irrelevant. Doing it this way might not matter in some cases, but here it does. The unnecessary interim steps move you away from your starting point, and here, in the wrong direction.

Another instructive point : this exercise provides an opportunity to distinguish between an arbitrary convention (i.e., which way to go from the middle) and a reasoned choice (i.e., the correct approach to round).

MULTIPLICATION CAN INCREASES OR DECREASE A NUMBER

Question **: Does multiplication always increase a number ?**

Misconception	Correct
Yes it does; take the number 8, for example : $2 \times 8 = 16$ $3 \times 8 = 24$ $4 \times 8 = 32$ etc. In each it is getting larger, so, yes, multiplication clearly increase a number.	No–it increases a number only under certain conditions. Multiplying any positive number by a whole number greater than 1 will always increase its value – but consider $\frac{1}{2} \times 8 = 4$ here the number 8 is reduced. 

Further Explanation :

So, multiplying can have a *reducing* effect when multiplying a positive number by a fraction which is less than one. But this can still be confusing. While we accept the above, the concept of 'a number *times* 8' continues to be perceived as an increase. How then can we attach meaning to $\frac{1}{2} \times 8$ so that this will be perceived as decreasing ?

When multiplying by a whole positive number, e.g. 6 *times* 5, we understand this as being 5 added over and over again, how ever many *times* – six times in this example. But this interpretation of *times* does not quite work with fractions. If we ask *how many times*, the answer is "*not quite once*".

Again we need to put the term multiplying into a context with which we can can identify, and which will then make the situation meaningful.

We want to buy 30 roses which are sold in bunches of 5, so we ask for "6 *of* the 5-rose bunches". In this way, the word *times* also often means *of*. If we try using the word *of* when *times* appears to have an unclear meaning, we get $\frac{1}{2}$ *of* 8 rather than $\frac{1}{2}$ *times* 8. Indeed we know what $\frac{1}{2}$ *of* 8 means-namely 4.

So, by using *of* instead of *times* we are able to understand the concept of multiplying by a fraction and how this can have a reducing effect when the fraction is smaller than 1.

This also helps us to understand how we multiply by a fraction, and why the method works: the 4 which results from $\frac{1}{2} \times 8$ $\left(\text{or } \frac{1}{2} \textit{ of } 8\right)$ can be reached by dividing 8 by 2; similarly, the 5 which results from, $\frac{1}{3} \times 15 \left(\text{or } \frac{1}{3} \textit{ of } 15\right)$ (or a third *of* fifteen) can be reached by dividing 15 by 3.

Generalising this result gives : $\frac{1}{d} \times n$ is the same as $\frac{n}{d}$.

Negative numbers :

When your bank balance is + 4 Rs. you *have* Rs. 4. When your bank balance is – 4 Rs. you *owe* Rs. 4.

Owing is the opposite of *having*, so we find that we can associate the concept of 'minus' with '(*the*) opposite (*of*)'. This also works in reverse.

Thus, $(-4) \times 8$ means "*owing* Rs. 4, eight times over" or "owing Rs. 32" which is – Rs. 32.

Now–32 is smaller than 8, so we have illustrated another case where *multiplying* has a reducing effect, i.e. when multiplying by a negative number.

Note that, using the method shown above, it follows that $-1 \times 8 = -8$, and vice versa. ■

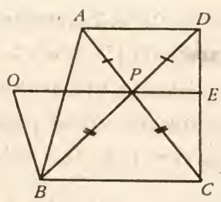
International Math Olympiad

PROBLEMS & SOLUTIONS

1. Segments AC and BD intersect in point P so that $PA = PD$, $PB = PC$. Let O be the circumcentre of triangle PAB. Prove that lines OP and CD are perpendicular.

Soln.: Because $PA = PD$, $PB = PC$ and $\angle APB = \angle DPC$, we get $\Delta PAB \equiv \Delta PDC$, so that

$\angle BAP = \angle CDP$ ... (i)



At least one of $\angle PAB$ and $\angle PBA$ is acute, so we may assume without loss of generality that $\angle PAB$ is acute. Since O is the circumcentre of ΔPAB we get $OB = OP$ and $\angle BOP = 2\angle BAP$, so that

$$\angle OPB = 90° - \frac{1}{2}\angle BOP = 90° - \angle BAP \quad \text{... (ii)}$$

Let E be the intersection of OP with CD. Then

$\angle EPD = \angle OPB$ ... (iii)

From (i), (ii) and (iii) we have

$\angle EPD = 90° - \angle CDP$

Thus $\angle EPD + \angle EDP = \angle EPD + \angle CDP = 90°$.

Therefore $OP \perp CD$.

Comment: Generally if A, B, C, D are concyclic, we have $OP \perp CD$ and this theorem is an extension of Brahmagupta's theorem.

2. Determine all functions f defined on the set of positive rational numbers, taking values in the same set which satisfy for every positive rational number x the conditions $f(x + 1) = f(x) + 1$ and $f(x^3) = (f(x))^3$.

Soln.: Let N and Q^+ denote the set of positive integers, and the set of positive rational numbers, respectively. We show that $f(x) = x$, for all $x \in Q^+$, is the only function satisfying the given conditions. First of all, by the first condition and an easy induction we see that $f(x + n) = f(x) + n$, for all $x \in Q^+$, and for all $n \in N$. Now for arbitrary $p/q \in Q^+$, where $p, q \in N$, we have

$$f\left(\left(\frac{p}{q}=q^2\right)^3\right) = f\left(\frac{p^3}{q^3}+3p^2+3pq^3+q^6\right)$$

$$= f\left(\left(\frac{p}{q}\right)^3\right)+3p^2+3pq^3+q^6 \quad \text{... (i)}$$

On the other hand,

$$f\left(\left(\frac{p}{q}+q^2\right)^3\right) = f\left(\left(\frac{p}{q}+q^2\right)\right)^3 = \left(f\left(\frac{p}{q}\right)+q^2\right)^3$$

$$= f\left(\left(\frac{p}{q}\right)\right)^3 + 3\left(f\left(\frac{p}{q}\right)\right)^2 q^2 + 3f\left(\left(\frac{p}{q}\right)\right)^2 \cdot q^4 + q^6 \quad \text{... (ii)}$$

Letting $t = f(p/q)$ and comparing (i) and (ii), we get since, $f\left(\frac{p^3}{q^3}\right) = \left(f\left(\frac{p}{q}\right)\right)^3$, $p(p + q^3) = q^2t^2 + q^4t$

or $q^2t^2 + q^4t - p(p + q^3) = 0$

or $(qt - p)(qt + p + q^3) = 0$.

Since $qt + p + q^3 > 0$, we must have $t = p/q$.

i.e. $f(p/q) = p/q$ and we are done.

3. Prove that the inequality

$$\sum_{n=1}^{r}\left(\sum_{m=1}^{r}\frac{a_m a_n}{m+n}\right) \geq 0$$

holds for any real numbers $a_1, a_2, \ldots, a_r$. Find conditions for equality.

Soln.: Consider the polynomial

$$p(x) = \sum_{n=1}^{r}\left(\sum_{m=1}^{r} a_m a_n x^{m+n-1}\right)$$

Then, $$x\,p(x) = \sum_{n=1}^{r}\sum_{m=1}^{r} a_m a_n x^{m+n}$$

$$= \left(\sum_{m=1}^{r} a_m x^m\right)\left(\sum_{n=1}^{r} a_n x^n\right) = \left(\sum_{i=1}^{r} a_i x^i\right)^2 \geq 0,$$

for all $x \in R$.

In particular $p(x) \geq 0$ for all $x \geq 0$. Hence

$$0 \leq \int_0^1 p(x)dx = \sum_{n=1}^{r}\left(\sum_{m=1}^{r}\frac{a_m a_n}{m+n}x^{m+n}\Big]_0^1\right)$$

$$= \sum_{n=1}^{r}\sum_{m=1}^{r}\frac{a_m a_n}{m+n}.$$

The inequality is strict unless $xp(x) \equiv 0$, that is $a_1 = a_2 = \ldots = a_r = 0$.

4. Define the sequence of functions $f_0, f_1, f_2, \ldots$ by $f_0(x) = 8$ for all $x \in R$, $f_{n+1}(x) = \sqrt{x^2 + 6f_n(x)}$ for $n = 0, 1, 2, \ldots$ and for all $x \in R$. For every positive integer n, solve the equation $f_n(x) = 2x$.

Soln.: Since $f_n(x)$ is positive, $f_n(x) = 2x$ has only positive solutions. We show that, for each n, $f_n(x) = 2x$ has a solution $x = 4$. Since $f_1(x) = \sqrt{x^2 + 48}$ is a solution of $f_2(x) = 2x$.

Now $f_{n+1}(4) = \sqrt{4^2 + 6f_n(4)} = \sqrt{4^2 + 6 \cdot 8} = 8 = 2 \cdot 4$ which completes the inductive step.

Next, induction on n gives us that for each n, $\dfrac{f_n(x)}{x}$ decreases as x increases in $(0, \infty)$. It follows that $f_n(x) = 2x$ has the unique solution $x = 4$.

5. Prove that, for every natural k, the number $(k^3)!$ is divisible by $(k!)^{k^2+k+1}$.

Soln.: Applying the well known fact that $(ab)!$ is divisible by $(a!)^b \cdot b!$ yields $(k^3)! = (k \cdot k^2)!$ is divisible by $(k!)^{k^2} \cdot (k^2)!$ and $(k^2)! = (k \cdot k)!$ is divisible by $(k!)^{k+1}$ from which the required result follows immediately.

6. Show that the polynomial

$$2(x^7 + y^7 + z^7) - 7xyz(x^4 + y^4 + z^4)$$

has $x + y + z$ as a factor.

Soln.: Let $f(x, y, z)$

$$= 2(x^7 + y^7 + z^7) - 7xyz(x^4 + y^4 + z^4)$$

If we can show that $f(x, y, z) = z$ when $x + y + z = 0$, we are done.

We know for $x + y + z = 0$, that $x^3 + y^3 + z^3 = 3xyz$.

Thus, $x^7 + y^7 + z^7 + x^3y^4 + x^3z^4 + y^3z^4 + y^3x^4 + z^3y^4 + z^3x^4$

$= (x^3 + y^3 + z^3)(x^4 + y^4 + z^4)$

$= 3xyz(x^4 + y^4 + z^4)$

so that, $x^7 + y^7 + z^7 = 3xyz(x^4 + y^4 + z^4) - x^3y^4 - x^3z^4 - y^3z^4 - y^3x^4 - z^3y^4 - z^3x^4$

Therefore, $f(x, y, z) = 2(x^7 + y^7 + z^7) - 7xyz(x^4 + y^4 + z^4)$

$= 6xyz(x^4 + y^4 + z^4) - 2(x^3y^4 + x^3z^4 + y^3z^4 + y^3x^4 + z^3y^4 + z^3x^4) - 7(xyz)(x^4 + y^4 + z^4)$

$= -xyz(x^4 + y^4 + z^4) - 2x^3y^3(x + y) - 2y^3z^3(y + z) - 2z^3x^3(z + x)$

$= -xyz(x^4 + y^4 + z^4) + 2x^3y^3z + 2xy^3z^2 + 2x^3yz^3$

$= -xyz(x^4 + y^4 + z^4 - 2x^2y^2 - 2y^2z^2 - 2z^2x^2)$

$= -xyz[(x^2 + y^2 + z^2)^2 - 4(x^2y^2 + y^2z^2 + z^2x^2)]$

Since $x^2 + y^2 + z^2 = -2(xy + yz + zx)$, we now have that

$f(x, y, z) = -xyz[4(xy + yz + zx)^2 - 4(x^2y^2 + y^2z^2 + z^2x^2)]$

$= -4xyz(2x^2yz + 2xy^2z + 2xyz^2)$

$= -8xyz[(xyz(x + y + z)] = 0$.

7. Prove that

$$(\sin(A) + \sin(B) + \sin(C))\left(\frac{1}{A} + \frac{1}{B} + \frac{1}{C}\right) \ge \frac{27\sqrt{3}}{2\pi},$$

where A, B, C are the angles (in radians) of a triangle.

Soln.: If $A = B = C = \pi/3$, equality obtains. It then suffices to show that each factor has an absolute minimum at the point. Note that $C = \pi - (A + B)$. Let $S = \{(A, B) : A > 0, B > 0, A + B < \pi\}$.

Let $f(A, B) = \dfrac{1}{A} + \dfrac{1}{B} + \dfrac{1}{\pi - (A + B)}$.

Then f is unbounded on S. So, if there is a unique critical point for f on S, it must be an absolute minimum.

Now, $F_A(A, B) = -\dfrac{1}{A^2} + \dfrac{1}{[\pi - (A + B)]^2} = 0$

implies that $\pi - (A + B) = A$.

Similarly, $F_B(A, B) = 0$ implies that $\pi - (A + B) = B$ and so that $A = B$.

Hence $A = B = \pi - (A + B) = \pi/3$.

Let $g(A, B) = \sin(A) + \sin(B) + \sin[\pi - (A + B)]$

$= \sin(A) + \sin(B) + \sin(A + B)$.

We now obtain that

$0 = g_A(A, B) = \cos(B) + \cos(A + B)$,

$0 = g_B(A, B) = \cos(A) + \cos(A + B)$

so that, $\cos(A) = \cos(B)$.

Since no two distinct angles in $(0, \pi)$ have equal cosines, we have that $A = B$.

Then $0 = \cos(A) + \cos(2A)$

$= 2\cos^2(A) + \cos(A) - 1$

$= [2\cos(A) - 1][\cos(A) + 1]$.

Since $\cos(A)$ cannot have the value -1, it must then have value $1/2$ and so we have

$$A = B = C = \frac{\pi}{3}.$$

■

See the pattern?

1. If the number to be squared is **666664**:
2. The square has:
   five 4's (same number as repeating 6's) 4 4 4 4 4
   next digit: 0 0
   four 8's (one fewer than repeating 6's) 8 8 8 8
   a final 96 9 6
3. So **the square of 666664 is 444,440,888,896**.

Squaring special numbers (6's and final 5)

1. Choose a number with repeating 6's and a final 5.
2. The square is made up of:
   - same number of 4's as repeating 6's
   - same number of 2's as repeating 6's
   - a final 25

Example:

1. If the number to be squared is **6665**:
2. The square has:
   three 4's (same number as repeating 6's) 4 4 4
   three 2's (same number as repeating 6's) 2 2 2
   A final 25 2 5
3. So **the square of 6665 is 44,422,225**.

See the pattern?

1. If the number to be squared is **666665**:
   five 4's (same number as repeating 6's) 4 4 4 4 4
   five 2's (same number as repeating 6's) 2 2 2 2 2
   A final 25 2 5
2. So **the square of 666665 is 444,442,222,225**.

Squaring special numbers (6's and final 7)

1. Choose a number with repeating 6's and a final 7.
2. The square is made up of:
   - The same number of 4's as there are digits in the number;
   - One fewer 8;
   - A final 9.

Example:

1. If the number to be squared is **6667**:
2. The square has:
   four 4's (number of digits in number) 4 4 4 4
   three 8's (one fewer) 8 8 8
   a final 9 9
3. So **the square of 6667 is 44448889**.

See the pattern?

1. If the number to be squared is **667**:
2. The square has:
   three 4's 4 4 4
   two 8's 8 8
   a final 9 9
3. So **the square of 667 is 444889**.

Squaring special numbers (6's and final 8)

1. Choose a number with repeating 6's and a final 8.
2. The square is made up of:
   - the same number of 4's as there are repeating 6's in the number;
   - one 6
   - the same number of 2's as repeating 6's;
   - a final 4.

Example:

1. If the number to be squared is **6668**:
2. The square has:
   three 4's (same as repeating 6's) 4 4 4
   one 6 6
   three 2's (same number as repeating 6's) 2 2 2
   a final 4 4
3. So **the square of 6668 is 44,462,224**.

See the pattern?

1. If the number to be squared is **666668**:
2. The square has:
   five 4's (same number as repeating 6's) 4 4 4 4 4
   one 6 6
   five 2's (same number as repeating 6's) 2 2 2 2 2
   a final 4 4

So **666668 × 666668 = 444,446,222,224**

5 CHALLENGING PROBLEMS

MAXIMA & MINIMA

1. Show that the height of an open cylinder of given surface and greatest volume is equal to the radius of its base.

2. Show that the radius of right circular cylinder of greatest curved surface which can be inscribed in a given cone is half that of the cone.

3. Find the surface of the right circular cylinder of greatest surface which can be inscribed in a sphere of radius r.

4. Prove that the least perimeter of an isosceles triangle in which a circle of radius r can be inscribed is $6r\sqrt{3}$.

5. A cone is circumscribed to a sphere of radius r; show that when the volume of the cone is least its altitude is $4r$ and its semi-vertical angle is $\sin^{-1}(1/3)$.

SOLUTIONS

1. Let r be the radius of the circular base; h, the height; S, the surface and V, the volume of the open cylinder so that

$$S = \pi r^2 + 2\pi r h \quad \text{... (i)}$$
$$V = \pi r^2 h \quad \text{... (ii)}$$

Here, as given, S is a constant and V is a variable. Also, h, r are variables. Substituting the value of h, as obtained from (i), in (ii), we get

$$V = \pi r^2\left(\frac{S-\pi r^2}{2\pi r}\right) = \frac{Sr - \pi r^3}{2} \quad \text{... (iii)}$$

which gives V in terms of one variable r.

As V must be necessarily non-negative, we have

$$Sr - \pi r^3 \geq 0 \Rightarrow Sr \geq \pi r^3 \Rightarrow r \leq \sqrt{(S/\pi)}$$

Also r is non-negative.

Thus r varies in the interval $\left[0, \sqrt{(S/\pi)}\right]$.

Now, $\dfrac{dV}{dr} = \dfrac{S - 3\pi r^2}{2}$,

so that $dV/dr = 0$ only when $r = \sqrt{(S/3\pi)}$: negative value of r being inadmissible. Thus V have only one stationary value.

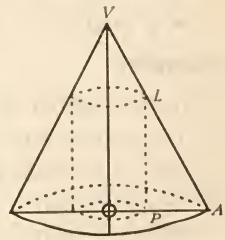
Now $V = 0$ for the end points $r = 0$ and $\sqrt{(S/\pi)}$ and positive for every other admissible value of r. Hence V is greatest for $r = \sqrt{(S/3\pi)}$.

Substituting this value of r in (i), we get

$$h = \frac{S - \pi r^2}{2\pi r} = \frac{S - \pi(S/3\pi)}{2\pi\sqrt{(S/3\pi)}} = \frac{2S}{3}\cdot\frac{1}{2\pi}\sqrt{\frac{3\pi}{S}} = \sqrt{\frac{S}{3\pi}}.$$

Hence $h = r$ for the cylinder of greatest volume and given surface.

2. Let r be the radius OA of the base and h, the height OV of the given cone.



We inscribe in it a cylinder, the radius of whose base is $OP = x$, as shown in the figure. We note that x may take up any value between 0 and r so that $x \in [0, r]$.

To determine the height PL, of this cylinder, we have

$$\frac{PL}{OV} = \frac{PA}{OA} \Rightarrow \frac{PL}{h} = \frac{r-x}{r} \Rightarrow PL = \frac{h(r-x)}{r}$$

If S be the curved surface of the cylinder, we have

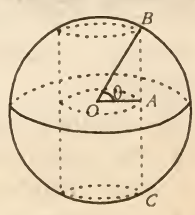
$$S = 2\pi\cdot OP\cdot PL = \frac{2\pi x h(r-x)}{r} = \frac{2\pi h}{r}(rx - x^2)$$

$$\Rightarrow \frac{dS}{dx} = \frac{2\pi h}{r}(r - 2x) = 0 \text{ for } x = r/2.$$

Thus S has only one stationary value:

Now S is 0 for $x = 0$ as well as for $x = r$ and is positive for values of x lying between 0 and r. Therefore S is greatest for $x = r/2$.

3. We construct a cylinder as shown in the figure. OA is the radius of the base and CB is the height of this cylinder.



Let $\angle AOB = \theta$, so that $\theta \in [0, \pi/2]$, we have

$$\frac{OA}{OB} = \cos\theta \Rightarrow OA = OB\cos\theta = r\cos\theta$$

Also $\frac{AB}{OB} = \sin\theta \Rightarrow AB = OB\sin\theta = r\sin\theta$.

If S be the surface, we have

$$S = 2\pi \cdot OA^2 + 2\pi \cdot OA \cdot BC$$
$$= 2\pi r^2(\cos^2\theta + \sin 2\theta) \quad \text{... (i)}$$

$$\Rightarrow \frac{dS}{d\theta} = 2\pi r^2(-2\cos\theta\sin\theta + 2\cos 2\theta)$$
$$= 2\pi r^2(2\cos 2\theta - \sin 2\theta)$$

$\Rightarrow dS/d\theta = 0$ for $2\cos 2\theta - \sin 2\theta = 0$

$\Leftrightarrow \tan 2\theta = 2$ ... (ii)

[It can be established that the equation $\tan 2\theta = 2$ admits of only one value of $\theta \in [0, \pi/2]$ as its solution.

Let $\theta_1 \in [0, \pi/2]$ be the root of $\tan 2\theta = 2$.

Thus S has only one extreme value.

Now $\tan 2\theta_1 = 2$

$\Rightarrow \sin 2\theta_1 = 2/\sqrt{5}$ and $\cos 2\theta_1 = 1/\sqrt{5}$.

From (i), we can see that

$\theta = 0 \Rightarrow S = 2\pi r^2$

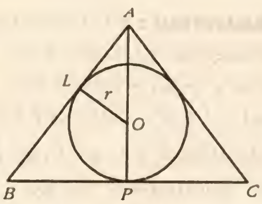
$\theta = \pi/2 \Rightarrow S = 0$

$$\theta = \theta_1 \Rightarrow S = 2\pi r^2\left(\frac{1+\cos 2\theta_1}{2} + \sin 2\theta_1\right)$$
$$= \frac{\pi r^2(5+5\sqrt{5})}{5}.$$

which is greater than $3\pi r^2$.

Hence $\frac{\pi r^2(5+5\sqrt{5})}{5}$ is the required greatest surface.

4. We take one vertex A of the triangle at a distance x from the centre O so that $OA = x$. Surely $x > r$. Let AO meet the circle at P. The two tangents from A and the tangent at P determine an isosceles triangle ABC circumscribing the given circle. We have $OL = r$.



$\therefore \quad AL = \sqrt{OA^2 - OL^2} = \sqrt{x^2 - r^2}$

Also $BP = AP\tan\angle BAP = AP\tan\angle LAO$

$$= AP \cdot \frac{OL}{AL} = (r+x)\cdot\frac{r}{\sqrt{x^2-r^2}}$$

If p denote the perimeter of the triangle, we have

$p = AB + AC + BC$

$= 2AB + 2BP = 2(AL + LB) + 2BP$

$= 2AL + 4BP$ (for $BL = BP$)

$$= 2\sqrt{x^2-r^2} + \frac{4(r+x)r}{\sqrt{x^2-r^2}} = 2\frac{(x+r)^2}{\sqrt{x^2-r^2}} \quad \text{... (i)}$$

$$\Rightarrow \frac{dp}{dx} = 2\frac{2(x+r)\sqrt{x^2-r^2} - x(x+r)^2(x^2-r^2)^{-1/2}}{(x^2-r^2)}$$
$$= 2\frac{2(x+r)(x^2-r^2) - x(x+r)^2}{(x^2-r^2)^{3/2}} = 2\frac{(x+r)^2(x-2r)}{(x^2-r^2)^{3/2}}$$

$\Rightarrow dp/dx = 0$ for $x = 2r$; negative value, $-r$ of x being inadmissible.

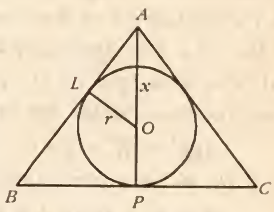
Now, x may take up any value $> r$ so that it varies in the interval $]r, \infty[$.

From (i), we see that $x \to r \Rightarrow p \to \infty$.

Also $x \to \infty \Rightarrow p \to \infty$.

Hence p is least for $x = 2r$. Putting this value of x in (i), we see that the least value of p is $6r\sqrt{3}$.

5. We take the vertex A of the cone at a distance x from the centre O of the sphere. By drawing tangent lines from A, as shown in the figure, we construct the cone circumscribing the sphere.



Let the semi-vertical angle BAP of the cone be θ.

Now, if V be the volume of the cone, we have

$$V = \frac{1}{3}\pi \cdot BP^2 \cdot AP,$$

which will now be expressed in terms of x.

We have, $AP = AO + OP = r + x$

Also $\sin\theta = \frac{OL}{OA} = \frac{r}{x} \Rightarrow \tan\theta = \frac{r}{\sqrt{x^2-r^2}}$

Again $BP/AP = \tan\theta$

$$\Rightarrow BP = AP\tan\theta = (r+x)\cdot\frac{r}{\sqrt{x^2-r^2}}$$

Thus, $$V = \pi\frac{(r+x)^2r^2}{3(x^2-r^2)}(x+r) = \frac{\pi r^2(x+r)^2}{3(x-r)}$$

$$\Rightarrow \frac{dV}{dx} = \frac{\pi r^2(x+r)(x-3r)}{3(x-r)^2}$$

$\Rightarrow \frac{dV}{dx}$ is 0 for $x = 3r$.

Here x can take up any value $\geq r$.

Also, $x \to r \Rightarrow V \to \infty$ and

$x \to \infty \Rightarrow V \to \infty$.

Thus V is least for $x = 3r$.

Hence, for least volume, the altitude of the cone

$= AP = r + 3r = 4r$.

and the semi-vertical angle θ

$$= \sin^{-1}\frac{r}{x} = \sin^{-1}\frac{r}{3r} = \sin^{-1}\frac{1}{3}.$$

■

ORTHOGONAL Circles

Contd. from December issue

By : **Sri K.A.N. Rao**, Secunderabad

Example 4 : If two circles are orthogonal prove that the polar of any point P on one circle with respect to the other circle passes through the other end of the diameter through P to the first circle.

Let the equations of the circles with centers C and C' be $x^2+y^2+2gx+2fy+c=0$ ...(1)

and $x^2+y^2+2g'x+2f'y+c'=0$ ...(2)

Since the two circles are orthogonal

$$2gg'+2ff'=c+c' \quad ...(3)$$

Let $P(h, k)$ be a point on circle (1), which implies

$$h^2+k^2+2gh+2fk+c=0 \quad ...(4)$$

Now polar of P with respect to the circle (2) is

$$xh+yk+g'(x+h)+f'(y+k)+c'=0$$

$$\Rightarrow \quad x(g'+h)+y(f'+k)+g'h+f'k+c'=0 \quad ...(5)$$

The coordinates of the other end of diameter PC are $(-2g-h,-2f-k)$

Substituting the above coordinates in the equation (5) we get

$$-(2g+h)(g'+h)-(2f+k)(f'+k)+g'h+f'k+c'$$

$$=-2gg'-g'h-2gh-h^2-2ff'-f'k-2fk-k^2+g'h+f'k+c'$$

$$=-(2gg'+2ff')-\left(h^2+k^2+2gh+2fk\right)+c'$$

$=-(c+c')-(-c)+c'=0$ ..... From (3) and (4)

Hence the polar passes through the other end of the diameter through P.

The above example gives us a method to find the equation of a circle, which is orthogonal to three given circles. Suppose $S=0$ is the circle which is orthogonal to the circles $S_1=0, S_2=0$ and $S_3=0$. If P is a point on the circles $S=0$, then the polars of P with respect to the three given circles are concurrent at the diametrically opposite point of P. Thus the circle $S=0$ can be considered as the locus of all points whose polars with respect to the given circles are concurrent.

Thus, if the equations of circles are $S_1=0, S_2=0$ and $S_3=0$ respectively i.e.

$$x^2+y^2+2g_1x+2f_1y+c_1=0$$

$$x^2+y^2+2g_2x+2f_2y+c_2=0$$

$$x^2+y^2+2g_3x+2f_3y+c_3=0$$

and the point P be (h, k) then the polars of P with respect to the circles will be

$$x(g_1+h)+y(f_1+k)+g_1h+f_1k+c_1=0$$

$$x(g_2+h)+y(f_2+k)+g_2h+f_2k+c_2=0$$

$$x(g_3+h)+y(f_3+k)+g_3h+f_3k+c_3=0$$

Since all these polars are concurrent, using determinants we can write

$$\begin{vmatrix} g_1+h & f_1+k & g_1h+f_1k+c_1 \\ g_2+h & f_2+k & g_2h+f_2k+c_2 \\ g_3+h & f_3+k & g_3h+f_3k+c_3 \end{vmatrix}=0$$

Hence the locus or the equation of $S=0$ is given by

$$\begin{vmatrix} g_1+x & f_1+y & g_1x+f_1y+c_1 \\ g_2+x & f_2+y & g_2x+f_2y+c_2 \\ g_3+x & f_3+y & g_3x+f_3y+c_3 \end{vmatrix}=0$$

Illustration : Find the equation of a circle which is orthogonal to each of the circles $x^2+y^2-4x-2y+6=0$, $x^2+y^2-2x+6y=0$ and $x^2+y^2-12x+2y+30=0$.

Substituting for $g_1, f_1, g_2, f_2, g_3, f_3, c_1, c_2, c_3$ values in the determinant we get

$$\begin{vmatrix} -2+x & -1+y & -2x-y+6 \\ -1+x & 3+y & -x+3y \\ -6+x & 1+y & -6x+y+30 \end{vmatrix}=0$$

$$\Rightarrow \begin{vmatrix} -2+x & -1+y & -2x-y+6 \\ 1 & 4 & x+4y-6 \\ -4 & 2 & -4x+2y+24 \end{vmatrix}=0$$

$$\Rightarrow 2\begin{vmatrix} -2+x & -1+y & -2x-y+6 \\ 1 & 4 & x+4y-6 \\ -2 & 1 & -2x+y+12 \end{vmatrix}=0$$

$$\Rightarrow 18\begin{vmatrix} -2+x & -1+y & -2x-y+6 \\ 1 & 4 & x+4y-6 \\ 0 & 1 & y \end{vmatrix}=0$$

Mathematical Challenges for I.I.T. MAINS

1. If $x > 0$, find the least value of the function

$$f(x) = 4x + \frac{9\pi^2}{x} + \sin x.$$

2. Find the values of the parameter k for which all the roots of $x^4 + 4x^3 - 8x^2 + k = 0$ are real and unequal.

3. Find the values of a for which the equation $x^3 + x^2 - 8x + a = 0$ has a repeated root. Solve the equation for these values of a.

4. Find all the integral values of a for which the quadratic equation $(x - a)(x - 10) + 1 = 0$ has integral roots.

5. A circle of radius 1 unit touches positive x-axis and positive y-axis at A and B respectively. A variable line passing through origin intersects the circle in two points D and E. Find the equation of the line for which area DEB is maximum.

6. Evaluate: $1^2 + 2^2x + 3^2x^2 + 4^2x^3 + \ldots + n^2x^{n-1}$ $[x \neq 1]$

7. Prove that the integer next above $(\sqrt{3}+1)^{2m}$ contains 2^{m+1} as a factor.

8. Evaluate the integral $\int \frac{\cos 7x - \cos 8x}{1 + 2\cos 5x} dx$.

9. If $f(x) = x + \int_0^1 (xy^2 + x^2y) f(y) dy$, find the minimum value of $f(x)$.

10. Let α, β are distinct real roots of the quadratic equation $ax^2 + bx + c = 0$, then show that

$$\underset{x \to \alpha}{\text{Lt}} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2} = \frac{b^2 - 4ac}{2}.$$

11. Let f be a real valued function satisfying

$$f(x) + f(x+6) = f(x+3) + f(x+9).$$

Prove that $\int_x^{x+12} f(t)\, dt$ is a constant function.

12.(a) If f and g are two continuous function being even and odd respectively, then show that $\int_{-a}^{a} \frac{f(x)}{g(x)+1}$ is independent of choice of g, a being any non-zero real number and b is a positive real number.

(b) Evaluate $\int_0^1 \frac{x^{\cos\alpha} - 1}{\ln x} dx$, α being a real number other than an odd multiple of π.

13. Prove that planes $x = y\cos C + z\cos B$, $y = z\cos A + x\cos C$ and $z = x\cos B + y\cos A$ always contains the line $\frac{x}{\sin A} = \frac{y}{\sin B} = \frac{z}{\sin C}$ if $A + B + C = \pi$.

14. Find the maximum and minimum areas of the triangle formed by x-axis, tangent and normal at point on the parabola $y = x^2 + 1$ $(1 \le x < 3)$.

15. Find the function $g : R \to R$ continuous in $[0, \infty)$, and positive in $(0, \infty)$, satisfying $g(1) = 1$ and $\frac{1}{2}\int_0^x g^2(t)\,dt = \frac{1}{x}\left(\int_0^x g(t)\,dt\right)^2$.

16. Find the smaller area enclosed by $y = f(x)$, when $f(x)$ is a polynomial of least degree satisfying $\lim_{x \to 0}\left(1 + \frac{f(x)}{x^3}\right)^{1/x} = e$ and the circle $x^2 + y^2 = 2$ above the x-axis.

SOLUTION

1. $f(x) = 4x + \frac{9\pi^2}{x} + \sin x$

$$= (2\sqrt{x})^2 + \left(\frac{3\pi}{\sqrt{x}}\right)^2 - 12\pi + 12\pi + \sin x$$

$$= \left(2\sqrt{x} - \frac{3\pi}{x}\right)^2 + 12\pi + \sin x$$

$\therefore$ $f(x)$ is minimum if $2\sqrt{x} - \frac{3\pi}{\sqrt{x}} = 0$ and $\sin x = -1$.

$\therefore$ $12\pi - 1$ is the minimum value of the function.

2. Let $f(x) = x^4 + 4x^3 - 8x^2 + k$

$\therefore\ f'(x) = 4x^3 + 12x^2 - 16x$

$= 4x(x^2 + 3x - 4) = 4x(x+4)(x-1)$

$\therefore\ f'(x) = 0$ has roots -4, 0, 1.

Now in order that $f(x) = 0$ has four real and distinct roots.

$f(-\infty)$	$f(-4)$	$f(0)$	$f(1)$	$f(\infty)$
+ve	−ve	+ve	−ve	+ve

$\Rightarrow k - 128 < 0 \qquad k > 0 \qquad k - 3 < 0$

$k < 128 \qquad\qquad k < 3$

$\therefore\ k \in (0, 3)$.

3. Let $f(x) = x^3 + x^2 - 8x + a$;

$\therefore\ f'(x) = 3x^2 + 2x - 8$

The given equation will have a repeated root if it has a root common with $f'(x) = 0$.

$f'(x) = 3x^2 + 2x - 8 = (x+2)(3x-4) = 0$

$\Rightarrow x = -2$ or $x = \frac{4}{3}$.

If -2 is also a root of the original equation

$-8 + 4 + 16 + a = 0 \Rightarrow a = -12$

and if 4/3 is a root of the original equation

we get $\left(\frac{4}{3}\right)^3 + \left(\frac{4}{3}\right)^2 - 8\cdot\frac{4}{3} + a = 0$

$\Rightarrow a = \frac{176}{27}$, these are the two required values of a.

If $a = -12$, -2 is a repeated root of the equation; thus $(x+2)^2$ is a factor of $x^3 + x^2 - 8x - 12$. By division, the other root is obtained by $x - 3 = 0$. $\therefore$ the roots in this case are $-2, -2, 3$. Similarly the roots can be obtained as $\frac{4}{3}, \frac{4}{3}, \frac{-11}{3}$ in other case *i.e.* when $a = \frac{176}{27}$.

4. Since a and x both are to be integers

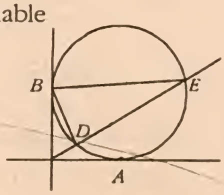
$(x-a)(x-10) = -1 \Rightarrow x - a = -1$ and $x - 10 = 1$ or $x - a = 1$ and $x - 10 = -1$ or $a = 12$ or $a = 8$.

5. Equation of the circle is $(x-1)^2 + (y-1)^2 = 1$

$\Rightarrow x^2 + y^2 - 2x - 2y + 1 = 0 \qquad \ldots(1)$

Evidently the equation of the variable straight line is

$y = mx \qquad \ldots(2)$



From (1) and (2) we get

$x^2 + m^2x^2 - 2x - 2mx + 1 = 0$

$\Rightarrow (1+m^2)x^2 - 2x(1+m) + 1 = 0$

Let $D \equiv (x_1, y_1)$ and $E(x_2, y_2)$

$$(x_2 - x_1)^2 = \frac{4(1+m^2) - 4(1+m^2)}{(1+m^2)} = \frac{8m}{(1+m^2)^2}$$

[Using difference of roots $= \frac{\sqrt{D}}{a}$]

Similarly $(y_2 - y_1)^2 = m^2(x_2 - x_1)^2 = \frac{8m^3}{(1+m^2)^2}$

A = area of ΔDEB

$= \frac{1}{2}DE \times$ (Distance of B from DE)

$$A^2 = \frac{1}{4}DE^2 \times \frac{1}{1+m^2}$$

$$= \frac{1}{4}\left[(x_2 - x_1)^2 + (y_2 - y_1)^2\right] \times \frac{1}{1+m^2}$$

$$= \frac{1}{4}\left[\frac{8m}{(1+m^2)^2} + \frac{8m^3}{(1+m^2)^2}\right] \times \frac{1}{1+m^2} = \frac{2m}{(1+m^2)^2}$$

$A'(m) = -6m^2 + 2 = 0 \Rightarrow m = \pm\frac{1}{\sqrt{3}}$

The readers are advised to show that for maximum area $m = \frac{1}{\sqrt{3}}$ $\qquad\therefore$ equation of line is $y = \frac{1}{\sqrt{3}}x$.

6. $1 + x + x^2 + \ldots + x^n = \frac{1 - x^{n+1}}{1 - x}$

Differentiating both sides

$1 + 2x + 3x^2 + \ldots + x^{n-1}$

$$= \frac{(1-x)(-\overline{n+1}\,x^n) + (1 - x^{n+1})}{(1-x)^2}$$

$$= \frac{nx^{n+1} - (n+1)x + 1}{(1-x)^2}$$

Multiplying both sides by x, we get

$$x + 2x^2 + \ldots + nx^n = \frac{nx^{n+2} - (n+1)x^{n+1} + x}{(1-x)^2}$$

Differentiating again both sides, we get

$1 + 2^2x + \ldots + n^2x^{n-1}$

$$= \frac{(1-x)^2[n(n+2)x^{n+1} - (n+1)^2x^n + 1] + [nx^{n+2} - (n+1)x^{n+1} + x]2(1-x)}{(1-x)^4}$$

$$= \frac{-n^2x^{n+2} + (2x^2 + 2n - 1)x^{n+1} - (n+1)x^n + x + 1}{(1-x)^3}.$$

7. Let $I + F = (\sqrt{3}+1)^{2m} = (3 + 1 + 2\sqrt{3})^m$

$= 2^m(2+\sqrt{3})^m \qquad 0 < F < 1 \qquad \ldots(1)$

Let $F' = 2^m(2-\sqrt{3})^m \qquad 0 < F' < 1 \qquad \ldots(2)$

Adding (1) and (2)

$$I + F + F' = 2^m\left[(2+\sqrt{3})^m + (2-\sqrt{3})^m\right]$$
$$= 2^m\left[2({}^mC_0 2^m + {}^mC_2 \cdot 2^{m-2}(\sqrt{3})^2 + ...\right]$$
$$= 2^{m+1} \text{ (integer) } \Rightarrow F + F' = 1$$

$\therefore\ I + 1 = 2^{m+1}$ (integer)

$\Rightarrow$ integer next above $(\sqrt{3}+1)^{2m}$

i.e. $I + 1$ has 2^{m+1} as factor.

8. Multiplying the numerator and denominator by $\sin\left(\frac{5x}{2}\right)$ we get

$$I = \int \frac{(\cos 7x - \cos 8x)\sin\frac{5x}{2}}{\sin\frac{5x}{2} + 2\sin\frac{5x}{2}\cdot\cos 5x}\,dx$$
$$= \int \frac{2\sin\frac{x}{2}\sin\frac{5x}{2}\cdot\sin\frac{15x}{2}}{\sin\frac{5x}{2} + \sin\frac{15x}{2}\cdot\sin\frac{5x}{2}}\,dx$$
$$= \int 2\sin\frac{x}{2}\cdot\sin\frac{5x}{2}\,dx = \int(\cos 2x - \cos 3x)dx$$
$$= \frac{\sin 2x}{2} - \frac{\sin 3x}{3} + k.$$

9. Since $f(x) = x + \int_0^1 x\,y^2 f(y)dy + x^2\int_0^1 y\,f(y)dy$

$$= x\left(1 + \int_0^1 y^2 f(y)dy\right) + x^2\int_0^1 y\,f(y)dy \quad ...(1)$$

$\therefore\ f(x)$ is of the form $ax + bx^2$ ...(2)

$\Rightarrow\ f(y) = ay + by^2.$

Putting $f(y)$ in (1) we get

$$f(x) = \left[\int_0^1(ay^3 + by^4)dy + 1\right]x + \left[\int_0^1(ay^2 + by^3)dy\right]x^2$$
$$= \left[\frac{a}{4} + \frac{b}{5} + 1\right]x + \left[\frac{a}{3} + \frac{b}{4}\right]x^2 \quad ...(3)$$

Comparing (2) and (3) we get

$$a = \frac{a}{4} + \frac{b}{5} + 1;\quad b = \frac{a}{3} + \frac{b}{4}$$

$\Rightarrow a = \frac{180}{119}$ and $b = \frac{80}{119}$ $\quad\therefore\ f(x) = \frac{80x^2 + 180x}{119}$

$\therefore\ f'(x) = 160x + 180 = 0$ for maximum or minimum

$\Rightarrow x = -\frac{9}{8}$ Hence minimum at $x = -\frac{9}{8}.$

10. $\lim\limits_{x\to\alpha} \frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2} = \lim\limits_{x\to\alpha} \frac{2\sin^2\frac{ax^2+bx+c}{2}}{(x-\alpha)^2}$

$$= \lim_{x\to\alpha} \frac{2\sin^2\frac{a(x-\alpha)(x-\beta)}{2}}{\left(\frac{a(x-\alpha)(x-\beta)}{2}\right)^2} \times \frac{a^2}{4}(x-\beta)^2$$
$$= \frac{a^2(\alpha-\beta)^2}{2} = \frac{a^2}{2}\left[(\alpha+\beta)^2 - 4\alpha\beta\right]$$
$$= \frac{a^2}{2}\left[\frac{b^2}{a^2} - \frac{4c}{a}\right] = \frac{b^2-4ac}{2}.$$

11. Given $f(x) + f(x+6) = f(x+3) + f(x+9)$ for $x = x+3$. We have

$f(x+3) + f(x+9) = f(x+6) + f(x+12)$

$\Rightarrow f(x) = f(x+12)$

$g(x) = \int_x^{x+12} f(t)dt \Rightarrow g'(x)[f(x+12) - f(x)] = 0$

$\therefore\ f(x)$ is a constant function.

12.(a) Since $\int_{-a}^{a} f(x)dx = \int_0^a f(x)dx + \int_0^a f(-x)\,dx$

Thus, $\int_{-a}^{a}\frac{f(x)}{b^{g(x)}+1}dx = \int_0^a \frac{f(x)}{b^{g(x)}+1}dx + \int_0^a \frac{f(x)}{b^{-g(x)}+1}dx$

as f is even and g is odd

$= \int_0^a f(x)dx,$ which is independent of g.

(b) $\int_0^1 \frac{x^{\cos\alpha}-1}{\ln x}dx$ is a function of α so let us call it $I(\alpha)$

$$\therefore I'(\alpha) = \int_0^1 \frac{x^{\cos\alpha}\ln x(-\sin\alpha)dx}{\ln x} = -\int_0^1 x^{x\cos\alpha}\sin\alpha\,dx$$
$$= -\frac{\sin\alpha}{\cos\alpha+1}$$
$$\therefore\ I'(\alpha) = -\int\frac{\sin\alpha}{\cos\alpha+1}d\alpha = \int\frac{d(\cos\alpha+1)}{\cos\alpha+1}$$

$= \ln|1+\cos\alpha| + c.$ As $\Rightarrow x = -\frac{9}{8}$

$\therefore\ I(\alpha) = \ln(1+\cos\alpha)$ where $\alpha \neq$ odd multiple of π.

13. The given planes are $x - y\cos C - z\cos B = 0$

$-x\cos C + y - z\cos A = 0$

$-x\cos B + y\cos A + z = 0$

For the above planes are to pass through a line, we get $\Delta = 0$ *i.e.* all the three homogenous equations must have infinitely many solutions.

$$\therefore\ \Delta = \begin{vmatrix} 1 & -\cos C & -\cos B \\ -\cos C & 1 & -\cos A \\ -\cos B & -\cos A & 1 \end{vmatrix} = 0$$

$$\Rightarrow 1-\cos^2 A-\cos^2 B-\cos^2 C-2\cos A\cos B\cos C=0$$

$$\Rightarrow 1-\left[\frac{3+\cos 2A+\cos 2B+\cos 2C \atop +4\cos A\cos B\cos C}{2}\right]=0$$

$$\Rightarrow 1-\left[\frac{3-1-4\cos A\cos B\cos C \atop +4\cos A\cos B\cos C}{2}\right]=0$$

Now let l, m, n be direction cosines of the line then

$l-m\cos C-n\cos B=0$

$-l\cos C+m-n\cos A=0$

$$\Rightarrow \frac{l}{\cos A\cos C+\cos B}=\frac{-m}{-\cos A-\cos B\cos C}$$

$$=\frac{n}{1-\cos^2 C}$$

$$\Rightarrow \cos A\cos C+\cos B=\cos A\cos C+\cos(\pi-\overline{A+C})$$

$$=\sin A\sin C$$

Similarly, $\cos A+\cos B\cos C=\sin B\sin C$

$$\Rightarrow \frac{l}{\sin A\sin C}=\frac{m}{\sin B\sin C}=\frac{n}{\sin^2 C}$$

$$\Rightarrow \frac{l}{\sin A}=\frac{m}{\sin B}=\frac{n}{\sin C}$$

Also the line passes through the origin

$\Rightarrow \dfrac{x}{\sin A}=\dfrac{y}{\sin B}=\dfrac{z}{\sin C}$ is equation of the line.

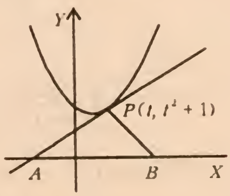
14. $y=x^2+1$. Any point in the parabola is (t, t^2+1)

$$\left.\frac{dy}{dx}\right]_{(t,\,t^2+1)}=2t$$

Equation of tangent at P, $y-(t^2+1)=2t(x-t)$

At A, $y=0$, so $x=t-\dfrac{t^2+1}{2t}$

Hence $A=\left(t-\dfrac{t^2+1}{2t},\ 0\right)$



Equation of normal at P:

$$y-(t^2+1)=-\frac{1}{2t}(x-t)$$

At B, $y=0$ so $x=t+2t(t^2+1)$

Hence $B=\left(2t(t^2+1)+t,\ 0\right)$

Now $AB=2t(t^2+1)+\dfrac{t^2+1}{2t}$

Area of $\Delta APB=A(t)$

$$=\frac{1}{2}(t^2+1)\left[2t(t^2+1)+\frac{t^2+1}{2t}\right]=\frac{(t^2+1)^2(4t^2+1)}{4t}$$

$$\therefore\ A'(t)=\frac{(t^2+1)(20t^4+7t^2-1)}{4t^2}>0$$

Hence $A(t)$ is increasing in $1\le x<3$

$\therefore$ Minimum area $A(1)=5$ sq.units

and maximum area $A(3)=\dfrac{925}{3}$ sq.units.

15. Let $F(x)=\int_0^x g(t)dt \Rightarrow F'(x)=g(x)$

$$\Rightarrow \frac{1}{2}\int_0^x (F'(t))^2dt=\frac{1}{2}\int_0^x g^2(t)dt=\frac{1}{x}\left[\int_0^x g(t)dt\right]^2$$

$$=\frac{1}{x}(F(x))^2.$$

Differentiating both sides with respect to x, we get

$$\frac{1}{2}[F'(x)]^2=\frac{x2F(x)F'(x)-(F(x))^2}{x^2}$$

$$\Rightarrow \frac{1}{2}[g(x)]^2x^2=x\cdot 2F(x)F'(x)-(F(x))^2$$

$$\Rightarrow \left(\frac{x\,g(x)}{F(x)}\right)^2=4\left(\frac{x\,g(x)}{F(x)}\right)-2$$

$$\Rightarrow t^2-4t+2=0 \text{ where } t=\frac{x\,g(x)}{F(x)}$$

$$\Rightarrow t=\frac{4\pm\sqrt{8}}{2}=2\pm\sqrt{2} \text{ or } x\frac{F'(x)}{F(x)}=2\pm\sqrt{2}$$

$$\Rightarrow \ln F(x)=(2\pm\sqrt{2})\ln x+c$$

$$\Rightarrow F(x)=c\,x^{2+\sqrt{2}} \text{ or } c\,x^{2-\sqrt{2}}$$

$$\Rightarrow g(x)=F'(x)=c'\,x^{1+\sqrt{2}} \text{ or } c''\,x^{1-\sqrt{2}}$$

where c' and c'' are constants.

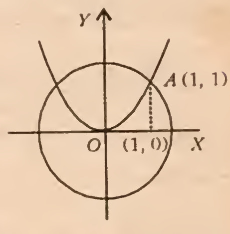
Hence $g(x)=c'\,x^{1+\sqrt{2}}$ or $c''\,x^{1-\sqrt{2}}$

But g is continuous in $(0,\infty)$ then $c''\,x^{1-\sqrt{2}}$ is ruled out. Hence $g(x)=c'\,x^{1+\sqrt{2}}$

As $g(1)=1 \Rightarrow c'=1 \Rightarrow g(x)=x^{1+\sqrt{2}}$.

16. Since $\lim\limits_{x\to 0}\left(1+\dfrac{f(x)}{x^3}\right)^{1/x}$ exists, $\lim\limits_{x\to 0}\dfrac{f(x)}{x^3}=0$

Thus $f(x)=a_4x^4+a_5x^5+\ldots+a_nx^n$, $a_n\ne 0$, $n\ge 4$



Since $f(x)$ is of least degree, $f(x)=a_4x^4$.

But $\underset{x\to 0}{\mathrm{Lt}}\left(1+\dfrac{f(x)}{x^3}\right)^{1/x}=e$

$\Rightarrow a_4=1\Rightarrow f(x)=x^4$. The graph of $f(x)=x^4$ and $x^2+y^2=2$, are shown in the figure.

$\therefore$ The required area $=2\int_0^1\left(\sqrt{2-x^2}-x^4\right)dx$

$$=\left(2\left[\frac{x}{2}\sqrt{2-x^2}+\sin^{-1}\frac{x}{\sqrt{2}}-\frac{x^5}{5}\right]_0^1\right)$$

$$=\frac{\pi}{2}-\frac{3}{5}\text{ sq. units.}$$

■

9. There are 5 addressed envelopes. Letters are written to the corresponding persons. But the letters are placed at random envelopes. Find the probability such that none of the letters enter into the correct envelope.

Soln.: This type of problem can be remembered as 'mismatch' problem of n objects.

Here $n = 5$.

Required probability $= \frac{1}{2!} - \frac{1}{3!} + \frac{1}{4!} - \frac{1}{5!} = \frac{11}{30}$.

10. A and B alternatively throw a pair of dice. A wins if he throw a sum equal to a prime number. B wins if he throws a sum which is an even number. They continue the game until one of them wins. Find the probability for A to win if he starts the game.

Soln.: P (A win in one throw) $= P_1 = 5/11$

P (B win in one throw) $= P_2 = 6/11$

Alternate chances game continued indefinitely.

$$P(A \text{ to win the game}) = \frac{P_1}{P_1 + P_2 - P_1P_2}$$

$$= \frac{5/11}{\frac{5}{11} + \frac{6}{11} - \frac{5}{11} \times \frac{6}{11}} = \frac{55}{91}.$$

Note : P (B to win the game) $= 1 - P(A) = \frac{36}{91}$.

11. Evaluate : $\int \frac{2e^x + 3e^{-x}}{3e^x + 4e^{-x}} dx$.

Soln.: $\int \frac{ae^x + be^{-x}}{ce^x + de^{-x}} dx$

$$= \frac{1}{2}\left[\left(\frac{a}{c} + \frac{b}{d}\right)x + \left(\frac{a}{b} - \frac{b}{d}\right)\log\left|ce^x + de^{-x}\right|\right] + \text{const.}$$

Here $a = 2$, $b = 3$, $c = 3$, $d = 4$

$$\therefore \quad \int \frac{2e^x + 3e^{-x}}{3e^x + 4e^{-x}} dx = \frac{1}{2}\left[\frac{17}{12}x - \frac{1}{12}\log\left|3e^x + 4e^{-x}\right|\right] + C.$$

12. Two forces F_1 and F_2 are acting at a point. The maximum and minimum resultants of those forces are in the ratio 7 : 3. Find the ratio of the magnitudes of the forces.

Soln.: Suppose $|F_1| > |F_2|$.

Maximum resultant force $|F_1| + |F_2|$

Minimum resultant force $|F_1| - |F_2|$

$(|F_1| + |F_2|) : (|F_1| - |F_2|) = 7 : 3$

$7|F_1| - 7|F_2| = 3|F_1| + 3|F_2|$

$4|F_1| = 10|F_2| \Rightarrow |F_1| : |F_2| = 5 : 2$.

13. Find the value of $(\underbrace{666....6}_{n \text{ digits}})^2 + (\underbrace{888...8}_{n \text{ digits}})^2$. in terms of n.

Soln.: Suppose $0 < a \leq 9$.

$\underbrace{aaa.... a}_{n \text{ digits}} = a\,[10^{n-1} + 10^{n-2} + ... + 1]$

$$= a\left(\frac{10^n - 1}{9}\right)$$

Using this result, $(666...6)^2 + (888...8)^2$

$$= \frac{36}{81}(10^n - 1)^2 + \frac{64}{81}(10^n - 1)^2 = \frac{100}{81}(10^n - 1)^2.$$

14. The circumcentre and centroid of a triangle are (2, 3) and (4, 5) respectively. Find the orthocentre of the triangle.

Soln.: Orthocentre divides the line segment joining circumcentre and centroid externally in the ratio 3 : 2.

$$\therefore \quad \text{Orthocentre} = \left(\frac{12-4}{3-2}, \frac{15-6}{3-2}\right) = (8,\ 9).$$

15. Find the $\lim_{n\to\infty} {}^nC_k \cdot p^k \cdot q^{n-k}$ if $np = \lambda$ and $p + q = 1$, where λ, k are constants.

Soln.: Let $x = {}^nC_k \cdot p^k \cdot q^{n-k}$.

Observe that x is a binomial variate. As $n \to \infty$ binomial variate becomes poisson variate.

Hence $\lim_{n\to\infty} x = \frac{\lambda^k \cdot e^{-\lambda}}{k!}$. ■

MATHS & MUSIC

One of the earliest known applications of mathematics in music is attributed to Pythagoras, a Greek mathematician best known for his theorem concerning right-angled triangles.

According to myth, Pythagoras was walking past a blacksmiths, listening to the sound of the hammers on the anvils. After a while, he realised that all but one of the hammers were sounding in harmony. Curious to know the reason, Pythagoras made a thorough examination of the hammers and discovered that when their masses were simple ratios i.e. 2 : 1 or 4 : 1, then the respective notes produced were in harmony. On the other hand, the mass of the hammer producing the discordant note wasn't in a simple ratio with any of the other hammers.

Although the absolute truth of this myth is questionable, it illustrates the first real application of the maths in music.

International Math Olympiad*

PROBLEMS & SOLUTIONS

1. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. The point P, where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is cyclic if and only if the triangles ABP and CDP have equal areas.

2. In a competition there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either "pass" or "fail". Suppose k is a number such that for any two judges their ratings coincide for at most k contestants. Prove $\frac{k}{a} \geq \frac{(b-1)}{2b}$.

3. a, b, c are non-zero real numbers and $x_1, x_2, \ldots, x_n$ satisfy the n equations:

$ax_i^2 + bx_i + c = x_{i+1}$, for $1 \leq i < n$

$ax_n^2 + bx_n + c = x_1$

Prove that the system has zero, 1 or >1 real solutions according as $(b-1)^2 - 4ac$ is < 0, $= 0$ or > 0.

4. Prove that every tetrahedron has a vertex whose three edges have the right lengths to form a triangle.

5. Let f be a real-valued function defined for all real numbers, such that for some $a > 0$ we have

$f(x+a) = \frac{1}{2} + \sqrt{f(x) - f(x)^2}$ for all x. Prove that f is periodic, and give an example of such a non-constant f for $a = 1$.

6. For every natural number n evaluate the sum

$$\left[\frac{(n+1)}{2}\right] + \left[\frac{(n+2)}{4}\right] + \left[\frac{(n+4)}{8}\right] + \ldots + \left[\frac{(n+2^k)}{2^{k+1}}\right] + \ldots,$$

where $[x]$ denotes the greatest integer $\leq x$.

SOLUTION

1. Let AC and BD meet at X. Let H, K be the feet of the perpendiculars from P to AC, BD respectively. We wish to express the areas of ABP and CDP in terms of more tractable triangles. There are essentially two different configurations possible. In the first, we have area PAB = area ABX + area PAX + area PBX, and area PCD = area CDX – area PCX – area PDX. So if the areas being equal is equivalent to: area ABX – area CDX + area PAX + area PCX + area PBX + area PDX. ABX and CDX are right-angled, so we may write their areas as $AX.BX/2$ and $CX.DX/2$. We may also put $AX = AM - MX = AM - PN$, $BX = BN - PM$, $CX = CM + PN$, $DX = DN + PM$. The other triangles combine in pairs to give area ACP + area BDP = $AC.PM + BD.PN$. This leads, after some cancellation to $AM.BN = CM.DN$. There is a similar configuration with the roles of AB and CD reversed.

The second configuration is area PAB = area ABX + area $PAX - PBX$, area PCD = area CDX + area PDX – area PCX. In this case $AX = AM + PN$, $BX = BN - PM$, $CX = CM - PN$, $DX = DN + PM$. But we end up with the same result: $AM \cdot BN = CM \cdot DN$.

Now if $ABCD$ is cyclic, then it follows immediately that P is the center of the circumcircle and $AM = CM$, $BN = DN$. Hence the areas of PAB and PCD are equal.

Conversely, suppose the areas are equal. If $PA > PC$, then $AM > CM$. But since $PA = PB$ and $PC = PD$ (by construction), $PB > PD$, so $BN > DN$. So $AM \cdot BN > CM \cdot DN$. Contradiction. So PA is not greater than PC. Similarly it cannot be less. Hence $PA = PC$. But that implies $PA = PB = PC = PD$, so $ABCD$ is cyclic.

2. Let us count the number N of triples (judge, judge, contestant) for which the two judges are distinct and rate the contestant the same. There

* *For similar problems and solutions read MTG's* ***Mathematical Olympiad Problems***

are $\frac{b(b-1)}{2}$ pairs of judges in total and each pair rates at most k contestants the same, so $N \le \frac{kb(b-1)}{2}$.

Now consider a fixed contestant X and count the number of pairs of judges rating X the same. Suppose x judges pass X, then there are $\frac{x(x-1)}{2}$ pairs who pass X and $\frac{(b-x)(b-x-1)}{2}$ who fail X, so a total of $\frac{(x(x-1)+(b-x)(b-x-1))}{2}$ pairs rate X the same. But

$$\frac{x(x-1)+(b-x)(b-x-1)}{2} = \frac{2x^2-2bx+b^2-b}{2}$$

$$= \left(\frac{x-b}{2}\right)^2 + \frac{b^4}{4} - \frac{b}{2} \ge \frac{b^2}{4} - \frac{b}{2} = \frac{(b-1)^2}{4} - \frac{1}{4}.$$

But $\frac{(b-1)^2}{4}$ is an integer (since b is odd), so the number of pairs rating X the same is at least $\frac{(b-1)^2}{4}$. Hence $N \ge \frac{a(b-1)^2}{4}$. Putting the two inequalities together gives $\frac{k}{a} \ge \frac{(b-1)}{2b}$.

3. Let $f(x) = ax^2 + bx + c - x$. Then $\frac{f(x)}{a} = \left(\frac{x+(b-1)}{2a}\right)^2 + \left(\frac{4ac-(b-1)^2}{4a^2}\right)$. Hence if $4ac - (b-1)^2 > 0$, then $f(x)$ has the same sign for all x. But $f(x) > 0$ means $ax^2 + bx + c > x$, so if $\{x_i\}$ is a solution, then either $x_1 < x_2 < \ldots < x_n < x_1$, or $x_1 > x_2 > \ldots > x_n > x_1$. Either way we have a contradiction. So if $4ac - (b-1)^2 > 0$ there cannot be any solutions.

If $4ac - (b-1)^2 = 0$, then we can argue in the same way that either $x_1 \le x_2 \le \ldots \le x_n \le x_1$, or $x_1 \ge x_2 \ge \ldots \ge x_n \ge x_1$. So we must have all x_i = the single root of $f(x) = 0$ (which clearly is a solution).

If $4ac - (b-1)^2 < 0$, then $f(x) = 0$ has two distinct real roots y and z and so we have at least two solutions to the equations: all $x_i = y$, and all $x_i = z$. We may, however, have additional solutions. For example, if $a = 1$, $b = 0$, $c = -1$ and n is even, then we have the additional solution $x_1 = x_3 = x_5 = \ldots = 0$, $x_2 = x_4 = \ldots = -1$.

4. The trick is to consider the longest side. That avoids getting into lots of different possible cases for which edge is longer than the sum of the other two.

So assume the result is false and let AB be the longest side. Then we have $AB > AC + AD$ and $BA > BC + BD$. So $2AB > AC + AD + BC + BC$. But by the triangle inequality, $AB < AC + CB$, $AB < AD + DB$, so $2AB < AC + CB + AD + DB$. Contradiction.

5. Directly from the equality given : $f(x+a) \ge \frac{1}{2}$ for all x, and hence $f(x) \ge \frac{1}{2}$ for all x. So

$$f(x+2a) = \frac{1}{2} + \sqrt{f(x+a) - f(x+a)^2}$$

$$= \frac{1}{2} + \sqrt{f(x+a)(1-f(x+a)}$$

$$= \frac{1}{2} + \sqrt{\frac{1}{4} - f(x) + f(x)^2}$$

$$= \frac{1}{2} + \left(f(x) - \frac{1}{2}\right) = f(x).$$

So f is periodic with period $2a$.

We may take $f(x)$ to be arbitary in the interval [0, 1).

For example, let $f(x) = 1$ for $0 \le x < 1$,

$$f(x) = \frac{1}{2} \text{ for } 1 \le x < 2.$$

Then use $f(x+2) = f(x)$ to define $f(x)$ for all other values of x.

6. For any real x we have $[x] = \left[\frac{x}{2}\right] + \left[\frac{(x+1)}{2}\right]$.

For if $x = 2n + 1 + k$, where n is an integer and $0 \le k < 1$, then

LHS = $2n + 1$, and RHS = $n + n + 1$.

Similarly, if $x = 2n + k$

Hence for any integer n, we have :

$$\left[\frac{n}{2^k}\right] - \left[\frac{n}{2^{k+1}}\right] = \left[\left(\frac{n}{2^k}+1\right)/2\right] = \left[\frac{(n+2^k)}{2^{k+1}}\right].$$

Hence summing over k, and using the fact that $n < 2^k$ for sufficiently large k, so that $\left[\frac{n}{2^k}\right] = 0$,

$$\therefore \left[\frac{(n+1)}{2}\right] + \left[\frac{(n+2)}{4}\right] + \left[\frac{(n+4)}{8}\right] + \ldots + \left[\frac{(n+2^k)}{2^{k+1}}\right] + \ldots = n$$

■

MBA MCA NDA

Quantitative Aptitude Test

NTSE GRE SAT

The aim of this test is to acclimatize you with real test-like experience and give you the required competitive edge.

1. On dividing a number by 3, 4 and 7, the remainders are 2, 1 and 4 respectively. If the same number is divided by 84 then the remainder is

(a) 80 (b) 76
(3) 53 (d) None of these.

2. There are three pieces of cake weighing 9/2 lbs, 27/4 lbs and 36/5 lbs. Pieces of the cake are equally divided and distributed in such a manner that every guest in the party gets one single piece of cake. Further the weight of the cake is as heavy as possible. What is the largest number of guests to whom we can distribute the cake?

(a) 54 (b) 72
(c) 20 (d) None of these.

3. For three real numbers x, y and z, $x+y+z=5$, and $xy+yz+xz=3$. What is the largest value which x can take ?

(a) $3\sqrt{13}$ (b) $\sqrt{19}$
(c) 13/3 (d) $\sqrt{15}$.

4. There are six persons sitting around a round table. Pankaj is sitting left of Dayanand who is facing Kundan and Ranjan is sitting right of Dayanand. Yash is sitting left of Pankaj and Abhishek is sitting right of Ranjan. If Pankaj and Ranjan swap their position and Yash and Abhishek also swap their position , then who will be to left of Abhishek?

(a) Kundan (b) Yash
(c) Dayanand (d) Pankaj.

5. A transport company charges for its vehicles in the following manner: If the driving is 5 hours or less, the company charges Rs. 60 per hour or Rs. 12 per km (which ever is larger). If driving is more than 5 hours, the company charges Rs. 50 per hour or Rs. 7.5 per km. If Anand drove it for 30 km and paid a total of Rs. 300, then for how many hours does he drive?

(a) 4 (b) 5.5
(c) 7 (d) 6

6. Only a single rail track exists between station A and B on a railway line. One hour after the north bound super fast train N leaves station A for station B, a south passenger train reaches station A from Station B. The speed of the super fast train is twice that of a normal express train E, while the speed of a passenger train S is half that of E. On a particular day N leaves for Station B from Station A, 20 minutes behind the normal schedule. In order to maintain the schedule both N and S increased their speed. If the super fast train doubles its speed, what should be the ratio (approximately) of the speed of passenger train to that of the super fast train, so that passenger train S reaches exactly at the scheduled time at the station A on that day

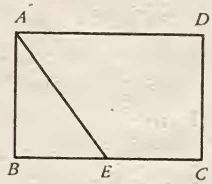
(a) 1 : 3 (b) 1 : 4
(c) 1 : 5 (d) 1 : 2.

7. If $x^2+5y^2+z^2=2y(2x+z)$ then which of the following statements are necessarily true ?

I. $x=2y$, II. $x=2z$ III. $2x=z$

(a) Only I (b) Only II
(c) Only III (d) Only I and II

8. In the following figure, the area of the isosceles right triangle ABE is 7 sq. cm. If $EC=3BE$, then the area of $ABCD$ is



(a) 64 (b) 82
(c) 26 (d) 56.

9. Number S is equal to the square of the sum of the digits of a 2 digit number D. If the difference between S and D is 27, then D is

(a) 32 (b) 54
(c) 64 (d) 52.

10. A boy finds the average of 10 positive integers. Each integer contains two digits. By mistake, the boy interchanges the digits of one number say ba for ab. Due to this, the average becomes 1.8 less than the previous one. What was the difference of

International Math Olympiad

PROBLEMS & SOLUTIONS

1. Determine the extreme values of $\frac{r_1}{h_1}+\frac{r_2}{h_2}+\frac{r_3}{h_3}+\frac{r_4}{h_4}$, where h_1, h_2, h_3, h_4 are the four altitudes of a given tetrahedron T, and r_1, r_2, r_3, r_4 are the corresponding signed perpendicular distances from any point in the space of T to the faces.

Soln.: If the face areas and volume of the tetrahedron are F_1, F_2, F_3, F_4 and V respectively, then

$$r_1F_1+r_2F_2+r_3F_3+r_4F_4 = 3V$$

and $\quad h_1F_1 = h_2F_2 = h_3F_3 = h_4F_4 = 3V.$

Now eliminating the F_i's, we get

$$\frac{r_1}{h_1}+\frac{r_2}{h_2}+\frac{r_3}{h_3}+\frac{r_4}{h_4} = 1 \text{ (a constant).}$$

2. Determine the minimum value of the product $P=(1+x_1+y_1)(1+x_2+y_2)...(1+x_n+y_n)$ where $x_i, y_i \ge 0$, and

$$x_1x_2...x_n = y_1y_2...y_n = a^n.$$

Soln.: More generally, consider

$$P=(1+x_1+y_1+...w_1)(1+x_2+y_2+...w_2)...(1+x_n+y_n+...w_n)$$

where $\quad x_1x_2...x_n=\xi^n$, $y_1y_2...y_n=\eta^n, ..., w_1w_2...w_n = \omega^n$, and $x_i, y_i, ..., w_i \ge 0$. Then by Holder's inequality

$$P^{1/n} \ge \left\{1+\Pi x_i^{1/n}+\Pi y_i^{1/n}+...+\Pi\omega_i^{1/n}\right\}$$

or $P \ge (1+\xi+\eta+....+\omega)^n$

In this case $\xi=\eta=a$, so $P \ge (1+2a)^n$.

3. Prove that if $F(x, y, z)$ is a concave function of x, y, z, then $\{F(x, y, z)\}^{-2}$ is a convex function of x, y, z.

Soln.: More generally $G(F)$ is a convex function where G is a convex decreasing function. By the convexity of G.

$$\lambda G\{F(x_1, y_1, z_1)\}+(1-\lambda)G\{F(x_2, y_2, z_2)\}$$
$$\ge G\{\lambda F(x_1, y_1, z_1)+(1-\lambda)F(x_2, y_2, z_2)\}.$$

By the concavity of F,

$$\lambda F(x_1, y_1, z_1)+(1-\lambda)F(x_2, y_2, z_2)$$
$$\le F([\lambda x_1+(1-\lambda)x_2], [\lambda y_1+(1-\lambda)y_1], [\lambda z_1+(1-\lambda)z_2])$$

Finally, since G is decreasing,

$$\lambda G\{F(x_1, y_1, z_1)\}+(1-\lambda)G\{F(x_2, y_2, z_2)\}$$
$$\ge G\{F[\lambda x_1+(1-\lambda)x_2], [\lambda y_1+(1-\lambda)y_2], [\lambda z_1+(1-\lambda)z_2]\}$$

More generally and more precisely, we have the following known result: if $F(X)$ is a concave function of $X=(x_1, x_2, ..., x_n)$ and $G(y)$ is a convex decreasing function of y, where y is a real variable and the domain of G contains the range of F, then $G\{F(X)\}$ is a convex function of X.

4. If a, b, c are sides of a given triangle of perimeter p, determine the maximum values of

(i) $(a-b)^2+(b-c)^2+(c-a)^2$,

(ii) $|a-b|+|b-c|+|c-a|$

(iii) $|a-b||b-c|+|b-c||c-a|+|c-a||a-b|$.

Soln.: (i) $(a-b)^2+(b-c)^2+(c-a)^2$

$$=2\left(\sum a^2-\sum bc\right)\le kp^2$$

Let $c=0$, so that $k \ge \frac{1}{2}$. We now show that $k=\frac{1}{2}$ suffices. Here,

$$2\left(\sum a^2-\sum bc\right) \le \frac{1}{2}(a+b+c)^2$$

reduces to $2bc+2ca+2ab-a^2-b^2-c^2 \ge 0$

The LHS is 16 times the square of the area of a triangle of sides $\sqrt{a}, \sqrt{b}, \sqrt{c}$ or

For more about this exam read MTG's **Math Olympiad Problems and Solutions**

$$(\sqrt{a}+\sqrt{b}+\sqrt{c})(\sqrt{a}+\sqrt{b}-\sqrt{c})(\sqrt{a}-\sqrt{b}+\sqrt{c})(-\sqrt{a}+\sqrt{b}+\sqrt{c})$$

There is equality if and only if the triangle is degenerate with one side 0.

(ii) $|a-b|+|b-c|+|c-a|\le kp$

Letting c = 0, $k \ge 1$. To show that k = 1 suffices, assume that $a \ge b \ge c$, so that

$$|a-b|+|b-c|+|c-a|=2a-2c \le a+b+c$$

and there is equality if and only if c = 0.

(iii) $|a-b||b-c|+|b-c||c-a|+|c-a||a-b|\le kp^2$

Letting $c = 0$, $k \ge 1/4$. To show that $k = 1/4$ suffices, let $a = y+z$, $b = z+x$, $c = x+y$ where $z \ge y \ge x \ge 0$. Our inequality then becomes

$$|x-y||z-y|+|y-z||z-x|+|z-x||x-y| \le (x+y+z)^2$$

or $$x^2-y^2+z^2+yz-3zx+xy \le x^2+y^2+z^2+2yz+2zx+2xy$$

or $$2y^2+5zx+1xy+1yz \ge 0$$

There is equality if and only if $x = y = 0$ or equivalently, $a = b$, and $c = 0$.

5. If A, B, C are three dihedral angles of trihedral angle, show that $\sin A$, $\sin B$, $\sin C$ satisfy the traingle inequality.

Soln.: Let a, b, c be the face angles of the trihedral angle opposite to A, B, C respectively. Since

$$\frac{\sin a}{\sin A}=\frac{\sin b}{\sin B}=\frac{\sin c}{\sin C}$$

by the Law of sines for spherical triangles, it suffice to show that $\sin b + \sin c > \sin a$, or

$$2\sin\frac{1}{2}(b+c)\cos\frac{1}{2}(b-c) > 2\sin\frac{1}{2}a\cos\frac{1}{2}a,$$

for any labelling of the angles. We now use the following properties of a, b and c;

(i) they satisfy the triangle inequality,

(ii) $0 < a + b + c < 2\pi$.

Hence, $\cos\frac{1}{2}(b-c) > \cos\frac{1}{2}a.$

To complete the proof, we show that

$$\sin\frac{1}{2}(b+c) > \sin\frac{1}{2}a.$$

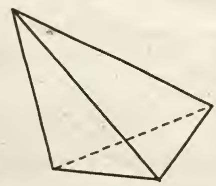
This follows immediately if $b+c \le \pi$; if $b+c > \pi$, then

$$\sin\frac{1}{2}(b+c) = \sin\left\{\pi-\frac{1}{2}(b+c)\right\} > \sin\frac{1}{2}a \quad \left(\text{since } \pi - \frac{b+c}{2} > \frac{a}{2}\right).$$

■

PERFECT PYRAMIDS

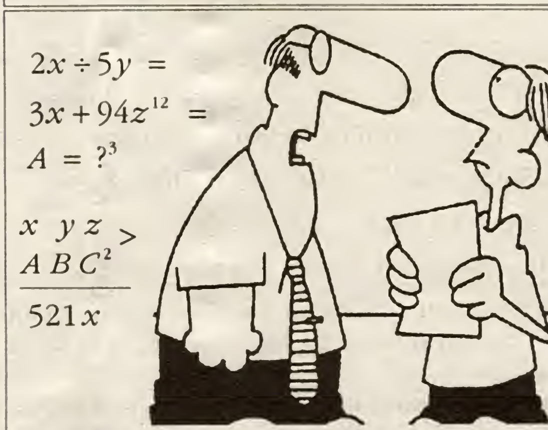
The tetrahedron is the simplest of all polyhedra – solids bounded by polygons. It has four triangular faces, four vertices, and six edges. If each edge has the same length and each face is an equilateral triangle, the result is a regular tetrahedron – one of the Platonic solids.



Another group of tetrahedra that some people consider special consists of those that have integer edge lengths, face areas, and volumes. Such a solid is sometimes called a Heronian tetrahedron or a perfect pyramid. The term "Heronian" refers to Heron of Alexandria (10-90). His name is also attached to a formula that relates the area of a triangle, A, to the lengths of its three sides, a, b, and c.

$$A = \sqrt{s(s-a)(s-b)(s-c)} \quad \text{where } s = \frac{a+b+c}{2}$$

There's an analogous formula for the volume of tetrahedron, given the lengths of its six edges. Tetrahedra with integer sides, face areas, and volumes are rare. For example, there's only one perfect pyramid with integer sides less than 157. It has sides of length 51, 52, 53, 80, 84, and 117; faces of area 1170, 1800, 1890, and 2016; and a volume of 18144.



"Why is it important for today's kids to learn algebra? Because I had to learn this junk in school and now it's your turn, that's why!"

Mathematical Challenges for I.I.T. MAINS

1. Calculate the number of integral roots of the equation $x^8 - 24x^7 - 18x^5 + 39x^2 + 1155 = 0$.

2. The amount of fuel consumed per hour by a certain train varies as the cube of its speed. When the speed is 15 miles per hour. The fuel consumed is $4\frac{1}{2}$ tons of coal per hour at Rs. 4 per ton. The other expenses are total Rs. 100 per hour. Find the most economical speed.

3. Find $\sum_{r=1}^{6N-1} \min\left(\left\{\frac{r}{3N}\right\}, \left\{\frac{r}{6N}\right\}\right)$; where $\{a\} = \min(a - [a], [a] - a + 1)$, i.e. the distance to the nearest integer, ($[a]$ represents greatest integer less than equal to a).

4. Show that the locus of a point which moves so that the chords of contact of tangents from the point to two fixed circles are perpendicular to each other is a circle with its centre at the mid point of the line joining the centres of the two given circles.

5. Let $I_n = \int_0^1 x^n \tan^{-1} x\, dx$. If $AI_n + BI_{n-2} = C$ then find the constants A, B and C.

6. Let $f(x)$ be a polynomial of degree 5 and with leading coefficients 2004. Suppose further, that $f(1) = 1, f(2) = 3, f(3) = 5, f(4) = 7$ and $f(5) = 9$. What is the value of $f(6)$?

7. If $f(x)$ be a continuous and differentiable real valued function with $f(0) = 0$ and $f'(x) + 2f(x) \le 1$. Prove that $f(x) \le \frac{1}{2}$.

8. If $\sum_{i=1}^{4} b_i = 0$ where b_i, are non zero real numbers, sum of no two being zero, and $\sum_{i=1}^{4} b_i z_i = 0$ where no three of the points with affixes z_1, z_2, z_3, z_4 are collinear then prove that the four points will be concyclic if $b_1 b_2 |z_1 - z_2|^2 = b_3 b_4 |z_3 - z_4|^2$.

9. Show that if a chord of the parabola $y^2 = 4ax$ touches the parabola $y^2 = 4bx$, then the tangents at its extremities meet on the parabola $by^2 = 4a^2x$ and the normals meet on the curve $(4a - b)^3 y^2 = 4b^2(x - 2a)^3$.

10. A triangle is circumscribed to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ and two of the vertices lie on the directrices. Prove that the locus of the third vertex is the ellipse $b^2 y^2 = a^2(e^2 + 1)^2(a^2 - x^2)$.

SOLUTION

1. Let $x = a$ be an integer root of the given equation, then a must divide 1155. This means a is an odd integer. But if x is odd then $x^8 + 39x^2 + 1155 - (24x^7 + 18x^5)$ is also odd, and therefore not 0. It follows that the given equation does not have any integral root.

2. Let the speed of train be v miles/hr. and the coal consumed per hour is $c = kv^3$ where $k = \frac{9}{2(15)^3}$. Let the distance d miles is to be covered, so $d = vt$ ...(1)

total expenses to cover this distance is

$p = (4kv^3 + 100)t$ Rs.

$$p = \left(\frac{18}{(15)^3} v^3 + 100\right) \cdot \frac{d}{v} \quad \text{(using (1) in it)}$$

$$\frac{dp}{dv} = \left(\frac{36}{(15)^3} v - \frac{100}{v}\right) d$$

so $v = \left(\frac{100 \times (15)^3}{36}\right)^{1/3}$ is the point of minima of

By : **Shailendra Maheshwari**, Career point, Kota

p, so the most economical speed is $15\left(\frac{5}{3}\right)^{2/3}$ miles/hour.

3. Since the middle term of the sequence is $\left\{\frac{3N}{3N}\right\}=0$ and $\left\{\frac{6N-r}{3N}\right\}=\left\{\frac{r}{3N}\right\}$ & $\left\{\frac{6N-r}{6N}\right\}=\left\{\frac{r}{6N}\right\}$

so the required sum is $2\sum\limits_{r=1}^{3N-1} \min\left(\left\{\frac{r}{3N}\right\}, \left\{\frac{r}{6N}\right\}\right)$

$$= 2\sum_{r=1}^{2N}\left\{\frac{r}{6N}\right\} + 2\sum_{r=2N+1}^{3N-1}\left\{\frac{r}{3N}\right\}$$

$$= 2\sum_{r=1}^{2N}\frac{r}{6N} + 2\sum_{r=1}^{N-1}\left\{\frac{3N-r}{3N}\right\}$$

$$= 2\sum_{r=1}^{2N}\frac{r}{6N} + 2\sum_{r=1}^{N-1}\frac{r}{3N}$$

$$= \frac{2}{6N}\cdot\frac{2N(2N+1)}{2} + \frac{2}{3N}\cdot\frac{(N-1)(N)}{2} = N.$$

4. Let the two fixed circles be

$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = 0$ and

$x^2 + y^2 + 2g_2x + 2f_2y + c_2 = 0$ and the variable point P be (h, k) chords of contact of point P w.r.t. these circles are

$(h+g_1)x + (k+f_1)y + g_1h + f_1k + c_1 = 0$ ...(1)

and

$(h+g_2)x + (k+f_2)y + g_2h + f_2k + c_2 = 0$ ...(2)

As they are at right angle, so

$$\left(-\frac{h+g_1}{k+f_1}\right)\left(-\frac{h+g_2}{k+f_2}\right) = -1$$

so $(x+g_1)(x+g_2)+(y+f_1)(y+f_2) = 0$ is the locus of point P which is the circle with two ends of diameter as $(-g_1,-f_1)$ and $(-g_2,-f_2)$.

5. $$I_n = \left(\frac{x^{n+1}\tan^{-1}x}{n+1}\right)_0^1 - \int_0^1 \frac{x^{n+1}}{(x^2+1)(n+1)}dx \quad \text{...(1)}$$

$$= \frac{\pi}{4(n+1)} - \frac{1}{n+1}\int_0^1 x^{n-1}dx + \frac{1}{n+1}\int_0^1 \frac{x^{n-1}}{1+x^2}dx$$

$$= \frac{1}{n+1}\left[\frac{\pi}{4}-\frac{1}{n}\right] + \frac{1}{n+1}\int_0^1 \frac{x^{n-1}}{1+x^2}dx \quad \text{...(2)}$$

from (1) $\int_0^1 \frac{x^{n+1}}{x^2+1}dx = \frac{\pi}{4}-(n+1)I_n$

so $\int_0^1 \frac{x^{n-1}}{x^2+1}dx = \frac{\pi}{4}-(n-1)I_{n-2}$

using it in (2)

$$I_n = \frac{1}{n+1}\left[\frac{\pi}{4}-\frac{1}{n}\right] + \frac{\pi}{4(n+1)} - \frac{n-1}{n+1}I_{n-2}$$

so $(n+1)I_n + (n-1)I_{n-2} = \frac{\pi}{2} - \frac{1}{n}$

so $A = n+1$, $B = n-1$ and $C = \frac{\pi}{2}-\frac{1}{n}$.

6. Let $g(x) = f(x)-(2x-1)$. So the given informations imply that

$$g(1) = g(2) = g(3) = g(4) = g(5) = 0.$$

As $g(x)$ must be the polynomial of degree 5 with leading coefficient 2004, so

$g(x) = 2004(x-1)(x-2)(x-3)(x-4)(x-5)$

$g(6) = 2004 \times \underline{|5} = 240480$ since $g(6) = f(6)-11$

so $f(6) = g(6)+11 = 240491$ $\therefore$ $f(6) = 240491$.

7. Let $y = f(x)$, so as given $\frac{dy}{dx} + 2y \le 1$

multiply bothe sides by e^{2x}

$$e^{2x}\frac{dy}{dx} + 2e^{2x}y \le e^{2x} \Rightarrow \frac{d}{dx}\left(e^{2x}y\right) \le e^{2x}$$

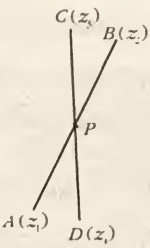
$$ye^{2x} \le \int_0^x e^{2t}dt \Rightarrow ye^{2x} \le \frac{1}{2}\left(e^{2x}-1\right)$$

so $y \le \frac{1}{2} - \frac{1}{2e^{2x}}$ so $y \le \frac{1}{2}$ so $f(x) \le \frac{1}{2}$.

8. $\sum\limits_{i=1}^{4} b_i = 0 \Rightarrow b_1 + b_2 = -(b_3 + b_4)$ ...(1)

$\sum\limits_{i=1}^{4} b_iz_i = 0 \Rightarrow b_1z_1 + b_2z_2 = -(b_3z_3 + b_4z_4)$ ...(2)

so $\frac{b_1z_1 + b_2z_2}{b_1 + b_2} = \frac{b_3z_3 + b_4z_4}{b_3 + b_4}$



so the point dividing z_1, z_2 in the ratio $b_1 : b_2$ and the point dividing z_3, z_4 in the ratio $b_3 : b_4$ are same.

Now if A, B, C, D are concyclic points then $AP \cdot BP = CP \cdot DP$

$$\frac{b_1}{b_1+b_2}|z_1-z_2| \cdot \frac{b_2}{b_1+b_2}|z_1-z_2| = \frac{b_3}{b_3+b_4}|z_3-z_4| \cdot \frac{b_4}{b_3+b_4}|z_3-z_4|$$

using (1) in it $b_1b_2|z_1-z_2|^2 = b_3b_4|z_3-z_4|^2$.

9. Let $P(t_1)$ and $Q(t_2)$ be two points on $y^2 = 4ax$

chord PQ : $y = \frac{2}{t_1+t_2}x + \frac{2at_1t_2}{t_1+t_2}$

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It is tangent to $y^2 = 4bx$ so

$$\frac{b}{m} = \frac{2at_1t_2}{t_1+t_2} \text{ where } m = \frac{2}{t_1+t_2}$$

so $b(t_1+t_2)^2 = 4at_1t_2$ ...(1)

If tangents at P and Q to $y^2 = 4ax$ meet at point $R(h, k)$ then $h = at_1t_2$ and $k = a(t_1+t_2)$

so from (1) : $b \cdot \frac{k^2}{a^2} = 4h$

thus locus of point R is $by^2 = 4a^2x$.

Now if normals at P and Q to $y^2 = 4ax$ meet at point $T(\alpha, \beta)$ then

$$\alpha = a(t_1^2+t_2^2+t_1t_2+2) = a\left[(t_1+t_2)^2 - t_1t_2 + 2\right]$$

and $\beta = -at_1t_2(t_1+t_2)$

using (1) in these results

$$\alpha = a(t_1+t_2)^2 - \frac{b(t_1+t_2)^2}{4} + 2a \text{ and}$$

$$\beta = -\frac{b(t_1+t_2)^3}{4}$$

eliminating t_1 and t_2 from these equations

$$4^3(\alpha-2a)^3 = (4a-b)^3\left(\frac{-4\beta}{b}\right)^2$$

so point T lies on the curve

$$4b^2(x-2a)^3 = (4a-b)^3y^2.$$

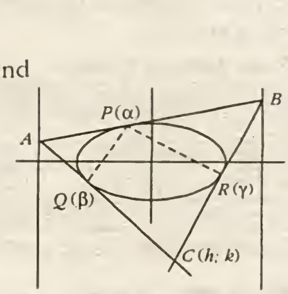
10. Let $C(h, k)$ be the point of intersection of tangents at $Q(\beta)$ and $R(\gamma)$. So

$$h = a\frac{\cos\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}} \quad \text{...(1)}$$

$$k = b\frac{\sin\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}} \quad \text{...(2)}$$

As the vertices A and B lie on directrices so PQ and PR must be focal chords. So

$$e = \frac{\cos\frac{\alpha-\gamma}{2}}{\cos\frac{\alpha+\gamma}{2}} \quad \text{...(3) and}$$



$$e = -\frac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}} \quad \text{...(4)}$$

dividing (3) by (4)

$$\frac{\cos\frac{\alpha-\gamma}{2}\cdot\cos\frac{\alpha+\beta}{2}}{\cos\frac{\alpha+\gamma}{2}\cdot\cos\frac{\alpha-\beta}{2}} = -1$$

$$\Rightarrow \cos\frac{2\alpha+\beta-\gamma}{2} + \cos\frac{\beta+\gamma}{2} = -\left(\cos\frac{2\alpha+\gamma-\beta}{2} + \cos\frac{\beta+\gamma}{2}\right)$$

$$\Rightarrow 2\cos\frac{\beta+\gamma}{2} + 2\cos\alpha\cdot\cos\frac{\beta-\gamma}{2} = 0$$

$$\Rightarrow -\cos\alpha = \frac{\cos\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}} \text{ (using (1) in it)}$$

$$\cos\alpha = -\frac{h}{a} \quad \text{...(5)}$$

Now again from (3) and (4)

$$\frac{\cos\frac{\alpha-\gamma}{2}\cdot\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\gamma}{2}\cdot\cos\frac{\alpha+\beta}{2}} = -e^2$$

$$\Rightarrow \frac{\cos\frac{2\alpha-\beta-\gamma}{2} + \cos\frac{\beta-\gamma}{2}}{\cos\frac{2\alpha+\beta+\gamma}{2} + \cos\frac{\beta-\gamma}{2}} = -e^2$$

using componendo and dividendo

$$\Rightarrow \frac{\cos\frac{2\alpha-\beta-\gamma}{2} - \cos\frac{2\alpha+\beta+\gamma}{2}}{\cos\frac{2\alpha-\beta-\gamma}{2} + \cos\frac{2\alpha+\beta+\gamma}{2} + 2\cos\frac{\beta-\gamma}{2}} = \frac{e^2+1}{e^2-1}$$

$$\frac{\sin\alpha\,\sin\frac{\beta+\gamma}{2}}{\cos\alpha\,\cos\frac{\beta+\gamma}{2} + \cos\frac{\beta-\gamma}{2}} = \frac{e^2+1}{e^2-1}$$

using (5) in it $\dfrac{\sin\alpha\,\sin\frac{\beta+\gamma}{2}}{\cos\frac{\beta-\gamma}{2}\left(1-\cos^2\alpha\right)} = \dfrac{e^2+1}{e^2-1}$

$$\frac{k}{b}\frac{\sin\alpha}{\sin^2\alpha} = \frac{e^2+1}{e^2-1} \quad \text{(using (2) in it)}$$

so $\sin\alpha = \left(\dfrac{e^2-1}{e^2+1}\right)\dfrac{k}{b}$ ...(6)

eliminate α from (5) and (6)

$$\frac{h^2}{a^2} + \left(\frac{e^2-1}{e^2+1}\right)^2\frac{k^2}{b^2} = 1$$

so required locus is $\dfrac{y^2}{b^2} = \dfrac{(e^2+1)^2}{a^2(e^2-1)^2}(-x^2+a^2)$

$$\frac{y^2}{b^2} = \frac{a^2(e^2+1)^2}{b^4}(-x^2+a^2)$$

$$b^2y^2 = a^2(e^2+1)^2(a^2-x^2).$$ ■

Critical Points

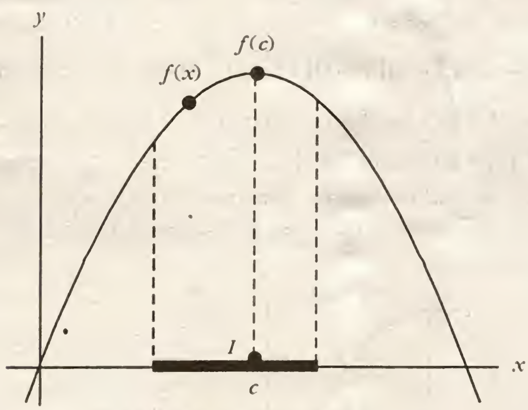
Before discussing what exactly critical points are, let us discuss the occurence of local maxima and local minima of a function.

Local maxima / local minima

Definition : A function $f(x)$ is said to have a local maximum at c iff there exists an interval I around c such that $f(c) \geq f(x)$ for all $x \in I$.

Analogously, $f(x)$ is said to have a local minimum at c iff there exists an interval I around c such that $f(c) \leq f(x)$ for all $x \in I$.

A local extremum is a local maximum or a local minimum.

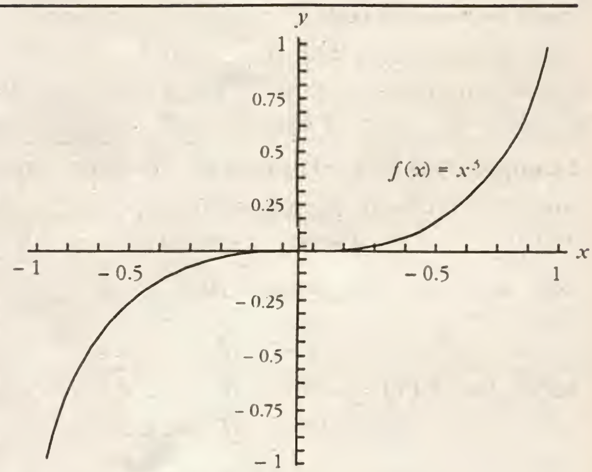


A local maximumat $x = c$

Using the definition of the derivative, we can easily show that :

> If $f(x)$ has a local extremum at c, then either $f'(c) = 0$ or $f'(c)$ does not exist.
> These points are called critical points.

Example : Consider the function $f(x) = x^3$. Then $f'(0) = 0$ but 0 is not a local extremum. Indeed, if $x < 0$, then $f(x) < f(0)$ and if $x > 0$, then $f(x) > f(0)$.



Therefore the conditions

$$f'(c) = 0 \text{ or } f'(x) \textbf{ does not exist.}$$

do not imply in general that c is a local extremum. So a local extremum must occur at a critical point, but the converse may not be true.

Example : Let us find the critical points of $f(x) = |x^2 - x|$.

Ans. We have $f(x) = \begin{cases} x^2 - x & \text{if} \quad x \leq 0 \\ -(x^2 - x) & \text{if} \quad 0 \leq x \leq 1 \\ x^2 - x & \text{if} \quad 1 \leq x \end{cases}$

$$f'(x) = \begin{cases} 2x - 1 & \text{if} \quad x < 0 \\ -2x + 1 & \text{if} \quad 0 < x < 1 \\ 2x - 1 & \text{if} \quad 1 < x \end{cases}$$

Clearly we have $f'(x) = 0$ iff $x = \frac{1}{2}$.

Also one may easily show that $f'(0)$ and $f'(1)$ do not exist. Therefore the critical points are $\frac{1}{2}$, 0, 1.

Let c be a critical point for $f(x)$. Assume that there exists an interval I around c, that is c is an interior point of I, such that $f(x)$ is increasing to the left

of c and decreasing to the right, then c is a local maximum. This implies that if $f'(x) \geq 0$ for $x \leq c$ (x close to c), and $f'(x) \leq 0$ for $x \geq c$ (x close to c), then c is a local maximum. Note that similarly if $f'(x) \leq 0$ for $x \leq c$ (x close to c), and $f'(x) \geq 0$ for $x \geq c$ (x close to c), then c is a local minimum.

> **First Derivative Test.** If c is a critical point for $f(x)$, such that $f'(x)$ changes its sign as x crosses from the left to the right of c, then c is a local extremum.

Example : Find the local extrema of $f(x) = |x^2 - x|$

Ans. Since the local extrema are critical points, then from the above discussion, the local extrema, if they exist, are among the points $\frac{1}{2}, 0, 1$.

Recall that $f'(x) = \begin{cases} 2x - 1 & \text{if} \quad x < 0 \\ -2x + 1 & \text{if} \quad 0 < x < 1 \\ 2x - 1 & \text{if} \quad 1 < x \end{cases}$

(1) For $x = \frac{1}{2}$, we have $\begin{cases} f'(x) > 0 & \text{if} \quad 0 < x < \frac{1}{2} \\ f'(x) < 0 & \text{if} \quad \frac{1}{2} < x < 1 \end{cases}$

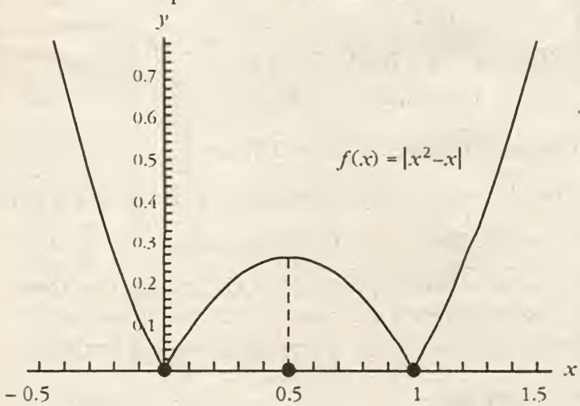
So the critical point $\frac{1}{2}$ is a local maximum.

(2) For $x = 0$, we have $\begin{cases} f'(x) < 0 & \text{if} \quad x < 0 \\ f'(x) > 0 & \text{if} \quad 0 < x < \frac{1}{2} \end{cases}$

So the critical point 0 is a local minimum.

(3) For $x = 1$, we have $\begin{cases} f'(x) < 0 & \text{if} \quad \frac{1}{2} < x < 1 \\ f'(x) > 0 & \text{if} \quad 1 < x \end{cases}$.

So the critical point 1 is a local minimum.

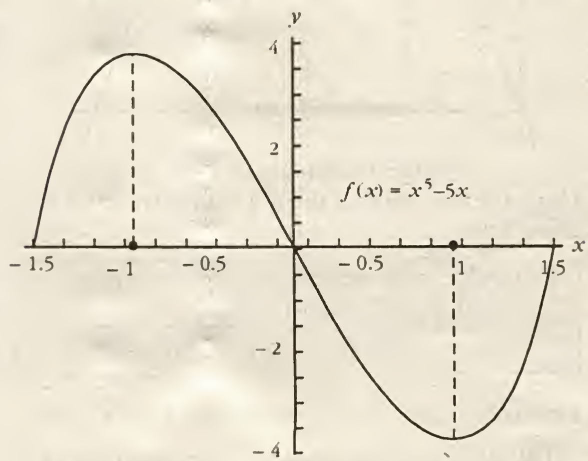


> Let c be a critical point for $f(x)$ such that $f'(c) = 0$.
>
> (i) If $f''(c) > 0$, then $f'(x)$ is increasing in an interval around c. Since $f'(c) = 0$, then $f'(x)$ must be negative to the left of c and positive to the right of c. Therefore, c is a local minimum.
>
> (ii) If $f''(c) < 0$, then $f'(x)$ is decreasing in an interval around c. Since $f'(c) = 0$, then $f'(x)$ must be positive to the left of c and negative to the right of c. Therefore, c is a local maximum. This test is known as the Second-Derivative Test.

Example : Find the local extrema of $f(x) = x^5 - 5x$.

Ans. : First let us find the critical points. Since $f(x)$ is a polynomial function, then $f(x)$ is continuous and differentiable everywhere. So the critical points are the roots of the equation $f'(x) = 0$, that is $5x^4 - 5 = 0$, or equivalently $x^4 - 1 = 0$. Since $x^4 - 1 = (x-1)(x+1)(x^2+1)$, then the critical points are 1 and -1. Since $f''(x) = 20x^3$, then $f''(1) = 20 > 0$ and $f''(-1) = -20 < 0$.

The second-derivative test implies that $x = 1$ is a local minimum and $x = -1$ is a local maximum.



■■

i.e. $\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 2f & 3f & 1f \end{pmatrix}$

$$\begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

Example :

In S_4 let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$, $g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}$

Then show that $fg \neq gf$.

Now we have given two permutations f and $g \in S_4$. To show the $fg \neq gf$ we will determine the two products individually and then show that they are not equal.

$$f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix};\ g = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}.$$

Step 1 : Determine fg by making the first row of g same as that of second row of f and deleting the common rows. i.e.

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ \cancel{3} & \cancel{4} & \cancel{1} & \cancel{2} \end{pmatrix}\begin{pmatrix} \cancel{3} & \cancel{4} & \cancel{1} & \cancel{2} \\ 1 & 3 & 2 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix}$$

Note that this rearrangement in g can be done easily by just writing first row g same as to second row of f and then write their corresponding image in g.

Step 2 : Determining gf by making the first row of f same as the second row of g and deleting the common rows

$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{pmatrix}\begin{pmatrix} 2 & 4 & 1 & 3 \\ 4 & 2 & 3 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$$

Step 3 :

Thus $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 2 & 4 \end{pmatrix} \neq \begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 2 & 3 & 1 \end{pmatrix}$ $\therefore fg \neq gf$

Example :

Let $f = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 4 & 1 & 2 \end{pmatrix}$, $g = \begin{pmatrix} 4 & 1 & 2 & 3 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

Then

$$fg = \begin{pmatrix} 1 & 2 & 3 & 4 \\ \cancel{3} & \cancel{4} & \cancel{1} & \cancel{2} \end{pmatrix}\begin{pmatrix} \cancel{3} & \cancel{4} & \cancel{1} & \cancel{2} \\ 1 & 2 & 3 & 4 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 \end{pmatrix} = I$$

Thus $fg = I$

$\therefore$ If $fg = I$ i.e. product of two permutations is an identity permutation then one is the inverse of another.

i.e. $g = f^{-1}$.

Inverse Permutation : Two permutations $f, g \in S$ such that $fg = gf = I$, identity permutation, are said to be inverse of each other. We write $f^{-1} = g$ or $f = g^{-1}$.

Working Rule : The inverse of a permutation f is given by simply interchanging two rows of f.

Let us know discuss one example :

Compute $a^{-1}ba$ where

$$a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix},\ b = \begin{pmatrix} 3 & 2 & 1 \\ 2 & 1 & 3 \end{pmatrix}.$$

For $a = \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 1 \end{pmatrix}$, $a^{-1} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$.

Note a^{-1} is determined by just interchanging the rows of a.

$$\therefore a^{-1}b = \begin{pmatrix} 2 & 3 & 1 \\ \cancel{1} & \cancel{2} & \cancel{3} \end{pmatrix}\begin{pmatrix} \cancel{1} & \cancel{2} & \cancel{3} \\ 3 & 1 & 2 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 3 & 1 & 2 \end{pmatrix}$$

$$\therefore a^{-1}ba = \begin{pmatrix} 2 & 3 & 1 \\ \cancel{3} & \cancel{1} & \cancel{2} \end{pmatrix}\begin{pmatrix} \cancel{3} & \cancel{1} & \cancel{2} \\ 1 & 2 & 3 \end{pmatrix} = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}$$

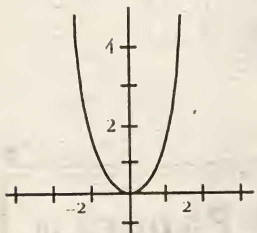
$$\therefore a^{-1}ba = \begin{pmatrix} 2 & 3 & 1 \\ 1 & 2 & 3 \end{pmatrix}.$$

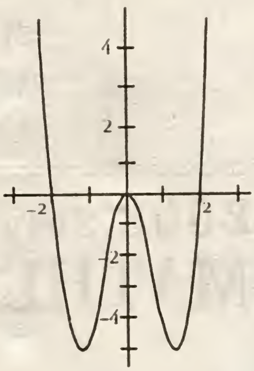
How to

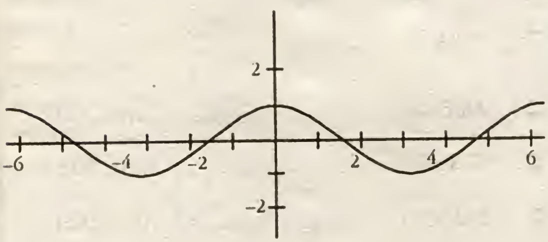
Test a Relation for SYMMETRY

WHEN you are graphing a relation given by an equation it is helpful to find out first whether its graph has any symmetries. The most common symmetries to look for are symmetry with respect to the x and y axes and the origin.

Here are some graphs that are symmetric with respect to the y axis.

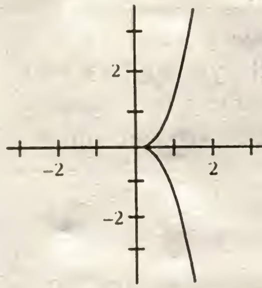


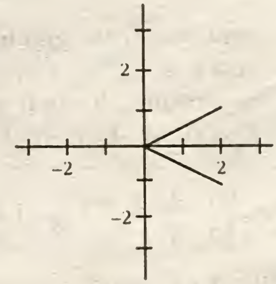


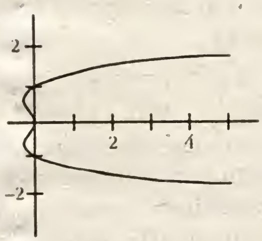


If you knew ahead of time that the graph of a relation was symmetric with respect to the y axis, it would cut your graphing work in half, because once you graphed the part where x was positive, the part to the right of the y axis, you could just reflect across the y axis to get the rest of the graph.

Here are some graphs that are symmetric with respect to the x axis.

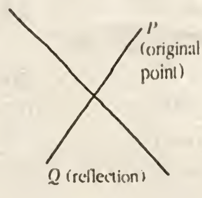




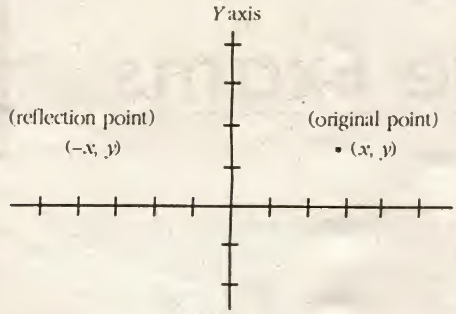


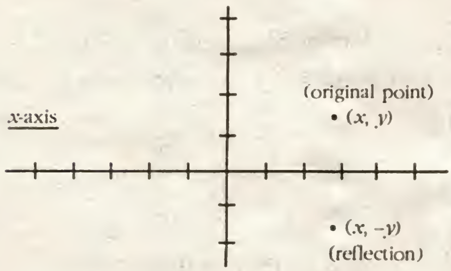
If you knew ahead of time that the graph of a relation was symmetric with respect to the x axis, this would also cut your graphing work in half, because once you graphed the part where y was positive, the part above the x axis, you could just reflect across the x axis to get the rest of the graph.

A useful way to characterize symmetry with respect to a line is that a set of point is symmetric with respect to a line if whenever a point is in it, its reflection across the line will be in it also. The reflection of a point P across a line is the point Q such that the line is perpendicular bisector of the segment PQ. This is just like a real world reflection that a mirror makes. If the line was a mirror, then the reflection is located at the place where the mirror image would be.

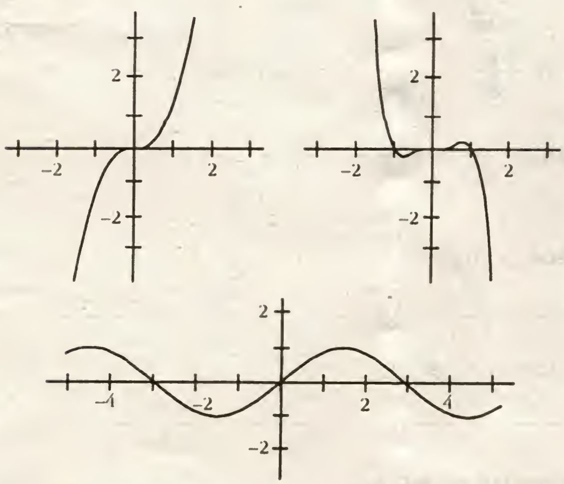


For the special lines, the y- axis and the x-axis there is a very easy way to find the reflection of any point. For the y axis you just change the sign of the x and for the x axis you change the sign of the y.

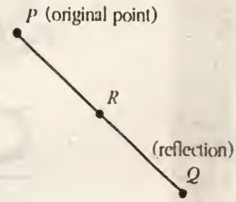




Here are some graphs that are symmetric with respect to the origin.



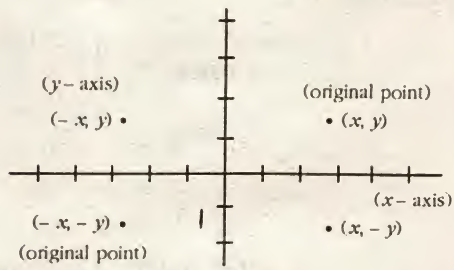
Symmetry with respect to the origin is a bit different from the others, because it is symmetry with respect to a point instead of a line, which is probably not so familiar. In this case what it comes out to mean is that if you know what the graph looks like for positive x's, so again you can cut your graphing work in half by knowing this symmetry ahead of time. The characterization of symmetry with respect to a point also has to do with reflections.



A set of points is symmetric with respect to a point if whenever a given point is in the set, its reflection across that point is also in the set, but the idea of the reflection across a point is probably not so familiar. The reflection of a point P across a point R is the point Q such that R is the midpoint of the segment PQ.

Like with the axes, when the point that you are reflecting across is the origin there is an easy way to find the reflection. This time you change the signs of both the x and the y coordinates.

Here is a picture of all of the reflection together for quick reference.



This characterization will help us find symmetries before we graph. The idea is that for there to be a given symmetry, whenever (x, y) is in the graph, its reflection has to be also. That means that whenever (x, y) satisfies the equation, it's reflection has to satisfy it as well. So to test for a symmetry what you need to do is replace (x, y) with its reflection and see if the new equation you get is equivalent to the original. Putting this together we get the following test.

Replace (x, y) with the following.

y axis	$(-x, y)$
x axis	$(x, -y)$
origin	$(-x, -y)$

If the relation simplifies to what it was originally, then it has that symmetry.

■■

SHORTcuts *for* Competitive Exams

– By M.Shrikant, Visakhapatnam

1. If ${}^nC_{r-1} = 15$, ${}^nC_r = 20$ and ${}^nC_{r+1} = 15$ then $r =$

(a) 3 (b) 2
(c) 1 (d) 4

Soln.: (a) If ${}^nC_{r-1} = a$, ${}^nC_r = b$ and ${}^nC_{r+1} = C$ then

$r = a(b+c)/(b^2 - ac)$

$r = 15(20+15)/(400-225) = 3$.

Similar problems :

2. If ${}^nC_{r-1} = 35$, ${}^nC_r = 35$ and ${}^nC_{r+1} = 21$ then $r =$

(a) 3 (b) 4
(c) 5 (d) 2

3. If ${}^nC_{r-1} = 28$, ${}^nC_r = 56$ and ${}^nC_{r+1} = 70$ then $r =$

(a) 3 (b) 2
(c) 4 (d) 5

4. If ${}^nC_{r-1} = 84$, ${}^nC_r = 126$ and ${}^nC_{r+1} = 126$ then $r =$

(a) 1 (b) 4
(c) 5 (d) 2

5. If ${}^nC_{r-1} = 165$, ${}^nC_r = 330$ and ${}^nC_{r+1} = 462$ then $r =$

(a) 3 (b) 5
(c) 4 (d) none

6. The product of perpendiculars from (– 1, 2) to the pair of lines $2x^2 - 5xy + 2y^2 = 0$ is

(a) 4 (b) 3
(c) 8 (d) 5

Soln. : (a) The product of perpendiculars from (α, β) to the pair of line $ax^2 + 2hxy + by^2 = 0$ is

$$\left|\frac{a\alpha^2 + 2h\alpha\beta + b\beta^2}{\sqrt{(a-b)^2 + 4h^2}}\right|$$

$$\therefore \text{ distance } = \left|\frac{2(-1)^2 - 5(-1)(2) + 2(2)^2}{\sqrt{(2-2)^2 + (-5)^2}}\right| = 4.$$

Similar Problems :

7. The product of perpendiculars from (1, 1) to the pair of lines $x^2 + 4xy + 3y^2 = 0$ is

(a) 3 (b) 1
(c) $\frac{4}{\sqrt{5}}$ (d) $\frac{\sqrt{5}}{4}$

8. The product of perpendiculars from (2, – 1) to the pair of lines $2x^2 + 6xy + y^2 = 0$ is

(a) $\frac{3}{2}$ (b) $\frac{\sqrt{3}}{2}$
(c) $\frac{3}{37}$ (d) $\frac{3}{\sqrt{37}}$

9. The product of perpendiculars from (1, 0) to the pair of lines $x^2 + 2xy + 3y^2 = 0$ is

(a) $\frac{1}{\sqrt{5}}$ (b) $\frac{2}{\sqrt{5}}$
(c) $\frac{1}{2\sqrt{5}}$ (d) none

10. $\frac{{}^{11}C_1}{{}^{11}C_0} + 2\frac{{}^{11}C_2}{{}^{11}C_1} + 3\frac{{}^{11}C_3}{{}^{11}C_2} + \ldots + 11\frac{{}^{11}C_{11}}{{}^{11}C_{10}} =$

(a) 110 (b) 66
(c) 45 d) 2^{10}

Soln.: (b)

$$\frac{{}^nC_1}{{}^nC_0} + 2\frac{{}^nC_2}{{}^nC_1} + 3\frac{{}^nC_3}{{}^nC_2} + \ldots + n\frac{{}^nC_n}{{}^nC_{n-1}} = \frac{n(n+1)}{2}$$

Here $n = 11$

$$\frac{{}^{11}C_1}{{}^{11}C_0} + 2\frac{{}^{11}C_2}{{}^{11}C_1} + \ldots + 11\frac{{}^{11}C_{11}}{{}^{11}C_{10}} = \frac{11(11+1)}{2} = 66.$$

Similar problems

11. $\frac{{}^{12}C_1}{{}^{12}C_0} + 2\frac{{}^{12}C_2}{{}^{12}C_1} + 3\frac{{}^{12}C_3}{{}^{12}C_2} + \ldots\ldots + 12\frac{{}^{12}C_{12}}{{}^{12}C_{11}} =$

(a) 78 (b) 66
(c) 45 (d) 2^{10}

12. $\frac{{}^{13}C_1}{{}^{13}C_0} + 2\frac{{}^{13}C_2}{{}^{13}C_1} + 3\frac{{}^{13}C_3}{{}^{13}C_2} + \ldots\ldots + 13\frac{{}^{13}C_{13}}{{}^{13}C_{12}} =$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10 Mathematical Challenges

– *By : prof. B.L. Sharma, Jaipur*

1. The co-ordinates of the feet of the perpendiculars from the vertices of a triangle on the opposite sides are (20, 25), (8, 16) and (8, 9). Find the co-ordinates of the vertices of the triangle and also, the number of such triangles.

2. Show that the straight lines

$x^2\left(C^2 - A^2r^2\right) + y^2\left(C^2 - B^2r^2\right) - 2ABr^2xy = 0$, where $r^2\left(A^2 + B^2\right) > C^2$ form with the line $Ax + By + C = 0$ isosceles triangles as r varies. In case $r^2 = \dfrac{4C^2}{3\left(A^2 + B^2\right)}$ then the triangle is equilateral.

3. If a circle be described on the line joning the centres of similitude of two given circles as diameter, prove that the tangents drawn from any point on it to the two circles are in the ratio of the corresponding radii.

4. If ABC be any triangle, and $A'B'C'$ be the triangle formed by the polars of the points A, B, C with respect to a circle, so that $B'C'$ is the polar of A, $C'A'$ is the polar of B, and $A'B'$ is the polar of C; prove that the three lines AA', BB', CC' meet in a point.

5. Show that the equation

$$\cos 3\alpha\left(x^3 - 3xy^2\right) + \sin 3\alpha\left(y^3 - 3x^2y\right) + 3a\left(x^2 + y^2\right) - 4a^3 = 0$$

represents three straight lines forming an equilateral triangle.

6. Tangents are drawn to an ellipse at four points, which are such that the normals at those points co-intersect; and four rectangles are constructed each having two adjacent sides along the axes of the ellipse, and one of the tangents for a diagonal. Prove that the distant extremities of the other diagonals lie in one straight line.

7. $ABCD$ is a rectangle circumscribing an ellipse whose foci are S and H; show that the circle ABS or ABH is equal to the auxiliary circle.

8. A circle and a rectangular hyperbola intersect in four points and one of their common chords is a diameter of the hyperbola. Show that the other chord is a diameter of the circle.

9. A circle through the centre of a rectangular hyperbola cuts the curve in the points A, B, C, D. Prove that the circle circumscribing the triangle formed by the tangents at A, B, C pass through the centre of the hyperbola and has its centre at the point on the hyperbola diametrically opposite to D.

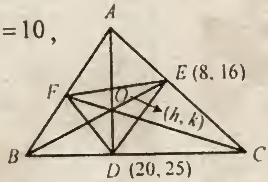
10. If a length PQ be taken in the normal at any point P of an ellipse whose centre is C, equal in length to the semi-diameter which is conjugate to CP, show that Q lies on one or other of two circles.

SOLUTIONS

1. We use the fact that the orthocentre O of the ΔABC is the incentre of the pedal ΔDEF.

$ED = \sqrt{(20-8)^2 + (25-16)^2} = 15$ also, $FD = 20$ and

$EF = 7$. $h = \dfrac{160+140+120}{7+20+15} = 10$,



$k = 15$, O (10, 15).

OE is perpendicular to AC.

Thus the equation of AC

$y - 16 = 2(x - 8)$ or $y - 2x = 0$ ...(1)

Similarly equation of AB.

$y - 9 = -\frac{1}{3}(x - 8)$ or $3y + x - 35 = 0$ ...(2)

Solving (1) and (2), we get A (5, 10)

Similarly equation of BC

$y + x - 45 = 0$ ...(3)

Solving (1) and (3), we get C (15, 30)

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Solving (2) and (3), we get B (50, – 5).

Thus we have co-ordinates of four points.

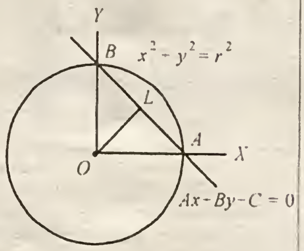
(10, 15), (5, 10), (50, – 5), (15, 30).

Thus there are four triangles formed by taking any three points, out of four, satisfying the condition given in the question.

2. Let us take the circle

$x^2+y^2=r^2$

Making this equation homogeneous with $Ax+By+c=0$, we have



$$x^2+y^2=r^2\left(\frac{Ax+By}{-C}\right)$$

or

$$x^2\left(C^2-A^2r^2\right)+y^2\left(C^2-B^2r^2\right)-2ABr^2xy=0 \quad ...(1)$$

$OL=\dfrac{C}{\sqrt{A^2+B^2}} \Rightarrow$ if $r^2\left(A^2+B^2\right)>C$, then the circle intersects the line $Ax+By+C=0$ at two points A and B. Also $OA=OB=r$. Hence OAB is an isoceles triangle. In case ΔAOB is equilateral then $\angle AOB=60°$.

$$\therefore\ \tan 60=\sqrt{3}=\frac{2\sqrt{A^2B^2r^4-(C^2-B^2r^2)(C^2-A^2r^2)}}{2C^2-r^2(A^2+B^2)}$$

or $\left[r^2\left(A^2+B^2\right)-4C^2\right]\left[3r^2\left(A^2+B^2\right)-4C^2\right]=0$

If $r^2=\dfrac{4C^2}{3(A^2+B^2)}$, then straight lines.

$x^2\left(A^2-3B^2\right)+y^2\left(B^2-3A^2\right)+8ABxy=0$ are the sides of an equilateral triangle.

If $r^2=\dfrac{4C^2}{A^2C^2}$ then the straight lines

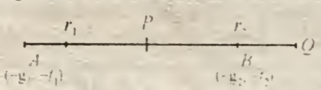
$x^2\left(B^2-3A^2\right)+y^2\left(A^2-3B^2\right)-8ABxy=0$ are the sides of a isosceles triangle i.e. $\angle AOB=120°$.

3. Let the two circles be

$$x^2+y^2+2g_1x+2f_1y+c_1=0 \quad ...(1)$$

$$x^2+y^2+2g_2x+2f_2y+c_2=0 \quad ...(2)$$

Now P and Q are the centre of similitude.



Hence $P\left(\dfrac{-g_1r_2-r_1g_2}{r_1+r_2}, \dfrac{-f_1r_2-f_2r_1}{r_1+r_2}\right)$

and $Q\left(\dfrac{g_1r_2-r_1g_2}{r_1-r_2}, \dfrac{f_1r_2-f_2r_1}{r_1-r_2}\right)$.

The equation of the circle having PQ as diameter is

$$\left(x+\frac{g_1r_2+r_1g_2}{r_1+r_2}\right)\left(x-\frac{g_1r_2-r_1g_2}{r_1-r_2}\right)+\left(y+\frac{f_1r_2+f_2r_1}{r_1+r_2}\right)\left(y-\frac{f_1r_2-f_2r_1}{r_1-r_2}\right)=0$$

or $x^2+y^2+2\left(\dfrac{r_1^2g_2-g_1r_2^2}{r_1^2-r_2^2}\right)x+2\left(\dfrac{f_2r_1^2-f_1r_2^2}{r_1^2-r_2^2}\right)y$

$$-\frac{g_1^2r_2^2-g_2^2r_1^2+f_1^2r_2^2-f_2r_1^2}{r_1^2-r_2^2}=0 \text{ or}$$

$$x^2+y^2+2\left(\frac{r_1^2g_2-g_1r_2^2}{r_1^2-r_2^2}\right)+2\left(\frac{f_2r_1^2-f_1r_2^2}{r_1^2-r_2^2}\right)y+\frac{c_2r_1^2-c_1r_2^2}{r_1^2-r_2^2}=0,$$

using $r_1^2=g_1^2+f_1^2-c_1,\ r_2^2=g_2^2+f_2^2-c_2$ ...(1)

(1) can be written as

$$\left(x^2+y^2+2g_1x+2f_1y+c_1\right)r_2^2=\left(x^2+y^2+2g_2x+2f_2y+c_2\right)r_1^2$$

Let (α, β) be any point on the circle.

Thus $\dfrac{\alpha^2+\beta^2+2g_1\alpha+2f_1\beta+c_1}{\alpha^2+\beta^2+2g_2\alpha+2f_2\beta+c_2}=\dfrac{r_1^2}{r_2^2}=\dfrac{L_1^2}{L_2^2}$,

where L_1 and L_2 are the lengths of the tangents from any point on the circle to (1) and (2) respectively.

Hence $\dfrac{L_1}{L_2}=\dfrac{r_1}{r_2}$.

4. Let the equation of the circle be

$$x^2+y^2=a^2 \quad ...(1)$$

Let $A(x_1,y_1)$, $B(x_2,y_2)$ and $C(x_3,y_3)$ be co-ordinates.

The equations of the lines $B'C', C'A', A'B'$ will be

$$xx_1+yy_1=a^2 \quad ...(2)$$

$$xx_2+yy_2=a^2 \quad ...(3)$$

$$xx_3+yy_3=a^2 \quad ...(4)$$

The line AA' is the intersection of (3) & (4).

Thus $xx_2+yy_2-a^2+\lambda\left(xx_3+yy_3-a^2\right)=0$,

it passes through A.

Thus $\lambda=-\left(x_1x_2+y_1y_2-a^2\right)/\left(x_1x_3+y_1y_3-a^2\right)$.

Thus equation of AA' is

$$\left(xx_2+yy_2-a^2\right)\left(x_1x_3+y_1y_3-a^2\right)$$
$$-\left(xx_3+yy_3-a^2\right)\left(x_1x_2+y_1y_2-a^2\right)=0 \qquad \text{...(5)}$$

The other equations are written down by symmetry.

$$\left(xx_3+yy_3-a^2\right)\left(x_2x_1+y_2y_1-a^2\right)$$
$$-\left(xx_1+yy_1-a^2\right)\left(x_2x_3+y_2y_3-a^2\right)=0 \qquad \text{...(6)}$$

and $\left(xx_1+yy_1-a^2\right)\left(x_3x_2+y_3y_2-a^2\right)$

$$-\left(xx_2+yy_2-a^2\right)\left(x_3x_1+y_3y_1-a^2\right)=0 \qquad \text{...(7)}$$

Since the equation (5), (6) and (7), when added together vanish identically. Hence the lines AA', BB', CC' meet in a point.

5. Let us consider in equation

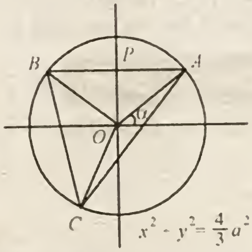
$$\cos 3\alpha\left(x^3-3xy^2\right)+\sin 3\alpha\left(y^3-3x^2y\right)$$
$$+\lambda\left[x^2+y^2-\frac{4}{3}a^2\right]=0, \qquad \text{...(1)}$$

where λ is constant.

(1) represents a curve passing through the points A, B, C for any value of λ.

$$\cos 3\alpha\left(x^3-3xy^2\right)+\sin 3\alpha\left(y^3-3x^2y\right)=0 \qquad \text{...(2)}$$

is the equation of the lines OA, OB and OC.



Put $\dfrac{y}{x}=m$ in (2), we have

$$\tan 3\alpha=\frac{3m-m^3}{1-3m^2}=\tan 3\theta$$

where $m=\tan\theta$ = slope of the line.

$\theta=\alpha, \alpha+60, \alpha+120°$ are the angles which the lines OA, OC and OB make with x-axis.

Hence ΔABC is equilateral.

Now we show that for $\lambda=3a$, (1) represents the straight lines AB, BC and CA.

$$A\left(\frac{2a}{\sqrt{3}}\cos\alpha, \frac{2a}{\sqrt{3}}\sin\alpha\right),$$

$$B\left(\frac{2a}{\sqrt{3}}\cos(\alpha+120), \frac{2a}{\sqrt{3}}\sin(\alpha+120)\right)$$

The equation of AB is

$$y-\frac{2a}{\sqrt{3}}\sin\alpha=\cot(\alpha+60)\left(x-\frac{2a}{\sqrt{3}}\cos\alpha\right).$$

This lines meets the y-axis in the point.

$$P\left(0, \frac{a}{\sin(\alpha+60)}\right)$$

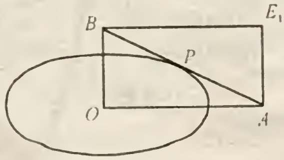
It is enough to show that the $P\left(0, \dfrac{a}{\sin(\alpha+60)}\right)$ satisfy (1), when $\lambda=3a$. Putting the value, we have

$$\sin 3\alpha\frac{a^3}{\sin^3(\alpha+60)}+3\frac{a^3}{\sin^2(\alpha+60)}-4a^3=0$$

$\sin 3\alpha+3\sin(\alpha+60)-4\sin^3(\alpha+60)=0$ or

$\sin 3\alpha+\sin(3\alpha+180)=0$ or $\sin 3\alpha-\sin 3\alpha=0$.

6. Let the co-ordinate of P be $(a\cos\alpha, b\sin\alpha)$. The equation of AB is



$$\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1.$$

Hence $A\left(\dfrac{a}{\cos\alpha}, 0\right)$ and $B\left(0, \dfrac{b}{\sin\alpha}\right)$.

$\therefore$ $E_1\left(\dfrac{a}{\cos\alpha}, \dfrac{b}{\sin\alpha}\right)$. Similarly the co-ordinates of other extremeties of the diagonals are

$$E_2\left(\frac{a}{\cos\beta}, \frac{b}{\sin\beta}\right), E_3\left(\frac{a}{\cos\gamma}, \frac{b}{\sin\gamma}\right), E_4\left(\frac{a}{\cos\delta}, \frac{b}{\sin\delta}\right).$$

We have to prove that E_1, E_2, E_3, E_4 are collinear. E_1, E_2, E_3 are vertices of a Δ (If they are not collinear). Thus

$$\text{Area}=\frac{1}{2}\begin{vmatrix}\frac{a}{\cos\alpha} & \frac{b}{\sin\alpha} & 1\\ \frac{a}{\cos\beta} & \frac{b}{\sin\beta} & 1\\ \frac{a}{\cos\gamma} & \frac{b}{\sin\gamma} & 1\end{vmatrix}$$

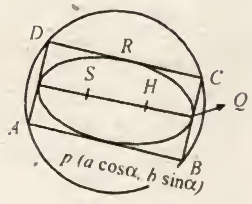
$$=\frac{ab}{2}\begin{vmatrix}\sin\alpha & \cos\alpha & \sin 2\alpha\\ \sin\beta & \cos\beta & \sin 2\beta\\ \sin\gamma & \cos\gamma & \sin 2\gamma\end{vmatrix} \qquad \text{...(1}$$

Since normals at P, Q, R, S intersect, thus

$$\begin{vmatrix} \sin\alpha & \cos\alpha & \sin 2\alpha \\ \sin\beta & \cos\beta & \sin 2\beta \\ \sin\gamma & \cos\gamma & \sin 2\gamma \end{vmatrix} = 0 \Rightarrow \text{ the points } E_1, E_2 \text{ and } E_3$$

are collinear. Similarly we can show that E_1, E_2 and E_4 are collinear.

7. $ABCD$ is a rectangle. Thus the points A, B, C, D are the points of intersection of two perpendicular tangents to the ellipse



$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Hence A, B, C, D, lie on the director circle $x^2+y^2=a^2+b^2$

The equation of AB is $\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha=1$.

The equation of a circle passing through A and B is $x^2+y^2-a^2-b^2+\lambda\left(\frac{x}{a}\cos\alpha+\frac{y}{b}\sin\alpha-1\right)=0$, it passes through H $(ea, 0)$. Thus

$$\lambda=\frac{-2b^2}{1-e\cos\alpha}$$

Hence the equation of the circle ABH is

$$x^2+y^2-\frac{2b^2\cos\alpha}{a(1-e\cos\alpha)}x-\frac{2b^2\sin\alpha}{b(1-e\cos\alpha)}y+\frac{2b^2}{1-e\cos\alpha}-a^2-b^2=0$$

R^2 = (radius of the circle)^2

$$=\frac{b^4\cos^2\alpha}{a^2(1-e\cos\alpha)^2}+\frac{b^4\sin^2\alpha}{b^2(1-e\cos\alpha)^2}+a^2+b^2-\frac{2b^2}{1-e\cos\alpha}$$

$$=\frac{b^4\cos^2\alpha}{a^2(1-e\cos\alpha)^2}+\frac{b^4\sin^2\alpha}{b^2(1-e\cos\alpha)^2}+a^2+b^2-\frac{2b^2}{1-e\cos\alpha}$$

$$=b^2\left[\frac{b^2\cos^2\alpha+a^2\sin^2\alpha}{a^2(1-e\cos\alpha)^2}+1-\frac{2}{1-e\cos\alpha}\right]+a^2$$

$$=b^2\left[\frac{b^2\cos^2\alpha+a^2\left(1-\cos^2\alpha\right)}{a^2(1-e\cos\alpha)^2}-\frac{1+e\cos\alpha}{1-e\cos\alpha}\right]+a^2$$

$$=b^2\left[\frac{a^2\left(1-e^2\right)\cos^2\alpha+a^2\left(1-\cos^2\alpha\right)}{a^2(1-e\cos\alpha)^2}-\frac{1+e\cos\alpha}{1-e\cos\alpha}\right]+a^2$$

$= a^2$.

8. Let the equation of the circle be

$x^2+y^2+2gx+2fy+c=0$.

Rectangular hyperbola $xy = k^2$.

The equation

$$x^2+y^2+2gx+2fy+c+\lambda\left(xy-k^2\right)=0 \qquad ...(1)$$

represents two straight lines for some value of λ, then (1) represents the common chords of the circle and hyperbola. It is given that one common chord is a diameter of the hyperbola $\Rightarrow$ that (1) passes through the origin.

Thus $\lambda=\frac{c}{k^2}$

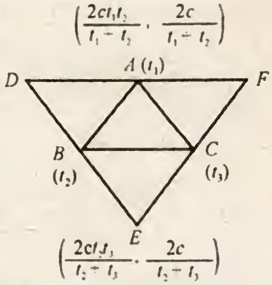
$$\therefore\ x^2+y^2+2gx+2fy+\frac{c}{k^2}xy=0 \qquad ...(2)$$

since (2) represents straight lines, thus

$$g^2+f^2-2\frac{c}{k^2}gh=0 \qquad ...(3)$$

The point $(-g, -f)$ satisfies (2) because of (3).

9. The equation of the circle passing through the centre of the rectangular hyperbola $xy = c^2$ is



$x^2+y^2+2gx+2fy=0$ ...(1)

Now $(ct, c/t)$ is any point on the hyperbola lie on (1).

$\therefore\ t^4-2\frac{g}{c}t^3-\frac{2f}{c}t+1=0$ ...(2)

and t_1, t_2, t_3, t_4 are the roots of (2)

$$t_1t_2+t_1t_4+t_2t_4+t_3\left(t_1+t_2+t_4\right)=0 \qquad ...(3)$$

$$t_1t_2t_3t_4=0 \qquad ...(4)$$

Now (3) can be written as

$$t_1t_2+t_4\left(t_1+t_2\right)+\frac{1}{t_1t_2t_4}\left(t_1+t_2+t_3\right)=0 \qquad ...(5)$$

(by using (4))

Also, $t_1t_3+t_4\left(t_1+t_3\right)+\frac{1}{t_1t_3t_4}\left(t_1+t_2+t_3\right)=0$ & ...(6)

$$t_2t_3+t_4\left(t_2+t_3\right)+\frac{1}{t_2t_3t_4}\left(t_1+t_2+t_3\right)=0 \qquad ...(7)$$

The co-ordinates of

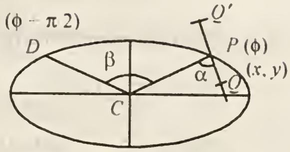
$D\left(\frac{2ct_1t_2}{t_1+t_2}, \frac{2c}{t_1+t_2}\right)$, $E\left(\frac{2ct_2t_3}{t_2+t_3}, \frac{2c}{t_2+t_3}\right)$ and $F\left(\frac{2ct_1t_3}{t_1+t_3}, \frac{2c}{t_1+t_3}\right)$.

Take $x=\frac{2ct_1t_2}{t_1+t_2}$, $y=\frac{2c}{t_1+t_2} \Rightarrow t_1+t_2=\frac{2c}{y}$ and $t_1t_2=\frac{x}{y}$. Putting these values in (5), we have

$$x^2+y^2+2ct_4x+\frac{2c}{t_4}y=0 \quad ...(8).$$

(8) is an equation of the circle, passing through D, E, F, centre $\left(-ct_4, -\frac{c}{t_4}\right)$.

10. PQ is a normal at P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. Take the points Q and Q' on the normal such that $PQ = CD$.



$$\tan\beta=\frac{\frac{b}{a}\tan\phi+\frac{b}{a}\cot\phi}{1-\frac{b^2}{a^2}}=\frac{ab}{a^2-b^2}\,\frac{1+\tan^2\phi}{\tan\phi}$$

$$\tan\alpha=\frac{\frac{a}{b}\tan\phi-\frac{b}{a}\tan\phi}{1+\tan^2\phi}=\frac{a^2-b^2}{ab}\,\frac{\tan\phi}{1+\tan^2\phi}$$

$\therefore\ \tan\alpha\tan\beta=1$ or $\cos(\alpha+\beta)=0$

$$\Rightarrow\ \alpha=\frac{\pi}{2}-\beta \quad ...(1)$$

$$\cos\alpha=\sin\beta \quad ...(2)$$

We know that $CP\cdot CD=\frac{ab}{\sin\beta}$ ...(3)

$$CP^2+CD^2=a^2+b^2 \quad ...(4)$$

Now in ΔCPQ

$$x^2+y^2=CP^2+CD^2-2CP\cdot CD\cos\alpha$$

$$=a^2+b^2-2\frac{ab}{\sin\beta}\cos\alpha=(a+b)^2$$

(using (2), (3), (4))

Hence the locus of Q is $x^2+y^2=(a+b)^2$.

Similarly the locus of Q' is $x^2+y^2=(a-b)^2$.

■■

ANSWERS : NDA-2003

1.	(c)	2.	(d)	3.	(b)	4.	(b)
5.	(c)	6.	(c)	7.	(a)	8.	(c)
9.	(c)	10.	(c)	11.	(b)	12.	(d)
13.	(b)	14.	(b)	15.	(a)	16.	(a)
17.	(d)	18.	(d)	19.	(d)	20.	(a)
21.	(a)	22.	(a)	23.	(d)	24.	(c)
25.	(a)	26.	(c)	27.	(c)	28.	(b)
29.	(a)	30.	(a)	31.	(b)	32.	(c)
33.	(c)	34.	(a)	35.	(b)	36.	(a)
37.	(c)	38.	(d)	39.	(a)	40.	(d)
41.	(b)	42.	(c)	43.	(a)	44.	(c)
45.	(a)	46.	(a)	47.	(b)	48.	(a)
49.	(a)	50.	(c)	51.	(c)	52.	(a)
53.	(c)	54.	(d)	55.	(a)	56.	(a)
57.	(d)	58.	(b)	59.	(a)	60.	(a)
61.	(c)	62.	(c)	63.	(b)	64.	(d)
65.	(b)	66.	(d)	67.	(b)	68.	(b)
69.	(a)	70.	(d)	71.	(c)	72.	(b)
73.	(c)	74.	(c)	75.	(d)	76.	(a)
77.	(d)	78.	(a)	79.	(a)	80.	(d)
81.	(b)	82.	(b)	83.	(d)	84.	(d)
85.	(a)	86.	(d)	87.	(a)	88.	(d)
89.	(c)	90.	(c)	91.	(b)	92.	(d)
93.	(b)	94.	(b)	95.	(a)	96.	(a)
97.	(c)	98.	(d)	99.	(d)	100.	(c)
101.	(c)	102.	(c)	103.	(b)	104.	(c)
105.	(a)	106.	(d)	107.	(b)	108.	(c)
109.	(c)	110.	(a)	111.	(b)	112.	(a)
113.	(a)	114.	(a)	115.	(a)	116.	(b)
117.	(d)	118.	(b)	119.	(b)	120.	(d)

EAXM ALERT !! AIEEE 2004

The CBSE will be conducting the third All India Engineering / Pharmacy Architecture Entrance Examination (AIEEE) for admission to degree level courses in Engineering, Pharmacy and Architecture for the year 2004.

Date of exam : 9th & 10th May, 2004

The Information Bulletin containing the Application Form and other details will be available for sale from **01.12.2003** to **12.01.2004** from designated branches of Syndicate Bank/NITs/ Deemed Universities/ Institutions and the Regional Offices of CBSE.

The last date for receipt of complete applications in CBSE will be **15.01.2004**.

For further details refer to Times of India - 17/11/2003 and Hindustan Times - 15/11/2003.

Now, probability of getting the tickets with numbers 1 and 2 in the selected lot of m tickets.

$= P_r$ (Number one on the first draw) $\times P_r$ (Number two in the remaining $(m-1)$ draws) $+ P_r$ (Number two on the first draw) $\times P_r$ (Number one in the remaining $(m-1)$ draws $+ P_r$ (Neither one nor two on the first draw) $\times P_r$ {1 & 2 in the remaining $(m-1)$ draws)

$$= \frac{2}{n(n+1)} \cdot \frac{^{m-1}C_1}{^{n-1}C_1} + \frac{4}{n(n+1)} \frac{^{m-1}C_1}{^{n-1}C_1} + \left\{1 - \frac{2}{n(n+1)} - \frac{4}{n(n+1)}\right\} \frac{^{m-1}C_2}{^{n-1}C_2}$$

$$= \frac{2}{n(n+1)}\left(\frac{m-1}{n-1}\right) + \frac{4}{n(n+1)}\left(\frac{m-1}{n-1}\right) + \frac{(n^2+n-6)}{n(n+1)} \cdot \frac{(m-1)(m-2)}{(n-1)(n-2)}$$

$$= \frac{(m-1)}{n(n+1)(n-1)}[2+4+(n+3)(m-2)]$$

$$= \frac{(m-1)}{n(n^2-1)}[mn+3m-2n].$$

9. Since z is satisfying

$$|z-8-6i| + |z-14-6i| = 10 \quad ...(1)$$

therefore, locus of z is ellipse with foci (8, 6) and (14, 6) and length of its major axis 10. Thus cartesian form of (1) is

$$\frac{(x-11)^2}{25} + \frac{(y-6)^2}{16} = 1$$

$\Rightarrow 16x^2 + 25y^2 - 352x - 300y + 2436 = 0$ ...(2)

Equation of pair of tangents drawn from origin to (2) is

$$2436 \times (16x^2 + 25y^2 - 352x - 300y + 2436) = 4(88x + 75y + 1218)^2$$

$$\Rightarrow 609\,(16x^2 + 25y^2 - 352x - 300y + 2436) = (88x + 75y + 1218)^2$$

$\Rightarrow 10x^2 - 66xy + 48y^2 = 0$ ...(3)

Since, maximum and minimum arguments of z are the slopes of lines given by (3)

Therefore $\max\{\arg(z)\} = \tan^{-1}\frac{33+\sqrt{609}}{48}$, and

$\min\{\arg(z)\} = \tan^{-1}\frac{33-\sqrt{609}}{48}$.

10. $$\Delta(m,p) = \begin{vmatrix} ^{m}C_p & ^{m}C_{p+1} & ^{m}C_{p+2} \\ ^{m+1}C_p & ^{m+1}C_{p+1} & ^{m+1}C_{p+2} \\ ^{m+2}C_p & ^{m+2}C_{p+1} & ^{m+2}C_{p+2} \end{vmatrix}$$

$$= \frac{1}{p(p+1)(p+2)} \begin{vmatrix} p \cdot {}^{m}C_p & (p+1)\cdot {}^{m}C_{p+1} & (p+2)\cdot {}^{m}C_{p+2} \\ p \cdot {}^{m+1}C_p & (p+1)\cdot {}^{m+1}C_{p+1} & (p+2)\cdot {}^{m+1}C_{p+2} \\ p \cdot {}^{m+2}C_p & (p+1)\cdot {}^{m+2}C_{p+1} & (p+2)\cdot {}^{m+2}C_{p+2} \end{vmatrix}$$

$$= \frac{1}{p(p+1)(p+2)} \begin{vmatrix} m \cdot {}^{m-1}C_{p-1} & m \cdot {}^{m-1}C_p & m \cdot {}^{m-1}C_{p+1} \\ (m+1)\cdot {}^{m}C_{p-1} & (m+1)\cdot {}^{m}C_p & (m+1)\cdot {}^{m}C_{p+1} \\ (m+2)\cdot {}^{m+1}C_{p-1} & (m+2)\cdot {}^{m+1}C_p & (m+2)\cdot {}^{m+1}C_{p+1} \end{vmatrix}$$

$$= \frac{m(m+1)(m+2)}{p(p+1)(p+2)}\Delta(m-1, p-1)$$

$$= \frac{^{m+2}C_3}{^{p+2}C_3}\Delta((m-1),\ (p-1)) = \frac{^{m+2}C_3}{^{p+2}C_3}\Delta((m-1),\ (p-1)).$$

CALCULUS

10 Mathematical Challenges

– By : Prabir Paul, West Bengal

1. Find the domain of definition of the function in the region $[-\pi, 2\pi]$

$$f(x)=\left(|x-1|-[x]\right)^{-\frac{1}{2}}+\text{cosec}^{-1}([\sin x])+\sin^{-1}\left(1+\sqrt{[|\sin x|]}\right)$$

(where [ ] denotes the greatest integer function).

2. Evaluate the limit :

$$\lim_{x\to 0}\left\{\lim_{n\to\infty}\frac{\left[1^2(\sin x)^x\right]+\left[2^2(\sin x)^x\right]+\ldots+\left[n^2(\sin x)^x\right]}{n^3}\right\}$$

(where [ ] denotes the greatest integer function).

3. Draw the graph of $y=\log_e[\sin x]$, [ ] denotes the greatest integer function.

4. If $f(x)$ be a real valued and differentiable function on R and $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)}$. Show that, $\dfrac{\tan^{-1}f(x)}{x}$ is constant.

5. $f(x)=\int|x|\,dx$ and $f(0)=0$ then, draw the graph of $f(x)$.

6. $\displaystyle\int_{-1}^{1}\frac{dx}{x^2}=\left[-\frac{1}{x}\right]_{-1}^{1}=-2$, what is wrong with the integeral ?

7. Check the continuity of the function

$$f(x)=\underset{t\to\infty}{lt}\left[\frac{(1+\sin\pi x)^t-1}{(1+\sin\pi x)^t+1}\right] \text{ at } x = 0, 1, 2, 3, \ldots..$$

8. Evalute : $\displaystyle\int\frac{\left(\sin^3\frac{x}{2}\right)\left(\sec\frac{x}{2}\right)dx}{\left[\cos^3 x+\cos^2 x+\cos x\right]^{\frac{1}{2}}}$.

9. Using calculus show that, $\dfrac{\log 5}{\log 6}>\dfrac{5}{(37)^{\frac{1}{2}}}$.

10. Evaluate the integral :

$$\int_0^4\frac{\sqrt{x}\,dx}{\left(1+\sin\{[x]\}\right)^5}+\int_0^{\frac{\pi}{4}}\sin(x-[x])\,d\left(x-[x]^5\right).$$

(where $[x]$ and $\{x\}$ denote respectively the integral and fractional part of x).

SOLUTIONS

1. $f(x)=\left(|x-1|-[x]\right)^{-\frac{1}{2}}$

$$+\text{cosec}^{-1}([\sin x])+\sin^{-1}\left(1+\sqrt{[|\sin x|]}\right).$$

obviously $f(x)$ is defined when

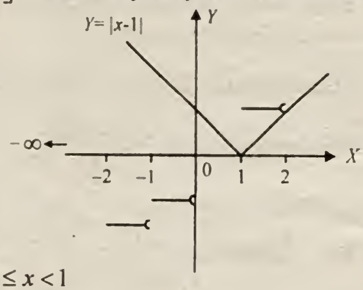
(i) $\left(|x-1|-[x]\right)^{-1/2}$ is defined.

(ii) $\text{cosec}^{-1}([\sin x])$ is defined.

(iii) $\sin^{-1}\left(1+\sqrt{[|\sin x|]}\right)$ is defined.

Now, (i) is defined when $[x]-|x-1|<0$

i.e. when $|x-1|>[x]$. This inequality can be solved graphically. From graph it is obvious that when $x\in(-\infty, 1)$ the condition $|x-1|>[x]$ is satisfied but as



$x\in[-\pi, 2\pi]$ $\therefore$ $-\pi\le x<1$

i.e. $x\in[-\pi, 1)$.

i) We know that $\text{cosec}^{-1}\theta$ exists when $|\theta|\ge 1$

e. when $\theta\ge 1$ or $\theta\le -1$.

o, clearly $\text{cosec}^{-1}([\sin x])$ exists when either

$\sin x]\ge 1$ or $[\sin x]\le -1$ when $[\sin x]\ge 1$

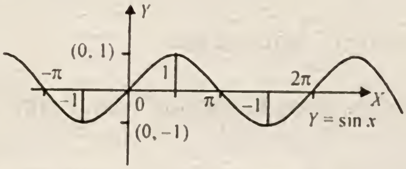
$\sin x\ge 1$, but for any real x, $\sin x\le 1$

$\sin x=1$ or $x=\frac{\pi}{2}$ $[\because\ x\in[-\pi, 2\pi]]$

d if $[\sin x]\le -1$

, $\sin x<0$ but $-1\le\sin x$, $\therefore\ -1\le\sin x<0$

, $x\in(\pi, 2\pi)$ $[\because\ x\in[-\pi, 2\pi]]$ and $x\in(-\pi, 0)$



i) For any real x, $|\sin x|\le 1$

en $|\sin x|<1$, $[|\sin x|]=0$ $\therefore\ \sqrt{[|\sin x|]}=0$

en $|\sin x|=1$ (i.e. $\sin x=\pm 1$) $\therefore\ [|\sin x|]=1$

t if $[|\sin x|]=1$, $\sqrt{[|\sin x|]}=1$,

en $\sin^{-1}(1+1)=\sin^{-1}2$ does not exist as $\sin^{-1}u$

ists only when $-1\le u\le 1$ $\therefore\ \sin^{-1}\left(1+\sqrt{[|\sin x|]}\right)$

ists for all real x except $|\sin x|=1$.

except $x=\pm\frac{\pi}{2}, \frac{3\pi}{2}$, as $x\in[-\pi, 2\pi]$

so, the domain of definition of $f(x)$ satisfying all

nditions is $-\pi<x<-\frac{\pi}{2}, -\frac{\pi}{2}<x<0$

terms set, $x\in\left(-\pi, \frac{-\pi}{2}\right)\cup\left(\frac{-\pi}{2}, 0\right)$)

$$0\left\{\underset{n\to\infty}{Lt}\frac{[1^2(\sin)^x]+[2^2(\sin x)^x]+\ldots+[n^2(\sin x)^x]}{n^3}\right\}$$

iously $x-1<[x]\le x$

$$\therefore\ 1^2(\sin x)^x-1<\left[1^2(\sin x)^x\right]\le 1^2(\sin x)^x$$

$$2^2(\sin x)^x-1<\left[2^2(\sin x)^x\right]\le 2^2(\sin x)^x$$

$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$

$$n^2(\sin x)^x-1<\left[n^2(\sin x)^x\right]\le n^2(\sin x)^x$$

Adding we get,

$$\left(1^2+2^2+3^2+\cdots+n^2\right)(\sin x)^x-(1+1+1+\cdots n\text{ terms})$$

$$<\sum_{r=1}^{n}\left[r^2(\sin x)^x\right]<(\sin x)^x\sum n^2$$

or, $$\frac{n(n+1)(2n+1)}{6n^3}(\sin x)^x-\frac{n}{n^3}<\sum_{n=1}^{n}\frac{\left[n^2(\sin x)^x\right]}{n^3}$$

$$<(\sin x)^x\frac{n(n+1)(2n+1)}{6n^3}$$

or, $$\frac{n(n+1)(2n+1)}{6n^3}(\sin x)^x-\frac{1}{n^2}<\sum_{n=1}^{n}\frac{\left[n^2(\sin x)^x\right]}{n^3}$$

$$<(\sin x)^x\frac{n(n+1)(2n+1)}{6n^3}$$

or, $$\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)(\sin x)^x-\frac{1}{n^2}<\frac{\sum\left[n^2(\sin x)^x\right]}{n^3}$$

$$<\frac{(\sin x)^x}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)$$

or, $$\underset{n\to\infty}{Lt}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)(\sin x)^x-\underset{n\to\infty}{Lt}\frac{1}{n^2}$$

$$<\underset{n\to\infty}{Lt}\frac{\sum\left[n^2(\sin x)^x\right]}{n^3}<\underset{n\to\infty}{Lt}\frac{(\sin x)^x}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right).$$

Now, $$\underset{n\to\infty}{Lt}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)-\frac{1}{n^2}=\frac{1}{6}(1+0)(2+0)-0=\frac{1}{3}$$

$$\underset{n\to\infty}{Lt}\frac{1}{6}\left(1+\frac{1}{n}\right)\left(2+\frac{1}{n}\right)=\frac{1}{3}$$

$$\therefore\ \frac{1}{3}(\sin x)^x<\underset{n\to\infty}{Lt}\frac{\sum_{n=1}^{n}\left[n^2(\sin x)^x\right]}{n^3}<\frac{1}{3}(\sin x)^x$$

using sandwitch theorem,

$$\underset{n\to\infty}{Lt}\frac{\sum_{n=1}^{n}\left[n^2(\sin x)^x\right]}{n^3}=\frac{1}{3}(\sin x)^x$$

Now, $$\underset{x\to 0}{Lt}\left(\underset{n\to\infty}{Lt}\frac{\sum_{1}^{n}\left[n^2(\sin x)^x\right]}{n^3}\right)=\underset{x\to 0}{Lt}\frac{1}{3}(\sin x)^x$$

$$=\frac{1}{3}e^{\underset{x\to 0}{Lt}\,x\log(\sin x)}\qquad=\frac{1}{3}e^{\underset{x\to 0}{Lt}\frac{\log(\sin x)}{\frac{1}{x}}}$$

$$=\frac{1}{3}e^{\underset{x\to 0}{Lt}\frac{\cot x}{\frac{-1}{x^2}}}=\frac{1}{3}e^{\underset{x\to 0}{Lt}\left(-x^2\cot x\right)}$$

$$=\frac{1}{3}e^{\underset{x\to 0}{Lt}-x^2\times\frac{1}{\tan x}}=\frac{1}{3}e^{\underset{x\to 0}{Lt}\frac{-x}{\tan x}\times\underset{x\to 0}{Lt}x}$$

$$=\frac{1}{3}e^{-\left(\underset{x\to 0}{Lt}\frac{x}{\tan x}\right)\times\left(\underset{x\to 0}{Lt}x\right)}=\frac{1}{3}e^{-1\times 0}=\frac{1}{3}e^{0}=\frac{1}{3}.$$

$$\left[\because\ \underset{x\to 0}{Lt}\frac{x}{\tan x}=\underset{x\to 0}{Lt}\frac{x}{\sin x}\cdot\left(\underset{x\to 0}{Lt}\cos x\right)\right.$$

$$\left.=\frac{1}{\underset{x\to 0}{Lt}\frac{\sin x}{x}}\times\underset{x\to 0}{Lt}\cos x=\frac{1}{1}\times 1=1\right].$$

$$\therefore\ \underset{x\to 0}{Lt}\left\{\underset{n\to\infty}{Lt}\frac{\left[\begin{array}{r}1^2(\sin x)^x+[2^2(\sin x)^x]+[3^2(\sin x)^x]\\+\ldots+[n^2(\sin x)^x]\end{array}\right]}{n^3}\right\}$$

$$=\frac{1}{3}.$$

3. $y=\log_e[\sin x]$ ...(1)

Let us at first investigate the real values of x for which (1) is defined i.e. for conic (1) can assume real values.

We know that, for all real x, $-1\le\sin x\le 1$

Now, when, $-1\le\sin x<0$, then $[\sin x]=-1$

when $\sin x=0$, then $[\sin x]=0$

when $0<\sin x<1$, then $[\sin x]=0$

and $[\sin x]=1$ only when $\sin x=1$ (in $-1\le x\le 1$).

But we know that, $\log_e f(x)$ is defined only when, $f(x)>0$.

So, $\log_e[\sin x]$ is defined only when $[\sin x]>0$ i.e. only when, $\sin x=1, [\sin x]=1$.

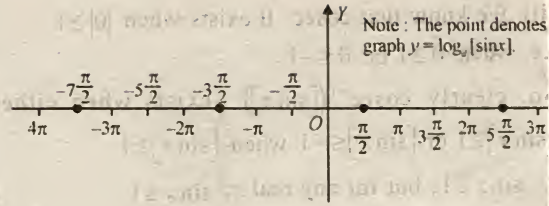
So, for only $\sin x=1$, $\log_e[\sin x]$ is defined.

When $\sin x=1, x=(4n+1)\frac{\pi}{2}$ so, $y=\log_e[\sin x]$ is defined only when $x=(4n+1)\frac{\pi}{2}$ $(n=0,\pm 1,\pm 2)$.

Now, $\log_e[\sin x]=0$ when $x=(4n+1)\frac{\pi}{2}$.

So, the range of y is given by $y=0$ and domain $x=(4n+1)\frac{\pi}{2}$.

It is very easy now to draw the graph as follows,



4. $f(x+y)=\dfrac{f(x)+f(y)}{1-f(x)f(y)}$

Put, $x=y=0$, $f(0)=\dfrac{f(0)+f(0)}{1-[f(0)]^2}$

or, $\left[1-[f(0)]^2\right]f(0)=2f(0)$

or, $f(0)\left[1-[f(0)]^2-2\right]=0$ or, $f(0)\left[[f(0)]^2+1\right]=$

Now, $[f(0)]^2$ being real $\neq -1$, $\therefore f(0)=0$.

$\left[\therefore\ 1+[f(0)]^2\neq 0\right]$.

Put, $y=-x$, $f(0)=\dfrac{f(x)+f(-x)}{1-f(x)f(-x)}$

or, $0=\dfrac{f(x)+f(-x)}{1-f(x)f(-x)}$

or, $f(x)+f(-x)=0$ or, $f(-x)=-f(x)$

Now, $\dfrac{f(y-x)}{y-x}=\dfrac{f(y)+f(-x)}{1-f(-x)f(y)}\times\dfrac{1}{y-x}$

$=\dfrac{f(y)-f(x)}{1+f(y)f(x)}\times\dfrac{1}{y-x}$ $\left[\because\ f(-x)=-f(x)\right]$

$\therefore\ \underset{y\to x}{Lt}\dfrac{f(y-x)}{y-x}=\underset{y\to x}{Lt}\dfrac{f(y)-f(x)}{y-x}\times\underset{y\to x}{Lt}\dfrac{1}{1+f(y)f(}$

or, $\underset{y\to x}{Lt}\dfrac{f(y-x)-f(0)}{(y-x)}=\underset{y\to x}{Lt}\dfrac{f(y)-f(x)}{y-x}\times\dfrac{1}{1+f^2(}$

$\left[\because f(0)=0\right.$

or, $f'(0)=f'(x)\left[\dfrac{1}{1+f^2(x)}\right]$

Let us denote $f(x)$ by f; $\therefore\ f'(0)=f'\times\dfrac{1}{1+f^2}$

or, $\dfrac{df}{1+f^2}=f'(0)dx$; or $\int\dfrac{df}{1+f^2}=f'(0)\int dx$

$\tan^{-1}(f) = x f'(0) + k$.

hen, $x = 0, f = 0, k = 0 \left[\because \tan^{-1} f = 0\right]$.

$\tan^{-1} f = x f'(0)$ or, $\tan^{-1} f(x) = x f'(0)$;

$\frac{\tan^{-1} f(x)}{x} = f'(0)$ = constant because when, $f(x)$

nown $f'(0)$ is constant, $\therefore$ $\frac{\tan^{-1} f(x)}{x}$ is constant.

$$f(x) = \int |x| dx,\ |x| = x,\ x \geq 0$$
$$= -x,\ x < 0$$

$(x) = \int x dx$ (when $x \geq 0$)

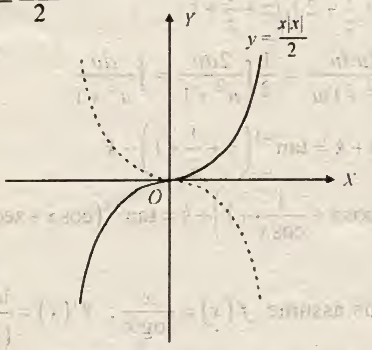
$$\frac{x^2}{2} + c = \frac{x \cdot x}{2} + c = \frac{x|x|}{2} + c \quad (|x| = +x, x \geq 0)$$

$(x) = \int -x\, dx$ (when $x < 0$)

$$\frac{-x^2}{2} + c_1 = \frac{(-x)x}{2} + c_1 = \frac{x|x|}{2} + c_1 \quad (|x| = -x, x < 0)$$

ow, $f(0) = 0$, $\therefore$ $c = 0,\ c_1 = 0$

$$f(x) = \frac{x|x|}{2}$$



te : $f(x) = \frac{x|x|}{2}$; $\therefore$ $f(-x) = \frac{(-x)(|-x|)}{2}$

$f(-x) = -\frac{x|x|}{2}$ $[\because |-x| = |x|]$ or, $f(-x) = -f(x)$

nce $f(x)$ is an odd function. Hence, the graph shown ove is symmetrical about origin.

$$\int_{-1}^{1} \frac{1}{x^2} dx = \left[-\frac{1}{x}\right]_{-1}^{1} = -[1-(-1)]$$

- 2, This computation is wrong.

e above integral looks very simple but really tough evaluate. when evaluating definite integral using Newton – Libnitz formula.

i.e. $\int_a^b f(x) dx = \phi(b) - \phi(a)$, $f(x) = \phi'(x)$, attention should be paid to the conditions of its use.

This method may be applied to compute definite integral of $f(x)$ continuous on $a \leq x \leq b$ only when $f(x) = \phi'(x)$ is entirely satisfied in the whole interval $[a, b]$. In particular the antiderivative must be a continuous function on $[a, b]$.

But obviously here, $f(x) = \frac{1}{x^2}$, $\phi(x) = -\frac{1}{x}$.

Now, $\frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2}$ is not satisfied (i.e. holds good) in the entire interval $[-1, 1]$ $\therefore$ $\frac{d}{dx}\left(-\frac{1}{x}\right) = \frac{1}{x^2}$ does not have any meaning at $x = 0$.

So, Newton - Libnitz formula cannot be applied directly to it to compute the value.

7. $f(x) = \underset{t \to \infty}{Lt} \left[\frac{(1+\sin \pi x)^t - 1}{(1+\sin \pi x)^t + 1}\right]$ at $x = 0, 1, 2, 3, \ldots;$

the continuity of $f(x)$ is to be checked.

obviously $\sin \pi x = 0$ at $x = 0, 1, 2, 3, \ldots$

$$f(x) = \underset{t \to \infty}{Lt} \frac{\left[(1+0)^t - 1\right]}{\left[(1+0)^t + 1\right]} = \frac{1-1}{1+1} = 0$$

Case I. Suppose $2m < x < 2m+1$, for all $m \in N$.

$\therefore$ $\sin 2m\pi < \sin \pi x < \sin(2m+1)\pi$

i.e. πx is either in first or second quadrant, obviously $\sin \pi x$ is positive,

$\therefore$ $1 + \sin \pi x > 0 + 1$ or, $(1 + \sin \pi x) > 1$

$\therefore$ $(1+\sin \pi x)^t \to \infty$ as $t \to \infty$.

$$\therefore\ f(x) = \underset{t \to \infty}{Lt} \frac{(1+\sin \pi x)^t - 1}{(1+\sin \pi x)^t + 1}$$

$$= \underset{t \to \infty}{Lt} \frac{1 - (1+\sin \pi x)^{-t}}{1 + (1+\sin \pi x)^{-t}} \left[\begin{array}{l} \because\ (1+\sin \pi x)^t \to \infty \\ \therefore\ (1+\sin \pi x)^{-t} \to 0 \end{array}\right]$$

$$= \frac{1-0}{1+0} = 1.$$

Case II. Suppose $2m+1<x<2m+2$

$\Rightarrow (2m+1)\pi<\pi x<(2m+2)\pi$

So, πx in either third or fourth quadrant.

Hence, $\sin\pi x$ is negative.

$-1<\sin\pi x<0$ or $(1+\sin\pi x)<1$ $\left[\therefore\ 0<(1+\sin\pi x)<1\right]$.

Hence, $(1+\sin\pi x)^t\to 0$ as $t\to\infty$

$$\therefore\ f(x)=\underset{t\to\infty}{Lt}\frac{(1+\sin\pi x)^t-1}{(1+\sin\pi x)^t+1}=\frac{0-1}{0+1}=-1$$

(Note : if x is assumed as $2m-1<.\ \ 2m$ the result will be same as above i.e. $f(x)=-1$)

$\therefore\ f(x)=0$ if $x=0,1,2,\ldots.$

$=1$ if $2m<x<2m+1$, m being an integer.

$=-1$ if $2m+1<x<2m+2$, m being an integer.

$\left(\text{or, } f(x)=-1 \text{ if } (2m-1)<x<2m\right)$

Obviously at $x=2m$, $f(x)=0$

but $\underset{x\to 2m+0}{Lt} f(x)=1$; $\underset{x\to 2m-0}{Lt} f(x)=-1$

at $x=2m-1$, $f(x)=0$ but $\underset{x\to(2m-1)^+}{Lt} f(x)=1$;

$\underset{x\to(2m-1)^-}{Lt} f(x)=-1$

Therefore $f(x)$ is discontinuous at $x=2m,\ 2m-1$

obviously, $f(x)$ is discontinuous at $x=c\in R$ when c is an integer.

Note : Each point of discontinuity is a point of jump discontinuity.

8. $$\int\frac{\sin^3\frac{x}{2}\cdot\left(\sec\frac{x}{2}\right)dx}{\left(\cos^3x+\cos^2x+\cos x\right)^{\frac{1}{2}}}$$

Let, $\cos x=z$ or, $-\sin x\,dx=dz$

Now,

$$\sin^3\frac{x}{2}\sec\frac{x}{2}=-\frac{\sin^2\frac{x}{2}\cdot\sin\frac{x}{2}}{\cos\frac{x}{2}}=\frac{\sin^2\frac{x}{2}}{\cos^2\frac{x}{2}}\times\left(\sin\frac{x}{2}\cos\frac{x}{2}\right)$$

$$=\frac{1}{2}\times\left(\frac{2\sin^2\frac{x}{2}}{2\cos^2\frac{x}{2}}\right)\times\left(2\sin\frac{x}{2}\cos\frac{x}{2}\right)=\frac{1}{2}\times\left(\frac{1-\cos x}{1+\cos x}\right)\times\sin x$$

$$\therefore\ I=\frac{1}{2}\int\frac{1-\cos x}{1+\cos x}\cdot\frac{\sin x\,dx}{\left(\cos^3x+\cos^2x+\cos x\right)^{\frac{1}{2}}}$$

$$=\frac{1}{2}\int\frac{1-z}{1+z}\frac{\sin x\,dx}{\left(z^3+z^2+z\right)^{\frac{1}{2}}}=\frac{1}{2}\int\frac{z-1}{z+1}\times\frac{dz}{\left(z^3+z^2+z\right)^{\frac{1}{2}}}$$

Put, $z+\frac{1}{z}+1=u^2$ $\therefore\ \left(1-\frac{1}{z^2}\right)dz=2u\,du$

$$\text{Now, } I=\frac{1}{2}\int\frac{z-1}{z+1}\cdot\frac{dz}{\sqrt{z^2\left(z+\frac{1}{z}+1\right)}}$$

$$=\frac{1}{2}\int\frac{z-1}{z(z+1)}\cdot\frac{dz}{\sqrt{z+\frac{1}{z}+1}}=\frac{1}{2}\int\frac{z^2-1}{z(z+1)^2}\cdot\frac{dz}{\sqrt{z+\frac{1}{z}+1}}$$

$$=\frac{1}{2}\int\frac{\frac{z^2-1}{z^2}}{\frac{(z+1)^2}{z}}\cdot\frac{dz}{\sqrt{z+\frac{1}{z}+1}}=\frac{1}{2}\int\frac{\left(1-\frac{1}{z^2}\right)}{\frac{z^2+2z+1}{z}}\cdot\frac{dz}{\sqrt{z+\frac{1}{z}+1}}$$

$$=\frac{1}{2}\int\frac{\left(1-\frac{1}{z^2}\right)dz}{\left(z+\frac{1}{z}+2\right)\sqrt{z+\frac{1}{z}+1}}$$

$$=\frac{1}{2}\int\frac{2u\,du}{(u^2+1)u}=\frac{1}{2}\int\frac{2du}{u^2+1}=\int\frac{du}{u^2+1}$$

$$=\tan^{-1}u+k=\tan^{-1}\left(z+\frac{1}{z}+1\right)+k$$

$$=\tan^{-1}\left(\cos x+\frac{1}{\cos x}+1\right)+k=\tan^{-1}(\cos x+\sec x+1)+$$

9. Let us assume $f(x)=\frac{x}{\log x}$; $f'(x)=\frac{\log x-1}{(\log x)^2}$

Now, for $f(x)$ to be decreasing $f'(x)<0$.

$\therefore\ \log x-1<0$ or $\log x<1$ or $\log x<\log e$ or $x<$

but, $\log x$ is defined for $x>0$.

$\therefore\ 0<x<e$, $f(x)$ is decreasing and obviously f $e<x<\infty$; $f(x)$ is increasing.

Now, $e=2\cdot718\ldots\ldots<5<6$

$\therefore\ e<x<\infty$ as, $f(x)$ is increasing

$\therefore\ f(6)>f(5)$; $\frac{6}{\log 6}>\frac{5}{\log 5}$

or, $\frac{\log 5}{\log 6}>\frac{5}{6}$ or $\frac{\log 5}{\log 6}>\frac{5}{\sqrt{36}}$ ...(

Now, $\sqrt{37} > \sqrt{36}$ or $\frac{1}{\sqrt{37}} < \frac{1}{\sqrt{36}}$

or $\frac{5}{\sqrt{37}} < \frac{5}{\sqrt{36}}$ ...(2)

From (1) and (2), $\frac{\log 5}{\log 6} > \frac{5}{\sqrt{37}}$.

10. $I = I_1 + I_2$

$$I_1 = \int_0^4 \frac{\{\sqrt{x}\}\,dx}{\left(1+\{[x]\}\right)^5}; \quad I_2 = \int_0^{\frac{\pi}{4}} \sin\left(x-[x]\right) d\left(x-[x]^5\right).$$

Now, $I_1 = \int_0^4 \frac{\{\sqrt{x}\}\,dx}{\left(1+\{[x]\}\right)^5}$

Now, obviously $\{[x]\} = 0$ for all real x (because, $[x]$ denotes the integral part of x hence fraction of an integer must be equal to zero, $\therefore \{[x]\} = 0$).

$$\therefore \; I_1 = \int_0^4 \frac{\{\sqrt{x}\}\,dx}{(1+0)^5} = \int_0^4 \{\sqrt{x}\}\,dx$$

$$= \int_0^1 \{\sqrt{x}\}\,dx + \int_1^4 \{\sqrt{x}\}\,dx$$

$$= \int_0^1 \sqrt{x}\,dx + \int_1^4 \left(\sqrt{x}-1\right) dx$$

$$\left[\because 0<x<1, 0<\sqrt{x}<1, 1<x<4, 1<\sqrt{x}<2 \therefore \{\sqrt{x}\} = \left(\sqrt{x}-1\right)\right]$$

$$= \int_0^1 \sqrt{x}\,dx - \int_1^4 dx = \frac{2}{3}\left[x^{\frac{3}{2}}\right]_0^4 - (4-1) = \frac{2}{3} \times 4^{\frac{3}{2}} - 3$$

$$= \frac{2}{3} \times 8 - 3 = \frac{16}{3} - 3 = \frac{7}{3}.$$

$$I_2 = \int_0^{\frac{\pi}{4}} \sin\left(x-[x]\right) d\left(x-[x]^5\right)$$

Now, for $0 < x < \frac{\pi}{4}$ $\because [x] = 0$

$$I_2 = \int_0^{\frac{\pi}{4}} \sin x\,dx = \left[-\cos x\right]_0^{\frac{\pi}{4}}$$

$$= -\cos\frac{\pi}{4} + \cos 0 = -\frac{1}{\sqrt{2}} + 1 = 1 - \frac{1}{\sqrt{2}}.$$

$$\therefore \; I = I_1 + I_2 = \frac{7}{3} + 1 - \frac{1}{\sqrt{2}} = \left(\frac{10}{3} - \frac{1}{\sqrt{2}}\right).$$

3rd Mathematical Challenge *for* I.I.T. MAINS

This section is designed to give IIT JEE aspirants a thorough grinding & exposure to variety of possible twists and turns of problems in mathematics that would be very helpful in facing IIT JEE. Each and every problem is well thought of in order to strengthen the concepts and we hope that this section would prove a rich resource for practicing challenging problems and enhancing the preparation level of IIT JEE aspirants.
The detailed solutions to these problems will be published in the next issue alongwith a new set of such problems.

1. If $a_1, a_2, a_3, \ldots, a_n$ are integers with $s = a_1 + a_2 + \ldots + a_n$, then prove that $\left[\frac{s}{a}\right] \geq \left[\frac{a_1}{a}\right] + \left[\frac{a_2}{a}\right] + \ldots + \left[\frac{a_n}{a}\right]$ for any integer $a > 0$, here $[x]$ represents greatest integer $\leq x$.

2. If x and y are prime numbers which satisfy $x^2 - 2y^2 = 1$, solve for x and y.

3. For all natural n, show that $\frac{1}{n+1} + \frac{1}{n+2} + \frac{1}{n+3} + \ldots + \frac{1}{3n+1} < 2$.

4. Prove using complex numbers that the sum of the finite series

$$\cos\theta + \cos 2\theta + \cos 3\theta + \ldots + \cos n\theta \text{ ; for } 0 < \theta < 2\pi \text{ is } \frac{\sin\frac{n\theta}{2}}{\sin\frac{\theta}{2}}\left(\cos(n+1)\frac{\theta}{2}\right).$$

5. Equilateral triangles are circumscribed to the parabola $y^2 = 4ax$. Prove that their angular points lie on the conic $(3x + a)(x + 3a) = y^2$.

6. Let the bisector of the angle C of a triangle ABC meet the side AB at D. Show that $CD^2 < AC \cdot BC$.

7. The position vectors of the focii of an ellipse are $\vec{b}$ and $-\vec{b}$ and the length of the major axis is $2a$. Prove that the equation of ellipse is, $a^4 - a^2(\vec{r}^2 + \vec{b}^2) + (\vec{b} \cdot \vec{r})^2 = 0$.

8. Let $f(x) = \frac{1}{2} - \tan\left(\frac{\pi x}{2}\right)$ and $g(x) = \sqrt{3 + 4x - 4x^2}$. If $g(f(x))$ is defined then find the range and domain of $f(x)$. Also find the range of $g(f(x))$.

9. Show that $\int_0^1 x^{m-1}(1-x)^{n-1}dx = \int_0^\infty \frac{x^{m-1}}{(1+x)^{m+n}}dx$; $(m, n > 0)$.

10. In an organization, number of women are λ times that of men. If k things are distributed among them, probability that number of things received by men are odd is $\left(\frac{1}{2} - \left(\frac{1}{2}\right)^{k+1}\right)$. Evaluate λ.

■

By : **Shailendra Maheshwari**, Career point, Kota

Now equation of MN is $y = \frac{b\sin\phi}{-a\cos\phi}(x - a\cos\phi)$

$$\frac{x}{a}\sec\phi + \frac{y}{b}\mathrm{cosec}\phi = 1 \quad ...(1)$$

Now equation to the normal at point $(a\cos\phi, b\sin\phi)$ with respect to any other concentric ellipse.

$\frac{x^2}{A^2}+\frac{y^2}{B^2} = 1$ is $Ax\sec\phi - By\,\mathrm{cosec}\phi = A^2 - B^2$

As (1) and (2) are similar, on comparing them, we have

$\frac{A}{1/a} = \frac{-B}{1/b} = A^2 - B^2$ or $Aa = -Bb = A^2 - B^2$

Solving we get $B = \frac{a^2b}{a^2-b^2}$ and $A = -\frac{ab^2}{a^2-b^2}$

So the line (1), i.e. $\left(\frac{x}{a}\right)\sec\phi + \left(\frac{y}{b}\right)\mathrm{cosec}\phi = 1$

is a normal to the fixed ellipse

$\frac{x^2}{A^2}+\frac{y^2}{B^2}=1$, where $A = \frac{-ab^2}{a^2-b^2}$ and $B = \frac{a^2b}{a^2-b^2}$,

both of them being constant.

Prove that the straight line $\frac{x}{a}-\frac{y}{b}=m$ and $\frac{x}{a}+\frac{y}{b}=\frac{1}{m}$ always meet on the hyperbola.

Soln.: Equation of straight line is $\frac{x}{a} - \frac{y}{b} = m \quad ...(1)$

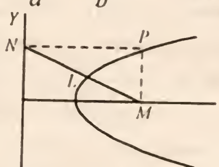
$$\frac{x}{a} + \frac{y}{b} = \frac{1}{m} \quad ...(2)$$

To find out the locus of the point of intersection of (1) and (2), we have to eliminate the variable m from these. So multiplying (1) and (2)

we get $\left(\frac{x}{a}-\frac{y}{b}\right)\left(\frac{x}{a}+\frac{y}{b}\right) = m\cdot\frac{1}{m}$

$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ which is clearly hyperbola.

The normal to the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ meet the axes on M and N and lines MP and NP are drawn at right angles to the axes. Prove that the locus of P is the hyperbola $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$



Soln.: Let the equation of the hyperbola be

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

Let L be any point on it having the co-ordinates $(a\sec\phi, b\tan\phi)$; then the equation of the normal at this point will be given by $ax\sin\phi + by = (a^2+b^2)\tan\phi$ ...(1)

This normal cut the axis of x at M whose co-ordinates $(x, 0)$ and the axis of y at N whose co-ordinates are $(0, y)$. Solving (1) with axis of x i.e., $y = 0$, we get

$x = \frac{a^2+b^2}{a\cos\phi}$ ...(2) Similarly, $y = \frac{(a^2+b^2)\tan\phi}{b}$ ...(3)

If PM and PN be the lines parallel to axes the co-ordinates of $P \equiv (x, y)$ will be clearly given by (2) and (3)

The required locus of P will be obtained by eliminating ϕ from (2) and (3). From (2),

$\cos\phi = \frac{a^2+b^2}{ax}$ hence $\tan\phi = \frac{\sqrt{a^2x^2-(a^2+b^2)^2}}{a^2+b^2}$

Putting this value in (3) and simplifying

we get the required locus as $a^2x^2 - b^2y^2 = (a^2 + b^2)^2$.

9. Prove that the chords of a hyperbola, which touch the conjugate hyperbola are bisected at the point of contact.

Soln.: Let the equation of hyperbola be

$\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$ ...(1), Its conjugate will be $\frac{y^2}{b^2}-\frac{x^2}{a^2}=1$ ...(2)

Any point on (2) is $(a\tan\phi, b\sec\phi)$. The equation to the tangent at this point to the curve (2) will be

$$\frac{y}{b}\sec\phi - \frac{x}{a}\tan\phi = 1 \quad ...(3)$$

Equation to the chord of (1) having the point $(a\tan\phi, b\sec\phi)$ as mid point will be

$$\frac{x\cdot a\tan\phi}{a^2} - \frac{y\,b\sec\phi}{b^2} = \frac{a^2\tan^2\phi}{a^2} - \frac{b^2\sec^2\phi}{b^2}$$

$\Rightarrow \frac{x\tan\phi}{a} - \frac{y\sec\phi}{b} = -1$ or, $\frac{y\sec\phi}{b} - \frac{x}{a}\tan\phi = 1$

which is same as equation (3).

10. Show that the chord, which joins the points in which a pair of conjugate diameters meets the hyperbola and its conjugate, is parallel to one asymptote and is bisected by the other.

Soln.: If P and Q be the points where the pair of conjugate diameters meet the hyperbola and the conjugate respectively then if the co-ordinates of P be $(a\sec\phi, b\tan\phi)$ then by the same article the co-ordinate of D will be $(a\tan\phi, b\sec\phi)$.

Slope of $PD = \frac{b\sec\phi - b\tan\phi}{a\tan\phi - a\sec\phi} = \frac{-b}{a}$

which is same as of the asymptotes.

Again the middle point of PD is

$$\frac{1}{2}(a\sec\phi + a\tan\phi),\ \frac{1}{2}b(\sec\phi + \tan\phi)$$

This point satisfies the equation to the other asymptotes $y = \left(\frac{b}{a}\right)x$ hence lies on it. ■

Complex Numbers

10 Mathematical Challenges

– *By : Prof. B.L. Sharma, Jaipur*

1. The coefficients a, b in the quadratic polynomial $p(z) = z^2 + az + b$ are complex numbers and $a^2 \neq 4b$. Prove that there are two real values of z and two purely imaginary values of z for all four of which $p(z)$ is purely imaginary if $\{R_e(a)\}^2 > 4\, R_e(b) > -\{I_m(a)\}^2$.

2. Show that there are two complex numbers z such that $|z - 2 - i| = 1$ and $\arg z = \pi/4$ and find their moduli.

3. The reflection of the point z_1 in the line $\theta = \alpha$ through the origin in the Argand diagram is z_2 and reflection of z_2 in the line $\theta = \beta$ through the origin is z_3. Prove that

$$z_2 = \bar{z}_1 (\cos 2\alpha + i \sin 2\alpha)$$

and that $z_3 = z_1 [\cos(2\beta - 2\alpha) + i \sin(2\beta - 2\alpha)]$.

4. Show that

$$|z - 3 - i| < 1 \;\Rightarrow\; \sqrt{10} - 1 < |z| < \sqrt{10} + 1.$$

5. If $|z - 2 - i| < 2$ and $|\omega - 5 - 5i| < 1$, find the maximum and minimum values of $|z - \omega|$.

6. A point P representing the complex number z moves in the Argand diagram so that it lies always in the region defined by $|z - 1| \leq |z - i|$ and $|z - 2 - 2i| \leq 1$.
If P describes the boundary of this region find $|z|$ when $\arg(z)$ has the smallest value.

7. If θ is real and $z = \cos\theta + i\sin\theta$, determine a, b, c, d such that
$2^7 \cos^3\theta \sin^5\theta = a \sin 8\theta + b \sin 6\theta + c \sin 4\theta + d \sin 2\theta$.

8. The affixes of the complex numbers $a = 3 + i$ and $b = 1 + 2i$ are A and B respectively. Find the complex numbers p, p', q, q' with affixes P, P', Q, Q' such that $ABQP$ and $ABQ'P'$ are squares. Find also complex numbers r, r' with affixes R, R' such that ABR, ABR' are equilateral triangles.

9. Find out how $\arg z(z - 1)$ will vary if the point z describes about a point O counter-clockwise, circle of radius 2 starting from the point $z = 2$.

10. The centre of the square is at the point $z_0 = 1 + i$ and one of its vertices is at the point $z_1 = 1 - i$. Find the complex numbers which corresponds to the other vertices of the square.

SOLUTIONS

1. **Case I :** z is real, *i.e.* $z = x$, $p(z) = p_1 + ip_2$
By given condition

$$ip_2 = x^2 + (a_1 + ia_2)x + b_1 + ib_2$$

$\Rightarrow\; x^2 + a_1 x + b_1 = 0.$

This is a quadratic equation with real roots. Thus

$$a_1^2 > 4\, b_1 \quad \text{or} \quad \{R_e(a)\}^2 > 4\, R_e(b) \qquad \text{... (i)}$$

Case II : If z is imaginary, *i.e.* $z = iy$
By hypothesis,

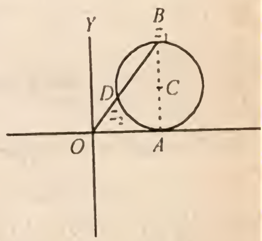
$$ip_2 = -y^2 + i(a_1 + ia_2)y + b_1 + ib_2$$

$\Rightarrow\; y^2 + a^2 y - b_1 = 0$

or $\{I_m(a)\}^2 + 4R_e(b) > 0$ ... (ii)

from (i) and (ii), we get the result.

2. $|z - 2 - i| = 1$ is a circle, centre at $(2, 1)$ and $R = 1$. Join B and D.
$AB = OA = 2$.
Hence $\angle AOB = \pi/4$.
Thus the points B and D have arguments equal to $\pi/4$.

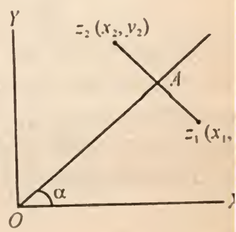


$AB^2 + OA^2 = BD^2 \;\Rightarrow\; BD = 2\sqrt{2},\; OD = \sqrt{2}$

$\therefore\; \arg(z_1) = \arg(z_2) = \pi/4$

and $|z_1| = 2\sqrt{2}$ and $|z_2| = \sqrt{2}$.

3. Let $m = \tan\alpha$, z_1 is a reflection of z_2. Thus,
$y_1 + y_2 = m(x_1 + x_2)$ ... (i)
$m(y_2 - y_1) = x_1 - x_2$ ... (ii)
Solving (i) and (ii), we have



$$x_2 = x_1 \frac{1-m^2}{1+m^2} + y_1 \frac{2m}{1+m^2}$$

$$y_2 = x_1 \frac{2m}{1+m^2} - y_1 \frac{1-m^2}{1+m^2}$$

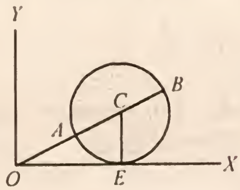
$\therefore \quad z_2 = x_2 + iy_2 = (x_1 - iy_1)\cos 2\alpha + (ix_1 + y_1)\sin 2\alpha$

$\quad = x_1 e^{2i\alpha} - iy_1 e^{2i\alpha} = \bar{z}_1 (\cos 2\alpha + i\sin 2\alpha)$.

Similarly, $z_3 = \bar{z}_2 e^{2i\beta} = z_1 e^{-2i\alpha} \cdot e^{2i\beta}$

$\quad = z_1[\cos(2\beta - 2\alpha) + i \sin(2\beta - 2\alpha)]$.

4. $|z - 3 - i| < 1$, it represents the points z lying inside the circle, centre (3, 1) and radius 1.



It is clear from the diagram that $OA < |z| < OB$

Now $OC = \sqrt{10}$

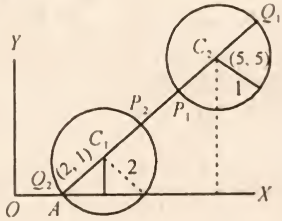
$\therefore \quad \sqrt{10} - 1 < |z| < \sqrt{10} + 1$.

5. $\overrightarrow{P_1P_2} = z - \omega$; $\left|\overrightarrow{P_1P_2}\right| = |z - \omega|$

$$= \sqrt{(5-2)^2 + (5-1)^2} - 3 = 2.$$

Minimum value of

$|z - \omega| = 2$



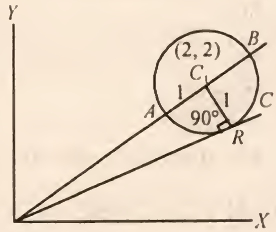
$\overrightarrow{Q_1Q_2} = z - \omega$;

$\left|\overrightarrow{Q_1Q_2}\right| = |z - \omega| = 5 + 2 + 1$

Maximum value of

$|z - \omega| = 8$.

6. It is clear from the diagram that the region defined by $|z - 1| \le |z - i|$ and $|z - 2 - 2i| \le 1$ is $ACBC_1$, *i.e.* the point lying inside and boundary of the semi-circle ACB. The points lying on the diameter AC_1B. If P describes the boundary of this region. If OR is the tangent to the circle $\Rightarrow$ arg(R) is the smallest.



$$|z| = OR = \sqrt{OG^2 - 1^2} = \sqrt{8-1} = \sqrt{7}.$$

7. $z = \cos\theta + i \sin\theta$; $\frac{1}{z} = \cos\theta - i\sin\theta$

$$z - \frac{1}{z} = 2i\sin\theta,\ z^2 - \frac{1}{z^2} = 2i\sin 2\theta, \ldots.$$

$$z + \frac{1}{z} = 2\cos\theta,\ z^2 + \frac{1}{z^2} = 2\cos 2\theta, \ldots..$$

$$(2\cos\theta)^3(2i\sin\theta)^5 = \left(z - \frac{1}{z}\right)^5\left(z + \frac{1}{z}\right)^3$$

$$= \left(z^8 - \frac{1}{z^8}\right) - 2\left(z^6 - \frac{1}{z^6}\right) - 2\left(z^4 - \frac{1}{z^4}\right) + 6\left(z^2 - \frac{1}{z^2}\right)$$

$= 2i \sin 8\theta - 4i \sin 6\theta - 4i \sin 4\theta + 12 i\sin 2\theta$

$\therefore \quad 2^7\cos^3\theta \sin^5\theta = \sin 8\theta - 2\sin 6\theta - 2\sin 4\theta + 6\sin 2\theta$

$\therefore \quad a = 1, b = -2, c = -2, d = 6$.

8. $\overrightarrow{AB} = b - a = -2 + i$. To obtain $\overrightarrow{AP}$ and $\overrightarrow{AP'}$, we must rotate $\overrightarrow{AB}$ through $\pm \pi/2$. Then

$\overrightarrow{AP} = i(-2+i) = -1-2i$; $\overrightarrow{AP'} = 1 + 2i$

$\therefore \quad p = a + (-1 - 2i) = 2 - i$

$p' = 3 + i + 1 + 2i = 4 + 3i$

Similarly, $q = b + (-1 - 2i) = 0$

$q' = 1 + 2i + 1 + 2i = 2 + 4i$.

To rotate $\overrightarrow{AB}$ through angle $\pm \pi/3$, we seek complex number z such that $|z| = 1$ and arg $z = \pm \pi/3$. Thus

$$z = \cos\frac{\pi}{3} \pm i\sin\frac{\pi}{3} = \frac{1}{2}(1 \pm i\sqrt{3})$$

$$\therefore \quad \overrightarrow{AR} = \frac{1}{2}(1 + i\sqrt{3})(b - a)$$

$$\therefore \quad r = a + \frac{1}{2}(1 + i\sqrt{3})(b - a) = 2 - \frac{1}{2}\sqrt{3} + \left(\frac{3}{2} - \sqrt{3}\right)$$

$$= \frac{7}{4} - \frac{3}{2}\sqrt{3}$$

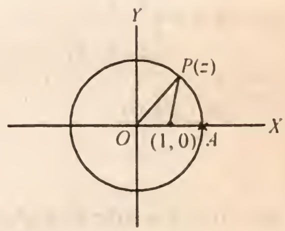
Also, $r' = \frac{7}{2} + \frac{3}{2}\sqrt{3}$.

9. arg $(z)(z - 1)$

$= \arg z + \arg(z - 1)$

Change in the arg $z(z - 1)$

$= 2\pi + 2\pi = 4\pi$.



10. Using the property that diagonals bisect each other.

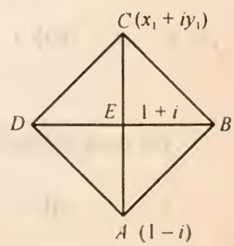
$$\therefore \quad \frac{x_1 + iy_1 + 1 - i}{2} = 1 + i$$

$x_1 + iy_1 + 1 - i = 2 + 2i$

$\therefore \quad x_1 + iy_1 = 1 + 3i$,

similarly B and D

$B = 3 + i$, $D = -1 + i$.



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Very Similar

MODEL TEST PAPER

for CBSE - AIEEE 2004

Exam on 9th May 2004

1. The area of the parallelogram whose adjacent sides are $2\hat{i}-3\hat{k}$ and $4\hat{j}+2\hat{k}$ is
(A) $2\sqrt{14}$ (B) $4\sqrt{14}$ (C) $16\sqrt{14}$ (D) $\sqrt{14}$.

2. A unit vector perpendicular to the plane of a and b where $\vec{a} = -\hat{i}+\hat{j}+\hat{k}$ and $\vec{b} = \hat{i}-\hat{j}+\hat{k}$ is
(A) $\frac{i+j}{\sqrt{2}}$ (B) k (C) $\frac{j+k}{\sqrt{2}}$ (D) $\frac{i-j}{\sqrt{2}}$.

3. The value of p such that the vectors $\hat{i}+3\hat{j}-2\hat{k}$, $2\hat{i}-\hat{j}+4\hat{k}$ and $3\hat{i}+2\hat{j}+p\hat{k}$ are coplanar is
(A) 4 (B) 2 (C) 8 (D) 10.

4. $\begin{vmatrix} a & b & c \\ b & c & a \\ c & a & b \end{vmatrix} =$
(A) $a+b+c+abc$
(B) $a^2+b^2+c^2+ab+bc+ca$
(C) $3abc-a^3-b^3-c^3$ (D) $a^3+b^3+c^3-3abc$.

5. If $A = \begin{bmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{bmatrix}$, $A^2 =$
(A) A (B) $2A$
(C) unit matrix (D) $3A$.

6. If ω is a cube root of unity $\begin{vmatrix} 1 & \omega & \omega^2 \\ \omega & \omega^2 & 1 \\ \omega^2 & \omega & 1 \end{vmatrix} =$
(A) 0 (B) 1 (C) ω (D) ω^2.

7. The roots of the equation $\begin{vmatrix} x-1 & 1 & 1 \\ 1 & x-1 & 1 \\ 1 & 1 & x-1 \end{vmatrix} = 0$ are
(A) 1, 2 (B) −1, 2 (C) 1, −2 (D) −1, −2.

8. The inverse of $\begin{bmatrix} 3 & -2 \\ -7 & 5 \end{bmatrix}$ is
(A) $\begin{bmatrix} 3 & -7 \\ -2 & 5 \end{bmatrix}$ (B) $\begin{bmatrix} 5 & -2 \\ -7 & 3 \end{bmatrix}$
(C) $\begin{bmatrix} 5 & 2 \\ 7 & 3 \end{bmatrix}$ (D) $\begin{bmatrix} -3 & 7 \\ -2 & 5 \end{bmatrix}$.

9. k is a scalar and A is an n - square matrix. Then $|kA| =$
(A) $k|A|^n$ (B) $k|A|$ (C) $k^n|A|^n$ (D) $k^n|A|$.

10. From the matrix equation $AB = AC$ we can conclude $B = C$ provided
(A) A is singular (B) A is non - singular
(C) A is symmetric (D) A is square.

11. Let $A = \begin{bmatrix} 1 & 2 \\ 0 & 1 \end{bmatrix}$. Then $A^n =$
(A) $\begin{bmatrix} 1 & 2n \\ 0 & 1 \end{bmatrix}$ (B) $\begin{bmatrix} 2 & n \\ 0 & 1 \end{bmatrix}$
(C) $\begin{bmatrix} 1 & 2^n \\ 0 & 1 \end{bmatrix}$ (D) $\begin{bmatrix} 1 & n \\ 0 & 2 \end{bmatrix}$.

12. In a triangle ABC if $a = 2$, $B = 60°$ and $C = 75°$, then $b =$
(A) $\sqrt{3}$ (B) $\sqrt{6}$
(C) $\sqrt{9}$ (D) $1+\sqrt{2}$.

13. If $\tan^{-1} 2x + \tan^{-1} 3x = \frac{\pi}{4}$, then $x =$
(A) 1 (B) − 1 (C) 1/6 (D) 1/4.

14. The general solution of $\tan 2\theta \tan \theta = 1$ is $\theta =$
(A) $(2n+1)\frac{\pi}{3}$ (B) $(2n+1)\frac{\pi}{4}$
(C) $(2n+1)\frac{\pi}{2}$ (D) $(2n+1)\frac{\pi}{6}$.

15. $\cos^{-1}\frac{2}{\sqrt{5}} + \tan^{-1}\frac{1}{3} =$

*Also useful for **IIT (Screening), DCE, UPSEAT, Orissa JEE, MP-PET, Kerala PET, Karnataka CET, ...***

Mathematics Olympia for IIT-JEE 2004 (Mains)

By : **Er. Akhlak Ahmad**, ABC Classes, Gorakhpur

1. Let $f(x)$ is a periodic function, with period $T > 0$.

Prove that $\int_0^{\infty} e^{-mt} f(t)dt = \dfrac{\int_0^T e^{-mt} f(t)dt}{1-e^{-mT}}$, where $m > 0$.

2. If $f(x)$ is a polynomial function of lowest degree that passes through three non-collinear points (distinct) $(a, (a-b)(a-c))$, $(b, (b-c)(b-a))$ and $(c, (c-a)(c-b))$ then prove that there always exists some x for which $f(x) < 0$.

3. Solve for x : $\sin^{-1}\left(\sin\left(\dfrac{2x^2+4}{1+x^2}\right)\right) < \pi - 3$.

4. Suppose a, b, c, d are fixed real numbers such that a quadrilateral can be formed with sides a, b, c, d in clock-wise order. Prove that the vertices of the quadrilateral of maximum area lie on a circle.

5. If $g(x) > 0 \ \forall \ x \in [a, b]$ and $f(x)$ is continuous in $[a, b]$ and $f(x)$ has least and greatest value m, M respectively. Prove that

$$m\int_a^b g(x)dx \le \int_a^b f(x)\,g(x)dx \le M\int_a^b g(x)dx.$$

6. If origin and z_0 lie to opposite side of the line $a\bar{z} + \bar{a}z - 4 = 0$ in a complex plane. Prove $\text{Re}(a) \cdot \text{Re}(z_0) + \text{Im}(a) \cdot \text{Im}(z_0) > 2$. (where a and z_0 are complex numbers).

7. If a and b are chosen randomly from the set $S = \{1, 2, 3, 4, ..., 10\}$ with replacement. Find the probability that $\lim\limits_{t\to 0}\left(\dfrac{a^t+b^t}{2}\right)^{2/t} = 10$.

8. Consider the region S_0 which is enclosed by the curve $y \ge \sqrt{1-x^2}$ and $\max\{|x|, |y|\} \le 4$. If slope of a family of lines is defined as $m(t) = \cos t$; where point $(t, 2t + 0.4)$ lies inside the region S_0.
Any member of this family of lines is called as Akhlakian if it passes through $(\pi, \max\{t\})$

(a) Find the equation of Akhlakian straight line having maximum possible slope.

(b) Compute the area of region 'S_0'.

9. Let a, b, c be the sides of a triangle for usual notation. If $a^2 + b^2 = 1993c^2$, find $\dfrac{\cot C}{\cot A + \cot B}$.

10. If $\alpha^{13} = 1$ and $(\alpha \ne 1)$, find the quadratic equation whose roots are $\alpha + \alpha^3 + \alpha^4 + \alpha^{-4} + \alpha^{-3} + \alpha^{-1}$ and $\alpha^2 + \alpha^5 + \alpha^6 + \alpha^{-6} + \alpha^{-5} + \alpha^{-2}$.

SOLUTION

1. If $f(x)$ is periodic with period T, then

$$f(x) = f(T+x) = f(2T+x) = ...$$

$$\int_0^{\infty} e^{-mt} f(t)dt = \int_0^T e^{-mt} f(t)dt + \int_T^{2T} e^{-mt} f(t)dt + \int_{2T}^{3T} e^{-mt} f(t)dt + ...\infty$$

and let $I = \int_0^T e^{-mt} f(t)\,dt$

$$\int_T^{2T} e^{-mt} f(t)\,dt = \int_0^T e^{-m(y+T)} f(T+y)dy \quad \begin{bmatrix} t = y+T \\ dt = dy \end{bmatrix}$$

$$= e^{-mT}\int_0^T e^{-my} f(y)\,dy \qquad [\because f(T+y) = f(y)]$$

$$= e^{-mT}\int_0^T e^{-mt} f(t)dt\ ; \quad \int_T^{2T} e^{-mt} f(t)dt = e^{-mT} I$$

Similarly; $\int_{2T}^{3T} e^{-mt} f(t)dt = e^{-2mT} I$

so, $\int_0^{\infty} e^{-mt} f(t)dt = I + e^{-mT} I + (e^{-mT})^2 I + ...\infty$

$$= \frac{I}{1-e^{-mT}} \qquad \text{(infinite G.P. sum)}$$

Hence $\int_0^{\infty} e^{-mt} f(t)\,dt = \dfrac{\int_0^T e^{-mt} f(t)dt}{1-e^{-mT}}$.

2. It is obvious that

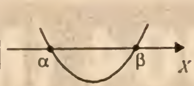
$f(x) = (x-a)(x-b) + (x-b)(x-c) + (x-c)(x-a)$

$f(x) = 3x^2 - 2(a+b+c)x + ab + bc + ca$

discriminant $= \Delta = 4(a + b + c)^2 - 12(ab + bc + ca)$

$= a^2 + b^2 + c^2 - ab - bc - ca$

$\Delta = \frac{1}{2}\left[(a-b)^2 + (b-c)^2 + (c-a)^2\right]$



Since a, b, c are distinct and real

$\Rightarrow$ $\Delta > 0 \Rightarrow$ roots of $f(x) = 0$ are real and distinct.

$\Rightarrow$ there is some x for which $f(x) < 0$.

3. Let $t = \dfrac{2x^2+4}{x^2+1} \Rightarrow x^2 = \dfrac{4-t}{t-2}$

$\because\ x^2 \geq 0$ we get $2 < t \leq 4$

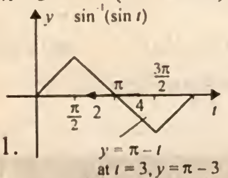
we have to solve $\sin^{-1}(\sin t) < \pi - 3 \qquad (2 < t \leq 4)$

$\Rightarrow$ to satisfy $\sin^{-1}(\sin t) < \pi - 3$

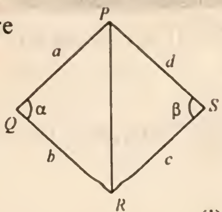
and $2 < t \leq 4$

Simultaneously $3 < t \leq 4$

$\Rightarrow 3 < \dfrac{2x^2+4}{x^2+1} \leq 4 \Rightarrow -1 < x < 1.$



4. Consider the following figure



In ΔPQR and ΔPSR

$PR^2 = a^2 + b^2 - 2ab\cos\alpha$

and $PR^2 = c^2 + d^2 - 2cd\cos\beta$

$\Rightarrow a^2 + b^2 - 2ab\cos\alpha$

$= c^2 + d^2 - 2cd\cos\beta \qquad \text{...(i)}$

Let A denotes the area of quadrilateral $PQRS$

A = area of ΔPQR + area of ΔPSR

$$A = \frac{1}{2}ab\sin\alpha + \frac{1}{2}cd\sin\beta$$

$$\Rightarrow \frac{dA}{d\alpha} = \frac{1}{2}ab\cos\alpha + \frac{1}{2}cd\cos\beta\frac{d\beta}{d\alpha} \qquad \text{...(ii)}$$

differentiating both sides of (i) w.r.t. α, we get

$$2ab\sin\alpha = 2cd\sin\beta\frac{d\beta}{d\alpha} \Rightarrow \frac{d\beta}{d\alpha} = \frac{ab\sin\alpha}{cd\sin\beta}$$

substituting $\dfrac{d\beta}{d\alpha}$ in (ii)

$$\frac{dA}{d\alpha} = \frac{1}{2}ab\cos\alpha + \frac{1}{2}cd\cos\beta\left(\frac{ab\sin\alpha}{cd\sin\beta}\right)$$

$$\frac{dA}{d\alpha} = \frac{1}{2}ab\left[\frac{\sin\alpha\cos\beta + \cos\alpha\sin\beta}{\sin\beta}\right]$$

$$\frac{dA}{d\alpha} = \frac{1}{2}ab\frac{\sin(\alpha+\beta)}{\sin\beta}$$

For maximum or minimum values of A

$$\frac{dA}{d\alpha} = 0 \Rightarrow \sin(\alpha+\beta) = 0$$

$\Rightarrow \alpha + \beta = \pi$, it is easy to check that

$\dfrac{d^2A}{d\alpha^2} < 0 \Rightarrow$ quadrilateral will be concyclic.

5. Since $f(x) - m \geq 0\ \forall\ x \in [a, b]$

$\Rightarrow (f(x) - m)g(x) \geq 0 \qquad (\because g(x) > 0)$

$\Rightarrow \int_a^b (f(x)-m)\, g(x)dx \geq 0$

$\Rightarrow \int_a^b f(x)g(x)dx - m\int_a^b g(x)\,dx \geq 0$

$\Rightarrow m\int_a^b g(x)dx \leq \int_a^b f(x)g(x)dx$ further $f(x) - M \leq 0$;

$f(x)g(x) - Mg(x) \leq 0 \qquad (\because g(x) \geq 0)$

$\Rightarrow \int_a^b f(x)g(x)dx - M\int_a^b g(x)dx \leq 0$

$\Rightarrow \int_a^b f(x)g(x)dx \leq M\int_a^b g(x)dx$. (Hence proved)

6. $\because$ Origin and z_0 lie in opposite side of the line

$a\bar{z} + \bar{a}z - 4 = 0$

$\because$ origin gives -4 (negative)

$\Rightarrow a\bar{z}_0 + \bar{a}z_0 - 4 > 0$

$\left.\begin{array}{l} a = A + iB \\ z_0 = x_0 + iy_0 \end{array}\right\}$ Here A, B, x_0, y_0 are real

$\bar{a} = A - iB$

$\bar{z}_0 = x_0 - iy_0$

$\therefore (A+iB)(x_0 - iy_0) + (A-iB)(x_0+iy_0) - 4 > 0$

we get $Ax_0 + By_0 > 2$

$\Rightarrow \text{Re}(a)\cdot\text{Re}(z_0) + \text{Im}(a)\cdot\text{Im}(z_0) > 2.$

7. a can be chosen in 10 ways

b can be chosen in 10 ways $\quad$ ($\because$ with replacement)

Total number of ways of chosing a and b is = 100

$$\lim_{t\to 0}\left(\frac{a^t + b^t}{2}\right)^{2/t} = e^{\lim_{t\to 0}\left(\frac{a^t+b^t-2}{t}\right)}$$

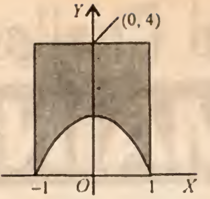
$$= e^{\lim_{t\to 0}\frac{a^t\ln a + b^t\ln b}{1}} = e^{\ln a + \ln b} = e^{\ln(ab)} = ab = 10$$

$$\left.\begin{array}{cc} a & b \\ 1 & 10 \\ 10 & 1 \\ 2 & 5 \\ 5 & 2 \end{array}\right\} \text{4 ways (favourable ways)}$$

$\Rightarrow p = \dfrac{4}{100} = \dfrac{1}{25}.$

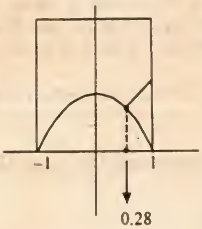
8. Required region is shown. We can not select $x > 1$

or $x < -1$ because $y = \sqrt{1-x^2}$
(y becomes imaginary if $x > 1$
or $x < -1$)



(b) area of region

$$S_0 = 2\times 4 - \frac{\pi \cdot 1^2}{2} = 8 - \frac{\pi}{2}\text{sq.units}$$



it is given that point $(t, 2t + 0.4)$ lies inside the region $S_0 \Rightarrow t$ lies on line $y = 2x + 0.4$

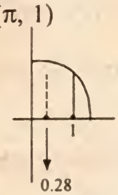
$$\text{Solve } \begin{cases} x^2 + y^2 = 1 \\ y = 2x + 0.4 \end{cases}$$

$x^2 + (2x + 0.4)^2 = 1$

$5x^2 + 1.6x - 0.84 = 0 \Rightarrow x = 0.28 \Rightarrow 0.28 \le t \le 1.$

(a) every Akhlakian line passes through $(\pi, 1)$

$m(t) = \cos\ t$ (slope)



maximum slope = $\cos(0.28)$

$\Rightarrow$ equation of Akhlakian line will be

$$\frac{y-1}{x-\pi} = \cos(0.28).$$

9. Since $a^2 + b^2 = 1993c^2$, $a^2 + b^2 - c^2 = 1992c^2$

$$a^2+b^2-c^2 = 2ab\cos C \Rightarrow \cos C = \frac{1992c^2}{2ab} \Rightarrow \cos C = \frac{996c^2}{ab}$$

$$\frac{\cot C}{\cot A + \cot B} = \frac{\cos C \sin A \sin B}{\sin C(\sin(A+B))} \quad (\because\ A+B+C = \pi)$$

$$= \frac{\cos C \sin A \sin B}{\sin^2 C} = \frac{\cos C(ab/4R^2)}{(c/2R)^2}$$

$$\left(\text{by sine rule } \frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = 2R\right)$$

$$= \frac{996c^2}{ab} \times \frac{ab}{c^2} = 996.$$

10. Let $A = \alpha + \alpha^3 + \alpha^4 + \alpha^{-4} + \alpha^{-3} + \alpha^{-1}$

$$\boxed{A = \alpha + \alpha^3 + \alpha^4 + \alpha^9 + \alpha^{10} + \alpha^{12}} \quad (\because \alpha^{13} = 1)$$

$$B = \alpha^2 + \alpha^5 + \alpha^6 + \alpha^{-6} + \alpha^{-5} + \alpha^{-2}$$

$$\boxed{B = \alpha^2 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^{11}}$$

$$A + B = \alpha + \alpha^2 + \alpha^3 + \alpha^4 + \ldots + \alpha^{12}$$

$$A + B = \frac{\alpha(\alpha^{12}-1)}{\alpha - 1} = \frac{\alpha^{13}-\alpha}{\alpha-1} = -1 \quad (\because \alpha^{13} = 1)$$

$$A \times B = (\alpha + \alpha^3 + \alpha^4 + \alpha^9 + \alpha^{10} + \alpha^{12}) \times (\alpha^2 + \alpha^5 + \alpha^6 + \alpha^7 + \alpha^8 + \alpha^{11})$$

$$= 3[\alpha + \alpha^2 + \ldots + \alpha^{12}]$$

$AB = -3 \Rightarrow$ equation is $x^2 + x - 3 = 0.$ ■

5^{th} Mathematical Challenge *for* I.I.T. MAINS

This section is designed to give IIT JEE aspirants a thorough grinding & exposure to variety of possible twists and turns of problems in mathematics that would be very helpful in facing IIT JEE. Each and every problem is well thought of in order to strengthen the concepts and we hope that this section would prove a rich resource for practicing challenging problems and enhancing the preparation level of IIT JEE aspirants.

The detailed solutions to these problems will be published in the next issue alongwith a new set of such problems.

1. Show that the six planes through the middle point of each edge of a tetrahedron perpendicular to the opposite edge meet in a point.

2. Prove that if the graph of the function $y = f(x)$, defined throughout the number scale, is symmetrical about two lines $x = a$ and $x = b$, $(a < b)$, then this function is a periodic one.

3. Show that an equilateral triangle is a triangle of maximum area for a given perimeter and a triangle of minimum perimeter for a given area.

4. Let $az^2 + bz + c$ be a polynomial with complex coefficients such that a and b are non-zero. Prove that the zeros of this polynomial lie in the region $|z| \le \left|\frac{b}{a}\right| + \left|\frac{c}{b}\right|$.

5. An isosceles triangle with its base parallel to the major axis of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ is circumscribed with all the three sides touching the ellipse. Find the least possible area of the triangle.

6. If one of the straight lines given by the equation $ax^2 + 2hxy + by^2 = 0$ coincides with one of those given by $a'x^2 + 2h'xy + b'y^2 = 0$ and the other lines represented by them be perpendicular, show that $\frac{ha'b'}{b'-a'} = \frac{h'ab}{b-a}$.

7. Prove that $\binom{n}{0}\binom{m}{n} + \binom{n}{1}\binom{m+1}{n} + \binom{n}{2}\binom{m+2}{n} + \ldots\ldots$ to $(n+1)$ terms

$$= \binom{n}{0}\binom{m}{0} + \binom{n}{1}\binom{m}{1}2 + \binom{n}{2}\binom{m}{2}2^2 + \ldots\ldots \text{ to } (n+1) \text{ terms}$$

8. If $n \ge 2$ and $I_n = \int_{-1}^{1} (1-x^2)^n \cos mx\, dx$, then show that $m^2 I_n = 2n(2n-1)I_{n-1} - 4n(n-1)I_{n-2}$.

9. Find the sum to infinite terms of the series : $\frac{3}{4} + \frac{5}{36} + \frac{7}{144} + \frac{9}{400} + \frac{11}{900} + \ldots\ldots\ldots\ldots \infty$.

10. ABC is a triangle inscribed in a circle. Two of its sides are parallel to two given straight lines. Show that the locus of foot of the perpendicular from the centre of the circle on to the third side is also a circle, concentric to the given circle.

■

By : **Shailendra Maheshwari**, Career point, Kota

6. $b^2+c^2-2bc\cos\alpha=a^2$ ...(1)

By the given hypothesis

$$a^2=Rr=\frac{2s}{a+b+c}\cdot\frac{a}{2\sin A}=\frac{a}{(a+b+c)\sin A}\times$$

$$\frac{1}{2}bc\sin A=\frac{abc}{2(a+b+c)}$$

or $bc-2a(b+c)=2a^2$ ...(2)

we rewrite (1) as

$$(b+c)^2-2bc(1+\cos\alpha)=a^2$$

and substituting the expression for bc from (2), we have a quadratic in $b+c$ i.e.

$$(b+c)^2-8a\cos^2\frac{\alpha}{2}(b+c)-\left(8a^2\cos^2\frac{\alpha}{2}+a^2\right)=0$$

or $b+c=a\left(1+8\cos^2\frac{\alpha}{2}\right)$ ...(3)

we reject the – ve sign, it has no meaning, using (3) in (2), we get

$bc=4a^2\left(1+4\cos^2\frac{\alpha}{2}\right)$ ...(4)

from (3) and (4), it is clear that b and c are the roots of the equation.

$$x^2-a\left(1+8\cos^2\frac{\alpha}{2}\right)x+4a^2\left(1+4\cos^2\frac{\alpha}{2}\right)=0$$

Solving the equation, we get the required result.

7. The necessary condition is met if the numbers $f(1)$ and $f(2)$ are negative i.e.

$$f(x)=x^2+mx+m^2+6m$$

$f(1)=m^2+7m+1\le 0$ ...(1)

$f(2)=m^2+8m+4\le 0$ ...(2)

Solving (1) and (2) on the number line, we get

$$-\frac{7+3\sqrt{5}}{2}\le m\le -4+2\sqrt{3}.$$

8. (a) We denote $\sin x$ by y in the second equation, we have

$$2y^2-y=0$$

$y_1=0,\ y_1=\frac{1}{2}$. We replace $\sin 3x=3y-4y^3$ then the first equation becomes

$\left[4y^2-(4-2|a|)y+a-3\right]y=0$ ...(1)

We find values of a such that (1) has the roots 0, $\frac{1}{2}$ and its third root is either 0, $\frac{1}{2}$ or > 1 in absolute term.

$y=0$ is a root of (1), hence (1) becomes

$4y^2-(4-2|a|)y+a-3=0$ ...(2)

(2) must have $\frac{1}{2}$ as root. Substituting $y=\frac{1}{2}$ in (2), we find $y=\frac{1}{2}$ is root if $a=|a|$ or $a\ge 0$.

The second root is $\frac{a-3}{2}\Rightarrow$

(i) $\frac{a-3}{2}=0$ (ii) $\frac{a-3}{2}=\frac{1}{2}$ (iii) $\left|\frac{a-3}{2}\right|>1$

(Remember $a\ge 0$)

Thus we have

$$a=3,\ a=4,\ 0\le a<1,\ a>5.$$

(b) Since $\cos^7 x\le\cos^2 x$ and $\sin^4 x\le\sin^2 x$, the left member of the given equation does not exceed unity and is equal to unity only when the equality occurs in both the above inequalities. It means the following system should hold i.e.

$\cos^7 x=\cos^2 x$ ...(A)

$\sin^4 x=\sin^2 x$ ...(B)

(A) is satisfied by $\cos x=0$ and $\cos x=1$. (B) is also satisfied by these equation i.e. $\cos x=0\Rightarrow\sin^2 x=1$. $\cos x=1\Rightarrow\sin x=0$. Hence the solution is

$$x=\frac{\pi}{2}+n\pi;\ x=2n\pi$$

where n is any integer.

9. The value of the determinant is not zero. Thus it represents the area of a Δ whose vertices are $A(x_1, y_1)$, $B(x_2, y_2)$, $C(x_3, y_3)$. The lengths of the sides are a, b, c. The result is obvious.

10. If $A+B+C=\pi$, then

(i) $\cos 2A+\cos 2B+\cos 2C=-1-4\cos A\cos B\cos C$ ...(1)

Now, $\cos A\cos B\cos C$ is maximum if $A=B=C=60°$

$\therefore$ maximum of $\cos A\cos B\cos C=\frac{1}{8}$.

Minimum value of

$$\cos 2A+\cos 2B+\cos 2C=-1-\frac{1}{2}=-\frac{3}{2}$$

$\therefore\ \cos 2A+\cos B+\cos C>-\frac{3}{2}$.

(ii) $\cos A+\cos B+\cos C=1+4\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ ...(2)

$\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}$ has maximum value if $\frac{A}{2}=\frac{B}{2}=\frac{C}{2}=30°$

Maximum value of $\sin\frac{A}{2}\sin\frac{B}{2}\sin\frac{C}{2}=\frac{1}{8}$.

$\therefore\ \cos A+\cos B+\cos C<\frac{3}{2}$

$\sin\frac{A}{2}\sin\frac{B}{2}+\cos\frac{C}{2}$ is always +ve in ΔABC.

$\therefore\ \cos A+\cos B+\cos C>1$. Hence the result.

■■

6^{th} Mathematical Challenge *for* I.I.T. MAINS

This section is designed to give IIT JEE aspirants a thorough grinding & exposure to variety of possible twists and turns of problems in mathematics that would be very helpful in facing IIT JEE. Each and every problem is well thought of in order to strengthen the concepts and we hope that this section would prove a rich resource for practicing challenging problems and enhancing the preparation level of IIT JEE aspirants.
The detailed solutions to these problems will be published in the next issue alongwith a new set of such problems.

1. Given that $\phi(x) = \frac{(x-b)(x-c)}{(a-b)(a-c)}f(a) + \frac{(x-c)(x-a)}{(b-c)(b-a)}f(b) + \frac{(x-a)(x-b)}{(c-a)(c-b)}f(c) - f(x)$ where $a < c < b$ and $f''(x)$ exists at all points in (a, b). Prove that there exists a number μ, $a < \mu < b$, such that $\frac{f(a)}{(a-b)(a-c)} + \frac{f(b)}{(b-c)(b-a)} + \frac{f(c)}{(c-a)(c-b)} = \frac{1}{2}f''(\mu)$.

2. An unbiased die is tossed until it lands the same way up twice running. Find the probability that it requires r tosses.

3. Given the base of a triangle and the sum of its sides prove that the locus of the centre of its incircle is an ellipse.

4. Let $f(x) = ax^2 + bx + c$ & $g(x) = cx^2 + bx + a$, such that $|f(0)| \le 1, |f(1)| \le 1$ and $|f(-1)| \le 1$, prove that $|f(x)| \le 5/4$ and $|g(x)| \le 2$.

5. In order to find the dip of an oil bed below the surface of the ground, vertical borings are made from the angular points, A, B, C of a triangle ABC which is in horizontal plane. The depth of the bed at these points are found to be x, $x + y$ and $x + z$ respectively. Show that the dip θ (angle with horizontal) of the oil bed which is assumed to be a plane is given by $\tan\theta \cdot \sin A = \sqrt{\frac{y^2}{c^2} + \frac{z^2}{b^2} - \frac{2yz}{bc}\cos A}$ where b and c are the lengths of the sides CA and AB respectively and A is the angle between CA and AB.

6. Evaluate : $\int \frac{\cos 8x - \cos 7x}{1 + 2\cos 5x} dx$.

7. Let $f(x)$ be an even function such that $f'(x)$ is continuous, find y for which $\frac{d^2y}{dx^2} = \int_{-x}^{x} f(t)dt$.

8. Prove the inequality $(a^\alpha + b^\alpha)^{1/\alpha} < (a^\beta + b^\beta)^{1/\beta}$, for $a > 0$, $b > 0$ & $\alpha > \beta > 0$.

9. A circle of radius 1 rolls (without sliding) along the x-axis so that its centre is of the form $(t, 1)$ with increasing. A certain point P touches the x-axis at the origin as the circle rolls. As the circle rolls further the point P passes through the point $(x, 1/2)$. Find x, when it passes through $(x, 1/2)$ first time.

10. Find all positive integers n for which $\sqrt{n-1} + \sqrt{n+1}$ is rational.

By : **Shailendra Maheshwari**, Career point, Kota

Mathematics Olympia for IIT-JEE 2004 (Mains)

By : Er. Akhlak Ahmad, ABC Classes, Gorakhpur

1. Let T_n, $n = 1, 2, 3, 4$ represent four distinct positive real numbers other than unity such that each pair of the logarithm of T_n and the reciprocal of logarithm denotes a point on a circle, whose centre lies on y-axis. Prove that $T_1 + T_2 + T_3 + T_4 \ge 4$.

2. A circle passes through the points $A(3, 4)$, $B(-3/5, -4/5)$ and $C(-6/5, 8/5)$. Find the point which lies on the circle and also lies on the line joining point C and the origin.

3. Through the point $P(x_0, y_0, z_0)$ a plane is drawn at right angles to OP meeting the axes in A, B, C. Prove that the area of ΔABC is $\dfrac{t_0^5}{2x_0y_0z_0}$, t_0 being the length OP and O is origin.

4. A variable plane forms a tetrahedron of constant volume v_0 with the three co-ordinate planes. Find the locus of centroid of tetrahedron.

5. Let $\vec{r}$ is a position vector of a variable point in Cartesian OXY plane such that $\vec{r}\cdot(10\hat{j} - 8\hat{i} - \vec{r}) = 40$ and $p_1 = \max\left\{\left|\vec{r} + 2\hat{i} - 3\hat{j}\right|^2\right\}$, $p_2 = \min\left\{\left|\vec{r} + 2\hat{i} - 3\hat{j}\right|^2\right\}$.

A tangent line is drawn to the curve $y = 8/x^2$ at point A with abscissa 2 if $p_1 + p_2 \in$ even otherwise a normal line is drawn at the same point. The drawn line cuts x-axis at a point B. Find $\overrightarrow{AB}\cdot\overrightarrow{OB}$.

6. In a triangle if sides a, b be constant and the base angles A, B vary. Show that $\dfrac{dA}{dB} = \sqrt{\dfrac{a^2 - b^2\sin^2 A}{b^2 - a^2\sin^2 B}}$.

7. Find the area between the curves $y = x^2 + x - 2$ and $y = 2x$, for which $|x^2 + x - 2| + |2x| = |x^2 + 3x - 2|$ is satisfied.

8. Evaluate : $I = \displaystyle\int_{-1/2}^{1/2} \frac{\cos^{-1}(4x^3 - 3x)}{\sqrt{1 - x^2}}\,dx.$

9. Let $f_n(x) = \dfrac{f_{n-1}(x)}{1 + a_{n-1} f_{n-1}(x)}$ where $n \in N$. If $f_0(x) = \pi^x$ and $\lambda = \sum\limits_{i=0}^{n-1} a_i$, then prove that $f_n'(x) = \dfrac{\pi^x \ln\pi}{(1 + \lambda\pi^x)^2}$.

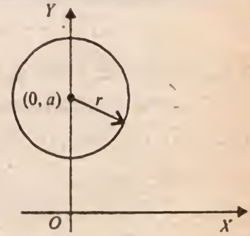
10. Consider the circle $\left|z - a - \dfrac{b}{2}i\right|^2 + |z|^2 = \dfrac{4a^2 + b^2}{4}$ on argand plane where $a, b \in \mathbb{R}$. Find conditions on a and b if two distinct chords each bisected by real axis can be drawn to the circle from $a + \dfrac{b}{2}i$.

SOLUTION

1. Equation of circle is

$$x^2 + (y - a)^2 = r^2$$

$$x^2 + y^2 - 2ay + a^2 - r^2 = 0$$



in general point $\left(\ln T_i, \dfrac{1}{\ln T_i}\right)$ lies on circle

$$(\ln T_i)^2 + \frac{1}{(\ln T_i)^2} - \frac{2a}{\ln T_i} + a^2 - r^2 = 0$$

Simplifying,

$$(\ln T_i)^4 + (0)(\ln T_i)^3 + (a^2 - r^2)(\ln T_i)^2 - 2a(\ln T_i) + 1 = 0$$

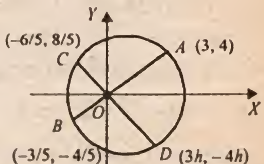
Sum of roots = 0, $\quad \ln T_1 + \ln T_2 + \ln T_3 + \ln T_4 = 0$

$$\ln(T_1\cdot T_2\cdot T_3\cdot T_4) = 0 \Rightarrow T_1\cdot T_2\cdot T_3\cdot T_4 = 1$$

A.M. $\ge$ G.M., $\quad \dfrac{T_1 + T_2 + T_3 + T_4}{4} \ge (T_1\,T_2\,T_3\,T_4)^{1/4}$

$$\Rightarrow T_1 + T_2 + T_3 + T_4 \ge 4.$$

2. The point A, B lies on $y = (4/3)x$ (which passes through origin)



($\because$ $h > 0$ it lies in 4th quadrant)

Equation of CO, $y = -\dfrac{4}{3}x$

$AO = 5$, $BO = 1$, $CO = 2$,

$DO = 5h$; $AO\cdot OB = CO\cdot OD$;

$5\times 1 = 2\times 5h \Rightarrow h = \dfrac{1}{2}$; $\quad D = \left(\dfrac{3}{2}, -2\right)$.

3. The direction ratios of OP are

$$(x_0-0,\ y_0-0,\ z_0-0)\ \text{i.e. } (x_0, y_0, z_0)$$

$OP = t_0 \Rightarrow$ direction cosines of OP are $\left(\frac{x_0}{t_0}, \frac{y_0}{t_0}, \frac{z_0}{t_0}\right)$

Hence equation of plane is

$$\frac{x_0}{t_0}x + \frac{y_0}{t_0}y + \frac{z_0}{t_0}z = t_0 \quad (\because\ lx+my+n = p)$$

hence we have $\dfrac{x}{\left(\frac{t_0^2}{x_0}\right)} + \dfrac{y}{\left(\frac{t_0^2}{y_0}\right)} + \dfrac{z}{\left(\frac{t_0^2}{z_0}\right)} = 1$

$$\Rightarrow OA = \frac{t_0^2}{x_0},\ OB = \frac{t_0^2}{y_0},\ OC = \frac{t_0^2}{z_0}$$

Now, projection of ΔABC on the y–z plane in the ΔOBC;

whose area $= \frac{1}{2}(OB)(OC) = \frac{1}{2}\frac{t_0^2}{y_0}\cdot\frac{t_0^2}{z_0} = \frac{t_0^4}{2y_0z_0}$

$\Rightarrow$ Let area of $\Delta ABC = A$

$A\cdot\cos\alpha = \dfrac{t_0^4}{2y_0z_0}$ but $\cos\alpha = \dfrac{x_0}{t_0} \Rightarrow A = \dfrac{t_0^5}{2x_0y_0z_0}$.

4. Let the variable plane be $\frac{x}{\alpha}+\frac{y}{\beta}+\frac{z}{\gamma} = 1$, where $OA = \alpha$, $OB = \beta$, $OC = \gamma$. Then vertices of O, A, B, C of the tetrahedron are respectively $(0, 0, 0)$, $(\alpha, 0, 0)$, $(0, \beta, 0)$, $(0, 0, \gamma)$. Let centroid of tetrahedron be (x_1, y_1, z_1).

Then $x_1 = \frac{1}{4}(0+\alpha+0+0) \Rightarrow \alpha = 4x_1,\ \beta = 4y_1,\ \gamma = 4z_1$

volume $= v_0$

$$\frac{1}{6}\begin{vmatrix}\alpha & 0 & 0\\ 0 & \beta & 0\\ 0 & 0 & \gamma\end{vmatrix} = v_0 \Rightarrow \alpha\beta\gamma = 6v_0$$

$(4x_1)(4y_1)(4z_1) = 6v_0 \Rightarrow 32x_1y_1z_1 = 3v_0$

Locus : $32xyz = 3v_0$.

5. $\vec{r} = x\hat{i} + y\hat{j}$ (in a plane)

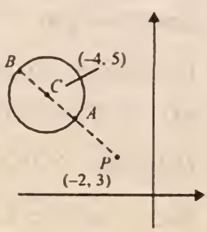
$\vec{r}\cdot(10\hat{j} - 8\hat{i} - \vec{r}) = 40$, gives

$x^2+y^2+8x-10y+40 = 0$

it is a circle with centre $(-4, 5)$ and radius $= 1$

$p_1 = \max\{(x+2)^2+(y-3)^2\}$

$p_2 = \min\{(x+2)^2+(y-3)^2\}$



$CP = 2\sqrt{2}$, $CA = 1$

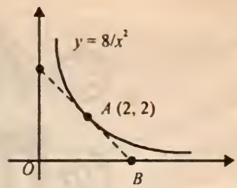
(x, y) is any point on the circle.

$\Rightarrow p_2 = AP^2 = (2\sqrt{2}-1)^2$

$\Rightarrow p_1 = BP^2 = (2\sqrt{2}+1)^2$,

$p_1 + p_2 = 18$ (even)

$\Rightarrow$ tangents is drawn.



Slope of $AB = \left(\frac{dy}{dx}\right)_{(2,\,2)} = -2$

Equation of AB, $2x + y = 6$

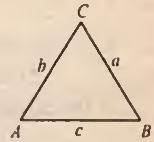
$B = (3, 0)$

$\overrightarrow{OA} = 2\hat{i}+2\hat{j};\ \overrightarrow{OB} = 3\hat{i},\ \overrightarrow{AB} = \overrightarrow{OB} - \overrightarrow{OA}$

$\overrightarrow{AB} = \hat{i} - 2\hat{j},\quad \overrightarrow{AB}\cdot\overrightarrow{OB} = (\hat{i}-2\hat{j})(3\hat{i}) = 3.$

6. By sine rule; $\dfrac{\sin A}{a} = \dfrac{\sin B}{b}$

differentiating w.r.t. B, $\dfrac{\cos A}{a}\dfrac{dA}{dB} = \dfrac{\cos B}{b}$



$$\frac{dA}{dB} = \frac{a}{b}\frac{\cos B}{\cos A} = \sqrt{\frac{a^2\cos^2 B}{b^2\cos^2 A}}$$

$$\frac{dA}{dB} = \sqrt{\frac{a^2(1-\sin^2 B)}{b^2(1-\sin^2 A)}}$$

$$= \sqrt{\frac{a^2-a^2\sin^2 B}{b^2-b^2\sin^2 A}} = \sqrt{\frac{a^2-b^2\sin^2 A}{b^2-a^2\sin^2 B}}.$$

$(\because\ b\sin A = a\sin B)$

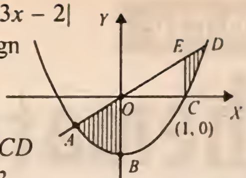
7. $|x^2 + x - 2| + |2x| = |x^2 + 3x - 2|$

$\Rightarrow x^2 + x - 2,\ 2x$ have same sign

$\because |A| + |B| = |A + B|$

The required region is shown

A = area of $OABO$ + area of ECD



$$= \int_{-1}^{0} [2x - (x^2 + x - 2)]\,dx + \int_{1}^{2} [2x - (x^2 + x - 2)]\,dx$$

$A = 7/3$.

8. We known that

$$\cos^{-1}(4x^3 - 3x) = \begin{cases} 3\cos^{-1}x - 2\pi & -1 \le x \le -\frac{1}{2} \\ 2\pi - 3\cos^{-1}x & -\frac{1}{2} \le x \le \frac{1}{2} \\ 3\cos^{-1}x & \frac{1}{2} \le x \le 1 \end{cases}$$

$$I = \int_{-1/2}^{1/2} \left(\frac{2\pi - 3\cos^{-1}x}{\sqrt{1-x^2}}\right) dx$$

$$= \frac{2\pi^2}{3} + \frac{\pi^2}{2} = \frac{4\pi^2 + 3\pi^2}{6} = \frac{7\pi^2}{6}$$

For I_1 : put $\cos^{-1}x = t,\ -\dfrac{dx}{\sqrt{1-x^2}} = dt$

$$I_1 = +3\int_{\cos^{-1}(-1/2)}^{\cos^{-1}(1/2)} t\,dt;\ I_1 = 3\int_{\pi/3}^{2\pi/3} t\,dt$$

$$= 3\left(\frac{t^2}{2}\right)_{\pi/3}^{2\pi/3} = \frac{\pi^2}{2}.\ \left(\because \cos^{-1}\frac{1}{2} = \frac{\pi}{3},\ \cos^{-1}\left(-\frac{1}{2}\right) = \frac{2\pi}{3}\right)$$

9. We have $a_{n-1} = \dfrac{1}{f_n(x)} - \dfrac{1}{f_{n-1}(x)}$

$$\therefore\ a_0 + a_1 + a_2 + \ldots + a_{n-1} = \frac{1}{f_n(x)} - \frac{1}{f_0(x)}$$

$$\lambda = \frac{1}{f_n(x)} - \frac{1}{\pi^x};\ \lambda + \frac{1}{\pi^x} = \frac{1}{f_n(x)};\ \frac{\lambda\pi^x + 1}{\pi^x} = \frac{1}{f_n(x)}$$

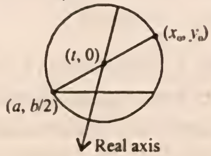
$$\Rightarrow f_n(x) = \frac{\pi^x}{(\lambda\pi^x + 1)} \Rightarrow f_n'(x) = \frac{\pi^x \ln\pi}{(1 + \lambda\pi^x)^2}.$$

10. The circle can be, if $z = x + iy$

$$\left|(x-a) + i\left(y - \frac{b}{2}\right)\right|^2 + |x + iy|^2 = \frac{4a^2 + b^2}{4}$$

it is very easy to see that

$$z = a + \frac{b}{2}i$$



satisfies the equation of circle.

$$\frac{x_0 + a}{2} = t \Rightarrow x_0 = 2t - a$$

$y_0 = -\dfrac{b}{2}$, (x_0, y_0) lies on circle.

$$(x-a)^2 + \left(y - \frac{b}{2}\right)^2 + x^2 + y^2 = \frac{4a^2 + b^2}{4}$$

$$(2t - 2a)^2 + b^2 + (2t - a)^2 + \frac{b^2}{4} = a^2 + \frac{b^2}{4}$$

$$8t^2 - 12ta + 4a^2 + b^2 = 0$$

$\because$ t is real and distinct (as chords are real and distinct)

$B^2 - 4AC > 0,\ 144a^2 - 4\cdot 8\cdot(4a^2 + b^2) > 0$

$a^2 - 2b^2 > 0 \Rightarrow a^2 > 2b^2$. ■

Probability

10 Mathematical Challenges

– *By : Prof. B.L. Sharma, Jaipur*

1. A marks a piece of paper with either a plus or a minus sign, the probability of his writing a plus sign is 1/3. A passes the slip to B who may either leave it alone or change the sign before passing it to C. Next C passes to a referee after perhaps changing the sign. The referee sees a plus sign on the slip. It is known that B and C each change the slip with probability 2/3. Find the probability that A originally wrote a plus sign.

2. A man has three coins A, B and C, A is unbiased; the probability that a head will result when B is tossed is 2/3; the probability that a head will result when C is tossed is 1/3. If one of the coins, chosen at random, is tossed three times, giving a total of two heads and one tail, find

(i) the probability that the chosen coin was A

(ii) the probability that a fourth toss of the same coin will give a head.

3. Five bags each contain 4 white and 8 black balls and two other each contains 10 white and 10 black balls. A ball is drawn at random from each bag. Calculate the probability that the 7 balls drawn include exactly three white balls. Find whether the probability of this event will altered if all the balls were into one bag and seven balls are drawn simultaneously at random from the bag.

4. An air battle between a bomber and two attaching fighter, is going on. The bomber is the first to fire; it fires once at each fight, and brings it down with probability p_1. If a fighter is not brought down, it fires at the bomber irrespective of the denstiny of the other fighter and brings it down with probability p_2. Find the probability of the following events.

A = {the bomber is brought down}
B = {both fighter are brought down}
C = {at least one fighter is brought down}
D = {at least one aircraft is brought down}
E = {exactly one fighter is brought down}
F = {exactly one aircraft is brought down}

5. **(a)** A fair coin is tossed 15 times. What is the probability of getting heads exactly as many times in the first ten throws as in the last throws.

(b) There are n test papers, each of which contains two questions. A student does not know the answers to all the $2n$ questions but only knows those of $k < 2n$ questions. Find the probability p that he will pass the test if he either answers both questions in one test papers chosen at random or one question in his paper and one question (which the examiner chosen) in another paper.

6. **(a)** A card is drawn from a pack, the cards is specified and the pack shuffled. If this is done six times, what is the chance that the cards drawn are 2 hearts, 2 diamonds, and two black cards.

(b) A 'hand' of six cards is dealt from a pack in the ordinary way. Find the probability that it consists of 2 hearts, 2 diamonds and 2 black cards.

7. **(a)** From a bag containing p balls, m balls are drawn simultaneously and replaced, then n balls are drawn. Find the probability that exactly r balls are common to two drawings.

(b) A bag contains 6 black balls and unknown numbers not greater than 6 of white balls; three balls are drawn successively and not replaced and all found to be white. Prove that the probability that a black ball be drawn next is 677/909.

8(a). A player tosses a coin and is to score one point for every head turned up and two for every tail. He is to play on until his scene reaches or passes n. If P_n is the probability for obtaining exactly n points, show that $P_n = \frac{1}{2}(P_{n-1} + P_{n-2})$ and hence find the value of P_n.

(b) Each of the n urns contains a white and b black

balls. A ball is selected randomly into the second, another one from the second to the third and so on. Finally, a ball is drawn at random from the n^{th} urn. What is the probability to be white? How does the probability change when the first ball transferred is known to be white.

9. **(a)** An urn contains n balls each of different colour of which one is white. Two independent observers, each with probability 0.1 of telling the truth, assert that a ball drawn from the urn is white. Prove that the probability that the ball is, infact white is $\frac{n-1}{n+80}$. Also show that if $n < 20$, this probability is less than the probability that at least one of the observers is talking the truth.

(b) We have two urns. The first contains m white and n black balls and the second contains p white and q black balls. Three balls are drawn from the first urn and put into the second (it is assumed that $m \geq 3$, $n \geq 3$). Then a ball is drawn from the second urn. Find the probability that the ball is white.

10. **(a)** In a game of craps, each player in turn acts as 'banker'. The banker throws with two dice numbered 1 to 6. If he throws 7 or 11 he wins. If he throws 2, 3, or 12 he loses. If he throws any other number, he throws again and continues to throw until either the number he threw first or 7 turns up. In the first case he wins, in the second case he loses. Show that the odds against the bank are 251 to 244.

(b) The digits of a number are 1, 2, 3, 4, 5, 6, 7, 8, 9 written in any order. Show that the odds are 115 to 11 against the number being divisible by 11.

SOLUTIONS

1. $E = \{A_+ \cap B_+ \cap C_+,\ A_+ \cap B_- \cap C_+,\ A_- \cap B_- \cap C_+,\ A_- \cap B_+ \cap C_+\}$

$$P(E) = \frac{1}{3}\left[\frac{2}{3}\times\frac{2}{3}+\frac{1}{3}\times\frac{1}{3}\right]+\frac{2}{3}\left[\frac{1}{3}\times\frac{2}{3}+\frac{2}{3}\times\frac{1}{3}\right] = \frac{13}{3^3}$$

$P(A) = 1/3;\quad P(E/A) = \frac{2}{3}\times\frac{2}{3}+\frac{1}{3}\times\frac{1}{3} = \frac{5}{9}.$

We have to calculate $P(A/E)$.

$$P(A/E) = \frac{P(A)P(E/A)}{P(E)} = \frac{(1/3)\times(5/9)}{13/3^3} = \frac{5}{13}.$$

2.(i) Let E denote the event of giving two heads and one tail.

$$P(E/A) = {}^3C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right) = \frac{3}{8};\quad P(A) = P(B) = P(C) = \frac{1}{3}$$

$$P(E/B) = {}^3C_2\left(\frac{2}{3}\right)^2\left(\frac{1}{3}\right) = \frac{4}{9};\ P(E/C) = {}^3C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right) = \frac{2}{9}$$

We have to calculate

$$P(A/E) = \frac{P(A)P(E/A)}{P(A)P(E/A)+P(B)P(E/B)+P(C)P(E/C)}$$
$= 9/25.$

(ii) If the fourth toss of A will give head

$= \frac{1}{2}P(A)P(E/A)$ Similarly, for B and C,

i.e., $\frac{2}{3}P(B)P(E/B),\ \frac{1}{3}P(C)\,P(E/C)$

The required probability

$$= \frac{1}{2}\times\frac{1}{3}\times\frac{3}{8}+\frac{1}{3}\times\frac{4}{9}\times\frac{2}{3}+\frac{1}{3}\times\frac{2}{9}\times\frac{1}{3} = \frac{241}{1296}.$$

3. E_1 = {three white balls from the first five bags}
E_2 = {two white balls from the first five bags and one white ball from the last two bags}
E_3 = {one white ball from the first five bags and two white balls from the last two balls}
E = {seven balls drawn include exactly three white balls}

$$P(E_1) = {}^5C_3\left(\frac{1}{3}\right)^3\left(\frac{2}{3}\right)^2 \times {}^2C_2\left(\frac{1}{2}\right)^0\left(\frac{1}{2}\right)^2 = \frac{10}{3^5}$$

$$P(E_2) = {}^5C_2\left(\frac{1}{3}\right)^2\left(\frac{2}{3}\right)^3 \times {}^2C_1\left(\frac{1}{2}\right)\left(\frac{1}{2}\right) = \frac{40}{3^5}$$

$$P(E_3) = {}^5C_1\left(\frac{1}{3}\right)\left(\frac{2}{3}\right)^4 \times {}^2C_2\left(\frac{1}{2}\right)^2\left(\frac{1}{2}\right)^0 = \frac{20}{3^5}$$

$P(E) = P(E_1) + P(E_2) + P(E_3) = 70/243.$

In case all the balls are put in one bag. The probability will change.

The required probability $= \dfrac{{}^{40}C_3 \times {}^{60}C_4}{{}^{100}C_7}$.

4. The probability of one of the fighter bringing down the bomber $= (1 - p_1)p_2$

The probability that neither of them brings down the bomber $= [1 - (1 - p_1)p_2]^2$

$P(A) = 1 - [1 - (1 - p_1)p_2]^2$; $P(B) = p_1^2$

$P(C) = 1 - (1 - p_1)^2$; $P(D) = 1 - (1 - p_1)^2(1 - p_2)^2$

$P(E) = p_1(1 - p_1) + (1 - p_1)p_1 = 2p_1(1 - p_1)$

F_1 = {the bomber is brought down and both fighters are safe}

F_2 = {the first fighter is brought down and the second fighter and the bomber are safe}

F_3 = {the second fighter is brought down and the first fighter and the bomber are safe}

$P(F) = P(F_1) + P(F_2) + P(F_3)$
$= (1 - p_1)^2\,[1 - (1 - p_2)^2] + 2p_1(1 - p_1)(1 - p_2)]$

5(a) Sample space $= 2^{15}$

Favourable cases $= {}^{10}C_0 \cdot {}^5C_0 + {}^{10}C_1 \cdot {}^5C_1 + {}^{10}C_2 \cdot {}^5C_2 + {}^{10}C_3 \cdot {}^5C_3 + {}^{10}C_4 \cdot {}^5C_4 + {}^{10}C_5 \cdot {}^5C_5 = 3003$

Required probability = $\dfrac{3003}{2^{15}}$.

(b) The hypothesis are

H_1 = {the student knows the answer to both questions in his paper}

H_2 = {the student knows the answer to one question out of two questions in his paper}

$$p=\frac{k(k-1)}{2n(2n-1)}\cdot 1+\frac{2k(2n-k)}{2n(2n-1)}\cdot\frac{k-1}{2n-2}$$
$$=\frac{k(k-1)(3n-k-1)}{2n(2n-1)(n-1)}.$$

6. **(a)** A = {a heart is drawn}, $P(A)$ = $\dfrac{13}{52}=\dfrac{1}{4}$

B = {a diamond is drawn}, $P(B)$ = $\dfrac{13}{52}=\dfrac{1}{4}$

C = {a black card is drawn}, $P(C)$ = $\dfrac{26}{52}=\dfrac{1}{2}$

The experiment is repeated six times. It is a question of generalised binomial experiment.

Required probability = $\dfrac{6!}{2!2!2!}\left(\dfrac{1}{4}\right)^2\left(\dfrac{1}{4}\right)^2\left(\dfrac{1}{2}\right)^2=\dfrac{45}{512}$.

(b) P (first card dealt is a heart) = $\dfrac{13}{52}=\dfrac{1}{4}$

P(second card dealt is a heart) = $\dfrac{12}{51}=\dfrac{4}{17}$

P(third card dealt is a diamond) = 13/50

P(fourth card dealt is a diamond) = 12/49

P(fifth card dealt is a black) = 26/48

P(sixth card dealt is a black) = 25/47

Hence the probability that 2 hearts, 2 diamonds and 2 black cards are dealt in this or any specified order is

$$\frac{13}{52}\cdot\frac{12}{51}\cdot\frac{13}{50}\cdot\frac{12}{49}\cdot\frac{26}{48}\cdot\frac{25}{44}$$

Required probability

$$=\frac{6!}{2!2!2!}\times\frac{13}{52}\times\frac{12}{51}\times\frac{13}{50}\times\frac{12}{49}\times\frac{26}{48}\times\frac{25}{47}=\frac{7605}{78302}$$

7. **(a)** Sample space = ${}^pC_n$

Favourable cases = ${}^mC_r\times{}^{p-m}C_{n-2}$

Required probability = $\dfrac{{}^mC_r\times{}^{p-m}C_{n-r}}{{}^pC_n}$.

(b) We consider seven bags.

B_1	B_2	B_3	B_4	B_5	B_6	B_7
$B=6$	$B=6$	$B=6$	$B=6$	$B=6$	$B=6$	$B=6$
$W=0$	$W=1$	$W=2$	$W=3$	$W=4$	$W=5$	$W=6$

$P(B_1)=P(B_2)=P(B_3)=P(B_4)=P(B_5)=P(B_6)=P(B_7)=1/7$.

E be the event of drawing 3 white balls.

$P(E/B_1) = P(E/B_2) = P(E/B_3) = 0$

$$P(E/B_4)=\frac{3}{{}^9C_3}=\frac{1}{84};\quad P(E/B_5)=\frac{4}{{}^{10}C_3}=\frac{1}{30}$$
$$P(E/B_6)=\frac{{}^5C_3}{{}^{11}C_3}=\frac{2}{33};\quad P(E/B_7)=\frac{{}^6C_3}{{}^{12}C_3}=\frac{1}{11}$$
$$P(E)=\frac{1}{7}\left[0+0+0+\frac{1}{84}+\frac{1}{30}+\frac{2}{33}+\frac{1}{11}\right]=\frac{1}{7}\times\frac{909}{4620}.$$

Using Baye's theorem, we have

$P(B_1/E) = P(B_2/E) = P(B_3/E) = 0$,

$$P(B_4/E)=\frac{P(B_4)P(E/B_4)}{P(E)}=\frac{(1/7)\times(1/84)}{(1/7)\times(909/4620)}$$
$$= 55/909.$$

Similarly, $P(B_5/E)$ = 154/909, $P(B_6/E)$ = 280/909, $P(B_7/E)$ = 420/909.

Three white balls have not been replaced. The number of balls in the different boxes are

B_1	B_2	B_3	B_4	B_5	B_6	B_7
$B=6$	$B=6$	$B=6$	$B=6$	$B=6$	$B=6$	$B=6$
$W=0$	$W=1$	$W=2$	$W=0$	$W=1$	$W=1$	$W=3$

E_2 be the event of now drawing a black ball.

$P(E_2)=P(B_1/E)$ · drawing a black ball from B_1 + $P(B_2/E)$· drawing a black ball from B_2 + ... + $P(B_7/E)$ · drawing a black ball from B_7

$$=0+0+0+\frac{55}{909}+\frac{154}{909}\cdot\frac{6}{7}+\frac{280}{909}\cdot\frac{3}{4}+\frac{420}{909}\cdot\frac{2}{3}=\frac{677}{909}.$$

8. **(a)** There are two possible ways in which the player can score n points.

(i) He throws a head when he has $n - 1$ points.

(ii) He throws a tail when he has $n - 2$ points.

There are mutually exclusive events. Thus

$$p_n=\frac{1}{2}p_{n-1}+\frac{1}{2}p_{n-2}$$

Rearranging the terms, $p_n+\dfrac{1}{2}p_{n-1}=p_{n-1}+\dfrac{1}{2}p_{n-2}$

$$=p_{n-2}+\frac{1}{2}p_{n-3}=.....=p_2+\frac{1}{2}p_1.$$

Now $p_1=\dfrac{1}{2}$ and $p_2=\dfrac{1}{2}+\dfrac{1}{2}\cdot\dfrac{1}{2}=\dfrac{3}{4}$, since p_2 can be obtained by throwing a tail in the beginning or two heads successively in first throws.

$$p_n+\frac{1}{2}p_{n-1}=p_2+\frac{1}{2}p_1=1;\quad\therefore\ p_n-\frac{2}{3}=-\frac{1}{2}\left(p_{n-1}-\frac{2}{3}\right)$$

For $n - 1$, $n - 2$, ... 2

$$p_n-\frac{2}{3}=-\frac{1}{2}\left(p_{n-1}-\frac{2}{3}\right);\quad p_{n-1}-\frac{2}{3}=-\frac{1}{2}\left(p_{n-2}-\frac{2}{3}\right)$$

....

....

$$p_3-\frac{2}{3}=-\frac{1}{2}\left(p_2-\frac{2}{3}\right).....;\quad p_2-\frac{2}{3}=-\frac{1}{2}\left(p_1-\frac{2}{3}\right).....$$

Multiplying and cancelling the common terms equations, we have

$$p_n-\frac{2}{3}=\left(-\frac{1}{2}\right)^{n-1}\left(p_1-\frac{2}{3}\right) \quad \therefore \quad p_n=\frac{1}{3}\left[2+(-1)^n\frac{1}{2^n}\right].$$

(b) Let the urn be numbered 1, 2, ... n and P_r be the probability of drawing a white ball from the r^{th} urn.

(i) From the $(r-1)^{th}$ urn a black ball has been transferred.

(ii) From the $(r-1)^{th}$ urn a white ball has been transferred.

Thus, $P_r=P_{r-1}\frac{a+1}{a+b+1}+(1-P_{r-1})\frac{a}{a+b+1}$,

$r = 2, 3, ..., n$ ... (i)

We have $P_1 = a/(a+b)$

From (i), $P_2=P_1\cdot\frac{a+1}{a+b+1}+(1-P_1)\frac{a}{a+b+1}$

$$=\frac{a(a+1)}{(a+b)(a+b+1)}+\frac{ba}{(a+b)(a+b+1)}=\frac{a}{a+b}$$

$\therefore \quad P_n = a/(a+b)$.

In the second case $P_1 = 1$,

$$P_2=\frac{a+1}{a+b+1}=\frac{a}{a+b}+\frac{b}{(a+b)(a+b+1)}$$

$$P_3=\frac{(a+1)^2}{(a+b+1)^2}+\frac{ab}{(a+b+1)^2}=\frac{a}{a+b}+\frac{b}{(a+b)(a+b+1)^2}$$

Proceeding in this manner, we get

$$P_n=\frac{a}{a+b}+\frac{b}{(a+b)(a+b+1)^n}.$$

9. **(a)** Since the balls are of different colours we mark the balls $a_1, a_2, ..., a_n$.

We assume that a_1 is a white ball.

P (A says a_k/a_k *i.e.* the ball drawn is a_k and A says a_k)

$= 1/10 = P(B \text{ says } a_k/a_k)$

$P(A \text{ says } a_r/a_s) = P(B \text{ says } a_r/a_s) = \frac{9}{10}\times\frac{1}{n-1}$

$$\therefore \quad P(a_1/A, B \text{ says } a_1)=\frac{P(a_1,(A, B \text{ say } a_1))}{P(A, B \text{ say } a_1)}$$

$$=\frac{P(A, B \text{ say } a_1/a_1)P(a_1)}{\sum_{s=1}^{n} P(A, B \text{ say } a_1/a_s)\,P(a_s)}$$

$$=\frac{\left(\frac{1}{10}\right)^2\frac{1}{n}}{\left(\frac{1}{10}\right)^2\frac{1}{n}+(n-1)\left(\frac{9}{10}\right)^2\left(\frac{1}{n-1}\right)^2\frac{1}{n}}=\frac{n-1}{n+80}.$$

Probability of at least one of A and B telling the truth

$$=1-\frac{81}{100}=\frac{19}{100} \quad \therefore \quad \frac{19}{100}>\frac{n-1}{n+80} \Rightarrow n<20.$$

(b) We consider two hypothesis. E_1 = (the ball drawn from the second urn belonged to the first urn}

E_2 = {the ball drawn from the second urn belonged to the second urn}

$$\therefore \quad P(E_1)=\frac{3}{p+q+3}, \; P(E_2)=\frac{p+q}{p+q+3}$$

Then the conditional probability of the event A = {a white ball is drawn from the second urn} on the hypothesis E_1 and E_2.

Note: The probability of an appearance of a white ball belonging to the first urn doesnot depend on whether the ball was drawn directly from the first urn or it had been transferred to the second urn.

$$P(A/E_1)=\frac{m}{m+n}, \; P(A/E_2)=\frac{p}{p+q}$$

$$P(A)=\frac{3}{p+q+3}\cdot\frac{m}{m+n}+\frac{p+q}{p+q+3}\cdot\frac{p}{p+q}.$$

10. **(a)** E_1 = {the event that banker wins}

E_2 = {the event that banker loses}

$P(E_1) = P(7) + P(11) = 8/36$

In the first throw he may lose.

$P(E_2) = P(2) + P(3) + P(12) = 4/36$

He throws any other number, 4, 10, 5, 9, 6, 8.

$P(4) = P(10) = 3/36$; $P(5) = P(9) = 4/36$

$P(6) = P(8) = 5/36$

If he throws 4 initially, E_4 denotes the event of his win.

$$P(E_4)=\left(\frac{3}{36}\right)^2+\left(\frac{3}{36}\right)\left(\frac{27}{36}\right)\left(\frac{3}{36}\right)+\left(\frac{3}{36}\right)\left(\frac{27}{36}\right)^2\left(\frac{3}{36}\right)+...$$

$$=\frac{1}{36}=P(E_{10})$$

Similarly, $P(E_5) = P(E_9) = 2/45$; $P(E_6) = P(E_8) = 5/36$

Probability of his win if he throws initially (4, 5, 6, 8, 9, 10) in the first throw.

$=P(E_4)+P(E_5)+P(E_6)+P(E_8)+P(E_9)+P(E_{10})$

$= 134/495. \quad P(\text{winning}) = \frac{8}{36}+\frac{134}{495}=\frac{224}{495}.$

Odds against the banker are 251 : 244.

(b) A number is divisible by 11, if the difference of the sum of its digit occupying even places and the sum of odd places is divisible by 11.

Sum = 1 + 2 + ... + 9 = 45

$x + y = 45$ ... (i) ; $x - y = 11, 22, 33$ ... (ii)

The students should show that $x - y \neq 22, 33$.

Solving (i) and (ii), $x = 28$, $y = 17$.

The following are the cases in which the sum of any four given number is 17.

9 + 5 + 2 + 1, 9 + 4 + 3 + 1, 8 + 6 + 2 + 1, 8 + 5 + 3 + 1, 8 + 4 + 3 + 2, 7 + 6 + 3 + 1, 7 + 5 + 4 + 1, 7 + 5 + 3 + 2, 6 + 5 + 4 + 2

Total 9 cases. In case $x - y = -11$, then $y = 28$.

There are two cases only: 9 + 8 + 7 + 4, 9 + 8 + 6 + 5

Favourable cases = 11 · 5! · 4! ; Sample space = 9!

$\text{Probability}=\frac{11\cdot 5!\cdot 4!}{9!}=\frac{11}{126}.$ Odds against = 115 : 11.

■

Very Similar
MODEL TEST PAPER
for KERALA PET - 2004

Exam on 26^{th} & 27^{th} April

1. In a, b, c are in A.P. and x, y, z are in G.P. Then x^{b-c}, y^{c-a}, z^{a-b} is equal to
(a) 1 (b) 2 (c) 0 (d) 3

2. The number of solution of the equations $|x|^2 + 3|x| + 2 = 0$ are
(a) 4 (b) 3 (c) 2 (d) 0

3. If the ratio of the roots of the equation $ax^2 + bx + c = 0$ is r then $\frac{(r+1)^2}{r}$ is equal to
(a) a^2/bc (b) b^2/ca (c) c^2/ab (d) $1/abc$

4. If α, β, γ are the roots of the equation $x^3 - px^2 + qx - r = 0$. Then $(\alpha + \beta)(\beta + \gamma)(\gamma + \alpha)$ is equal to
(a) $pq - r$ (b) $p - rq$
(c) $p(q + r)$ (d) none of these

5. If $x = a + b$, $y = a\omega + b\omega^2$, $z = a\omega^2 + b\omega$. Then $x^3 + y^3 + z^3$ is equal to
(a) $3(a^3 + b^3)$ (b) $3(a^3 - b^3)$
(c) 0 (d) $x^3 + y^3 + z^3 - 3$

6. Given 5 different green dyes, four different blue dyes and three different red dyes. The number of combinations of dyes which can be chosen taking at least one green and one blue dye is
(a) 3600 (b) 3720
(c) 3800 (d) none of these

7. If a, b, c are different and
$$\begin{vmatrix} a & a^2 & a^3-1 \\ b & b^2 & b^3-1 \\ c & c^2 & c^3-1 \end{vmatrix} = 0; \text{ Then}$$
(a) $abc = 1$ (b) $ab + bc + ca = 0$
(c) $a + b + c = 0$ (d) none of these

8. A Telegraph has 5 arms and each arm is capable of 4 distinct position including the position of rest. Then the total number of signals that can be made are
(a) 1022 (b) 2310 (c) 1023 (d) 2013

9. In a geometrical progression first term and common ratio are both $\left(\frac{\sqrt{3}+i}{2}\right)$. Then the absolute value of the nth term of the progression is
(a) 1 (b) 2 (c) 3/2 (d) 1/2.

10. In a right-angled triangle the hypotenuse is $2\sqrt{2}$ times the length of $\perp^r$ drawn from the opposite vertex on the hypotenuse, then the other two angles are
(a) $\pi/3$, $\pi/6$ (b) $\pi/4$, $\pi/4$
(c) $\pi/8$, $3\pi/8$ (d) $\pi/12$, $5\pi/12$

11. $2\sin^2 x + \sin 2x = 2$, $-\pi < x < \pi$ then x is equal to
(a) $\pm\pi/2$ (b) $\pm\pi/4$
(c) $\pm 3\pi/4$ (d) all of the above

12. The set of values of x for which
$$\frac{\tan 3x - \tan 2x}{1 + \tan 3x \tan 2x} = 1 \text{ is}$$
(a) ϕ (b) $\pi/4$
(c) $\{n\pi + (\pi/4), n = 1, 2, 3 ....\}$
(d) $2n\pi + \pi/4$, $n = 1, 2, 3$ .....

13. If $\Delta = a^2 - (b - c)^2$ where Δ is the area of the triangle ABC. Then $\tan A$ is equal to
(a) 15/16 (b) 8/15 (c) 8/17 (d) 1/2

14. If $c^2 + a^2 - b^2 = ac$. Then $\angle B$ is equal to
(a) $\pi/3$ (b) $\pi/4$ (c) $\pi/6$ (d) $\pi/2$.

15. If $c^2 = a^2 + b^2$ Then $4s(s-a)(s-b)(s-c)$ is equal to
(a) s^4 (b) b^2c^2 (c) c^2a^2 (d) a^2b^2

16. In an equilateral triangle of side $2\sqrt{3}$ cm, the circum radius is
(a) 1 cm (b) 2 cm
(c) $\sqrt{3}$ cm (d) none of these

17. If in a triangle ABC, $\cos A \cos B + \sin A \sin B \sin C = 1$ then the sides are proportional to
(a) $1 : 1 : \sqrt{2}$ (b) $1 : \sqrt{2} : 1$
(c) $\sqrt{2} : 1 : 1$ (d) none of these.

18. Area of the circle and the area of a regular polygon of n sides with perimeter equal to that of the circle are in the ratio
(a) $\tan\frac{\pi}{n} : \frac{\pi}{n}$ (b) $\cos\frac{\pi}{n} : \frac{\pi}{n}$
(c) $\sin\frac{\pi}{n} : \frac{\pi}{n}$ (d) $\cot\frac{\pi}{n} : \frac{\pi}{n}$

3. This gives 7 as quotient and 1 as the remainder.

The quotient 7 is added to the left hand column product and the remainder 1 is multiplied by the unit size 3 and then added to the right hand column product. Thus,

Left hand column product becomes $48 + 7 = 55$,

Right hand column product becomes $1 \times 3 + 2 = 5$.

6 yards	2 feet
8 yards	1 foot
55 yards	5 feet

The required area is 55 square yards, 5 square feet.

Example 9: To find the area of a rectangular plot with sides 9 feet 10 inches and 12 feet 9 inches.

As above we come across three products obtained from two columns after writing the lengths of the sides one below the other. We have ,

9 feet	10 inches
12 feet	9 inches

Three products are:

1. Vertical product of rightmost column, given by $10 \times 9 = 90$
2. Cross product of two columns $(9 \times 9) + (12 \times 10) = 201$
3. Vertical product of leftmost column, $12 \times 9 = 108$

We have,

9 feet	10 inches
12 feet	9 inches
108 / 201 / 90	

For the middle part $201 = 12 \times 16 + 9$, which is obtained by dividing the middle part by the unit size 12. This gives 16 as quotient and 9 as the remainder.

The quotient 16 is added to the left hand column product and the remainder 9 is multiplied by the unit size 12 and then added to the right hand column product. Thus,

Left hand column product becomes $108 + 16 = 124$,

Right hand column product becomes $9 \times 12 + 90 = 198$.

This is again larger than square unit 144 and hence $198 - 144 = 54$ is the answer in the rightmost column. Therefore 1 is added to the leftmost column giving $124 + 1 = 125$.

We get,

9 feet	10 inches
12 feet	9 inches
125 feet	54 inches

The required area is 125 square feet, 54 square inches.

to be contd...

10 Challenging Problems

Co-ordinate Geometry

By: **Rajesh Gupta, Jaipur**

1. The base of a triangle passes through a fixed point (f, g) and its sides are respectively bisected at right angles by the lines $y^2 - 8xy - 9x^2 = 0$. Determine the locus of its vertex.

2. If $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ and $ax^2 + 2hxy + by^2 - 2gx - 2fy + c = 0$ each represents a pair of straight lines, prove that the area of the parallelogram enclosed by them is $\dfrac{2|c|}{\sqrt{h^2 - ab}}$

3. Find the intervals of values of 'a' for which the line $x + y = 0$ bisects two chords drawn from a point $\left(\dfrac{1+\sqrt{2}\,a}{2}, \dfrac{1-\sqrt{2}\,a}{2}\right)$ to the circle

$2x^2+2y^2 - (1+\sqrt{2}\,a)\,x-(1-\sqrt{2}\,a)\,y = 0$

4. Prove that the limiting points of the coaxial system of circles given by $x^2+y^2+2gx+c+\lambda\ (x^2+y^2+2fy+k) = 0$ subtend a right angle at the origin if $\dfrac{k}{f^2}+\dfrac{c}{g^2} = 2$.

5. Prove that the shortest normal chord of the parabola $y^2 = 4ax$ is $6a\sqrt{3}$ and that its inclination to the axis is $\tan^{-1}(\sqrt{2})$

6. From a point A common tangents are drawn to the circle $x^2 + y^2 = \dfrac{a^2}{2}$ and parabola $y^2 = 4ax$. Find the area of the quadrilateral formed by the common tangents, the chord of contact of the circle and the chord of contact of the parabola.

7. From any point on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=4$, tangents are drawn to the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, prove that the normals at the points of contact meet on the ellipse $a^2x^2+b^2y^2=\dfrac{1}{4}\left(a^2-b^2\right)^2$

8. If α and β are the angles subtended by the major axis of an ellipse at the extremities of a pair of conjugate diameters, prove that $\cot^2\alpha + \cot^2\beta =$ constant.

9. Find the range of parameter 'a' for which a unique circle will pass through the points of intersection of the rectangular hyperbola $x^2 - y^2 = a^2$ and the parabola $y = x^2$. Find also the equation of the circle.

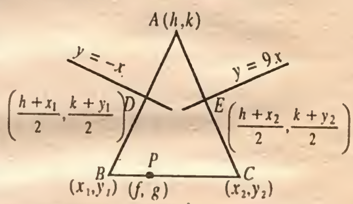
10. Find the locus of the poles of the normal chords of the rectangular hyperbola $xy = c^2$.

SOLUTIONS

1. Given equation gives two straight lines as

$y = -x$ and $y = 9x$

Let the vertex of triangle ABC be $A\ (h,\ k)$. Also let $B(x_1, y_1)$ and $C(x_2, y_2)$. So D and E are the mid points of sides AB and AC.



$$\therefore \quad D\left(\frac{h+x_1}{2}, \frac{k+y_1}{2}\right) \text{and } E\left(\frac{h+x_2}{2}, \frac{k+y_2}{2}\right)$$

Since D lies on $x+y = 0$ i.e. $\dfrac{h+x_1}{2}+\dfrac{k+y_1}{2}=0$

$\Rightarrow\ x_1+y_1 = -h-k$ ... (1)

and (Slope of AB)$\times(-1) = -1$

$\Rightarrow\ \dfrac{y_1-k}{x_1-h} = 1 \Rightarrow x_1-y_1 = h-k$ ... (2)

$\therefore$ (1) & (2) $\Rightarrow x_1 = -k$ and $y_1 = -h$.

Again E lies on $9x = y$ i.e. $9\left(\dfrac{x_2+h}{2}\right)=\dfrac{y_2+k}{2}$

$\Rightarrow\ -9x_2 + y_1 = 9h - k$ ... (3)

and (Slope of AC) $\times$ (9) $= -1$

$\Rightarrow\ \dfrac{y_2-k}{x_2-h}\cdot 9 = -1 \Rightarrow x_2 + 9y_2 = h + 9k$ ... (4)

$\therefore$ (3) & (4) $\Rightarrow x_2 = \dfrac{9k-40h}{41}$, $y_2 = \dfrac{40k+9h}{41}$

Also B, P, C are collinear then

$f(y_1 - y_2) + g(x_2 - x_1) + (x_1y_2 - x_2y_1) = 0$

$$\Rightarrow \quad -f\left(\frac{40k+9h}{41}+h\right)+g\left(\frac{9k-40h}{41}+k\right)$$

$$+(-k)\left(\frac{40k+9h}{41}\right)-\left(\frac{9k-40h}{41}\right)(-h)=0$$

$$\Rightarrow -f\left(\frac{40k+50h}{41}\right)+g\left(\frac{50k-40h}{41}\right)-\frac{40\left(h^2+k^2\right)}{41}=0$$

$\Rightarrow$ $4(h^2 + k^2) + h(5f - 4g) + k(4f - 5g) = 0$

$\therefore$ locus of (h, k) is

$4(x^2 + y^2) + x(5f - 4g) + y(4f - 5g) = 0$

2. Let $ax^2 + 2hxy + by^2 + 2gx + 2fy + c = 0$ ... (1)
represents two straight lines as

$y = m_1x + c_1$ and $y = m_2x + c_2$.

$\therefore$ $(y - m_1x - c_1)(y - m_2x - c_2) = 0$

$\Rightarrow$ $m_1m_2x^2 - xy(m_1 + m_2) + y^2 + x(m_1c_2 + m_2c_1)$
$- y(c_1 + c_2) + c_1c_2 = 0$ ... (2)

On comparing (1) and (2) we get

$$\frac{m_1m_2}{a}=\frac{-(m_1+m_2)}{2h}=\frac{1}{b}=\frac{m_1c_2+m_2c_1}{2g}=\frac{-(c_1+c_2)}{2f}=\frac{c_1c_2}{c}$$

$$\therefore \quad m_1+m_2=\frac{-2h}{b},\ m_1m_2=\frac{a}{b},\ c_1c_2=\frac{c}{b}$$

and let θ be the angle between the lines given by (1) is

$$\tan\theta=\frac{2\sqrt{h^2-ab}}{a+b} \quad \Rightarrow \sin\theta=\frac{2\sqrt{h^2-ab}}{\sqrt{(a-b)^2+4h^2}}$$

Area of the parallelogram = $\frac{p_1p_2}{\sin\theta}$, where p_1 and p_2 are the distances between the pairs of the parallel lines.

Now p_1 = distance between the lines $y = m_1x + c_1$ and $y = m_1x - c_1$

$$\Rightarrow \quad p_1=\frac{|2c_1|}{\sqrt{1+m_1^2}}, \text{ similarly } p_2=\frac{|2c_2|}{\sqrt{1+m_2^2}}$$

$$\therefore \quad p_1p_2=\frac{4c_1c_2}{\sqrt{(m_1m_2)^2+(m_1+m_2)^2-2m_1m_2+1}}$$

$$=\frac{\left|4\,{}^{c}\!/_{b}\right|}{\sqrt{\frac{a^2}{b^2}+\frac{4h^2}{b^2}-\frac{2a}{b}+1}} \Rightarrow p_1p_2=\frac{4c}{\sqrt{4h^2+(a-b)^2}}$$

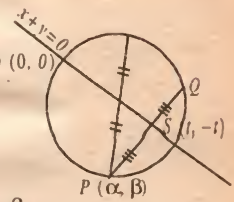
$$\therefore \quad S=\frac{p_1p_2}{\sin\theta}=\frac{|4c|}{\sqrt{4h^2+(a-b)^2}}\times\frac{\sqrt{(a-b)^2+4h^2}}{2\sqrt{h^2-ab}}$$

$$\Rightarrow \quad S=\frac{2|c|}{\sqrt{h^2-ab}}$$

3. $x^2 + y^2 - \left(\frac{1+\sqrt{2}a}{2}\right)x - \left(\frac{1-\sqrt{2}a}{2}\right)y = 0$

$\Rightarrow$ $x^2 + y^2 - \alpha x - \beta y = 0$ ... (1)

where $\alpha=\frac{1+\sqrt{2}a}{2}$ & $\beta=\frac{1-\sqrt{2}a}{2}$ Clearly circle is passing through (α, β) and let the mid point of PQ is $S(t, -t)$ then equation of chord is



$$T = S' \text{ i.e. } xt + y(-t) - \frac{\alpha}{2}(x+t)$$

$$-\frac{\beta}{2}(y-t) = t^2 + t^2 - t\alpha + t\beta$$

which is passing through (α, β), so

$$\alpha t-\beta t-\frac{\alpha}{2}(\alpha+t)-\frac{\beta}{2}(\beta-t)=2t^2-\alpha t+\beta t$$

$$\Rightarrow \quad 2t^2-t\left(\frac{3\alpha}{2}-\frac{3\beta}{2}\right)+\frac{\left(\alpha^2+\beta^2\right)}{2}=0$$

$\Rightarrow$ $4t^2 - 3(\alpha - \beta)t + (\alpha^2 + \beta^2) = 0$

Since t is real so $D>0$

$\Rightarrow$ $9(\alpha - \beta)^2 - 16(\alpha^2 + \beta^2) > 0$

$\Rightarrow$ $9(2a^2) - 8(1 + 2a^2) > 0$

$\Rightarrow$ $2a^2 - 8 > 0$

$\Rightarrow$ $a^2 > 4 \Rightarrow a \in (-\infty, -2) \cup (2, \infty)$

4. Given coaxial system of circles is

$(x^2 + y^2 + 2gx + c) + \lambda(x^2 + y^2 + 2fy + k) = 0$

$$\Rightarrow \quad x^2+y^2+\frac{2g}{1+\lambda}x+\frac{2f\lambda}{1+\lambda}y+\frac{c+k\lambda}{1+\lambda}=0 \quad \text{... (1)}$$

$$\therefore \quad \text{Centre is } \left(\frac{-g}{1+\lambda}, \frac{-f\lambda}{1+\lambda}\right)$$

$$\text{and radius } = \sqrt{\frac{g^2+f^2\lambda^2-(c+k\lambda)(1+\lambda)}{(1+\lambda)^2}}$$

For the limiting points radius = 0

$\Rightarrow$ $g^2 + f^2\lambda^2 - (c + k\lambda)(1 + \lambda) = 0$

$\Rightarrow$ $\lambda^2(f^2 - k) - \lambda(k + c) + (g^2 - c) = 0$ ... (2)

Let, λ_1 and λ_2 be the roots of this equation then

$$\lambda_1+\lambda_2=\frac{c+K}{f^2-k} \text{ and } \lambda_1+\lambda_2=\frac{g^2-c}{f^2-k}$$

If coordinates of limiting points $L_1\left(\frac{-g}{1+\lambda_1}, \frac{-f\lambda_1}{1+\lambda}\right)$

and $L_2\left(\frac{-g}{1+\lambda_2}, \frac{-f\lambda_2}{1+\lambda_2}\right)$ then according the problem

(slope of OL_1) × (slope of OL_2) = –1

$$\Rightarrow \frac{f\lambda_1}{g}\times\frac{f\lambda_2}{g}=-1 \quad \Rightarrow f^2(\lambda_1\lambda_2)=-g^2$$

$$\Rightarrow f^2\left(\frac{g^2-c}{f^2-k}\right)=-g^2 \quad \Rightarrow f^2(g^2-c)=-g^2(f^2-k)$$

$$\Rightarrow 2g^2f^2=g^2k+f^2c \quad \Rightarrow \frac{k}{f^2}+\frac{c}{g^2}=2$$

5. We know that if normal at $P(at_1^2, 2at_1)$ cuts the parabola at $Q\left(at_2^2, 2at_2\right)$, then $t_2=-t_1-\frac{2}{t_1}$ and

$$PQ=a(t_2-t_1)\sqrt{(t_1+t_2)^2+4}$$

$$\Rightarrow PQ^2=a^2\left(-2t_1-\frac{2}{t_1}\right)^2\left(\frac{4}{t_1^2}+4\right)=z \text{ (say)}$$

$$\Rightarrow z=\frac{16a^2\left(t_1^2+1\right)^3}{t_1^4}$$

$$\therefore \frac{dz}{dt_1}=16a^2\left[\frac{6(t_1^2+1)^2t_1}{t_1^4}-\frac{4}{t_1^5}\left(t_1^2+1\right)^3\right]$$

$$=16a^2\frac{\left(t_1^2+1\right)^2}{t_1^5}\left(6t_1^2-4\left(t_1^2+1\right)\right)$$

$$\Rightarrow \frac{dz}{dt_1}=\frac{32a^2\left(t_1^2+1\right)^2\left(t_1^2-2\right)}{t_1^5}$$

For extremum points $\frac{dz}{dt_1}=0$

$$\therefore \frac{32a^2\left(t_1^2+1\right)^2}{t_1^5}\left(t_1^2-2\right)=0 \Rightarrow t_1=\pm\sqrt{2}$$

Since sign of $\frac{dz}{dt_1}$ is change from negative to positive in the neighbourhood of $t_1=\sqrt{2}$

$$\therefore z_{\min}=16a^2\frac{(3)^3}{4} \Rightarrow PQ=6\sqrt{3}\,a$$

and slope of normal (tanθ) = $-t_1 \Rightarrow \theta=\tan^{-1}\left(-\sqrt{2}\right)$

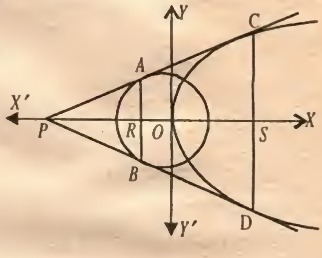
6. Equation of tangent to the parabola $y^2 = 4ax$ is $y = mx + \frac{a}{m}$ which is touches the circle $x^2+y^2=\frac{a^2}{2}$

$$\therefore \left(\frac{a}{m}\right)^2=\left(\frac{a}{\sqrt{2}}\right)^2\left(1+m^2\right)$$

$$\Rightarrow m^4+m^2-2=0 \Rightarrow (m^2+2)(m^2-1)=0$$

$\therefore$ $m=\pm 1$

$\therefore$ Equation of tangents are $y = x + a$ and $y = -x-a$ and also clear that $P(-a, o)$ so equation of chord of contact to the circle and parabola are $x=\frac{-a}{2}$ and $x = a$ respectively.

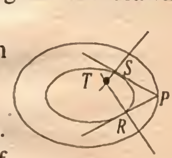


$\therefore$ Area of trapezium $ABCD$

$$=\frac{1}{2}(AB+CD)\cdot RS$$

$$=\frac{1}{2}(a+4a).\frac{3a}{2}=\frac{15a^2}{4} \text{ square units.}$$

7. It is clear from figure that if tangents are drawn from P to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ touch it at $S(a\cos\alpha, b\sin\alpha)$ and $R(a\cos\beta, b\sin\beta)$. Then P is the point of intersection of tangents at S and R to the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$



So the coordinate of P are

$$\left(a\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}, b\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}\right)$$

which is lies on $\frac{x^2}{a^2}+\frac{y^2}{b^2}=4$

$$\therefore \cos^2\left(\frac{\alpha+\beta}{2}\right)+\sin^2\left(\frac{\alpha+\beta}{2}\right)=4\cos^2\left(\frac{\alpha-\beta}{2}\right)$$

$$\Rightarrow \cos\left(\frac{\alpha-\beta}{2}\right)=\pm\frac{1}{2}$$

$$\therefore \cos(\alpha-\beta)=2\cos^2\left(\frac{\alpha-\beta}{2}\right)-1=-\tfrac{1}{2}$$

Let $T(h, k)$ be the point of intersection of the normals to the ellipse at S and R.

$$\therefore h=\left(\frac{a^2-b^2}{a}\right)\cos\alpha\cdot\cos\beta\frac{\cos\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}$$

$$\Rightarrow \frac{ah}{a^2-b^2}=\pm 2\cos\alpha\cos\beta\cdot\cos\left(\frac{\alpha+\beta}{2}\right) \qquad \text{... (1)}$$

and $k=-\frac{(a^2-b^2)}{b}\sin\alpha\sin\beta\frac{\sin\left(\frac{\alpha+\beta}{2}\right)}{\cos\left(\frac{\alpha-\beta}{2}\right)}$

$\Rightarrow \frac{-bk}{a^2-b^2}=\pm 2\sin\alpha\sin\beta\sin\left(\frac{\alpha+\beta}{2}\right)$ ... (2)

$\therefore$ (1) $\Rightarrow \frac{ah}{a^2-b^2}=\pm\{\cos(\alpha+\beta)+\cos(\alpha-\beta)\}\cos\left(\frac{\alpha+\beta}{2}\right)$

$\Rightarrow \frac{ah}{a^2-b^2}=\pm\left(\cos(\alpha+\beta)-\frac{1}{2}\right)\cos\left(\frac{\alpha+\beta}{2}\right)$ ... (3)

and (2) $\Rightarrow \frac{-bk}{a^2-b^2}=\pm(\cos(\alpha-\beta)-\cos(\alpha+\beta))\sin\left(\frac{\alpha+\beta}{2}\right)$

$\Rightarrow \frac{-bk}{a^2-b^2}=\pm\left(-\frac{1}{2}-\cos(\alpha+\beta)\right)\sin\left(\frac{\alpha+\beta}{2}\right)$ ... (4)

$\therefore \frac{a^2h^2}{(a^2-b^2)^2}+\frac{b^2k^2}{(a^2-b^2)^2}=\frac{1}{4}+\cos^2(\alpha+\beta)$

$-\cos(\alpha+\beta)\cos^2\left(\frac{\alpha+\beta}{2}\right)+\cos(\alpha+\beta)\sin^2\left(\frac{\alpha+\beta}{2}\right)$

$\Rightarrow \frac{a^2h^2+b^2k^2}{(a^2-b^2)^2}=\frac{1}{4}+\cos^2(\alpha+\beta)$

$-\cos(\alpha+\beta).\left(\cos^2\left(\frac{\alpha+\beta}{2}\right)-\sin^2\left(\frac{\alpha+\beta}{2}\right)\right)$

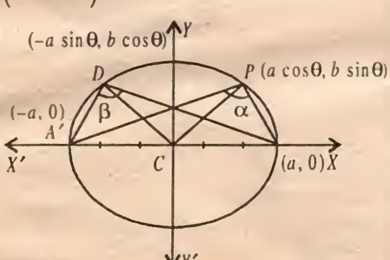
$\Rightarrow \frac{a^2h^2+b^2k^2}{(a^2-b^2)^2}=\frac{1}{4}+\cos^2(\alpha+\beta)-\cos^2(\alpha+\beta)$

$a^2h^2+b^2k^2=\frac{1}{4}(a^2-b^2)^2$

so locus of (h, k) is

$a^2x^2+b^2y^2=\frac{1}{4}(a^2-b^2)^2$

8. Let CP and CD be a pair of conjuate semidiameters of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$.



Let

P (a cosθ, b sinθ) and D ($-a$ sinθ, b cosθ)

Now slope of AP (m_1) $=-\frac{b\sin\theta}{a-a\cos\theta}=\frac{-b}{a}\cot(\theta/2)$

and slope of $A'P$ (m_2) $=\frac{b\sin\theta}{a\cos\theta+a}=\frac{b}{a}\tan(\theta/2)$.

$\therefore \tan\alpha=\left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$\Rightarrow \tan\alpha=\left|\frac{-\frac{b}{a}\cot(\theta/2)-\frac{b}{a}\tan(\theta/2)}{1-\frac{b^2}{a^2}}\right|$

$=\frac{ab}{a^2-b^2}(\cot\theta/2+\tan\theta/2)$

$\Rightarrow \tan\alpha=\frac{2ab}{a^2-b^2}\frac{1}{\sin\theta}$

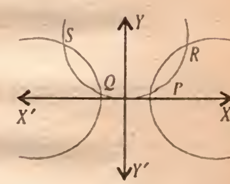
according the problem θ is replacing by $\pi/2+\theta$, then

$\tan\beta=\frac{2ab}{a^2-b^2}\frac{1}{\cos\theta}$

$\therefore \cot^2\alpha+\cot^2\beta=\left(\frac{a^2-b^2}{2ab}\right)^2.(\cos^2\theta+\sin^2\theta)$

$\Rightarrow \cot^2\alpha+\cot^2\beta=\left(\frac{a^2-b^2}{2ab}\right)^2$ which is constant.

9. We know that for determine a equation of circle uniquely, we required three points. The points of intersection of two curves $x^2-y^2=a^2$ and $y=x^2$ is $y-y^2=a^2$



$\Rightarrow y^2-y+a^2=0 \Rightarrow y=\frac{1\pm\sqrt{1-4a^2}}{2}$ ... (1)

Since y is real so $1-4a^2\geq 0 \Rightarrow \frac{-1}{2}\leq a\leq\frac{1}{2}$

and $y_1+y_2=1$, $y_1y_2=a^2$

$\therefore$ a unique circle will pass through the intersection of two curves if $\frac{-1}{2}\leq a\leq\frac{1}{2}$.

The coordinate of the points of intersection of two curves are $P(\sqrt{y_1}, y_1), Q(-\sqrt{y_1}, y_1), R(\sqrt{y_2}, y_2)$ and $S(-\sqrt{y_2}, y_2)$.

Let required equation of circle be

$$x^2 + y^2 + 2gx + 2fy + c = 0 \qquad \ldots (2)$$

Since equation (2) is passing through P, Q, R and S

$$\therefore \quad y_1 + y_1^2 + 2g\sqrt{y_1} + 2fy_1 + c = 0 \qquad \ldots (3)$$

$$y_1 + y_1^2 - 2g\sqrt{y_1} + 2fy_1 + c = 0 \qquad \ldots (4)$$

$$y_2 + y_2^2 + 2g\sqrt{y_2} + 2fy_2 + c = 0 \qquad \ldots (5)$$

and $y_2 + y_2^2 - 2g\sqrt{y_2} + 2fy_2 + c = 0 \qquad \ldots (6)$

From equations (3) and (4) we get

$$g = 0 \text{ and } y_1 + y_1^2 + 2fy_1 + c = 0 \qquad \ldots (7)$$

And equations (5) and (6) gives

$$g = 0 \text{ and } y_2 + y_2^2 + 2fy_2 + c = 0 \qquad \ldots (8)$$

Now equation (8) is subtracted from equation (7) then

$$y_1 - y_2 + y_1^2 - y_2^2 + 2f(y_1 - y_2) = 0$$

$$\Rightarrow \quad (y_1 - y_2)\left[1 + (y_1 + y_2) + 2f\right] = 0$$

$$\therefore \quad 1 + 1 + 2f = 0 \quad \Rightarrow f = -1$$

and by adding (7) and (8) we get

$$y_1 + y_2 + y_1^2 + y_2^2 + 2c - 2(y_1 + y_2) = 0$$

$$\Rightarrow \quad 1 + 1 - 2a^2 + 2c - 2 = 0$$

$$\Rightarrow \quad c = a^2$$

$\therefore$ Equation of circle is

$$x^2 + y^2 - 2y + a^2 = 0$$

10. The equation of the normal at $P\left(ct, \frac{c}{t}\right)$ to the rectangular hyperbola $xy = c^2$ is

$$xt^3 - yt = c\,(t^4 - 1) \qquad \ldots (1)$$

Let $P(h, k)$ be the pole. Then equation of polar is

$$xh + yk = 2c^2 \qquad \ldots (2)$$

$\therefore$ Equation (1) and (2) represent the same line so

$$\frac{t^3}{k} = \frac{-t}{h} = \frac{c\left(t^4 - 1\right)}{2c^2} \quad \therefore \quad t^2 = \frac{-k}{h} \text{ and } -t = \frac{h}{2c}\left(t^4 - 1\right)$$

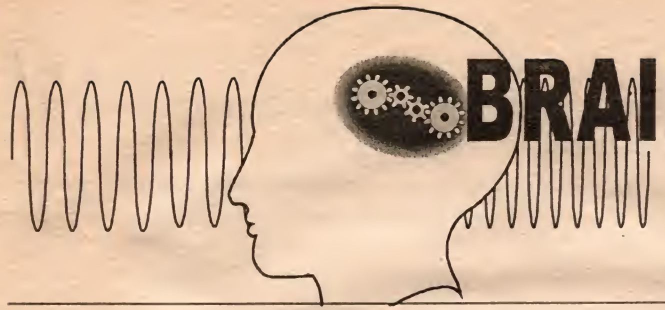
$$\therefore \quad \frac{-k}{h} = \frac{h^2}{4c^2}\left(\frac{k^2}{h^2} - 1\right)^2 \Rightarrow -4c^2hk = (k^2 - h^2)^2$$

$$\Rightarrow \quad (h^2 - k^2) + 4c^2hk = 0$$

$$\therefore \quad \text{locus of } (h, k) \text{ is } (x^2 - y^2)^2 + 4c^2xy = 0$$

TOP 10 Engineering Colleges

1. Indian Institute of Technology, Kanpur
2. Indian Institute of Technology, Delhi
3. Indian Institute of Technology, Mumbai
4. Indian Institute of Technology, Chennai
5. Indian Institute of Technology, Guwahati
6. Indian Institute of Technology, Roorkee
7. Indian Institute of Technology, Kharagpur
8. BITS, Pilani
9. Anna University, Chennai
10. Delhi College of Engineering, Delhi



BRAIN TWISTERS
Functions

By : S.K. Tiwari, VISION 2000, Gwalior

1. Draw the graph of following:

(a) $f(x)=\sin\{x\}$ (b) $f(x)=\log\{x\}$

(c) $f(x)=e^{\sin x}$ (d) $f(x)=\ln(\sin x)$

(e) $f(x)=x+\sqrt{\{x\}}$ (f) $f(x)=x+\{x\}^2$

(g) $f(x)=\sin|x|+|x|$ (h) $f(x)=\sin|x|-|x|$

(i) $f(x)=\cos|x|+|x|$ (j) $f(x)=\cos|x|-|x|$

2. $f:R\to R$

$$f(x)=\cos\left\{\sin\ln\left(\frac{x^2+e}{x^2+1}\right)\right\}+\sin\left\{\cos\ln\left(\frac{x^2+e}{x^2+1}\right)\right\}$$

check the one-one and ontoness of function.

3. Find out the domain of function

$$f(x)=\sqrt{\frac{1}{[|x-9|]+[|1-x|]-8}}$$

4. Let $f(x,y)$ be a periodic function, satisfying the condition $f(x,y) = f(2x+2y,\ 2y-2x)$ $\forall\, x, y\in R$ and let $g(x)$ be a function defined as $g(x) = f(2^x, 0)$ Prove that $g(x)$ is a periodic function. Find its period.

5. Find out the range of function

$$f(x)=\log\left\{\cot^{-1}\left(x^2-x\right)\right\}$$

6. $\phi(x)$ be a function in the form of $\frac{a}{b}=\phi(x)$ where 'a' and 'b' are the functions of x then

$f:R-\{0,1\}\to A$

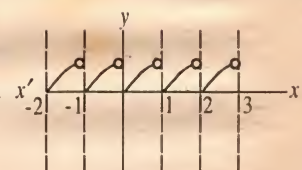
$$f(x)=\frac{\frac{d}{dx}(a)}{b+1}$$ is an onto function. Find out the range of A, if $\phi(x)$ satisfying the condition.

$$\phi(x)+\phi\left(\frac{x-1}{x}\right)=1+x \qquad \forall\, x\in R-\{0,1\}$$

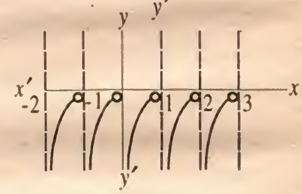
7. $f:\left(0,\frac{\pi}{2}\right)\to A$, $f(x)=\log_e\left(\sin x^{\sin x}+1\right)$ check if the function is one-one, Many-one and if $f(x)$ is an onto function find out the range of A.

SOLUTIONS

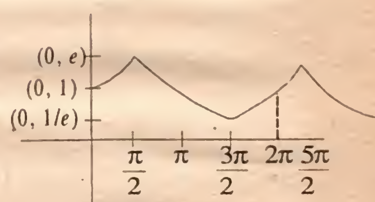
1. (a): $f(x) = \sin\{x\}$



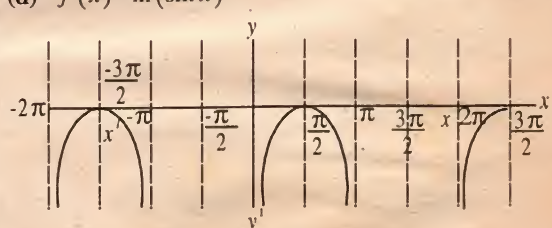
(b) $f(x)=\ln\{x\}$



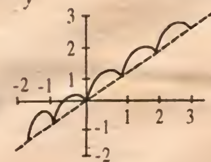
(c) $f(x)=e^{\sin x}$



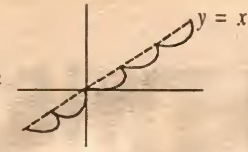
(d) $f(x)=\ln(\sin x)$



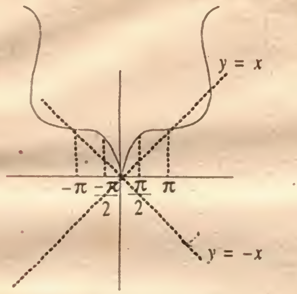
(e) $f(x)=x+\sqrt{\{x\}}$



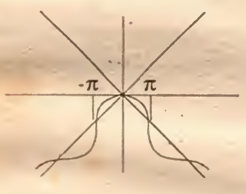
(f) $f(x)=x+\{x\}^2$



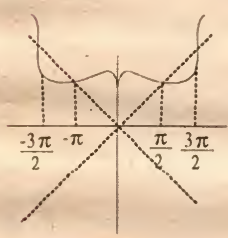
(g) $f(x)=\sin|x|+|x|$



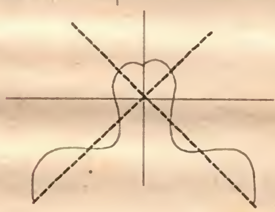
(h) $f(x)=\sin|x|-|x|$



(i) $f(x)=\cos|x|+|x|$



(j) $f(x)=\cos|x|-|x|$



2. Firstly range $1\le\frac{x^2+e}{x^2+1}\le e$, i.e. $0\le\ln\left(\frac{x^2+e}{x^2+1}\right)\le 1$

then the range of

$$f(x)=\cos\left(\sin\left(\ln\left(\frac{x^2+e}{x^2+1}\right)\right)\right)+\sin\left(\cos\left(\ln\left(\frac{x^2+e}{x^2+1}\right)\right)\right)$$

$f_{\min}=1+\sin 1$ at $\ln\left(\frac{x^2+e}{x^2+1}\right)=0$

$f_{\max}=\cos(\sin 1)+\sin(\cos 1)$

Then range

$[\cos(\sin 1)+\sin(\cos 1),\ 1+\sin 1]$

Codomain $\Rightarrow R$

$\therefore$ Range $\neq$ codomain so that function will not be onto

$\therefore$ $f(x)$ is even function so that this is not one-one function. This is many-one into function.

3. $[|x-9|]+[|1-x|]-8\neq 0$ and

$[|x-9|]+[|1-x|]-8>0$

Then three cases arises

(a) $x<1$ then

$[9-x]+[1-x]-8>0$

$9+[-x]+1+[-x]-8>0$

$2[-x]>-2$

$[-x]>-1$ then

$-x\ge 0$

$x\le 0$

(b) If $1<x<9$ then

$[9-x]+[x-1]-8>0$

$9+[-x]+[x]-1-8>0\ \therefore\ [-x]+[x]>0$

$\therefore$ We know that $[x]+[-x]$ either 0 or -1 according x is integer or non integer. Then there is no value of k between (1, 9)

(c) If $x>9$

$[x-9]+[x-1]-8>0$

$2[x]>18$

$[x]>9$

$x\ge 10$

Then domain of function will be $(-\infty,0]\cup[10,\infty)$

4. $f(x,y)=f(2x+2y,2y-2x)$

$=f\{2(2x+2y)+2(2y-2x),2(2y-2x)-2(2x+2y)\}$

$f(x,y)=f(8y,-8x)=f\{8(-8x),-8(8y)\}$

$=f(-64x,-64y)=f\{-64(-64x),-64(-64y)\}$

$=f\left(2^{12}x,2^{12}y\right)$

$\therefore$ $g(x)=f(2^x,0),=f\left(2^{12}\cdot 2^x,2^{12}\cdot 0\right)=f\left(2^{x+12},0\right)$

$=g(x+12)$

Hence $g(x)$ is periodic with period 12.

5. $f(x)=\log\left\{\cot^{-1}\left(x^2-x\right)\right\}$

Firstly the range of (x^2-x) is

$$\left(x-\frac{1}{2}\right)^2-\frac{1}{4}\geq-\frac{1}{4}$$

$$\frac{-1}{4}\leq x^2-x<\infty$$

$$0<\cot^{-1}\left(x^2-x\right)\leq\cot^{-1}\left(-\frac{1}{4}\right)$$

$$-\infty<\log\left\{\cot^{-1}\left(x^2-x\right)\right\}\leq\log\left\{\cot^{-1}\left(-\frac{1}{4}\right)\right\}$$

Then range $\left(-\infty,\log\cot^{+}\left(\frac{-1}{4}\right)\right]$

6. $f(x)+f\left(\frac{x-1}{x}\right)=1+x\ \forall x\in R-\{0,1\}$ ... (i)

$$f\left(\frac{x-1}{x}\right)+f\left(\frac{\frac{x-1}{x}-1}{\frac{x-1}{x}}\right)=1+\frac{x-1}{x}$$

[Replace x by $\frac{x-1}{x}$]

$$\Rightarrow\ f\left(\frac{x-1}{x}\right)+f\left(\frac{-1}{x-1}\right)=1+\frac{x-1}{x}\quad\text{... (ii)}$$

eqn. (i) – eqn. (ii)

(f=φ)

$$f(x)-f\left(\frac{-1}{x-1}\right)=x-\frac{x-1}{x}\quad\text{... (iii)}$$

Replacing x by $\frac{-1}{x-1}$ in eqn (i) we get

$$f\left(\frac{-1}{x-1}\right)+f\left(\frac{\frac{-1}{x-1}-1}{\frac{-1}{x-1}}\right)=1-\frac{1}{x-1}$$

$$=\ f\left(\frac{-1}{x-1}\right)+f(x)=1-\frac{1}{x-1}\quad\text{... (iv)}$$

eqn. (iii) + eqn. (iv)

$$2f(x)=x-\frac{x-1}{x}+1-\frac{1}{x-1}$$

$$\boxed{f(x)=\frac{x^3-x^2-1}{2x^2-2x}}$$

Here $a = x^3 - x^2 - 1$

$$\frac{d(a)}{dx}=3x^2-2x$$

$$\phi(x)=\frac{\frac{d(a)}{dx}}{b+1}=\frac{3x^2-2x}{2x^2-2x+1}=y\ \text{(Let)}$$

For onto, range of $\phi(x)$ should be equal to codomain. Then for A we have to find the range.

$$3x^2-2x=2x^2y-2xy+y$$

$$x^2(3-2y)-2x(1-y)-y=0$$

$$4(1-y)^2+4y(3-2y)\geq0$$

$$1+y^2-2y+12y-8y^2\geq0$$

$$7y^2-10y-1\leq0$$

$$\left\{y-\left(5+4\sqrt{2}\right)\right\}\left\{y-\left(5-4\sqrt{2}\right)\right\}<0$$

Range $y\in\left[5-4\sqrt{2},\ 5+4\sqrt{2}\right]$

For onto

$$A\to\left[5-4\sqrt{2},\ 5+4\sqrt{2}\right]$$

7. Range of $\log_e\left(\sin x^{\sin x}+1\right)$

for $0<x<\frac{\pi}{2}$ equivalent to $\log_e(x^x+1)$

for $0<x<1$

Let $h(x) = (x^x + 1)$

$h'(x) = x^x(1 + \log_e x)$

$\therefore\ h'(x)>0$ If $x>\frac{1}{e}$

It means if $x>\frac{1}{e}$ function will be increasing then it will be one-one for $x\in\left(0,\frac{\pi}{2}\right)$

$$\therefore\ h'(x)<0 \text{ if } x<\frac{1}{e}$$

It means $x=\frac{1}{e}$ there is minima and maxima for $x\in\left(0,\frac{\pi}{2}\right)$ will be at $x=\frac{\pi}{2}$

$$f_{\text{min.}}=\log_e\left(\left(\frac{1}{e}\right)^{\frac{1}{e}}+1\right),\quad f_{\text{max.}}=\log_e 2$$

$$\text{Range of } A\Rightarrow\left[\log_e\left[\left(\frac{1}{e}\right)^{1/e}+1\right],\log_e 2\right]$$

■

ALGEBRA *for* J.E.E SCREENING

In IIT-JEE algebra has a weightage of about 23% of the total marks. Students should emphasise practising only those topics that have been well understood by them. Since problems on Algebra are trickier, students must practice solving as many different types of problems as they can.

Note that the JEE is not about scoring 100% in a subject or the examination. Your aim should be to attain an optimal score in each subject, depending on your strengths and weaknesses.

Time : 1 hr *Max. Marks : 120*

1. If z_1, z_2 are two complex numbers satisfying the equation $\left|\dfrac{z_1+z_2}{z_1-z_2}\right|=1$ then z_1/z_2 is a number which is

(a) positive real (b) negative real
(c) zero (d) purely imaginary

2. Let $P(x)$ and $Q(x)$ be two polynomials. Suppose that $f(x) = P(x^3) + xQ(x^3)$ is divisible by $x^2 + x + 1$, then

(a) $P(x)$ is divisible by $(x-1)$ but $Q(x)$ is not divisible by $x - 1$
(b) $Q(x)$ is divisible by $(x-1)$ but $P(x)$ is not divisible by $x - 1$
(c) Both $P(x)$ and $Q(x)$ are divisible by $x - 1$
(d) $f(x)$ is divisible by $x - 1$.

3. The value of the expression

$$S=\sum_{p=1}^{32}(3p+2)\left(\sum_{q=1}^{10}\left(\sin\frac{2q\pi}{11}-i\cos\frac{2q\pi}{11}\right)\right)^{4p} \text{ is}$$

(a) 1648 (b) 1646
(c) 1645 (d) zero

4. If $ax + by = 1$, $cx^2 + dy^2 = 1$ have only one solution, then

(a) $\dfrac{a^2}{c}+\dfrac{b^2}{d}=1$ (b) $x=\dfrac{a}{c}$
(c) $y=\dfrac{b}{d}$ (d) none of these.

5. The equation $3^{x-1} + 5^{x-1} = 34$ has

(a) no solution (b) one solution
(c) two solutions (d) more than two solutions

6. If $c \neq 0$ and the equation $\dfrac{p}{2x}=\dfrac{a}{x+c}+\dfrac{b}{x-c}$ has two equal roots, then p can be

(a) $\left(\sqrt{a}-\sqrt{b}\right)^2$ (b) $\left(\sqrt{a}+\sqrt{b}\right)^2$
(c) $a + b$ (d) $a - b$

7. If a, b, c are in G.P., and $\log a - \log 2b$, $\log 2b - \log 3c$ and $\log 3c - \log a$ are in A.P., then a, b, c are the lengths of the sides of a triangle which is

(a) acute-angled (b) obtuse-angled
(c) right-angled (d) equilateral

8. If b_1, b_2, b_3 ($b_1 > 0$) are three successive terms of a G.P. with common ratio r, the value of r for which the inequality $b_3 > 4b_2 - 3b_1$ holds is given by

(a) $r > 3$ (b) $r < 1$
(c) $r = 3.5$ (d) $r = 5.2$

9. The sum to n terms of the series

$$\left(\frac{2n+1}{2n-1}\right)+3\left(\frac{2n+1}{2n-1}\right)^2+5\left(\frac{2n+1}{2n-1}\right)^3+\dots \text{ is}$$

(a) $n^2 + 4n$ (b) $n^2 + n$
(c) $2n^2$ (d) none of these

10. If $\sum_{r=1}^{n} r(r+1)(2r+3)$ $= an^4 + bn^3 + cn^2 + dn + e$, then

(a) $a = 1/2$ (b) $b = 8/3$
(c) $c = 9/2$ (d) $e = 0$

11. $\lim_{n\to\infty}\sum_{r=1}^{n}\tan^{-1}\left(1/2r^2\right)$ is

(a) $\pi/2$ (b) π
(c) $\pi/4$ (d) 3π

12. The number of five-digit telephone numbers having at least one of their digits repeated is

(a) 90000 (b) 100000
(c) 30240 (d) 69760

or, $m^2 - am + \frac{m}{a} - 1 = 0$ or, $m(m-a) + \frac{1}{a}(m-a) = 0$

or, $m(m-a) + \frac{1}{a}(m-a) = 0$

or, $(m-a)\left(m+\frac{1}{a}\right) = 0 \quad \therefore\ m = a, -\frac{1}{a}$

$\therefore$ The general solution is $y = c_1e^{ax} + c_2e^{-\frac{1}{a}x}$

$$\frac{dy}{dx} = ac_1e^{ax} - \frac{c_2}{a}e^{-\frac{1}{a}x}$$

Given $y = b$ and $\frac{dy}{dx} = 0$ when $x = 0$ $\therefore\ b = c_1 + c_2$

$0 = ac_1 - \frac{c_2}{a} \quad \therefore\ ac_1 = \frac{c_2}{a} \quad \therefore\ c_2 = a^2c_1$

$\therefore\ b = c_1(1+a^2) \Rightarrow c_1 = \frac{b}{1+a^2}$ and $c_2 = \frac{a^2b}{1+a^2}$

The particular solution is $y = \frac{b}{1+a^2}\left(e^{ax} + a^2e^{-\frac{1}{a}\cdot x}\right)$.

(b) The equation of hyperbola is $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

Differentiating, $\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$

or, $\frac{y}{b^2}\frac{dy}{dx} = \frac{x}{a^2} \quad \therefore\ \frac{dy}{dx} = \frac{b^2}{a^2}\cdot\frac{x}{y}$

$\therefore$ Slope of the normal at (a secθ, b tanθ) is

$$-\frac{1}{\left[\frac{dy}{dx}\right]_{(a\sec\theta,\ b\tan\theta)}} = -\frac{a}{b}\cdot\frac{\tan\theta}{\sec\theta}$$

$\therefore$ The equation of normal at $(a\sec\theta, b\tan\theta)$ is

$$y - b\tan\theta = -\frac{a}{b}\cdot\frac{\tan\theta}{\sec\theta}(x - a\sec\theta)$$

or, $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

The equations of normals at (a secθ, b tanθ) and (a secϕ, b tanϕ) are $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

or, $ax + by\cdot\frac{\sec\theta}{\tan\theta} = (a^2+b^2)\sec\theta$

and $ax + by\cdot\frac{\sec\phi}{\tan\phi} = (a^2+b^2)\sec\phi$

Since they pass through (h, k) and $\phi = \frac{\pi}{2} - \theta$

$\therefore\ ah + bk\cdot\frac{\sec\theta}{\tan\theta} = (a^2+b^2)\sec\theta$

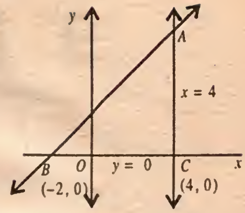
$ah + bk\cdot\frac{\operatorname{cosec}\theta}{\cot\theta} = (a^2+b^2)\operatorname{cosec}\theta$

Subtracting $bk\left(\frac{1}{\sin\theta} - \frac{1}{\cos\theta}\right) = (a^2+b^2)\left(\frac{1}{\cos\theta} - \frac{1}{\sin\theta}\right)$

or, $bk(\operatorname{cosec}\theta - \sec\theta) = (a^2+b^2)(\sec\theta - \operatorname{cosec}\theta)$

$\therefore\ bk = -(a^2+b^2) \quad \therefore\ k = -\frac{a^2+b^2}{b}$.

(c) $2y - 3x = 6$ or $\frac{x}{-2} + \frac{y}{3} = 1$



The area of the triangle ABC is $\int_{-2}^{4} y\,dx$ where y is the ordinate of a point on the line $2y - 3x = 6$ or, $y = \frac{6+3x}{2}$.

$$= \int_{-2}^{4}\frac{6+3x}{2}dx = \int_{-2}^{4}\left(3+\frac{3}{2}x\right)dx$$

$$= \left[3x + \frac{3}{4}x^2\right]_{-2}^{4} = (12+12) - (-6+3) = 27 \text{ sq. unit.}$$

10.(a) $y = 1 - x^4 \quad \therefore\ \frac{dy}{dx} = -4x^3$

$\therefore\ \frac{dy}{dx} < 0$ when $x > 0$

> 0 when $x < 0$

$\therefore$ The function is decreasing for $x > 0$ and increasing for $x < 0$. As $\frac{dy}{dx}$ changes its sign from positive to negative when x changes from negative to positive, therefore there is a maximum value at $x = 0$ and $\frac{dy}{dx} = 0$ at $x = 0$.

And the maximum value of y is $1 - 0 = 1$.

(b) Let $f(x) = \sin x(1+\cos x)$

$\therefore\ f'(x) = \cos x(1+\cos x) - \sin x\cdot\sin x = \cos x + \cos 2x$

$f''(x) = -\sin x - 2\sin x$

$\therefore$ At $x = \frac{\pi}{3}$ $f'(x) = 0$, $f''(x) = -\frac{\sqrt{3}}{2} - 2\cdot\frac{\sqrt{3}}{2} < 0$

$\therefore\ f(x)$ has maximum value at $x = \frac{\pi}{3}$.

(c) The distance of the particle from a fixed point on a given line after time t is given by

$s = 12t - 15t^2 + 4t^3 \quad \therefore\ \frac{ds}{dt} = 12 - 30t + 12t^2$

When $\frac{ds}{dt} = 0$, $12 - 30t + 12t^2 = 0$ or, $2t^2 - 5t + 2 = 0$

or, $(2t-1)(t-2) = 0 \quad \therefore\ t = 1/2, 2$.

$\therefore$ Its velocity becomes zero after times 1/2 sec and 2 sec.

Now, $\frac{d^2s}{dt^2} = -30 + 24t$

At $t = \frac{1}{2}$sec, $\frac{d^2s}{dt^2} = -30 + 12 < 0$

At $t = 2$ sec, $\frac{d^2s}{dt^2} = -30 + 48 = 18 > 0$

$\therefore$ The direction of acceleration after time 1/2 secs is opposite to the positive direction of displacement and after time 2 secs the direction of acceleration is towards the positive direction of displacement.

Distance travelled between these two times

$$= \left[12\cdot\frac{1}{2} - 15\cdot\frac{1}{4} + 4\cdot\frac{1}{8}\right] - \left[12\cdot 2 - 15\cdot 2^2 + 4\cdot 2^3\right]$$

$$= \frac{11}{4} + 4 = \frac{27}{4} = 6\frac{3}{4} \text{ units.}$$

■

9^{th} Mathematical Challenge *for* I.I.T. MAINS

This section is designed to give IIT JEE aspirants a thorough grinding & exposure to variety of possible twists and turns of problems in mathematics that would be very helpful in facing IIT JEE. Each and every problem is well thought of in order to strengthen the concepts and we hope that this section would prove a rich resource for practicing challenging problems and enhancing the preparation level of IIT JEE aspirants.
The detailed solutions to these problems will be published in the next issue alongwith a new set of such problems.

1. The number of $2\cos(10°)$ is a root of the equation $f(x) = 0$, where $f(x)$ is a polynomial of degree 6 having integer coefficients. Find $f(x)$?

2. Find unsymmeterical form of the projection of the line $3x - y + 2z - 1 = 0 = x + 2y - z - 2$ on the plane $3x + 2y + z = 0$.

3. Let f be a continuous function on $[a, b]$. Prove that there exists a number $x \in [a, b]$ such that $\int_a^x f(t)dt = \int_x^b f(t)dt$.

4. $OABC$ is a regular tetrahedron. D is mid point of edge OC and E is circumcentre of ΔOAB. Using vectors, show that the distance DE is half of the edge length.

5. A number is chosen at random from the set $\{1, 2, 3, ..., 2004\}$. What is the probability that it has no prime factor in common with 10!.

6. If $p = \sin\theta - \cos\theta$, and $q = \operatorname{cosec}\theta - \sin\theta$. Show that $(p^2 + 1)(q^2 + 1) = (p + q)^2(3 - p^2)$.

7. If $q\sqrt{h^2 + k^2} = pa$. A point divides the join of (h, k) and the variable point on the circle of radius a, with centre at the origin, in the ratio $p : q$. Prove that the point lies on other circle whose radius is independent of h and k. Also find its centre.

8. Let AB and CD, two focal chords of a parabola $y^2 = 4ax$, are at right angles. Find their slopes if the area of the quadrilateral $ABCD$ is minimum.

9. Let $f(x) = x^4 + ax^3 + bx^2 + cx - c$. If it shares two distinct integral zeros with its derivative $f'(x)$ and $abc \neq 0$ then determine it completely.

10. Find the area bounded by the two curves $y = |\cos 3x|$ and $y = |\sin 3x|$; $0 \leq x \leq \pi$.

By: **Shailendra Maheshwari**, Career point, Kota

Mathematics Olympia for IIT-JEE 2005

By : Er. Akhlak Ahmad, ABC Classes, Gorakhp

1. If $a_1, a_2, ..., a_n$ are in A.P. and $a_k > 0 \ \forall \ k \geq 1$, then prove that $\sum_{k=1}^{n} a_k \geq n\sqrt{a_1^2 + (n-1)da_1}$, where d is the common difference of the A.P.

2. Two circles, the sum of whose radii is a are placed in the same plane with their centre $2a$ apart. An endless string fully stretched so as partly to surround the circle and to cross between them. Prove that length of string is $\left(\frac{4\pi}{3} + 2\sqrt{3}\right)a$.

3. If A, A_1, A_2, A_3 are the areas of the inscribed and escribed circles respectively of a ΔABC. Prove that

$$\frac{1}{\sqrt{A}} = \frac{1}{\sqrt{A_1}} + \frac{1}{\sqrt{A_2}} + \frac{1}{\sqrt{A_3}}.$$

4. If the terms of the A.P. $\sqrt{a-x}, \sqrt{x}, \sqrt{a+x}, ...$ are all integers where $a, x > 0$, then find the least composite odd integral value of a.

5. Let S_n be the sum of first n terms of an A.P. with non zero common difference. Find the ratio of first term to common difference if $\frac{S_{n_1 n_2}}{S_{n_1}}$ is independent of n_1.

6. Find the sum $\sum_{k=1}^{n} \tan^{-1}\left[\frac{2k}{2 + k^2 + k^4}\right]$

7. According to Suheb, if in a ΔABC, $a^2 + b^2 + c^2 - ac - \sqrt{3}ab = 0$ then triangle is equilateral. Predict whether Suheb is right or wrong.

8. The area of cyclic quadrilateral $ABCD$ is $\frac{3\sqrt{3}}{4}$. The radius of the circle circumscribing it is 1. If $AB = 1$, $BD = \sqrt{3}$, then evaluate $BC \cdot CD$.

9. In an acute triangle, prove that

$$\sum (\tan A \tan B)^n \geq 3^{n+1} \ (n > 1).$$

10. Two consecutive numbers from 1, 2, 3, ..., n a removed then arithmetic mean of the remaining numb is 105/4. Find n and three removed numbers.

SOLUTIONS

1. $\because \sum_{k=1}^{n} a_k = a_1 + a_2 + ... + a_n = \frac{n}{2}[a_1 + a_n]$ ...(1

(a_1, a_n are +ve

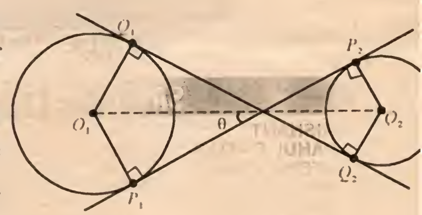
Applying, A.M. $\geq$ G.M.

$$\frac{a_1 + a_n}{2} \geq \sqrt{a_1 a_n} \Rightarrow n\left(\frac{a_1 + a_2}{2}\right) \geq n\sqrt{a_1 a_n}$$

Using (1)

$$\sum_{k=1}^{n} a_k \geq n\sqrt{a_1^2 + (n-1)da_1} \quad (\because a_n = a_1 + (n-1)d$$

2. Let the centre of circle be O_1 and O_2, and their radii be r_1 and r_2 respectively.



$r_1 + r_2 = a$; $O_1O_2 = 2a$

$\angle O_1OP_1 = \theta \Rightarrow \angle P_2OO_2 = \theta$

Now, $O_1O_2 = O_1O + OO_2 = r_1 \operatorname{cosec}\theta + r_2 \operatorname{cosec}\theta$

$O_1O_2 = (r_1 + r_2) \operatorname{cosec}\theta$

$\Rightarrow \operatorname{cosec}\theta = 2 \Rightarrow \theta = \pi/6$.

Now $P_1O = r_1\cot\theta$, $OP_2 = r_2 \cot\theta$

$P_1P_2 = (r_1 + r_2)\cot\theta = a\sqrt{3}$

Central angle $\angle Q_1O_1P_1 = 2\left(\frac{\pi}{2} - \theta\right) = \pi - 2\theta$

$\angle P_2O_2Q_2 = \angle Q_1O_1P_1 = \pi - 2\theta$

$\Rightarrow$ Total string length $= 2P_1P_2 + (r_1 + r_2)(2\pi - (\pi - 2$

$= 2a\sqrt{3} + a\left(\pi + \frac{\pi}{3}\right) = \left(\frac{4\pi}{3} + 2\sqrt{3}\right)a.$

3. $A = \pi r^2$, $A_1 = \pi r_1^2$, $A_2 = \pi r_2^2$, $A_3 = \pi r_3^2$

$$\frac{1}{\sqrt{A_1}}+\frac{1}{\sqrt{A_2}}+\frac{1}{\sqrt{A_3}} = \frac{1}{\sqrt{\pi}}\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}\right)$$

$$= \frac{1}{\sqrt{\pi}}\left[\frac{s-a}{\Delta}+\frac{s-b}{\Delta}+\frac{s-c}{\Delta}\right] = \frac{1}{\sqrt{\pi}}\left[\frac{3s-(a+b+c)}{\Delta}\right]$$

$$= \frac{1}{\sqrt{\pi}}\frac{s}{\Delta} = \frac{1}{\sqrt{\pi}\, r} = \frac{1}{\sqrt{\pi r^2}} = \frac{1}{\sqrt{A}}.$$

4. $\because \sqrt{a-x}, \sqrt{x}, \sqrt{a+x}, \ldots$ are in A.P.

$\Rightarrow 2\sqrt{x} = \sqrt{a-x} + \sqrt{a+x}$

simplifying we get $x = 0$ or $x = 4a/5$

$(\because x>0 \Rightarrow x \neq 0)$

hence $x = 4a/5 \Rightarrow \sqrt{\frac{a}{5}}, 2\sqrt{\frac{a}{5}}, 3\sqrt{\frac{a}{5}}, \ldots$

(which are all integers)

$a = 5n^2;\ n \in \mathrm{N}$ ... (i)

If (1) $n = 1 \Rightarrow a = 5$ (which is not composite)

$n = 2 \Rightarrow a = 20$

(which is composite but not odd)

$n = 3 \Rightarrow a = 45$ (which is least composite odd).

5. $$\frac{S_{n_1n_2}}{S_{n_1}} = \frac{\left(\frac{n_1n_2}{2}\right)[2a+(n_1n_2-1)d]}{\left(\frac{n_1}{2}\right)[2a+(n_1-1)d]}$$

$$= n_2\frac{[(2a-d)+n_1n_2d]}{[(2a-d)+n_1d]}$$

for $\frac{S_{n_1n_2}}{S_n}$ is to be independent of n_1;

$2a-d = 0 \Rightarrow \frac{a}{d} = \frac{1}{2}.$

6. $T_n = \tan^{-1}\left[\frac{2n}{2+n^2+n^4}\right]$

$$T_n = \tan^{-1}\left[\frac{(n^2+n+1)-(n^2-n+1)}{1+(n^2+n+1)(n^2-n+1)}\right]$$

$T_n = \tan^{-1}(n^2+n+1)-\tan^{-1}(n^2-n+1)$

$T_1 = \tan^{-1}(3) - \tan^{-1}(1)$

$T_2 = \tan^{-1}7 - \tan^{-1}3$

$T_3 = \tan^{-1}13 - \tan^{-1}7$

$\vdots \qquad \vdots$

$T_n = \tan^{-1}(n^2+n+1) - \tan^{-1}(n^2-n+1)$

on addition we get;

$T_1 + T_2 + \ldots + T_n = \tan^{-1}(n^2+n+1) - \frac{\pi}{4}.$

7. $\because\ a^2 - a(c+\sqrt{3}b) + b^2 + c^2 = 0$

$2a = c + \sqrt{3}b \pm \sqrt{(c+\sqrt{3}b)^2 - 4(b^2+c^2)}$

$2a = c + \sqrt{3}b \pm \sqrt{-(\sqrt{3}c-b)^2}$ ...(i

$\Rightarrow \sqrt{3}c-b=0 \Rightarrow b=\sqrt{3}c$ ...(ii

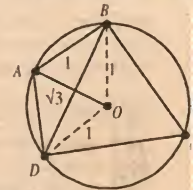
$2a = c + \sqrt{3}b$ ...(iii) (from (i)

$a = 2c$ (using (ii) and (iii))

$a = 2c,\ b=\sqrt{3}c,\ c = c$

$b^2 + c^2 = a^2 \Rightarrow \Delta$ is right angled $\Rightarrow$ Suheb is wrong

8. In ΔBOD, $\angle BOD = 2C$



$$\cos 2C = \frac{1^2+1^2-(\sqrt{3})^2}{2\cdot 1\cdot 1}$$

$\cos 2C = -\frac{1}{2} \quad \angle C = 60°$

also $\angle A+\angle C = 180° \Rightarrow \angle A = 120°$

$$\cos 120° = \frac{1^2+AD^2-(\sqrt{3})^2}{2AD\cdot 1} \Rightarrow AD = 1$$

ar$(ABCD)$ = ar(ΔABD) + ar(ΔBCD)

$\frac{3\sqrt{3}}{4} = \frac{1}{2}\cdot 11\sin 120° + \frac{1}{2}BC \cdot DC\sin 60°$

$\Rightarrow BC\cdot DC = 2.$

9. $(\tan A\cdot\tan B)^n+(\tan B\cdot\tan C)^n+(\tan C\cdot\tan A)^n$

$\geq 3^{n+}$

apply AM $\geq$ GM

$$\frac{(\tan A\cdot\tan B)^n+(\tan B\cdot\tan C)^n+(\tan C\cdot\tan A)^n}{3}$$

$\geq (\tan^2 A\ \tan^2 B\ \tan^2 C)^{n/}$

Now; use $\tan A + \tan B + \tan C = \tan A \cdot \tan B \cdot \tan C$

We easily get the required inequality.

10. Let p and $(p + 1)$ be removed numbers

$$\frac{105}{4} = \frac{\frac{n(n+1)}{2} - (2p+1)}{n-2}$$

$\Rightarrow 2n^2 - 103n - 8p + 206 = 0$

Since n, p are integers so n must be even let $n = 2$

$\Rightarrow p = \frac{4r^2+103(1-r)}{4}$

$\because\ p$ is an integer $\Rightarrow 1 - r$ must be integer

$\Rightarrow r = 1 + 4t$, we get

$n = 2 + 8t$, $p = 16t^2 - 95t + 1$

Now $1 \leq p < n \Rightarrow 1 \leq 16t^2 - 95t + 1 < 8t + 2$

$t = 6$, $n = 50$ and $p = 7$

Hence, removed numbers are 7 and 8.

BRAIN TWISTERS
Functions

By : S.K. Tiwari, VISION 2000, Gwalior

1. If $f(x)$ is a continuous function with $\int_0^x f(t)\,dt \to \infty$ as $|x| \to \infty$ then show that every line $y = mx$ intersects the curve $y^2 + \int_0^x f(t)\,dt = 2$

2. Prove that

$$\lim_{h\to\infty} \frac{1}{\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\sqrt{\frac{1}{2}}}} \ldots n \text{ terms}}$$

3. $S_n = 1 - \frac{1}{4} + \frac{1}{6} - \frac{1}{9} + \frac{1}{11} - \frac{1}{14} + \ldots\ldots\ldots\ldots\ n$ terms.

find $\lim_{n\to\infty} S_n$.

4. Let $f(x) = x^3 - 3x^2 + 6 \ \forall x \in R$

and $g(x) = \begin{cases} \max f(t) & x+1 \le t \le x+2 \quad -3 \le x \le 0 \\ 1 - x & \text{for} \quad x \ge 0 \end{cases}$

test the differentibility of $g(x)$ for $x \in [-3, 1]$

5. A sportsman walks in a horizontal circle round the foot of pole which is inclined to the vertical. The foot of the pole is at the centre of circle the greatest and least angles which the pole subtends at his eyes are $\tan^{-1}\left(\frac{9}{5}\right)$ and $\tan^{-1}\left(\frac{6}{5}\right)$ respectively and when he is midway between the corresponding position, the angle is θ. If the man's height be neglected, find the length of pole and θ. If the radius of the circles is a and length of pole $> a$.

6. An ellipse is inscribed in an isosceles triangle of height h and base $2k$ and having one axis lying along the perpendicular. From the vertex of triangle to the base. Find maximum area of triangle.

7. Evaluate $\lim_{n\to\infty} \sum_{n=0}^{n} \frac{1}{n!}\left\{\sum_{r=0}^{n} (r+1)\int_0^1 2^{-(r+1)x}\,dx\right\}$

8. Evaluate

$$Sgn\left[\lim_{n\to\infty} \frac{\left\{1\sum_{r=1}^{n} r + 2\sum_{r=1}^{n-1} r + 3\sum_{r=1}^{n-2} r + \ldots\ldots\ldots\ldots n-1\right\}}{n^4}\right]$$

$$+Sgn\left[\lim_{n\to\infty}\left\{\left(151\times 4^{100}\right)^n + \left(\sum_{r=0}^{300} r\cdot b_r\right)^n\right\}^{1/n} - 152\times 4^{400}\right]$$

where $\left(1 + x + x^2 + x^3\right)^{100} = \sum_{r=0}^{300} b_r x^r$ where sgn stands for signom function.

9. Find $f(x)$ if $\left\{f^2(x) + 4f'(x)\cdot f(x) + (f'x)^2\right\} = 0$

10. $\lim_{a\to b} \frac{(\sin a)^{\sin b} - (\sin b)^{\sin a}}{(\sin a)^{\sin a} - (\sin b)^{\sin b}}$

SOLUTIONS

1. Putting $y = mx$ in $y^2 + \int_0^x f(t)\,dt$ we get

$m^2x^2 + \int_0^x f(t)\,dt = \phi(x)$ (Let)

$\therefore$ $\int_0^x f(t)$ continuous & m^2x^2 (poly nomial) continuous then $\phi(x)$ will be continuous

If $\int_0^x f(t)\,dt \to \infty$ as $|x| \to \infty$

$m^2x^2 \to \infty$ as $|x| \to \infty$

Then $\phi(x) \to \infty$ as $|x| \to \infty$

Here $\phi(x)$ lies in set $[0, \infty)$ for all $x \in R$

$\therefore$ $\phi(x)$ will also attain the value 2 for some real x and all real m

$\therefore$ $\phi(x) = 2$, $\quad m^2x^2 + \int_0^x f(t)\,dt = 2$

will have real solution.

2. $\sin x=2\sin\frac{x}{2}\cdot\cos\frac{x}{2}$

$$=2\left(\cdot 2\sin\frac{x}{2^2}\cdot\cos\frac{x}{2^2}\right)\cdot\cos\frac{x}{2}$$

$$2^2\left(2\sin\frac{x}{2^3}\cdot\cos\frac{x}{2^3}\right)\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2}$$

$$\sin x=2^n\cdot\sin\frac{x}{2^n}\left(\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}.....\cos\frac{x}{2^n}\right)$$

$$\Rightarrow \lim_{n\to\infty}\frac{2^n\sin\frac{x}{2^n}}{\sin x}=\frac{1}{\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}\cdot\cos\frac{x}{2^3}.......\cos\frac{x}{2^n}}$$

$$=\lim_{n\to\infty}\frac{2^n\dfrac{\sin\frac{x}{2^n}}{\frac{x}{2^n}}\times\frac{x}{2^n}}{\sin x}=\frac{x}{\sin x}=\frac{1}{\left(\cos\frac{x}{2}\cdot\cos\frac{x}{2^2}.......\cos\frac{x}{2^n}\right)}$$

Put $x=\frac{\pi}{2}$ both sides

$$\frac{\pi}{2}=\frac{1}{\sqrt{\frac{1}{2}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}\cdot\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}+\frac{1}{2}\sqrt{\frac{1}{2}}}}} \quad \text{proved.}$$

3. $\lim_{n\to\infty} S_n=1-\frac{1}{4}+\frac{1}{6}-\frac{1}{9}+\frac{1}{11}-\frac{1}{14}......$ upto n terms

this can be written as follows

$$=\int_0^1\left(1-x^3+x^5-x^8+x^{10}-x^{13}........\infty\right)dx$$

$$=\int_0^1\left[\left(1-x^3\right)+x^5\left(1-x^3\right)+x^{10}\left(1-x^3\right)+........\infty\right]dx$$

$$=\int_0^1\frac{\left(1-x^3\right)}{\left(1-x^5\right)}dx=\int_0^1\frac{\left(x^2+x+1\right)}{x^4+x^3+x^2+x+1}dx$$

$$=\int_0^1\frac{adx}{x^2+cx+1}+\int_0^1\frac{bdx}{x^2+dx+1}$$

From partial fractions we can calculate

$$a=\frac{5+\sqrt{5}}{10},\ b=\frac{5-\sqrt{5}}{10},\ c=\frac{1-\sqrt{5}}{2},\ d=\frac{1+\sqrt{5}}{2}$$

from integration we can calculate.

$$S_n=\frac{\pi}{50}\left[\sqrt{10+2\sqrt{5}}+\sqrt{10-2\sqrt{5}}\right]$$

4. $f'(x)=3x^2-6x \quad =3x(x-2)$

+ − +

$-\infty$ 0 2 $+\infty$

Increasing in $(-\infty 0)\cup(2,\infty)$

Decreasing in (0, 2)

If $x+2\le 0 \quad \Rightarrow x\le -2$

Then $g(x)=f(x+2) \quad -3\le x\le -2$

If $x+1<0$ and $0<x+2<2$

$x<-1 \quad -2<x<0$

$g(x)=f(e) \quad 2<x<-1$

Now for $x+1\ge 0 \quad x+2\le 2$

$g(x)=f(x+1) \quad -1\le x\le 0$

$$g(x)=\begin{cases} f(x+2) & -3\le x<-2\\ f(0) & -2\le x<-1\\ f(x+1) & -1\le x<0\\ 1-x & x\ge 0\end{cases}$$

continuous in [–3, 0] then

$$g(x)=\begin{cases} x^3+3x^2+2 & -3\le x<-2\\ 6 & -2\le x<-1\\ x^3-3x+4 & -1\le x<0\\ 1-x & x\ge 0\end{cases}$$

$$g'(x)=\begin{cases} 3x^2+6x & -3\le x<-2\\ 0 & -2\le x<-1\\ 3x^2-3 & -1\le x<0\\ -1 & x\ge 0\end{cases}$$

Differentiable everywhere except $x=0$

5. Let O be the centre of the circle and the pole OA be inclined to the vertical OZ at an angle α. Let $OA=l$. Take OX, OY, OZ as x, y, z axes. P be any point on the circle $\angle POY=\phi$ and $\angle OPA=\psi$. Now $A\,(0, l\sin\alpha, l\cos\alpha)$ and $P=(a\sin\pi, a\cos\pi, 0)$ where a is the radius

$$\overrightarrow{OP}=a\sin\phi\,\hat{i}+a\cos\phi\,\hat{j}$$

$$\overrightarrow{OA}=l\sin\alpha\,\hat{j}+l\cos\alpha\,\hat{k}$$

$$\overrightarrow{AP}=\overrightarrow{OP}-\overrightarrow{OA}$$

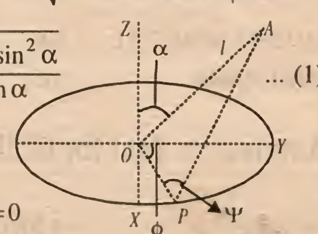
$$=a\sin\phi\,\hat{i}+\left(a\cos\phi-l\sin\alpha\right)\hat{j}-l\cos\alpha\,\hat{k}$$

$$|\overrightarrow{AP}|=\sqrt{a^2\sin^2\phi+\left(a\cos\phi-l\sin\alpha^2\right)+l^2\cos^2\alpha}$$

$$|\overrightarrow{AP}|=\sqrt{a^2+l^2-2al\cos\phi\cdot\sin\alpha}$$

$$\cos\psi=\frac{OP^2+AP^2-OA^2}{2OP\cdot AP}=\frac{a-l\cos\phi\cdot\sin\alpha}{\sqrt{a^2+l^2-2al\cos\phi\sin\alpha}}$$

$$\tan\psi=\frac{\sqrt{l^2-l^2\cos^2\phi\cdot\sin^2\alpha}}{a-l\cos\phi\cdot\sin\alpha} \quad \text{... (1)}$$



$$\Rightarrow \frac{d(\tan\psi)}{d\phi}=0$$

$$\sin\phi\left[a\cos\phi\cdot\sin\alpha-l\right]=0$$

$\Rightarrow \quad \sin\phi = o, \quad \Rightarrow \phi = 0, or\, \pi$

$$\cos\phi = \frac{l}{a\sin\alpha} > 1 \text{ (Impossible)}$$

Ψ is maximum when $\phi = 0$ and minimum when $\phi = \pi$ when $\phi = 0$ in (1)

$$\tan\psi = \frac{9}{5} = \frac{l\cos\alpha}{a - l\sin\alpha} \quad \text{... (2)}$$

When $\phi = \pi$; $\tan\psi = \frac{6}{5} = \frac{l\cos\alpha}{a + l\sin\alpha}$ ... (3)

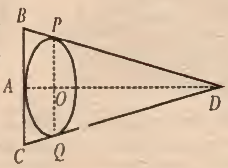
$$\cos\alpha = \frac{36}{\sqrt{1321}},\ \sin\alpha = \frac{5}{\sqrt{1321}},\quad l = \frac{\sqrt{1321}}{25}a$$

$\therefore$ When man is midway $\phi = \frac{\pi}{2}$ so from ...(1)

$$\tan\theta = \frac{l}{a} = \frac{\sqrt{1321}}{25}$$

$$\boxed{\theta = \tan^{-1}\left(\frac{\sqrt{1321}}{25}\right)}$$

6. Let BCD be a given triangle where $DA = h$ & $BC = 2k$. Let the equation of ellipse be $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$



where x-axis is along OA and y-axis perpendicular to OA. The equation of tangent to the ellipse at the point $(a\cos\theta, b\sin\theta)$ is $\frac{x}{a}\cos\theta + \frac{y}{b}\sin\theta = 1$ ... (i)

$\therefore$ point B $(-a, k)$ lies on tangent (i) therefore

$$\Rightarrow \quad b = \frac{k\sin\theta}{1+\cos\theta} \quad \text{... (ii)}$$

The point D $(h - a, 0)$ also lies on tangent (i) therefore

$$\Rightarrow \quad a = \frac{h\cos\theta}{1+\cos\theta}$$

Area of ellipse $= \pi\, ab$ $\quad \Rightarrow S = \frac{\pi hk\sin\theta\cdot\cos\theta}{(1+\cos\theta)^2}$

For maximum & minimum $\frac{dS}{d\theta} = 0 \Rightarrow \cos\theta = \frac{1}{2}\ \therefore \theta = \frac{\pi}{3}$

$$S_{max} = \frac{\sqrt{3}\,\pi hk}{9}$$

7. $\lim\limits_{n\to\infty} \sum\limits_{n=0}^{n} \frac{1}{n!}\left\{\sum\limits_{r=0}^{n} (r+1)\int_0^1 2^{-(r+1)x}\,dx\right\}$

$$\therefore \int_0^1 2^{-(r+1)x}\,dx = \left|\frac{2^{-(r+1)x}}{-(r+1)\log_e 2}\right|_0^1$$

$$= \frac{1}{(r+1)\log_e 2} - \frac{2^{-(r+1)}}{(r+1)\log_e 2}$$

Then $\sum\limits_{r=0}^{n} (r+1)\cdot\frac{1}{(r+1)}\cdot\frac{1}{\log_e 2}\left(1 - 2^{-(r+1)}\right)$

$$= \frac{1}{\log_e 2}\sum_{r=0}^{n}\left(1 - 2^{-(r+1)}\right) = \frac{1}{\log_e 2}\left[(n+1) - \frac{1}{2}\sum_{k=0}^{h}\left(\frac{1}{2}\right)^k\right]$$

$$\Rightarrow \lim_{n\to\infty}\frac{1}{\log_e 2}\sum_{n=0}^{n}\frac{1}{n!}\left[(n+1) - \frac{1}{2}\cdot\frac{1 - \frac{1}{2^{n+1}}}{1 - \frac{1}{2}}\right]$$

$$\Rightarrow \lim_{h\to\infty}\frac{1}{\log_e 2}\sum_{n=0}^{n}\frac{n + \frac{1}{2^{n+1}}}{n!}$$

$$= \lim_{h\to\infty}\frac{1}{\log_e 2}\left[\sum_{n=0}^{n}\frac{1}{(n-1)!} + \frac{1}{2}\sum_{n=0}^{n}\frac{\left(\frac{1}{2}\right)^n}{n!}\right]$$

$$= \frac{e + \frac{1}{2}e^{\frac{1}{2}}}{\log_e 2} = \frac{2e + \sqrt{e}}{2\log_e 2}$$

8. $Sgn\left[\lim\limits_{n\to\infty}\left\{\frac{1\cdot\sum\limits_{r=1}^{n} r + 2\sum\limits_{r=1}^{n-1} r + 3\sum\limits_{r=1}^{n-2} r + \text{........} n\cdot 1}{n^4}\right\}\right]$

$$Let\ T_k = k\cdot\sum_{r=1}^{n-k+1} r = k\left[1 + 2 + \text{..........} + (n-k+1)\right]$$

$$= k\left[\frac{(n-k+1)(n-k+2)}{2}\right]$$

$$T_k = \left(\frac{n^2}{2} + \frac{3n}{2} + 1\right)k - \left(n + \frac{3}{2}\right)k^2 + \frac{k^3}{2}$$

$$S_n = \sum_{k=1}^{n} T_k = \frac{n(n+1)^2(n+2)}{4} - \frac{n(n+1)(2n+1)(2n+3)}{12} + \frac{n^2(n+1)^2}{8}$$

$\lim\limits_{n\to\infty}\frac{S_n}{n^4} = \frac{1}{24}$ then $Sgn\left[\frac{1}{24}\right] = 1$ ... (1

Now (ii) Part

$$Sgn\left[\lim_{n\to\infty}\left\{\left(151\times 4^{100}\right)^n + \left(\sum_{r=1}^{300} rb_r\right)^n\right\}^{1/n} - 152\times 4^{400}\right]$$

Contd. on page no. 7

egative and positive accordingly. In this case, the right and part of the answer becomes negative which is to subtracted from the left hand part of the answer for taining the answer.

xample 14. Find the Product of 12 and 7.

$$\begin{array}{lll} 12 & : & 12+2 \\ 7 & : & 7-3 \end{array}$$

ght hand part of the answer is the product $) \times (-3) = -6$. Left hand part of the answer is either $2 - 3)$ or $(7 + 2) = 9$. We get

$$\begin{array}{lll} 12 & : & 12+2 \\ 7 & : & \underline{7-3} \\ 7 & : & 9/\bar{6} = 84 \text{ (using vinculum)} \end{array}$$

xample 15. Find the Product of 131 and 91 .

$$\begin{array}{lll} 131 & : & 131+31 \\ 91 & : & \underline{91-9} \\ & & 122/_{2}\bar{7}\bar{9} = (122-2)/\bar{7}\bar{9} \end{array}$$

quired product is $120\bar{7}\bar{9} = 12\bar{1}21$

$= 11921$ (use vinculum)

xample 16. Find the Product of 1023 and 997 .

$$\begin{array}{lll} 1023 & : & 1023+023 \\ 997 & : & \underline{997-003} \\ & & 1020\ /\ 0\bar{6}\bar{9} = 10200\bar{6}\bar{9} \end{array}$$

quired product is $10200\bar{6}\bar{9} = 1020\bar{1}31$

$= 102\bar{1}931 = 1019931$

ample 17. Find the Product of 10007 and 8965 .

$$\begin{array}{lll} 10007 & : & 10007+0007 \\ 8965 & : & \underline{8965-1035} \\ & & 8972\ /\ \bar{7}\bar{2}\bar{4}\bar{5} \\ & & = 8972\bar{7}\bar{2}\bar{4}\bar{5} = 89712755 \end{array}$$

to be continued.....

■

ANSWERS - KARNATAKA CET 2004

	(b)	2.	(b)	3.	(c)	4.	(c)	5.	(c)
	(a)	7.	(d)	8.	(a)	9.	(c)	10.	(b)
.	(b)	12.	(b)	13.	(a)	14.	(d)	15.	(c)
.	(b)	17.	(b)	18.	(c)	19.	(a)	20.	(d)
.	(d)	22.	(a)	23.	(c)	24.	(b)	25.	(b)
.	(b)	27.	(c)	28.	(c)	29.	(a)	30.	(b)
.	(d)	32.	(b)	33.	(a)	34.	(c)	35.	(b)
.	(a)	37.	(d)	38.	(b)	39.	(b)	40.	(c)
.	(a)	42.	(d)	43.	(b)	44.	(b)	45.	(c)
.	(d)	47.	(d)	48.	(c)	49.	(a)	50.	(b)
.	(a)	52.	(d)	53.	(c)	54.	(b)	55.	(b)
.	(c)	57.	(a)	58.	(b)	59.	(c)	60.	(d)

Contd. from page no. 16

$$\left(1+x+x^2+x^3\right)^{100} = b_0+b_1x+b_2x^2+\ldots\ldots\ldots b_{300}x^{300}$$

Differentiating both side

$$100\left(1+x+x^2+x^3\right)^{99}\left(1+2x+3x^2\right)$$

$$= b_1+2b_2x+\ldots..300b_{300}x^{299}$$

Put $x = 1$; $\quad 100\times 4^{99}\cdot 6 = 150\times 4^{100}$

$$\therefore \sum_{r=0}^{300} r\,b_r = 150\times 4^{100}$$

$$\lim_{n\to\infty}\left\{\left(151\times 4^{100}\right)^n+\left(150\times 4^{100}\right)^n\right\}^{1/n}$$

$$\lim_{n\to\infty}\left[\left(151\times 4^{100}\right)^n\left\{1+\left(\frac{150\times 4^{100}}{151\times 4^{100}}\right)^n\right\}\right]^{1/n}$$

$$= 151\times 4^{100}$$

$$\Rightarrow \ Sgn\left[151\times 4^{100}-152\times 4^{100}\right]$$

$$Sgn\left[-1\times 4^{100}\right] = -1$$

Now complete (i) + (ii) = 1 + (–1) = 0.

9. Notice that given equation is quadratic in $f'(x)$

$$f'(x) = \frac{-4f(x)\pm\sqrt{16f^2(x)-4f^2(x)}}{2}$$

$$= \frac{-4f(x)\pm 2\sqrt{3}f(x)}{2}$$

$$f'(x) = -2f(x)\pm\sqrt{3}f(x)\,; \qquad \frac{f'(x)}{f(x)} = -2\pm\sqrt{3}$$

Integrate both side

$$\log f(x) = \left(-2\pm\sqrt{3}\right)x+c\,; \qquad \boxed{f(x) = e^{c\left(-2\pm\sqrt{3}\right)x}}$$

10.

$$\lim_{a\to b}\frac{(\sin a)^{\sin b}-(\sin b)^{\sin b}+(\sin b)^{\sin b}-(\sin b)^{\sin a}}{(\sin a)^{\sin a}-(\sin b)^{\sin a}+(\sin b)^{\sin a}-(\sin b)^{\sin b}}$$

$$\lim_{a\to b}\frac{\left[\dfrac{(\sin a)^{\sin b}-(\sin b)^{\sin b}}{\sin a-\sin b}\right]-(\sin b)^{\sin b}\left[\dfrac{(\sin b)^{\sin a-\sin b}-1}{\sin a-\sin b}\right]}{\dfrac{(\sin a)^{\sin a}-(\sin b)^{\sin a}}{\sin a-\sin b}+(\sin b)^{\sin b}\left[\dfrac{(\sin b)^{\sin a-\sin b}-1}{\sin a-\sin b}\right]}$$

$$\Rightarrow \frac{\sin b\cdot(\sin b)^{\sin b-1}-(\sin b)^{\sin b}\log\sin b}{\sin a\cdot(\sin b)^{\sin a-1}+(\sin b)^{\sin b}\log(\sin b)}$$

$$= \frac{(\sin b)^{\sin b}[1-\log\sin b]}{(\sin b)^{\sin b}[1+\log\sin b]} = \frac{1-\log\sin b}{1+\log\sin b}$$

■

MATHS MUSING (SET - 18)

SOLUTION

1. Integrating by parts we get

$$\frac{1}{2}\int_0^x (x-t)^2 f'''(t)dt = -\frac{x^2}{2}f''(0) - xf'(0) - f(0) + f(x)$$

hence $f(x) = f(0) + xf'(0) + \frac{x^2}{2}f''(0) + \frac{1}{2}\int_0^x (x-t)^2 f'''(t)dt.$

2. Using the formula $\int_0^1 f(x)\,dx = \lim_{n\to\infty}\frac{1}{n}\sum_{r=1}^{n} f\left(\frac{r}{n}\right)$,

the limit becomes $\dfrac{\int_0^1 \sqrt{x}\,dx \cdot \int_0^1 \frac{dx}{\sqrt{x}}}{\int_0^1 x\,dx} = \frac{8}{3}.$

3. $1-2\sin^2\frac{A}{2},\ 1-2\sin^2\frac{B}{2},\ 1-2\sin^2\frac{C}{2}$ are in A.P.

$\therefore\ \sin^2\frac{A}{2},\ \sin^2\frac{B}{2},\ \sin^2\frac{C}{2}$ are in A.P.

or $\frac{(s-b)(s-c)}{bc}, \frac{(s-c)(s-a)}{ca}, \frac{(s-b)(s-a)}{ab}$ are in A.P. or $\frac{a}{s-a}, \frac{b}{s-b}, \frac{c}{s-c}$ are in A.P.

Adding 1 to each of the term and then multiplying by $\frac{\Delta}{s^2}$, we find $\tan\frac{A}{2}, \tan\frac{B}{2}, \tan\frac{C}{2}$ are in A.P.

4. Let the sides be $a, b, \sqrt{a^2+b^2}$.

It is given that $\frac{ab}{2} = a+b+\sqrt{a^2+b^2}$. Solving for b, we find $b = \frac{4(a-2)}{a-4}$. Since a and b are positive integers, we have $a = 5$, $b = 12$ and $a = 6$, $b = 8$. Hence the area is 30 or 24.

5. Let A be (x_1, y_1) and the line be $x = x_1 + r\cos\theta$, $y = y_1 + r\sin\theta$. Substituting in the equation of the circle, we get $r^2 + 2(x_1\cos\theta + y_1\sin\theta)r + x_1^2 + y_1^2 - a^2 = 0$ whose roots are $r_1 = AB$ and $r_2 = AC$. If $AP = r$, we have $2r = r_1 + r_2$ or $r^2 = -(r\cos\theta + r\sin\theta)$ or $x(x-x_1) + y(y-y_1) = 0$ which is the circle with AO as diameter.

6. Let P be (x_1, y_1). The equation of pair of tangents from P is

$$(y_1^2 - 4ax_1)(y^2 - 4ax) - (yy_1 - 2hx - 2ax_1)^2 = 0$$

If these lines meet the coordinate axes at concyclic points, then the coefficient of x^2 = coefficient of y^2 = i.e., $x_1 = a$ or the desired locus is $x = a$.

7. $$\int_0^{n^2}\{\sqrt{x}\}dx = \int_0^{x^2}\sqrt{x}dx - \int_0^{x^2}[\sqrt{x}]dx$$

$$= \frac{2}{3}n^3 - [1(2^2-1^2) + 2(3^2-2^2) + 3(4^2-3^2) + \dots + (n-1)(n^2-(n-1)^2)]$$

$$= \frac{2}{3}n^3 - n^2(n-1) + 1^2 + 2^2 + 3^2 + \dots + (n-1)^2$$

$$= \frac{n}{6}(3n+1).$$

8. For any positive integer n, $\sum_{r=1}^{n} r^3 = \left(\frac{n(n+1)}{2}\right)^2$

$$\therefore\ n^3 = \left(\frac{n(n+1)}{2}\right)^2 - \left(\frac{n(n-1)}{2}\right)^2 \qquad \text{..(i)}$$

If n is an odd integer, we have

$$n^3 = \left(\frac{n^3+1}{2}\right)^2 - \left(\frac{n^3-1}{2}\right)^2 \qquad \text{...(ii)}$$

From (i) and (ii), we get

$13^3 = 2197 = 91^2 - 78^2 = 1099^2 - 1098^2$.

9. The A.G.P. is of the form $a, (a+d)r, (a+2d)r^2, \dots$

Here $a = 1$, $(1+d)r = 3$, $(1+2d)r^2 = 8$

Solving we get $d = 1/2$, $r = 2$ or $d = -1/4$, $r = 4$

The progressions are 1, 3, 8, 20, 48, 112

1, 3, 8, 16, 0, –256

The required sum = 180 or –240.

10. Let $A = (1, 0, 0)$, $B = (1, 1, 0)$, $C = (0, 1, 0)$, $D = (0, 0, 0)$

The new position of D is $D'\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}\right)$

The unit vector perpendicular to AB and CD' i

$$\bar{n} = \frac{2}{\sqrt{3}}\left(\frac{-i}{\sqrt{2}} + \frac{k}{2}\right)$$

The shortest distance between AB and CD'

$$= \overline{AC}\cdot\bar{n} = (j-i)\cdot\bar{n} = \sqrt{\frac{2}{3}}.$$

See names of solution senders of Maths Musing – 18 on page No.52

$S_n = 1+25\cdot 5+81\cdot 5^2+169\cdot 5^3+.....+(4n+1)^2\cdot 5^{n-1}$

$5S_n = 1\cdot 5+25\cdot 5^2+81\cdot 5^3+(4n-3)^2 5^{n-1}+(4n+1)^2\cdot 5^n$

on subtraction we get,

$$(-4)S_n = 1+24\cdot 5+56\cdot 5^2+88\cdot 5^3+................ + (8n-2)\cdot 4\cdot 5^{n-1}-(4n+1)^2 5^n$$

$$\Rightarrow -4S_n = 1+24\cdot 5+56\cdot 5^2+88\cdot 5^3+........... + (8n-2)\cdot 4\cdot 5^{n-1}-(4n+1)^2 5^n$$

$$5(-4)S_n = 5+24\cdot 5^2+56\cdot 5^3++(8n-2)\cdot 2\cdot 5^n -(4n+1)^2 5^{n+1}$$

Again on subtraction we get;

$$16S_n = 1+23\cdot 5+\underbrace{32\cdot 5^2+32\cdot 5^3+.........}_{(n-2)times} - 8(4n-1)\cdot 5^n+(4n+1)^2\cdot 5^{n+1}$$

$$\Rightarrow 16S_n = 1+23\cdot 5+32\cdot 5^2\frac{5^{n-2}-1}{5-1}-8(4n-1)5^n +(4n+1)^2\cdot 5^{n+1}$$

$$\Rightarrow 16S_n = 116+8\cdot\left(5^n-5^2\right)-8(4n-1)5^n +(4n+1)^2 5^{n+1}$$

$$16S_n = (4n+1)^2 5^{n+1}+16\cdot 5^n-32\cdot n\cdot 5^n-84$$

$$\Rightarrow S_n = \frac{(4n+1)^2\cdot 5^{n+1}+16\cdot 5^n-32\cdot n\cdot 5^n-84}{16}$$

9. $\because \sin^2 x+\left(\sin^2 x\right)^2+\left(\sin^2 x\right)^3+........ = \dfrac{\sin^2 x}{1-\sin^2 x}$

$$= \tan^2 x \begin{cases} \because 0<x<\dfrac{\pi}{2} \\ \therefore 0<\sin x<1 \end{cases}$$

$$\therefore \exp\left[\left(\sin^2 x+\sin^4 x+\sin^6 x+.....\right)\log_e 2\right] = e^{\tan^2 x\cdot\log_e 2} = 2^{\tan^2 x}$$

Now, $x^2-9x+8=0 \Rightarrow x=1,8$

It is given that $2^{\tan^2 x}$ satisfies the quadratic equation $x^2-9x+8=0$

So $2^{\tan^2 x}=1$ & $2^{\tan^2 x}=8$

$\Rightarrow$ $\tan^2 x = 0$ & $\tan^2 x = 3$

$\Rightarrow$ $x = 0$, $x = \pi/3$

if $x = 0$ then $\dfrac{\sin x-\cos x}{\sin x+\cos x} = -1$

if $x = \dfrac{\pi}{3}$ then $\dfrac{\sin x-\cos x}{\sin x+\cos x} = \dfrac{\dfrac{\sqrt{3}}{2}-\dfrac{1}{2}}{\dfrac{\sqrt{3}}{2}+\dfrac{1}{2}} = \dfrac{\sqrt{3}-1}{\sqrt{3}+1} = 2-\sqrt{3}$

10. $\because f(x) = 7+2x\log_e 25-5^{x-1}-5^{2-x}$

$$\Rightarrow f'(x) = 2\log_e 25-5^{x-1}\log 5+5^{2-x}\log 5$$

$$\Rightarrow f'(x) = (\log_e 5)\left(4-5^{x-1}+5^{2-x}\right)$$

$$\Rightarrow f'(x) = \frac{-(\log_e 5)}{5^{x+1}}\left(5^{2x}-20\cdot 5^x-125\right)$$

$$\Rightarrow f'(x) = \frac{-(\log_e 5)}{5^{x+1}}\left(5^x-25\right)\left(5^x+5\right)$$

For maximum or minimum, we must have $f'(x)=0$ i.e. $5^x-25=0 \Rightarrow x=2$

clearly, $f'(x)$ changes its sign from $+ve$ to $-ve$ in the neighbourhood of $x = 2$. So $f(x)$ attains a local maximum at $x = 2$.

Thus, if a is the first term of the infinite G.P., then $a = 2$. Let r be the common ratio of the G.P., then

$$r = \underset{x\to 0}{Lt}\int_0^x \frac{t^2 dt}{x^2(\tan(\pi+x))} = \underset{x\to 0}{Lt}\frac{x^3}{3\cdot x^2\cdot\tan x}$$

$$= \frac{1}{3}\cdot\underset{x\to 0}{Lt}\frac{x}{\tan x} = \frac{1}{3}$$

$$\therefore \text{ Required sum } = \frac{a}{1-r} = \frac{2}{1-\frac{1}{3}} = \frac{2}{\frac{2}{3}} = 3$$

■

Puzzle ???

Suppose there are twin brothers, one which always tells the truth and one which always lies. (So in this case they know what is true and what is false, or as you put it, both are accurate in their knowledge). What one yes-no question could you ask to either one of the brothers to figure out which one he is?

Ans.: The question one could ask is, "If I were to ask your brother whether you always tells the truth, what would he say?" A reply of "no" means you are talking to the truth teller, a reply of "yes" means you are talking to the liar.

Another possible question is, "If I were to ask you whether you always tell the truth, what would you say?" In this case a reply of "yes" means you are talking to the truth teller and a reply of "no" means you are talking to the liar.

Both questions take advantage of the liar lying about what he or his brother would say, creating a double negative type situation.

10^{th} Mathematical Challenge for I.I.T. MAINS

This section is designed to give IIT JEE aspirants a thorough grinding & exposure to variety of possible twists and turns of problems in mathematics that would be very helpful in facing IIT JEE. Each and every problem is well thought of in order to strengthen the concepts and we hope that this section would prove a rich resource for practicing challenging problems and enhancing the preparation level of IIT JEE aspirants.
The detailed solutions to these problems will be published in the next issue alongwith a new set of such problems.

1. Let $I_n = \int_0^{\infty} e^{-x} \sin^n x dx$, if n is an even integer, prove that $I_n = \dfrac{(n)!}{\prod_{r=1}^{n/2}[(2r)^2+1]}$.

2. Show that all the members of the family of curves represented by $y'' - y\cot x + 2\text{cosec } x = 0$ intersect the lines $y = \pm 2$ at equally spaced infinite number of points.

3. Find the number of integers which lie between the numbers 786 and 999786 having the sum of their digits equal to 25?

4. Show that the cube roots of three distinct prime numbers can't be three terms of an A.P. (not necessarily consecutive).

5. Tangents are drawn from any point on the rectangular hyperbola $x^2 - y^2 = a^2 - b^2$ to the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$. Prove that these tangents are equally inclined to the asymptotes of the hyperbola.

6. Let $f(n, \theta) = \left(1 - \tan^2\dfrac{\theta}{2}\right)\left(1 - \tan^2\dfrac{\theta}{4}\right)\left(1 - \tan^2\dfrac{\theta}{8}\right)$.......... to n factors. Show that $\lim_{n\to\infty} f(n, \theta) = \theta\cot\theta$.

7. Without finding the vertices or angles of the triangle, show that the three straight lines $au + bv = 0$, $au - bv = 2ab$ and $u + b = 0$ form an isosceles triangle where $u = x + y - b$ and $v = x - y - a$ and $a, b \neq 0$.

8. Let a_0, a_1, a_2, a_3, ... be real such that $a_3 + 2a_1 = 0$ and $a_2 + 3a_0 = 0$, then prove that all the zeros of cubic polynomial $a_3x^3 + a_2x^2 + a_1x + a_0$ will be real.

9. If $x > 0$ and n is a natural number, show that $\dfrac{x^n}{1 + x + x^2 + ... + x^{2n}} \leq \dfrac{1}{1+2n}$.

10. Consider the non-zero vectors $\vec{a}, \vec{b}, \vec{c}$ and $\vec{d}$ such that no three of which are coplanar then prove that $\vec{a}[\vec{b}\ \vec{c}\ \vec{d}] + \vec{c}[\vec{a}\ \vec{b}\ \vec{d}] = \vec{b}[\vec{a}\ \vec{c}\ \vec{d}] + \vec{d}[\vec{a}\ \vec{b}\ \vec{c}]$. Hence prove that $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ represent the position vectors of the vertices of a plane quadrilateral if and only if $\dfrac{[\vec{b}\ \vec{c}\ \vec{d}] + [\vec{a}\ \vec{b}\ \vec{d}]}{[\vec{a}\ \vec{c}\ \vec{d}] + [\vec{a}\ \vec{b}\ \vec{c}]} = 1$.

By: **Shailendra Maheshwari**, Career point, Kota

BRAIN TWISTERS for IIT MAINS

By : S.K. Tiwari, Insight, Kota

Time : 2 hrs. *Max. Marks : 60*

1. A cubic equation $f(x) = 0$ has one real root α and two complex roots $\beta + i\gamma$ and $\beta - i\gamma$. Points A, B, C represents roots α, $\beta + i\gamma$ and $\beta - i\gamma$ respectively on the argand diagram. Show that the roots of the derived equation $f'(x) = 0$ are complex. If A falls inside one of the two equilateral triangles described on base BC.

2. Equilateral triangles are described externally on the sides BC, CA and AB of a given triangle ABC. Prove using complex numbers that their centroids form an equilateral triangle.

3. Let $g(x) = \begin{cases} 0 & -e \le x < 1 \\ \left\{1 + \frac{1}{3}\sin\left(\ln x^{2\pi}\right)\right\} & 1 \le x < e \end{cases}$ where

$\{.\}$ denotes fractional part of x and

$$f(x) = \begin{cases} xg(x) & \text{for } g(x) = 1 + \frac{1}{3}\sin(\ell n\, x^{2\pi}) \\ x(g(x)+1) & \text{otherwise} \end{cases}$$

discuss the continuity and differentiability of $f(x)$ over its domain.

4. A parabola is drawn such that each vertex of a given triangle is the pole of the opposite side. Show that the focus of the parabola lies on the nine point circle of the triangle and that the orthocentre of the triangle formed by joining the middle points of the sides lies on the directrix.

5. If the function $f(x) = x^3 - 9x^2 + 24x + C$ has three real and distinct roots α, β, γ then find the possible values of C. Hence or otherwise show that $[\alpha] + [\beta] + [\gamma]$ can take only two values and determine these values where [.] denotes greatest integer function.

6. Change in cartesian form of these two complex number :

(i) $\cot(x + iy)$ (ii) $e^{\sin(x + iy)}$

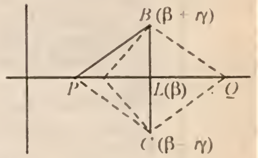
7. Let $P_n(x)$ be the polynomial $P_n(x) = 1 + 2x + 3x^2 + \ldots\ldots (n+1)x^n$. Show that $P_n(x)$ has no real root if n is even and exactly one real root if n is odd and the roots lie between -1 and 0.

8. If $S_n = 1 - \frac{1}{4} + \frac{1}{6} - \frac{1}{9} + \frac{1}{11} - \frac{1}{14} + \ldots\ldots\ldots$ upto n terms evaluate $\lim\limits_{n\to\infty} S_n$.

9. Find the sum of the series $\sum\limits_{m=1}^{\infty}\sum\limits_{n=1}^{\infty} \frac{m^2 n}{3^m(n\cdot 3^m + m\cdot 3^n)}$

SOLUTION

1. Since α, $\beta + i\gamma$ and $\beta - i\gamma$ are the roots of $f(x) = 0$ so let



$f(x) = k(x - \alpha)(x - \beta - i\gamma)(x - \beta + i\gamma)$

$= k(x - \alpha)\{(x - \beta)^2 + \gamma^2\}$

$\Rightarrow f'(x) = k.\{3x^2 - 2x(\alpha + 2\beta) + \beta^2 + \gamma^2 + 2\alpha\beta\}$

So $D = 4(\alpha + 2\beta)^2 - 12(\beta^2 + \gamma^2 + 2\alpha\beta)$

$= 4\{\alpha^2 + \beta^2 - 3\gamma^2 - 2\alpha\beta\}$

$|BC| = 2|\gamma|$; $|PL| = \sqrt{3}|\gamma|$

Clearly $A(\alpha)$ lies inside $A(\alpha, \phi)$ the equilateral triangle PBC and QBC.

If $AL < PL$

$\Rightarrow |\beta - \alpha| < \sqrt{3}|\gamma| \Rightarrow (\beta - \alpha)^2 < 3\gamma^2$

$\Rightarrow \beta^2 + \alpha^2 - 2\alpha\beta - 3\gamma^2 < 0$

$\Rightarrow D < 0$ So $f'(x) = 0$ has imaginary roots.

2. Let the complex numbers z_1, z_2, z_3 represents th

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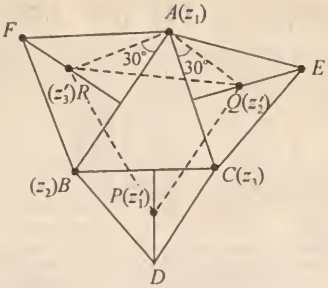
vertices A, B and C respectively of ΔABC. Equilateral triangles BCD, CAE and ABF are constructed on the side BC, CA and AB respectively. Let P, Q and R be the centroids of triangles BCD, CAE and ABF respectively.



We have $CP = \frac{2}{3}$ (Median of equilateral triangle BCD)

$$= \frac{2}{3}\left(\frac{\sqrt{3}}{2}BC\right), \quad = \frac{BC}{\sqrt{3}}$$

similarly $AQ = \dfrac{AC}{\sqrt{3}}$ and $AR = BR = \dfrac{AB}{\sqrt{3}}$

Let P, Q, R represents the complex numbers z'_1, z'_2 and z'_3 respectively. We have,

$$\angle CAQ = 30°, \Rightarrow z'_2 - z_1 = \frac{1}{\sqrt{3}}(z_3 - z_1)\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$$

$$\Rightarrow z'_2 - z_1 = \frac{1}{\sqrt{3}}(z_3 - z_1)\left(\frac{\sqrt{3}}{2} + \frac{1}{2}i\right),$$

$$\Rightarrow z'_2 - z_1 = \frac{1}{2\sqrt{3}}(z_3 - z_1)(\sqrt{3} + i) \qquad ...(i)$$

$$\angle BAR = 30°, \ \therefore z'_3 - z_1 = \frac{AR}{AB}(z_2 - z_1)e^{-i\pi/6}$$

$$\Rightarrow z'_3 - z_1 = \frac{1}{\sqrt{3}}(z_2 - z_1)\left(\cos\frac{\pi}{6} - i\sin\frac{\pi}{6}\right)$$

$$\Rightarrow z'_3 - z_1 = \frac{1}{2\sqrt{3}}(z_2 - z_1)(\sqrt{3} - i) \qquad ...(ii)$$

Now, $z'_2 - z'_3 = (z'_2 - z_1) - (z'_3 - z_1)$

$$= \frac{1}{2\sqrt{3}}(z_3 - z_1)(\sqrt{3} + i) - \frac{1}{2\sqrt{3}}(z_2 - z_1)(\sqrt{3} - i)$$

[using (i) and (ii)]

$$= \frac{1}{2}\{(z_3 - z_1)(z_2 - z_1)\} + \frac{1}{2\sqrt{3}}i(z_3 - z_1) + (z_2 - z_1)\}$$

$$= \frac{1}{2}(z_3 - z_2) + \frac{1}{2\sqrt{3}}i(z_3 - z_1) + (z_2 - z_1)\} \qquad ... (iii)$$

Similarly

$$z'_3 - z'_1 = \frac{1}{2}(z_1 - z_3) + \frac{i}{2\sqrt{3}}\{(z_3 - z_2) + (z_1 - z_2)\} \ ..... (iv)$$

Now, $(z'_1 - z'_3)e^{i\pi/3} = (z'_1 - z'_3)\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

$$= (z'_1 - z'_3)\left(\frac{1 + i\sqrt{3}}{2}\right)$$

$$= \left[\frac{1}{2}(z_3 - z_1) + \frac{i}{2\sqrt{3}}\{(z_2 - z_3) + (z_2 - z_1)\}\right]\left(\frac{1}{2} + \frac{i\sqrt{3}}{2}\right)$$

$$= \left[\frac{1}{4}(z_3 - z_1) - \frac{1}{4}\{(z_2 - z_3) + (z_2 - z_1)\}\right] + i$$

$$\left[\left\{\frac{\sqrt{3}}{4}(z_3 - z_1) + \frac{1}{4\sqrt{3}}\{(z_2 - z_3) + (z_2 - z_1)\}\right\}\right]$$

$$= \frac{1}{4}[(z_3 - z_1) - z_2 + z_3 - z_2 + z_1]$$

$$+ \frac{i}{4\sqrt{3}}[3(z_3 - z_1) + z_2 - z_3 + z_2 - z_1]$$

$$= \frac{1}{4}[2(z_3 - z_2)] + \frac{i}{2\sqrt{3}}[(z_3 - z_1) + (z_2 - z_1)]$$

$$= \frac{1}{4}(z_3 - z_2) + \frac{i}{2\sqrt{3}}[z_3 - z_1) + (z_2 - z_1)]$$

$$= (z'_2 - z'_3) \qquad ... \text{From (iii)}$$

Thus

$$\Rightarrow RQ = RP \text{ and } \angle PQR = \frac{\pi}{3}$$

$\Rightarrow \Delta PQR$ is an equilateral triangle.

---

3. Given $g(x) = \left\{1 + \frac{1}{3}\sin(\ln x^{2\pi})\right\}$ for $1 \le x \le e$

$= 0$ for $-e \le x < 1$

i.e., $g(x) = 1 + \frac{1}{3}\sin(\ln x^{2\pi}) - \left[1 + \frac{1}{3}\sin(\ln x^{2\pi})\right]$, $1 \le x \le e$

$= 0$, $-e \le x < 1$

where [.]denotes the greatest integer function.

Consider : $1 \le x \le e$

$$\Rightarrow (1)^{2\pi} \le x^{2\pi} \le e^{2\pi} \Rightarrow \ln(1) \le \ln\left(x^{2\pi}\right) \le \ln(e^{2\pi})$$

$$\Rightarrow 0 \le \ln(x^{2\pi}) \le 2\pi$$

Case (i) If $0 \le \ln(x^{2\pi}) \le \pi$ *i.e.*, $1 \le x \le \sqrt{e}$ then

$$0 \le \sin(\ln(x^{2\pi})) \le 1 \Rightarrow 0 \le \frac{1}{3}\sin(\ln(x^{2\pi})) \le \frac{1}{3}$$

$$\therefore \left[\frac{1}{3}\sin(\ln(x^{2\pi}))\right] = 0$$

$\therefore g(x) = \frac{1}{3}\sin(\ln x^{2\pi})$ for $1 \le x \le \sqrt{e}$

Case (ii) If $\pi < \ln(x^{2\pi}) < 2\pi$ *i.e.*, $\sqrt{e} < x < e$

then $-1 \le \sin(\ln(x^{2\pi})) < 0 \ \Rightarrow -\frac{1}{3} \le \frac{1}{3}\sin(\ln(x^{2\pi})) < 0$

$$\therefore \ \left[\frac{1}{3}\sin(\ln(x^{2\pi}))\right] = -1$$

$\therefore \ g(x) = 1 + \frac{1}{3}\sin(\ln(x^{2\pi}))$ for $\sqrt{e} < x < e$

Case (iii) If ln $(x^{2\pi}) = 2\pi$

$\Rightarrow x = e \Rightarrow g(x) = \{1\} = 0$

combining all cases, we get

$$f(x)=x\left(1+\frac{1}{3}\sin(\ln(x^{2\pi}))\right) \text{ for } \sqrt{e}<x<e$$

$$=x\left(1+\frac{1}{3}\sin(\ln(x^{2\pi}))\right) \text{ for } 1\le x\le\sqrt{e}$$

$$= x\ (1 + 0) \text{ for } -e \le x <1$$

$$= x\ (1 + 0) \text{ for } x = e$$

$$\Rightarrow f(x)=x\left(1+\frac{1}{3}\sin(\ln(x^{2\pi}))\right) \quad \text{for } 1\le x \le e$$

$$= x \quad \text{for } -e \le x < 1$$

$\therefore f$ is differentiable in $(-e, 1)$ and $(1, e)$.

Check the differentiability of $f(x)$ at $x = 1$:

$$\text{LF}'(1)=\lim_{h\to 0}\frac{f(1-h)-f(1)}{-h}=\lim_{h\to 0}\frac{(1-h)-1}{-h}=1$$

and $$\text{RF}'(1)=\lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$=\lim_{h\to 0}\frac{(1+h).\left(1+\frac{1}{3}\sin(\ln(1+h)^{2\pi})\right)-1}{h}$$

$$=\lim_{h\to 0}\frac{h+\frac{(1+h)}{3}\sin(\ln(1+h)^{2\pi})}{h}$$

$$=\lim_{h\to 0}\left(1+\frac{(1+h)}{3}\frac{\sin\left\{\ln(1+h)^{2\pi}\right\}}{h}\right)$$

$$=1+\lim_{h\to 0}\frac{(1+h)}{3}\lim_{h\to 0}\frac{\sin\{\ln(1+h)^{2\pi}\}}{h}$$

$$=1+\lim_{h\to 0}\frac{(1+h)}{3}\lim_{h\to 0}\frac{\sin\{2\pi\ln(1+h)\}}{2\pi\ln(1+h)}\cdot\frac{2\pi\ln(1+h)}{h}$$

$$=1+\left(\frac{1+0}{3}\right)\cdot 1\cdot 2\pi\cdot 1 \;=1+\frac{2\pi}{3}.$$

Thus f is not differentiable at $x = 1$. Hence f is continuous and differentiable for all $x \in$ domain of f except not differentiable at $x = 1$.

4. The circle circumscribing the triangle formed by joining middle points of the sides of the given triangle formed by the nine point circle. As the circle circumscribing the triangle formed by the three tangents on the parabola always passes through the focus of the parabola, hence, we have to prove that the lines joining the mid-points of the given triangle are the tangents on the parabola.

Let the equation of the parabola be $y^2 = 4ax$. .....(i)

Suppose the given triangle is ABC and let PQ be the chord of contact of A w.r.t. (1). Again as BC is given to the polar of A w.r.t. (i) BC and PQ must lie on the same line. Hence the line joining the mid points of AB and AQ also passes through the mid points of AB and AC.

Suppose the co-ordinates of P and Q be $(at_1^2, 2at_1)$ and $(at_2^2, 2at_2)$ respectively so that the co-ordinates of the point of intersection of the tangents at P and Q i.e., A are $[at_1t_2, a(t_1 + t_2)]$.

$\therefore$ Co-ordinates of mid-points of AP will be

$$\left\{\frac{at_1(t_1+t_2)}{2},\frac{a(3t_1+t_2)}{2}\right\}$$

Similarly, the co-ordinates of midpoint of AQ will be

$$\left\{\frac{at_2(t_1+t_2)}{2},\frac{a(t_1+3t_2)}{2}\right\}$$

The line joining the mid-points of AP and AQ is

$$y-\frac{a(3t_1+t_2)}{2}=\frac{\frac{a(3t_1+t_2)}{2}-a\frac{(t_1+3t_2)}{2}}{\frac{at_1(t_1+t_2)}{2}-\frac{at_2(t_1+t_2)}{2}}\left\{x-\frac{at_1(t_1+t_2)}{2}\right\}$$

or $$y=\frac{2}{t_1+t_2}x+\frac{a(t_1+t_2)}{2}$$

This is a tangent on the parabola as it is of the form $y = mx + \frac{a}{m}$.

Again the ortho-centre of the triangle formed by the tangents is on direcrix and we have proved that the lines joining the midpoints of the sides of the triangle are tangents on the parabola. Hence the ortho-centre of the triangle formed by joining the mid points of the sides of the triangle lies on directrix of the parabola.

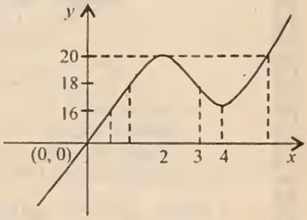
5. Let us consider

$y = x^3 - 9x^2 + 24x$,

$y_1 = 3(x - 2)(x - 4)$

$\therefore$ For turning point

$y_1 = 0 \therefore x = 2, 4$ are turning points.



$\therefore y(2) = 8 - 36 + 48 = 20$

$y(4) = 64 - 144 + 96 = 16$

From the graph we can see that x-axis will cut the graph 3 times, if it is shifted downward by 16 to 20 units.

$\therefore C \in (-20, -16)$ i.e. $-20 < c < -16$

For $[\alpha] + [\beta] + [\gamma]$

(i) If $C \in (-18, -16)$, $\alpha \in (-20, -16)$ i.e. $-20 < C < -16$

For $[\alpha] + [\beta] + [\gamma] = 1 + 3 + 4 = 8$

Contd. on page no. 80

Contd. from page no. 26

Again if $C \in (-20, -18)$ then $\alpha \in (1, 2)$, $\beta \in (2, 3)$, $\gamma \in (4, 5)$ $\therefore$ $[\alpha] + [\beta] + [\gamma] = 1 + 2 + 4 = 7$

Finally we can say

$$[\alpha]+[\beta]+[\gamma] = \begin{cases} 8 & \text{if } -18 < C < 16 \\ 7 & \text{if } -20 < C < -18 \end{cases}$$

6.(i) $$\cot(x+iy) = \frac{\cos(x+iy)}{\sin(x+iy)} = \frac{2\cos(x+iy)\cdot\sin(x-iy)}{2\sin(x+iy)\cdot\sin(x-iy)}$$

$$= \frac{\sin 2x - \sin(2iy)}{\cos(2iy) - \cos 2x} = \frac{\sin 2x - \left(\frac{e^{-2y} - e^{2y}}{2i}\right)}{\left(\frac{e^{-2y} + e^{2y}}{2}\right) - \cos 2x}$$

$$= \frac{2\sin 2x}{e^{-2y} + e^{2y} - 2\cos 2x} + i\left(\frac{e^{-2y} - e^{2y}}{e^{-2y} + e^{2y} - 2\cos 2x}\right).$$

(ii) $$e^{\sin(x+iy)} = e^{\sin x\cdot\cos(iy)} + \cos x \cdot \sin(iy)$$

$$= e^{\sin x}\cdot\left(\frac{e^{y} + e^{-y}}{2}\right) + \cos x \cdot \left(\frac{e^{-y} - e^{y}}{2i}\right)$$

$$= e^{\sin x}\cdot\left(\frac{e^{y} + e^{-y}}{2}\right)\cdot e^{i}\left(\frac{e^{y} - e^{-y}}{2}\right)\cos x$$

$$= e^{\sin x}\cdot\left(\frac{e^{y} + e^{-y}}{2}\right)\left[\cos\left\{\left(\frac{e^{y} - e^{-y}}{2}\right)\cos x\right\} + i\sin\left\{\left(\frac{e^{y} - e^{-y}}{2}\right)\cos x\right\}\right].$$

7. $P_n(x) = 1 + 2x + 3x^2 + 1 + \ldots (n + 1)x^n$

(where $x > 0$; $P_n(x) > 0$)

So, $P_n(x) = 0$ have no positive real root

$P_n(x) = 1 + 2x + 3x^2 + .. + (n + 1)x^n$

$xP_n(x) = x + 2x^2 + \ldots + nx^n + (n + 1)x^{n+1}$

$\Rightarrow$ $(1 - x)P_n(x) = 1 + x + x^2 + x^3 + \ldots + x^n - (n + 1) \cdot x^{n+1}$

$$= \frac{1(1-x^{n+1})}{1-x} - (n+1)x^{n+1}$$

$$\Rightarrow P_n(x) = \frac{1-(n+2)x^{n+1} + (n+1)x^{n+2}}{(1-x)^2}$$

For negative values of x, $P_n(x)$ will vanish whenever

$f(x) = 1 - (n + 2)x^{n+1} + (n + 1)x^{n+2}$

$f(-x) = 1 - (n + 2)(-x)^{n+1} + (n + 1)(-x)^{n+2}$

If n is even, there is no change of sign in this expression and so there is no negative real root. If n is odd, there is one change of sign, so there can be one negative real root.

In this case $f(-1) = 1(n + 2) - (n + 1) = -(n + 1)$

$= -(2n + 2) < 0$ and $f(0) > 0$

So we can say that when n is odd the real root lies between 0 and -1.

8. Above series can be written in form of integration as follows

$$S = \int_0^1 (1 - x^3 + x^5 - x^8 - x^{13} + \ldots)dx$$

$$\Rightarrow S = \int_0^1 ((1-x^3) + x^5(1-x^3) + x^{10}(1-x^3) + \ldots)dx$$

$$= \int_0^1 \frac{(1-x^3)}{(1-x^5)}dx = \int_0^1 \frac{(x^2+x+1)}{x^4+x^3+x^2+x+1}dx$$

$$= \int_0^1 \frac{adx}{(x^2+cx+1)} + \int_0^1 \frac{bdx}{(x^2+dx+1)}$$

so from partial fractions we can calculate a, b, c, d

$$a = \left(\frac{5+\sqrt{5}}{10}\right);\ b = \left(\frac{5-\sqrt{5}}{10}\right);$$

$$c = \left(\frac{1-\sqrt{5}}{2}\right);\ d = \left(\frac{1+\sqrt{5}}{2}\right)$$

From integration we can calculate

$$S = \frac{\pi}{50}\left[\sqrt[3]{(10+2\sqrt{5})} + \sqrt{(10+2\sqrt{5})}\right].$$

9. Let $$S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{m^2 n}{(n\cdot 3^m + m\cdot 3^n)}$$

Let $a_n = 3^n/n$

Then $$S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_m(a_m+a_n)} \quad \text{...(i)}$$

Interchanging m and n we can write

$$S = \sum_{m=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_n(a_m+a_n)} \quad \text{...(ii)}$$

Adding (i) and (ii)

$$2S = \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\left(\frac{1}{a_m(a_m+a_n)} + \frac{1}{a_n(a_m+a_n)}\right)$$

$$= \sum_{n=1}^{\infty}\sum_{n=1}^{\infty}\frac{1}{a_m\cdot a_n} = \left(\sum_{n=1}^{\infty}\frac{n}{3^n}\right)^2 = \left(\frac{3}{4}\right)^2$$

$$= \frac{9}{16} \qquad \left[\because \sum_{n=1}^{\infty}\frac{n}{3^n} = \frac{3}{4}\right] \Rightarrow S = 9/32.$$

■

(b) $abc + 2fgh = 1$ (c) $abc - 2fgh = 1$
(d) $abc + 2fgh - bg^2 - af^2 - ch^2 = 0$.

19. A regular polygon has 104 diagonals. Then the number of its sides are
(a) 11 (b) 12 (c) 16 (d) 17.

20. If $\cos\alpha + \sin\alpha = \sqrt{2}\cos\alpha$, then $\cos\alpha - \sin\alpha =$
(a) $\sqrt{2}\sin\alpha$ (b) $\sqrt{2}\cos\alpha$
(c) $\sqrt{2}\tan\alpha$ (d) $\sqrt{2}(\cos\alpha + \sin\alpha)$.

21. $\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right) =$
(a) 1/2 (b) 1/4 (c) 1/8 (d) 1/12.

22. The length of intercepted on the line $3x + 4y + 1 = 0$ by the circle $(x-1)^2 + (y-4)^2 = 25$ is
(a) 5 (b) 7 (c) 6 (d) 8.

23. A, B, C are three events such that $P(A) = P(B) = P(C) = 1/4$ and $P(AC) = 1/8$, then the probability that at least one of the events A, B, C occurs is
(a) 7/8 (b) 3/8 (c) 5/8 (d) 1/8.

24. For any complex number iz $\arg z + \arg \bar{z} =$
(a) $2n\pi$ (b) 2π (c) π (d) $\pi/2$.

25. If $(x - \alpha)$ is a factor of the polynomial $f(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots a_n$ repeated m times $(1 \le m \le n)$, then α is a root of $f(x) = 0$ repeated
(a) m^2 times (b) $(m - 1)$ times
(c) $(m + 1)$ times (d) none of these.

26. $\lim_{x\to\pi/2} \frac{\cos 5x}{x - \pi/2}$ equals
(a) -5 (b) -6 (c) π (d) $-\pi$.

27. If $[2f(x)]^2 = (x-2)^2$, then at the point $x = 2$, the function $f(x)$ is
(a) continuous and differentiable
(b) continuous but not differentiable
(c) discontinuous (d) none of these.

28. A curve has the parametric equation $x = \frac{a}{2t}(t^2+1)$ and $y = \frac{b}{2t}(t^2-1)$, then its equation in rectangular cartesian coordinates is
(a) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (b) $x^2 + y^2 = a^2b^2$
(c) $b^2x^2 - a^2y^2 = a^2b^2$ (d) none of these.

29. If the points z_1, z_2, z_3 are the vertices of an equilateral triangle in the argand plane, then
(a) $\frac{1}{z_1 - z_2} + \frac{1}{z_2 - z_3} + \frac{1}{z_3 - z_1} = 0$
(b) $z_1^2 + z_2^2 + z_3^2 = 3z_0$
(c) $(z_1 - z_2)^2 + (z_2 - z_3)^2 + (z_3 - z_2)^2 = 0$
(d) $z_1^3 + z_2^3 + z_3^3 = 3z_1z_2z_3$.

30. If E and F are independent events such that $0 < P(E) < 1$ and $0 < P(F) < 1$ then
(a) E and F are mutually exclusive
(b) $P(\bar{E})$ and $P(F)$ are equal
(c) E and $\bar{F}$ are independent
(d) $P(\bar{E})$ and $P(\bar{F})$ are equal.

Note : Blacken your choice with blue/black ball pen. Cut and send the answer sheet given below. Students between age 17-21 years are eligible. Copy of your 10th class board marksheet is a must. Last date of receipt of entries : **31st** March. The name of the **WINNERS** of this contest will be published in the April 2005 issue. **Send your entries to :** Amity Institute for Competitive Examinations, 65 A, Kalu Sarai, Sarvapriya Vihar, New Delhi-16.

MARCH AMITY ENTRANCE EDGE CONTEST

Name (Block Letters) and Complete Address : ....

....

Name & Address of School : ....

1. ⓐⓑⓒⓓ	7. ⓐⓑⓒⓓ	13. ⓐⓑⓒⓓ	19. ⓐⓑⓒⓓ	25. ⓐⓑⓒⓓ
2. ⓐⓑⓒⓓ	8. ⓐⓑⓒⓓ	14. ⓐⓑⓒⓓ	20. ⓐⓑⓒⓓ	26. ⓐⓑⓒⓓ
3. ⓐⓑⓒⓓ	9. ⓐⓑⓒⓓ	15. ⓐⓑⓒⓓ	21. ⓐⓑⓒⓓ	27. ⓐⓑⓒⓓ
4. ⓐⓑⓒⓓ	10. ⓐⓑⓒⓓ	16. ⓐⓑⓒⓓ	22. ⓐⓑⓒⓓ	28. ⓐⓑⓒⓓ
5. ⓐⓑⓒⓓ	11. ⓐⓑⓒⓓ	17. ⓐⓑⓒⓓ	23. ⓐⓑⓒⓓ	29. ⓐⓑⓒⓓ
6. ⓐⓑⓒⓓ	12. ⓐⓑⓒⓓ	18. ⓐⓑⓒⓓ	24. ⓐⓑⓒⓓ	30. ⓐⓑⓒⓓ

Mathematics Olympia

for IIT-JEE (MAINS) 2005

Exam on 22nd May 2005

By : **Er. Akhlak Ahmad**, ABC Classes, Gorakhpur

1. If z lies on the circle $|z-1| = 1$, show that $\frac{z-2}{z} = i\tan(\arg z)$.

2. Find the number of ways of choosing triplets (x, y, z) such that $z \geq \max\{x, y\}$ and $x, y, z \in \{1, 2, 3, ..., n+1\}$.

3. Find the value of the expression
$$T = \sum_{r<s}\sum (r+s)C_rC_s$$

4. If $T_0, T_1, T_2, T_3, ...$ be the terms in the expansion of $(x+a)^n$, then prove that
$$(T_0 - T_2 + T_4 - ...)^2 + (T_1 - T_3 + T_5 - .....)^2 = (x^2 + a^2)^n$$

5. Prove that $(1+x)^n - nx - 1$ is divisible by x^2 for all $n \in N$. Hence, deduce that $2^{3n} - 7^n - 1$ is divisible by 49.

6. If x, y, z are three positive real numbers such that $|x-y| \geq z$, $|y-z| \geq x$ then $|z-x| \geq y$; show that one of x, y, z is sum of the other two.

7. If x, y, z are positive real numbers, such that $x < y < z$, show that $\frac{x^2}{z} < \frac{x^2+y^2+z^2}{x+y+z} < \frac{z^2}{x}$.

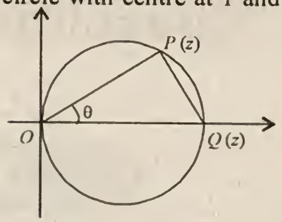
8. Prove that the lines $\frac{x+1}{2} = \frac{y+1}{3} = \frac{z+1}{4}$ and $x + 2y + 3z - 14 = 0 = 3x + 4y + 5z - 26$ are coplanar and find the co-ordinates of their point of intersection.

9. Let x, y, z be real variables satisfying the equations $x + y + z = 6$ and $xy + yz + zx = 7$. Find the range in which the varibales can lie.

10. Find the position vector of the point of intersection of the three planes $\vec{r}\cdot\vec{n}_1 = q_1$, $\vec{r}\cdot\vec{n}_2 = q_2$ and $\vec{r}\cdot\vec{n}_3 = q_3$; where $\vec{n}_1$, $\vec{n}_2$, and $\vec{n}_3$ are non-coplanar vectors.

SOLUTION

1. $|z-1| = 1$ represents a circle with centre at 1 and radius 1.



applying rotation

$$\frac{z-2}{z} = \frac{|z-2|}{|z|}e^{i(\pi/2)}$$

$$\frac{z-2}{z} = \frac{PQ}{OP}i$$

$\tan\theta = \frac{PQ}{OP}$ and $\theta = \arg(z) \Rightarrow \frac{z-2}{z} = i\tan(\arg z)$.

---

2. Let $z = n + 1 \Rightarrow$ we can choose x, y from $\{1, 2, 3, ..., n\}$

Thus when $z = n + 1 \Rightarrow x$ and y can be chosen in n^2 ways

Thus when $z = n \Rightarrow x$ and y can be chosen in $(n-1)^2$ ways

$\therefore$ Total number of ways of choosing the triplets

$$= n^2 + (n-1)^2 + ... + 1^2 = \frac{1}{6}n(n+1)(2n+1).$$

3.
$$T = \sum_{r<s}\sum (r+s)C_rC_s$$
$$= \sum_{r<s}\sum (n-r+n-s)C_{n-r}C_{n-s}$$
$$= \sum_{r<s}\sum 2nC_rC_s - \sum_{r<s}\sum (r+s)C_rC_s$$
$$2T = 2n\sum_{r<s}\sum C_rC_s \ ; \ T = \frac{n}{2}\left[2^{2n} - {}^{2n}C_n\right].$$

---

4. We know that

$$(x+a)^n = {}^nC_0x^n + {}^nC_1x^{n-1}a + {}^nC_2x^{n-2}a^2 + ...$$
$$= T_0 + T_1 + T_2 + ...$$

$a \to ai$

$$(x+ai)^n = {}^nC_0x^n + {}^nC_1x^{n-1}(ai) + {}^nC_2x^{n-2}(ai)^2 + ...$$
$$\Rightarrow (x+ai)^n = T_0 + T_2i + T_2i^2 + T_3i^3 + ...$$
$$(x+ai)^n = (T_0 - T_2 + T_4 ...) + i(T_1 - T_3 + T_5 ...)$$

Taking modulus on both sides we get,

$$|x+ai|^n = \sqrt{(T_0 - T_2 + T_4 - ...)^2 + (T_1 - T_3 + T_5 - ...)^2}$$
squaring we get answer.

5. We know that

$(1+x)^n = {}^nC_0 + {}^nC_1x + {}^nC_2x^2 + {}^nC_3x^3 + ... + {}^nC_nx^n$

$(1+x)^n - {}^nC_1x - {}^nC_0 = x^2[{}^nC_2 + {}^nC_3x + {}^nC_4x^2 + ... + {}^nC_nx^{n-2}]$

$(1+x)^n - nx - 1 = x^2\ [\ ]$

$\Rightarrow (1+x)^n - nx - 1$ is divisible by x^2.

If we take $x = 7$

$(1+7)^n - 7n - 1$ is divisible by 7^2

$8^n - 7n - 1$ is divisible by 49

$\Rightarrow\ 2^{3n} - 7n - 1$ is divisible by 49

6. We have, $z^2 - (x-y)^2 \le 0$

$x^2 - (y-z)^2 \le 0$

$y^2 - (z-x)^2 \le 0$

$\Rightarrow [(z-x+y)(z+x-y)(y+x-z)]^2 \le 0$

$\Rightarrow (z-x+y)(z+x-y)(y+x-z) = 0$

$\Rightarrow$ at least one of x, y, z is sum of other two.

7. $\because\ x^2 + x^2 + x^2 < x^2 + y^2 + z^2 < z^2 + z^2 + z^2$

$\Rightarrow 3x^2 < x^2 + y^2 + z^2 < 3z^2$ ...(1)

and $3x < x + y + z < 3z$

$\Rightarrow \frac{1}{3z} < \frac{1}{x+y+z} < \frac{1}{3x}$ ...(2)

Multiplying (1) and (2), we get

$$\frac{x^2}{z} < \frac{x^2+y^2+z^2}{x+y+z} < \frac{z^2}{x}.$$

8. Let $\frac{x+1}{2} = \frac{y+1}{3} = \frac{z+1}{4} = r$

$\Rightarrow x = 2r - 1,\ y = 3r - 1,\ z = 4r - 1$

hence any point on the first line

$(2r-1, 3r-1, 4r-1)$

If this point lies on the plane

$x + 2y + 3z = 14\ ;\ r = 1$

Thus point of intersection is (1, 2, 3)

Clearly, the point satisfies the other plane.

9. Eliminating z

we get, $y^2 + y(x-6) + x^2 - 6x + 7 = 0$ as y is real

$(x-6)^2 - 4(x^2 - 6x + 7) \ge 0$

we get $3x^2 - 12x - 8 \le 0$

$\Rightarrow\ 2 - \frac{2}{3}\sqrt{15} \le x \le 2 + \frac{2}{3}\sqrt{15}$

Hence $x, y, z \in \left[2 - \frac{2}{3}\sqrt{15},\ 2 + \frac{2}{3}\sqrt{15}\right]$.

10. The vectors $\vec{n}_1 \times \vec{n}_2$, $\vec{n}_2 \times \vec{n}_3$ and $\vec{n}_3 \times \vec{n}_1$ are non-coplanar vectors, so every vector can be written as

$$\vec{r} = a(\vec{n}_1 \times \vec{n}_2) + b(\vec{n}_2 \times \vec{n}_3) + c(\vec{n}_3 \times \vec{n}_1)$$

apply $\vec{r} \cdot \vec{n} = q_1$ we get

$$a = \frac{q_3}{[\vec{n}_1\ \vec{n}_2\ \vec{n}_3]},\ b = \frac{q_1}{[\vec{n}_1\ \vec{n}_2\ \vec{n}_3]} \text{ and } c = \frac{q_2}{[\vec{n}_1\ \vec{n}_2\ \vec{n}_3]}$$

$$\Rightarrow \vec{r} = \frac{1}{[\vec{n}_1\ \vec{n}_2\ \vec{n}_3]}[q_3(\vec{n}_1 \times \vec{n}_2) + q_1(\vec{n}_2 \times \vec{n}_3) + q_2(\vec{n}_3 \times \vec{n}_1)].$$

■

EXAM ALERT!

TAMIL NADU PROFESSIONAL COURSES ENTRANCE EXAMINATIONS (TNPCEE) 2005

TNPCEE 2005 would be held on April 23 and 24. This examination is applicable for undergraduate professional courses offered by Anna University; the Tamil Nadu Agricultural University, CBE; Tamil Nadu Veterinary and Animal Sciences University, Chennai; Government, Government Aided and Government Quota seats in unaided colleges coming under the Directorates of Technical Education, Medical Education and Indian Medicine and Homoeopathy; and the School of Engineering and Technology of Bharatidasan University, Tiruchi.

VELLORE INSTITUTE OF TECHNOLOGY

ADMISSION TO B.Tech Courses for 70% Seats - VIT Entrance Examination (VITEE) 2005. Application Form and Information Brochure will be issued from 15.02.2005 onwards. The application will also be available on website: www.vit.ac.in.

EAMCET 2005

ANDHRA PRADESH Engineering Agriculture Medical Common Entrance Test - EAMCET 2005

The Andhra Pradesh State Council of Higher Education (APSCHE) has announced the schedule of Common Entrance Tests (CET).It is to be conducted on April 29, 2005 (Friday).

Solution for Amity Edge Contest (Mathematics) February 2005

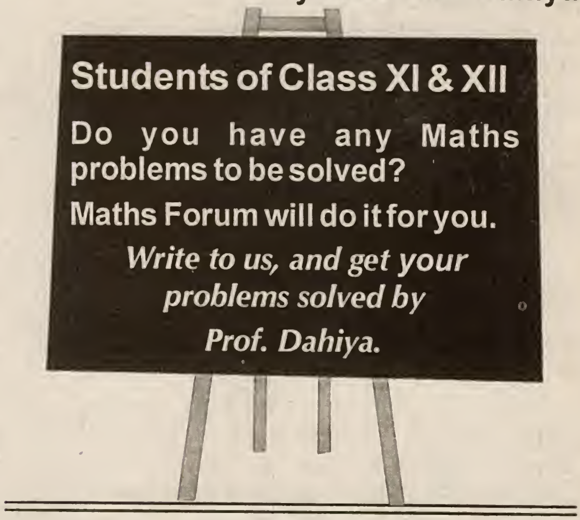
1.	(b)	2.	(c)	3.	(c)	4.	(c)	5.	(d)
6.	(b)	7.	(c)	8.	(c)	9.	(b)	10.	(c)
11.	(d)	12.	(b)	13.	(a)	14.	(c)	15.	(b)
16.	(c)	17.	(b)	18.	(c)	19.	(a)	20.	(a)
21.	(a)	22.	(d)	23.	(b)	24.	(a)	25.	(c)
26.	(c)	27.	(a)	28.	(a)	29.	(a)	30.	(c)

JANUARY WINNERS NAME

Debabrata Santra	Jyotish Kumar Patel	Gautam Kiesrwani
Village - Gongla	D.L.W. Colony	3, Sadar Gola Bazar
West Bengal	Varanasi, UP	Uttar Pradesh

MATHS FORUM

by Prof. S.S. Dahiya



1. In ΔABC, three circles of radii x, y, z are drawn touching the sides (AB, AC), (BC, BA), (CA, CB) and the inscribed circle of the ΔABC, then prove that $r = \sqrt{xy}+\sqrt{yz}+\sqrt{zx}$.

2. Consider the two equations in x

(i) $\sin\left(\dfrac{\cos^{-1}x}{y}\right) = 1$ (ii) $\cos\left(\dfrac{\sin^{-1}x}{y}\right) = 0$

The sets $X_1, X_2 \subseteq [-1, 1]$; $Y_1, Y_2 \subseteq [-1, 1]$ are such that X_1 is solution set of equation (i), X_2 is solution set of equation (ii)

Y_1 is set of integral values of y for which equation (i) possesses a solution

Y_2 is set of integral values of y for which equation (ii) possesses a solution

Let R_1 be the correspondence $X_1 \to Y_1$ such that xR_1y for $x \in X_1, y \in Y_1$, (x, y) satisfy equation (i) and R_2 be correspondence $X_2 \to Y_2$ such that for $x \in X_2, y \in Y_2$, (x, y) satisfy equation (ii) State with reasons if R_1 and R_2 are functions, if yes state whether these are bijective or into.

–Jayant Goyal, Meerut

3. How to solve these type of questions?

(i) $f(x) = \text{Max.}\{\sin t,\ 0 \le t \le x,\ 0 \le x \le 2\pi\}$, find $f(x)$

(ii) $f(x) = \text{Max.}\{\cos t,\ 0 \le t \le x,\ 0 \le x \le 2\pi\}$, find $f(x)$

(iii) $f(x) = \text{Min.}\{\sin t,\ x \le t \le x+\frac{\pi}{2},\ 0 \le x \le 2\pi\}$, find $f(x)$

(iv) $f(x) = \text{Max.}\{t^2-4t+3,\ x < t < x+1,\ 0 \le x \le 4\}$, find $f(x)$

4. Find greatest and least value of $f(x)$

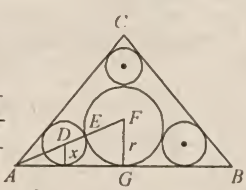
$$f(x)=\begin{cases}\text{Min}\{3t^4-8t^3-6t^2+24t,\ 1 \le t < x, 1 \le x < 2\}\\ \text{Max}\ \{3t+2+\frac{1}{4}\sin^2(\pi t),\ 2 \le t \le x,\ 2 \le x \le 4\}\end{cases}$$

– Nitesh Bhatia, Ajmer

SOLUTION

1. $DE = x,\ EF = r;\ DF = AF - AD$

$$x+r = \frac{r}{\sin\frac{A}{2}} - \frac{x}{\sin\frac{A}{2}}$$



$$\frac{r-x}{r+x} = \sin\left(\frac{A}{2}\right) \text{ or } \frac{x}{r} = \frac{1-\sin\frac{A}{2}}{1+\sin\frac{A}{2}}$$

$$\frac{x}{r} = \frac{\left(\cos\frac{A}{4}-\sin\frac{A}{4}\right)^2}{\left(\cos\frac{A}{4}-\sin\frac{A}{4}\right)^2} = \left(\frac{1-\tan\frac{A}{4}}{1+\tan\frac{A}{4}}\right)^2$$

$$= \tan^2\left(\frac{\pi}{4}-\frac{A}{4}\right) = \tan^2\left(\frac{\pi-A}{4}\right)$$

$\therefore\ x = r\tan^2\left(\dfrac{\pi-A}{4}\right)$, similarly $y = r\tan^2\left(\dfrac{\pi-B}{4}\right)$

and $z = r\tan^2\left(\dfrac{\pi-C}{4}\right)$

$$\sqrt{xy}+\sqrt{yz}+\sqrt{zx} = r\{\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha\}$$

where $\alpha = \dfrac{\pi-A}{4},\ \beta = \dfrac{\pi-B}{4},\ \gamma = \dfrac{\pi-C}{4}$ and

hence $\alpha+\beta+\gamma = \dfrac{\pi}{2}$, then

$$\tan\alpha\tan\beta+\tan\beta\tan\gamma+\tan\gamma\tan\alpha = 1$$

$$\therefore\ \sqrt{xy}+\sqrt{yz}+\sqrt{zx} = r.$$

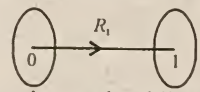
2. In given equations $y \ne 0, y \in Y_1, Y_1 = \{-1, +1\}$ $y \in Y_1, Y_2 = \{-1, +1\}$, hence value of y is either $+1$ or -1.

(i) $\sin\left(\dfrac{\cos^{-1}x}{y}\right) = 1$

when $y = 1$, $\cos^{-1}x = \dfrac{\pi}{2}$ $\therefore\ x = 0$

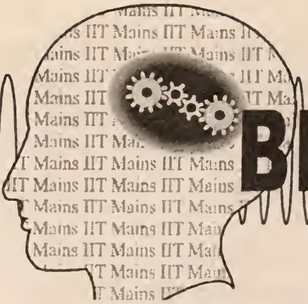
when $y = -1, \cos^{-1}x = \dfrac{-\pi}{2}$ which is impossible

$0 \le \cos^{-1}x \le \pi$ $\therefore$



Therefore R_1 is a function and a bijective function.

(ii) $\cos\left(\dfrac{\sin^{-1}x}{y}\right) = 0$



BRAIN TWISTERS for IIT MAINS

By : S.K. Tiwari, Insight, Kota

Time : 2 hrs. *Max. Marks : 60*

1. Show that

$$\int_0^\infty \frac{\tan^{-1}\alpha x \cdot \tan^{-1}\beta x}{x^2}dx = \frac{\pi}{2}\log\left[\frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^\alpha+\beta^\beta}\right]$$

2. Two functions $f(x)$ and $g(x)$ are defined such that

$$f(x)=\begin{cases}[h(x)]-\left\{\dfrac{h(x)}{2}\right\} & \text{for } x \in \text{ domain of } h\\ 0 & \text{for } x \notin \text{ domain of } h\end{cases}$$

$$g(x)=\begin{cases}\operatorname{sgn}(h(x)) & \text{for } x \in \text{ domain of } h\\ 0 & \text{for } x \notin \text{ domain of } h\end{cases}$$

where $h(x)=\dfrac{1}{\sqrt{b-a}}\cdot\dfrac{\sqrt{\dfrac{b-a}{a}}\sin 2x}{\sqrt{1+\left(\sqrt{\dfrac{b-a}{a}}\sin x\right)^2}}.\sqrt{a+b\tan^2 x}$

for $b > a > 0$

where [.] denotes greatest integer function and {.} denotes fractional part of x.

Discuss the continuity of f and g at $x = 0, \pi/2$ respectively.

3. Evaluate $\displaystyle\sum_{r=0}^{n}\frac{r(r-2)\left({}^nC_r\right)^2}{(n-r+1)(n-r+3)}$

4. Given that $\int_{-\infty}^{\infty}e^{-x^2}dx=\sqrt{\pi}$. Find $f'(t)$ explicity where $f(t)=\int\limits_{-\infty}^{\infty}e^{-tx^2}dx,\quad t>0$

5. (i) If $e^2 < 1$ then find the mean value of

$f(x) = \log(1 - e^2\sin^2\theta)$ on $\theta \in \left[0, \frac{\pi}{2}\right]$

(ii) Find the mean value of

$$f(x)=\frac{\log(1+\cos a\cdot\cos x)}{\cos x} \quad \text{on } x\in\left[0,\frac{\pi}{2}\right]$$

6. An urn contains 6 black balls and unknown number ($\leq$ 6) of white balls. Three balls are drawn successively and not replaced and are all found to be white. Prove that the chance that a black ball will be drawn in the next draw is $\frac{677}{909}$.

7. Through a fixed point O are drawn two straight lines OPQ and ORS to meet a circle in P and Q and R and S respectively. Prove that the locus of point of intersection of PS and QR is also that of the point of intersection of PR and QS is the polar of O w.r.t. the circle.

8. Circles of constant radius r are drawn to pass through the ends of a variable diameter of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

Prove that the locus of their centres is the curve

$$(x^2+y^2)(a^2x^2+b^2y^2+a^2+b^2)=r^2(a^2x^2+b^2y^2)$$

9. If A_1, A_2, A_3 and B_1, B_2, B_3 are two sets of collinear points, then prove that the points of intersection of the pair of lines A_1B_2; A_2B_1; A_2B_3, A_3B_2, A_3B_1, A_1B_3 are collinear.

10. If $Z_1 + Z_2 + \ldots + Z_n = 0$, then prove that any straight line passing through the origin separates the points representing $Z_1, Z_2 \ldots Z_n$, provided that they do not lie on this line.

Solutions

1. Let $I=\int_0^\infty\frac{\tan^{-1}\alpha x\cdot\tan^{-1}\beta x}{x^2}dx$

Then $\dfrac{dI}{d\alpha}=\int_0^\infty\dfrac{x}{1+\alpha^2x^2}\cdot\dfrac{\tan^{-1}\beta x}{x^2}dx$

$$\frac{dI}{d\alpha}=\int\limits_0^\infty\frac{\tan^{-1}\beta x}{x(1+\alpha^2x^2)}dx$$

Then $\dfrac{d}{d\beta}\cdot\left(\dfrac{dI}{d\alpha}\right)=\int_0^\infty\dfrac{x\,dx}{(1+\beta^2x^2)\cdot x(1+\alpha^2x^2)}$

$$=\frac{1}{(\alpha^2-\beta^2)}\int_0^\infty\left(\frac{\alpha^2}{1+\alpha^2x^2}-\frac{\beta^2}{1+\beta^2x^2}\right)dx$$

$$= \frac{1}{(\alpha^2-\beta^2)}\left[\alpha\cdot\tan^{-1}\alpha x - \beta\tan^{-1}\beta x\right]_0^{\infty}$$

$$= \frac{1}{(\alpha^2-\beta^2)}\left[\frac{(\alpha-\beta)\pi}{2}\right] = \frac{\pi}{2(\alpha+\beta)}$$

Integrating w.r.t. β then α

$$I = \frac{\pi}{2}\log\left[\frac{(\alpha+\beta)^{\alpha+\beta}}{\alpha^{\alpha}\beta^{\beta}}\right]$$

2. Since $h(x) = \frac{1}{\sqrt{b-a}}\cdot\frac{\sqrt{b-a}}{\sqrt{a}}\cdot\frac{\frac{\sin 2x\cdot\sqrt{a\cos^2 x+b\sin^2 x}}{|\cos x|}}{\frac{\sqrt{a\cos^2 x + b\sin^2 x}}{\sqrt{a}}}$

$$= \frac{\sin 2x}{|\cos x|}$$

Domain of h is $R - \left\{n\pi + \frac{\pi}{2},\quad n\in I\right\}$

$$\therefore\quad h(x) = \begin{cases} 2\sin x\ ; & x<\frac{\pi}{2} \\ -2\sin x\ ; & x>\frac{\pi}{2}\end{cases}$$

at $x = \frac{\pi}{2}$, $f\left(\frac{\pi}{2}\right) = 0 = g\left(\frac{\pi}{2}\right)$

it is clear that $h(x)$ is continuous at $x = 0$ and discontinuous at $x = \frac{\pi}{2}$.

Since $[x]$ and $\{x\}$ are discontinuous for all integral values of x and continuous for all $x \in R - I$.

$\because\quad f(x) = [h(x)] - \left\{\frac{h(x)}{2}\right\}$

Hence $f(x)$ is discontinous at $x = 0$ and $x = \pi/2$

($\because$ $h(x)$ is discontinuous at $x = \pi/2$)

and $q(x) = \text{sgn}\,(h(x))$

$$= \begin{cases} 1 & h(x)>0 \\ 0 & h(x)=0 \\ -1 & h(x)<0\end{cases}$$

$$= \begin{cases} 1 & 0<x<\pi/2 \\ -1 & x>\pi/2 \\ 0 & x=0 \\ 0 & x=\pi/2\end{cases}$$

Hence $g(x)$ is discontinuous at $x = 0$ and $x = \pi/2$

3. $\because\quad \sum_{r=0}^{n} {}^nC_r\cdot x^r = (1+x)^n \qquad \text{...(i)}$

Performing differentiation then division by x and then again differentiation on (i) we get

$$\sum_{r=0}^{n} r(r-2)\ {}^nC_r\cdot x^{r-3} = \frac{d}{dx}\left\{\frac{n\cdot(1+x)^{n-1}}{x}\right\}$$

$$= n(1+x)^{n-2}\left(\frac{nx-2x-1}{x^2}\right) \qquad \text{...(ii)}$$

Performing Integration then multiplication by x and again Integration of (i) we get

$$\sum_{r=0}^{n}\frac{{}^nC_{n-r}x^{n-r+3}}{(n-r+1)(n-r+3)} = \int\frac{x(1+x)^{n+1}}{n+1}\,dx$$

$$= \frac{(1+x)^{n+2}}{(n+1)(n+2)}\cdot\frac{nx+2x-1}{(n+3)} \qquad \text{...(iii)}$$

Multiplying (ii) and (iii) and compairing the coefficient of x^n on both sides, we get

$$\sum_{r=0}^{n}\frac{r(r-2)\left({}^nC_r\right)^2}{(n+1)(n+2)(n+3)}$$

$$= \text{cofficient of } x^n \text{ in } \frac{n(1+x)^{2n}}{n(n+1)(n+2)}[(nx-1)^2-4x^2]$$

$$= \frac{n}{(n+1)(n+2)(n+3)}[(n^2-4)\ {}^{2n}C_n - 2n\,{}^{2n}C_{n+1} + {}^{2n}C_{n+2}$$

4. $\because\ \int_{-\infty}^{\infty} e^{-x^2}\cdot dx = \sqrt{\pi}$

Then in $\int_{-\infty}^{\infty} e^{-tx^2}\cdot dx$ put $y = x\sqrt{t}$, $dx = \frac{dy}{\sqrt{t}}$

$$f(t) = \int_{-\infty}^{\infty} e^{-y^2}\cdot\frac{dy}{\sqrt{t}},\quad f(t) = \int_{-\infty}^{\infty}\frac{e^{-y^2}\cdot dy}{\sqrt{t}} = \sqrt{\frac{\pi}{t}}$$

$$f'(t) = \frac{\pi}{2t\sqrt{t}}$$

5. (i) Mean Value $= \frac{\int_0^{\pi/2}\log(1-e^2\sin^2\theta)}{\pi/2-0}$

so, $\int_0^{\pi/2}\log(1-e^2\sin^2\theta) = I = \pi\log_e\left[\frac{1+\sqrt{1-e^2}}{2}\right]$

By partial differentiation

By mean value $= \frac{\pi\log_e\left[\frac{1+\sqrt{1-e^2}}{2}\right]}{\pi/2}$

$$= 2\log_e\left[\frac{1+\sqrt{1-e^2}}{2}\right]$$

(ii) $I = \int_0^{\pi/2} \frac{\log(1+\cos a \cos x)}{\cos x}$

$\frac{dI}{da} = \int_0^{\pi/2} (1+\cos a \cos x)^{-1} \cdot \cos x(-\sin a)\ dx$

and by integration with respect to x and a

So mean value $= \frac{\frac{1}{2}\left(\frac{\pi^2}{4} - a^2\right)}{\pi/2} = \frac{\pi^2 - 4a^2}{4\pi}$

6. Clearly urn contains at least 3 white balls.

Let E_3 = the event of the urn containing 6 black ball and 3 white balls.

E_4 = The event of the urn containing 6 black balls and 4 white balls.

E_5 = The event of the urn containing 6 black balls and 5 white balls.

E_6 = The event of the urn containing 6 black balls and 6 white balls.

each of E_3, E_4, E_5, E_6 are equiporable and they are exhaustive.

Now $P(W/E_3)$ = Probability of 3 balls drawn being white

in case E_3 $= \frac{{}^3C_3}{{}^9C_3} = \frac{1}{84}$

Similarily $P(W/E_4) = \frac{{}^4C_3}{{}^{10}C_3} = \frac{1}{30}$

$P(W/E_5) = \frac{{}^5C_3}{{}^{11}C_3} = \frac{2}{33}$

$P(W/E_6) = \frac{{}^6C_3}{{}^{12}C_3} = \frac{1}{11}$

$\therefore$ by Bayes' theorem for equiporable events.

$$P(E_3/W) = \frac{P(W/E_3)}{P(W/E_3)+P(W/E_4)+P(W/E_5)+P(W/E_6)}$$

$$= \frac{55}{909}$$

$$P(E_4/W) = \frac{154}{909}$$

7. Let the two lines OPQ and ORS be taken as axes of x and y respectively, O as origin and $\angle ROP = \theta$.

Equation of the given circle may be taken as

$x^2 + y^2 + 2xy\cos\theta + 2gx + 2fy + c = 0$

Now equation of x axis is $y = 0$

So $x^2 + 2gx + C = 0$

Let the co-ordinates of P and Q $(x_1, 0)$ and $(x_2, 0)$ respectively

So, $x_1 + x_2 = -2g$

$x_1 x_2 = C$

Similarily co-ordinate of R and S be $(0, y_1)$ and $(0, y_2)$ respectively

$$y_1 + y_2 = -2f,\ y_1 y_2 = C$$

Equation of PS will be $\frac{x}{x_1} + \frac{y}{y_1} = 1$ ...(i)

Equation of QR will be $\frac{x}{x_2} + \frac{y}{y_2} = 1$ ...(ii)

The require locus by adding (i) and (ii)

$$x\left(\frac{x_1+x_2}{x_1x_2}\right) + y\left(\frac{y_1+y_2}{y_1y_2}\right) = 2$$

$gx + fy + c = 0$ ...(iii)

and polar of origin is also $gx + fy + c = 0$ ...(iv)

Equation (iii) and (iv) are same.

8. The ends of the variable diameter of the ellipse may be taken as $P(a\cos\theta, + b\sin\theta)$ and $(-a\cos\theta, -b\sin\theta)$

Circle with centre (h, k) and radius r is

$$(x-h)^2 + (y-k)^2 = r^2$$

As it passes through P and Q therefore

$(a\cos\theta - h)^2 + (b\sin\theta - k)^2 = r^2$ ...(i)

and $(a\cos\theta + h)^2 + (b\sin\theta + k)^2 = r^2$ ...(ii)

Adding equation (i) and (ii) we get

$h^2 + k^2 + a^2\cos^2\theta + b^2\sin^2\theta = r^2$ ...(iii)

Subtracting equation (i) from equation (ii) we get

$a h\cos\theta + bk\sin\theta = 0$, $\tan\theta = -ah/kb$ ...(iv)

use equation (iv) in equation (iii) we get

$$h^2 + k^2 + a^2\left(\frac{k^2b^2}{a^2h^2+k^2b^2}\right) + b^2\left(\frac{a^2h^2}{a^2h^2+k^2b^2}\right) = r^2$$

Hence required is

$$(x^2 + y^2)(a^2x^2 + b^2y^2 + a^2b^2) = r^2(a^2x^2 + b^2y^2)$$

9. Suppose the points A_1, A_2, A_3 lie on the line OA and the points B_1, B_2, B_3 lie on the line OB. Since any two vectors are always coplanar. So, let OA and OB be two coplanar lines. Taking O, the point of intersection of the given lines, as the origin of reference, let $\overline{OA} = \overline{a}$ and $\overline{OB} = \overline{b}$.

Let $\lambda_1\bar{a}$, $\lambda_2\bar{a}$, $\lambda_3\bar{a}$ be the position vector of A_1, A_2, A_3 respectively. Also, let the position vetors of points B_1, B_2, B_3 be $\mu_1\bar{b}, \mu_2\bar{b}, \mu_3\bar{b}$ respectively.

The vector equation of lines A_1B_2 and A_2B_1 are

$$\bar{r} = \lambda_1\bar{a} + l(\mu_2\bar{b} - \lambda_1\bar{a}) \text{ and } \bar{r} = \lambda_2\bar{a} + m(\mu_1\bar{b} - \lambda_2\bar{a})$$

Suppose A_1B_2 and A_2B_1 intersect at P_3. At their point of intersection, we have

$$\lambda_1\bar{a} + l(\mu_2\bar{b} - \lambda_1\bar{a}) = \lambda_2\bar{a} + m(\mu_1\bar{b} - \lambda_2\bar{a})$$

$$\Rightarrow (\lambda_1 - \lambda_2) - l\lambda_1 + m\lambda_2 = 0 \text{ and } l\mu_2 - \mu_1 m = 0$$

[$\because$ $\vec{a}$ and $\vec{b}$ are non collinear vectors]

$$\Rightarrow m = \frac{\mu_2(\lambda_2 - \lambda_1)}{\mu_2\lambda_2 - \mu_1\lambda_1}$$

Substituting the value of m in (ii), we see that the position vector $\vec{r_3}$ of the point of intersection of A_1B_2 and A_2B_1 is given by

$$\vec{r_3} = \lambda_2\bar{a} + \frac{\mu_2(\lambda_2 - \lambda_1)}{\mu_2\lambda_2 - \mu_1\lambda_1}(\mu_1\bar{b} - \lambda_2\bar{a})$$

or
$$\vec{r_3} = \frac{1}{\mu_1\lambda_1 - \mu_2\lambda_2}\left\{\lambda_2\lambda_1(\mu_1 - \mu_2)\vec{a} + \mu_1\mu_2(\lambda_1 - \lambda_2)\vec{b}\right\}$$

Similarily the position vector of r_2 point P

$$\vec{r_2} = \frac{1}{\mu_3\lambda_3 - \mu_1\lambda_1}\left\{\lambda_1\lambda_3(\mu_3 - \mu_1)\vec{a} + \mu_1\mu_3(\lambda_3 - \lambda_1)\vec{b}\right\}$$

The position vector $\vec{r_1}$ of the point P_1, the point of intersection of A_2B_3 and A_3B_2 is given by

$$\vec{r_1} = \frac{1}{\mu_2\lambda_2 - \mu_3\lambda_3}\left\{\lambda_2\lambda_3(\mu_2 - \mu_3)\vec{a} + \mu_2\mu_3(\lambda_2 - \lambda_3)\vec{b}\right\}$$

In order to prove that points P_1, P_2, P_3 are collinear, it is sufficient to show that there exist scalars x_1, x_2, x_3 such

that $\quad x_1\vec{r_1} + x_2\vec{r_2} + x_3\vec{r_3} = 0$

where $\quad x_1 + x_2 + x_3 = 0$

We observe that $x_1 + x_2 + x_3 = \sum\mu_1\lambda_1(\mu_2\lambda_2 - \mu_3\lambda_3) = 0$

10. If possible let there be a line passing through the origin $a\bar{z} + \bar{a}z = 0$. Such that the points representing $z_1, z_2 \ldots z_n$ lie on one side of it then,

$a\bar{z}_i + \bar{a}z_i > 0$ or $a\bar{z}_i + \bar{a}z_i < 0$ for all $i = 1, 2, 3, \ldots n$

$$\Rightarrow a\left(\sum_{i=1}^{n}\bar{z}_i\right) + \bar{a}\left(\sum_{i=1}^{n}z_i\right) > 0 \text{ or } a\left(\sum_{i=1}^{n}\bar{z}_i\right) + \bar{a}\left(\sum_{i=1}^{n}z_i\right) < 0$$

$$\Rightarrow a \times 0 + \bar{a} \times 0 > 0 \text{ or } a \times 0 + \bar{a} \times 0 < 0$$

This is contradiction so our proposition is wrong. Hence any straight line passing through origin seperates points $z_1\ z_2 \ldots z_n$.

$$P(E_5/W) = \frac{280}{909},\quad P(E_6/W) = \frac{140}{303}$$

Let B = the event of drawing a black ball in the next draw. Now the ball in the next draw may be black after the 3 white balls were drawn from the bag containing 6 black and 3 white balls. Its prabability is

$$P(E_3/W) \cdot P(B(E_3/W))$$

$\therefore$ By the theorem of total conditional probability,

$$P(B) = P(E_3/W) \cdot P(B/(E_3/W)) + P(E_4/W) \cdot P(B/(E_4W)) + P(E_5/W) \cdot P(B/(E_5/W)) + P(E_6/W) \cdot P[B/(E_6/W)] \quad \ldots\text{(i)}$$

Now $P\{B/(E_3/W)\}$ = The probability of drawing 1 black in case of E_3 when whites are already drawn = 1.

$$P\{B/(E_4/W)\} = \frac{{}^6C_1}{{}^7C_1} = \frac{6}{7},$$

$$P\{B/(E_4/W)\} = \frac{{}^6C_1}{{}^8C_1} = \frac{3}{4}$$

$$P\{B/(E_5/W)\} = \frac{{}^6C_1}{{}^9C_1} = \frac{2}{3}$$

Using (i) put the value so $= \dfrac{677}{909}$

■

FORM IV

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Mathematics Olympia for IIT-JEE (MAINS) 2005

Exam on 22nd May 2005

By : Er. Akhlak Ahmad, ABC Classes, Gorakhpur

1. Find the sum of the series $\sum_{r=0}^{n}\left(\frac{n-3r+1}{n-r+1}\right)\frac{^nC_r}{2^r}$.

2. Points 0, $\frac{\sqrt{3}+i}{2}$, z and i are the vertices of a rhombus, then find the values of Arg z and $|z|$.

3. Show that $5x \le 8\sin x - \sin 2x \le 6x$ for $0 \le x \le \pi/3$.

4. Let $f(n) = 1+\frac{1}{2}+\frac{1}{3}+\ldots+\frac{1}{n}$, then show that
$$f(n) = \int_0^{\pi/2}\cot\left(\frac{\theta}{2}\right)(1-\cos^n\theta)d\theta.$$

5. A function is such that it coincides with its derivative. Find all such possible functions. Using this result prove that $e^x = 1+\frac{x}{1!}+\frac{x^2}{2!}+\frac{x^3}{3!}+\ldots$

6. If the circle C_1 touches x-axis and the line $y = x\tan\theta$ ($\tan\theta > 0$) in first quadrant and circle C_2 touches the line $y = x\tan\theta$, y-axis and circle C_1 in such a way that ratio of the radius of C_1 and C_2 is 2 : 1 then find the value of $\tan(\theta/2)$.

7. Two dice are rolled simultaneously to define events
A = sum of the numbers appearing on the upper faces is 9.
B = sum is divisible by 4
C = 5 has appeared on one of the dice.
Find $P(A \cup (B \cap C))$.

8. If $\tan^{-1}(x+h) = \tan^{-1}x + (h\sin y)(\sin y) - (h\sin y)^2\frac{\sin 2y}{3}+(h\sin y)^3\frac{\sin 3y}{3}+\ldots\ldots$
where $x \in (0, 1)$, $y \in (\pi/4, \pi/2)$, then prove that $y = \cot^{-1}x$.

9. Find the area bounded by the curves $y = |\sin x| + |\cos x|$ and $y = \frac{1}{|\sin x|+|\cos x|}$ for $0 \le x \le 2\pi$.

10. If (a, b, c) is a point on the plane $3x + 2y + z = 7$, then find the least value of $a^2 + b^2 + c^2$, using vector method.

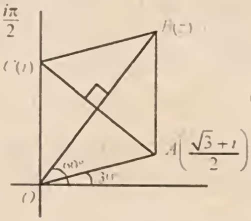
SOLUTION

1. Let $S = \sum_{r=0}^{n}\left(\frac{n-3r+1}{n-r+1}\right)\cdot\frac{^nC_r}{2^r}$

$$S = \sum_{r=0}^{n}\left(1-\frac{2r}{n-r+1}\right)\cdot\frac{^nC_r}{2^r}$$
$$= \sum_{r=0}^{n}\frac{^nC_r}{2^r}-\sum_{r=0}^{n}\frac{2r}{(n-r+1)}\cdot\frac{n!}{r!(n-r)!\cdot 2^r}$$
$$= \sum_{r=0}^{n}\frac{^nC_r}{2^r}-\sum_{r=0}^{n}\frac{^nC_{r-1}}{2^{r-1}} = \frac{1}{2^n}.$$

2. Since $\frac{\sqrt{3}+i}{2} = 1\left(\cos\frac{\pi}{6}+i\sin\frac{\pi}{6}\right) = e^{\frac{i\pi}{6}}$

and $i = 1\left(\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}\right) = e^{\frac{i\pi}{2}}$



Now, $|AC| = \left|\frac{\sqrt{3}-i}{2}\right| = 1,$

$AD = \frac{1}{2}$

$\Rightarrow OD = \sqrt{1-\frac{1}{4}} = \frac{\sqrt{3}}{2} \Rightarrow |z| = \sqrt{3}$, and Arg $z = 60°$.

3. Let $f(x) = 8\sin x - \sin 2x$

$f'(x) = 8\cos x - 2\cos 2x$

$f''(x) = -8\sin x - 4\sin 2x = -8\sin x(1-\cos x)$

from these we see that $f'(0) = 6$,

$f'\left(\frac{\pi}{3}\right) = 5$, $f(0) = 0$, $f''(x) < 0$ in $\left[0, \frac{\pi}{3}\right]$

Therefore $5 \le f'(x) \le 6$ in $\left[0, \frac{\pi}{3}\right]$

Integrating from 0 to x, gives

$\Rightarrow 5x \le f(x) \le 6x$ in $\left[0, \frac{\pi}{3}\right]$.

4. Let $g(x) = 1 + x + x^2 + ... + x^{n-1} = \dfrac{x^n - 1}{x-1}$

$$\therefore\ f(n) = \int_0^1 g(x)dx = \int_0^1 \frac{x^n-1}{x-1}dx$$

Put $x = \cos\theta \Rightarrow dx = -\sin\theta\ d\theta$

$$\therefore\ f(n) = \int_{\pi/2}^{0} \frac{(\cos^n\theta - 1)(-\sin\theta)}{(\cos\theta - 1)}d\theta$$

$$= \int_0^{\pi/2} \frac{(1-\cos^n\theta)2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{2\sin^2\frac{\theta}{2}}d\theta$$

$$= \int_0^{\pi/2} \cot\left(\frac{\theta}{2}\right)(1-\cos^n\theta)d\theta.$$

5. Given $f'(x) = f(x)$

$$\Rightarrow \frac{f'(x)}{f(x)}dx = dx \Rightarrow \ln f(x) = k + x \quad\text{or}\quad f(x) = k\cdot e^x$$

Consider a function whose derivative is the function itself and $f(0) = 1$

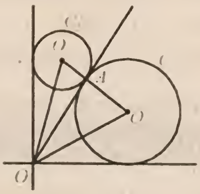
e^x is only such function from above result

but $f(x) = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ...$

also satisfies $f(0) = 1$ and $f'(x) = f(x)$. This is possible iff

$e^x = 1 + \dfrac{x}{1!} + \dfrac{x^2}{2!} + \dfrac{x^3}{3!} + ...$

6. Let $m = \tan\theta$, then



$\angle AOO_1 = \dfrac{\theta}{2}$, $\angle O_2OO_1 = 45°$

$\Rightarrow \angle AOO_2 = 45° - \dfrac{\theta}{2}$

$\tan\dfrac{\theta}{2} = \dfrac{r_1}{OA}$; $\tan\left(45° - \dfrac{\theta}{2}\right) = \dfrac{r_2}{OA}$

$$\Rightarrow \frac{1}{\tan\left(\frac{\theta}{2}\right)}\left(\frac{1-\tan\left(\frac{\theta}{2}\right)}{1+\tan\left(\frac{\theta}{2}\right)}\right) = \frac{r_2}{r_1} = \frac{1}{2}$$

$$\Rightarrow 2 - 2\tan\frac{\theta}{2} = \tan^2\frac{\theta}{2} + \tan\frac{\theta}{2}$$

$$\Rightarrow \tan^2\frac{\theta}{2} + 3\tan\frac{\theta}{2} - 2 = 0 \Rightarrow \tan\frac{\theta}{2} = \frac{-3+\sqrt{17}}{2}.$$

7. $(B \cap C) \Rightarrow$ sum is either 4, 8 or 12 and 5 has appeared on one of the dice i.e. (5, 3) (3, 5).

$A \cup (B \cap C) \Rightarrow$ either (5, 3) or (3, 5) or sum is 9 i.e. (6, 3), (3, 6), (5, 4), (4, 5), (5, 3), (3, 5).

$$P(A\cup(B\cap C)) = \frac{6}{36} = \frac{1}{6}.$$

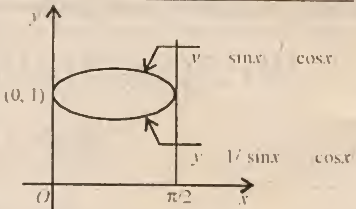
8. $$\frac{\tan^{-1}(x+h) - \tan^{-1}x}{h} = \sin^2 y - h\sin^2 y\cdot\frac{\sin 2y}{2} + ...$$

Now taking limits of both sides as $h \to 0$, we get

$$\frac{1}{1+x^2} = \sin^2 y \Rightarrow x^2 = \operatorname{cosec}^2 y - 1 = \cot^2 y$$

$\Rightarrow x = \cot y$, $y = \cot^{-1}(x)$.

9. The given function are periodic with period $\dfrac{\pi}{2}$.



$$A = 4\int_0^{\pi/2}\left((\sin x + \cos x) - \left(\frac{1}{\sin x + \cos x}\right)\right)dx$$

$$= 4[-\cos x + \sin x]_0^{\pi/2} - 4I_1,$$

where $I_1 = \displaystyle\int_0^{\pi/2} \frac{1}{\sin x + \cos x}dx$

$$= \int_0^{\pi/2} \frac{\sec^2\frac{x}{2}}{1 + 2\tan\frac{x}{2} - \tan^2\frac{x}{2}}dx = -\int_0^{\pi/2} \frac{\sec^2\left(\frac{x}{2}\right)dx}{\left(\tan\frac{x}{2} - 1\right)^2 - 2}$$

$$= -\int_0^1 \frac{2dy}{(y-1)^2 - 2} = \frac{1}{\sqrt{2}}\ln(3+2\sqrt{2})$$

$$\Rightarrow A = 8 - 2\sqrt{2}\ln(3+2\sqrt{2}).$$

10. (a, b, c) lies on the plane

$3x + 2y + z = 7 \Rightarrow 3a + 2b + c = 7$ ...(1)

We have

$(a\hat{i} + b\hat{j} + c\hat{k})\cdot(3\hat{i} + 2\hat{j} + \hat{k}) = 3a + 2b + c = 7$ ...(2)

Also $(a\hat{i} + b\hat{j} + c\hat{k})\cdot(3\hat{i} + 2\hat{j} + \hat{k})$

$= \sqrt{a^2 + b^2 + c^2}\cdot\sqrt{3^2 + 2^2 + 1^2}\cos\theta$ ...(3)

where θ is the angle between the vectors $a\hat{i} + b\hat{j} + c\hat{k}$ and $3\hat{i} + 2\hat{j} + \hat{k}$.

From (2) and (3) we get

$$a^2 + b^2 + c^2 = \frac{49}{14}\sec\theta \geq \frac{7}{2}$$

as L.H.S. is positive and $|\sec\theta| \geq 1$

Equality hold if $\sec\theta = 1$, which is the case when the vectors $a\hat{i} + b\hat{j} + c\hat{k}$ and $3\hat{i} + 2\hat{j} + \hat{k}$ are parallel.

Hence least value of $a^2 + b^2 + c^2$ is 7/2.

■

10 Challenging PROBLEMS

Co-ordinate Geometry

By : Alok Kumar, B.Tech, IIT Kanpur

1. A parallelogram is formed by the lines $ax^2 + 2hxy + by^2 = 0$ and the lines through (p, q) parallel to them. Prove that the equation of that diagonal of the parallelogram which doesn't pass through origin is
$(2x - p)(ap + hq) + (2y - q)(hp + bq) = 0$

2. If by an orthogonal transformation without change of origin, the equation $ax^2 + 2hxy + by^2 = c$ is changed into one, in which there is no term involving xy, prove that the transformed equation is
$(a + b + \lambda) x^2 + (a + b - \lambda) y^2 = 2c$,
$\lambda = [(a - b)^2 + 4h^2]^{1/2}$

3. Prove that the equation of the circle circumscribing the triangle formed by the three lines whose equations are
$a_i x + b_i y + c_i = 0,\ i = 1, 2, 3$ is
$$\begin{vmatrix} (a_1^2+b_1^2)/(a_1x+b_1y+c_1) & a_1 & b_1 \\ (a_2^2+b_2^2)/(a_2x+b_2y+c_2) & a_2 & b_2 \\ (a_3^2+b_3^2)/(a_3x+b_3y+c_3) & a_3 & b_3 \end{vmatrix} = 0$$

4. Find the equation of the circle whose radius is 5 and which touches the circle $x^2 + y^2 - 2x - 4y - 20 = 0$ at the point (5, 5).

5. Normals are drawn to the parabola $y^2 = 4ax$ at the points t_1, t_2, t_3 on it. Prove that the area of the triangle formed by the normals is
$$\frac{a^2}{2}(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)(t_1 + t_2 + t_3)^2$$

6. Find the locus of the poles of the normal chords of the parabola $y^2 = 4ax$.

7. Prove that the locus of the middle points of chords of an ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ whose length is constant, say, $2c$ is
$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)+\frac{c^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0$$

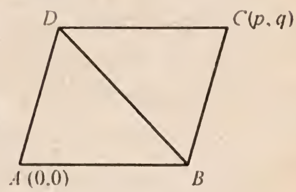
8. Prove that the area of triangle inscribed in an ellipse is $2ab \sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}$ where α, β, γ are the eccentric angles of the vertices.

9. Find the co-ordinates of the limiting points of the system of circles determined by the two circles.
$x^2 + y^2 + 5x + y + 4 = 0$ and $x^2 + y^2 + 10x - 4y - 1 = 0$

10. Three normals from a point to the parabola $y^2 = 4ax$ meet its axis in points whose abscissas are in A.P. Show that the locus of the point is $27ay^2 = 2(x - 2a)^3$.

SOLUTIONS

1. The combined equation of AB and AD is



$S_1 \equiv ax^2 + 2hxy + by^2 = 0$

Now the lines through (p, q) and parallel to the lines given by $S_1 \equiv 0$ are

$S_2 \equiv a(x - p)^2 + b(y - q)^2 + 2h(x - p)(y - q) = 0$

We have to find diagonal BD which passes through the intersection of S_1 and S_2.

For all values of $\lambda_1 S_1 + \lambda S_2 \equiv 0$ represents the curve through the intersection of $S_1 = 0$ and $S_2 = 0$. Choosing $\lambda = -1$, we get $S_1 - S_2 \equiv 0$ as the curve through the intersection of S_1 and S_2. But as S_1

and S_2 have the same terms of 2^{nd} degree, $S_1 - S_2 = 0$ reduces to linear equation which will represent a straight line and that will be the equation of diagonal BD.

$S_1 - S_2 = 0$ gives

$$a[-2xp + p^2] + 2h[-py - qx + pa] + b[-2yp + q^2] = 0$$

$$\Rightarrow -ap(2x-p) - hq(2x-p) - hp(2y-q) - bq(2y-q) = 0$$

i.e. $(2x-p)(ap+hq) + (2y-q)(bq+hp) = 0$

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2. The axes should be turned through an angle θ given by $\tan 2\theta = \dfrac{2h}{a-b}$ for xy terms to disappear.

or $\dfrac{\sin 2\theta}{2h} = \dfrac{\cos 2\theta}{a-b} = \dfrac{\sqrt{\sin^2 2\theta + \cos^2 2\theta}}{\sqrt{4h^2 + (a-b)^2}} = \dfrac{1}{\lambda}$

and the transformed equation is

$$a'x^2 + b'y^2 = c$$

$$\text{or } 2a'x^2 + 2b'y^2 = 2c$$

Where $2a' = (a+b) + [2h\sin 2\theta + (a-b)\cos 2\theta]$

$$= (a+b) + \frac{4h^2 + (a-b)^2}{\lambda} = (a+b) + \frac{\lambda^2}{\lambda}$$

$\therefore 2a' = (a+b) + \lambda$

Similarly $2b' = (a+b) - \lambda$

Hence the transformed equation is

$(a+b+\lambda)x^2 + (a+b-\lambda)y^2 = 2c$

$\lambda = [(a-b)^2 + 4h^2]^{1/2}$

---

3. From the equation

$$(a_1x + b_1y + c_1)(a_2x + b_2y + c_2) + \lambda(a_2x + b_2y + c_2)(a_3x + b_3y + c_3) + \mu(a_3x + b_3y + c_3)(a_1x + b_1y + c_1) = 0 \quad ...(1)$$

Above equation is satisfied by the points obtained by taking any two of the three given equations, i.e. it is satisfied by the vertices of the triangle formed by the given equations. For the equation to represent a circle, coefficient of x^2 and y^2 should be equal and the coefficient of xy should vanish.

$\therefore a_1a_2 + \lambda a_2a_3 + \mu a_3a_1 = b_1b_2 + \lambda b_2b_3 + \mu b_3b_1$

or

$(a_1a_2 - b_1b_2) + \lambda(a_2a_3 - b_2b_3) + \mu(a_3a_1 - b_3b_1) = 0 \ ...(2)$

and

$(a_1b_2 + a_2b_1) + \lambda(a_2b_3 + a_3b_2) + \mu(a_3b_1 + a_1b_3) = 0 \ ...(3)$

write (1) in the form

$$\frac{1}{a_3x + b_3x + c_3} + \lambda\frac{1}{a_1x + b_1y + c_1} + \mu\frac{1}{a_2x + b_2y + c_2} = 0 \ ...(4)$$

Eliminating λ and μ from (2), (3) and (4) we get

$$\begin{vmatrix} \dfrac{1}{a_3x + b_3y + c_3} & \dfrac{1}{a_1x + b_1y + c_1} & \dfrac{1}{a_2x + b_2y + c_2} \\ a_1a_2 - b_1b_2 & a_2a_3 - b_2b_3 & a_3a_1 - b_1b_3 \\ a_1b_2 + a_2b_1 & a_2b_3 + a_3b_2 & a_3b_1 + a_1b_3 \end{vmatrix} = 0$$

Again the coefficient of $\dfrac{1}{a_3x + b_3y + c_3}$ in the above determinant is

$(a_2a_3 - b_2b_3)(a_3b_1 + b_3a_1) - (a_3a_1 - b_3b_1)(a_2b_3 + a_3b_2)$

$= a_3^2(a_2b_1 - a_1b_2) + b_3^2(-a_1b_2 + b_1a_2)$

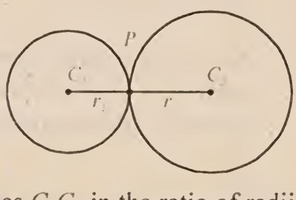
$= -(a_3^2 + b_3^2)(a_1b_2 - a_2b_1)$

Hence the equation of the circle can be given as

$$\begin{vmatrix} \dfrac{a_1^2 + b_1^2}{a_1x + b_1y + c_1} & a_1 & b_1 \\ \dfrac{a_2^2 + b_2^2}{a_2x + b_2y + c_2} & a_2 & b_2 \\ \dfrac{a_3^2 + b_3^2}{a_3x + b_3y + c_3} & a_3 & b_3 \end{vmatrix} = 0$$

---

4. Centre of the given circle is (1, 2) and $r_1 = 5$. If (α, β) be the centre of the other circle and $r_2 = 5$, the point of contact P (5, 5) divides C_1C_2 in the ratio of radii 5 : 5, so the point of contact is the midpoint of C_1C_2



$5 = \dfrac{1+\alpha}{2}$ and $5 = \dfrac{2+\beta}{2}$ $\therefore$ centre (α, β) is (9, 8)

Required circle is

$$(x-9)^2+(y-8)^2=5^2$$

i.e. $x^2+y^2-18x-16y+120=0$

---

5. The three points of intersection are

$[2a+a(t_1^2+t_1t_2+t_2^2), -at_1t_2(t_1+t_2)]$, etc

$$\therefore\ \Delta=\frac{1}{2}\begin{vmatrix} 2a+a(t_1^2+t_1t_2+t_2^2) & -at_1t_2(t_1+t_2) & 1 \\ 2a+a(t_2^2+t_2t_3+t_3^2) & -at_2t_3(t_2+t_3) & 1 \\ 2a+a(t_3^2+t_3t_1+t_1^2) & -at_3t_1(t_3+t_1) & 1 \end{vmatrix}$$

split the determinant into sum of two determinants. One of them is zero for two column, 1st and 3rd are identical.

$$\therefore\ \Delta=-\frac{a^2}{2}\begin{vmatrix} t_1^2+t_1t_2+t_2^2 & t_1t_2(t_1+t_2) & 1 \\ t_2^2+t_2t_3+t_3^2 & t_2t_3(t_2+t_3) & 1 \\ t_3^2+t_3t_1+t_1^2 & t_3t_1(t_3+t_1) & 1 \end{vmatrix}$$

$$=-\frac{a^2}{2}\begin{vmatrix} t_1^2+t_1t_2+t_2^2 & t_1t_2(t_1+t_2) & 1 \\ (t_3-t_1)(t_1+t_2+t_3) & t_2(t_3-t_1)(t_1+t_2+t_3) & 0 \\ (t_3-t_2)(t_1+t_2+t_3) & t_1(t_3-t_2)(t_1+t_2+t_3) & 0 \end{vmatrix}$$

$R_2 \to R_2 - R_1$

$R_3 \to R_3 - R_1$

$$=-\frac{a^2}{2}(t_3-t_1)(t_3-t_2)(t_1+t_2+t_3)^2\begin{vmatrix} 1 & t_2 \\ 1 & t_1 \end{vmatrix}$$

$$=-\frac{a^2}{2}(t_1-t_2)(t_2-t_3)(t_3-t_1)(t_1+t_2+t_3)^2$$

---

6. Let (h, k) be the pole or point of intersection of tangents, so that the equation of the chord is

$ky = 2a\,(x+h)$ ...(1)

Since it is a normal chord, its equation should be of the form

$y = mx - 2am - am^3$, $\quad m \in R$ ...(2)

comparing (1) and (2), we get

$$m=\frac{2a}{k} \quad \text{and} \quad h=-a(2+m^2)$$

Eliminating m between them yields

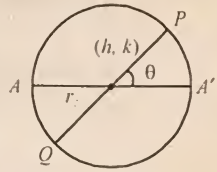
$$h+2a+a\frac{4a^2}{k^2}=0$$

Thus locus is

$y^2(x+2a)+4a^3=0$

---

7. Let (h, k) be the middle point of the chord whose inclination to the major axis be θ, then the extremities P and Q of the chord are $(h + c\cos\theta, k + c\sin\theta)$ and $(h - c\cos\theta, k - c\sin\theta)$



Both these points are on the ellipse, $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$

$$\therefore\quad \frac{(h+c\cos\theta)^2}{a^2}+\frac{(k+c\sin\theta)^2}{b^2}=1 \qquad ...(1)$$

$$\text{and}\quad \frac{(h-c\cos\theta)^2}{a^2}+\frac{(k-c\sin\theta)^2}{b^2}=1 \qquad ...(2)$$

To Eliminate θ between them, add (1) and (2)

$$\Rightarrow\ 2\left[\frac{h^2}{a^2}+\frac{k^2}{b^2}+c^2\left(\frac{\cos^2\theta}{a^2}+\frac{\sin^2\theta}{b^2}\right)\right]=2 \qquad ...(3)$$

Subtract (1) from (2)

$$\frac{4ch\cos\theta}{a^2}-\frac{4ck\sin\theta}{b^2}=0 \Rightarrow \frac{h^2\cos^2\theta}{a^4}=\frac{k^2\sin^2\theta}{b^4}$$

$$\Rightarrow \frac{\cos^2\theta}{a^4/h^2}=\frac{\sin^2\theta}{b^4/k^2}=\frac{\cos^2\theta+\sin^2\theta}{\frac{a^4}{h^2}+\frac{b^4}{k^2}}=\frac{1}{\frac{a^4}{h^2}+\frac{b^4}{k^2}} \qquad ...(4)$$

Plugging the values of (4) into (3).

$$\frac{h^2}{a^2}+\frac{k^2}{b^2}+c^2\,\frac{\frac{a^2}{h^2}+\frac{b^2}{k^2}}{\frac{a^4}{h^2}+\frac{b^4}{k^2}}=1$$

$$\Rightarrow\ \left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)+c^2\left(\frac{a^2k^2+b^2h^2}{a^4k^2+b^4h^2}\right)=0$$

$$\Rightarrow\ \left(\frac{h^2}{a^2}+\frac{k^2}{b^2}-1\right)+\frac{c^2a^2b^2}{a^4b^4}\left[\frac{\frac{h^2}{a^2}+\frac{k^2}{b^2}}{\frac{h^2}{a^4}+\frac{k^2}{b^4}}\right]=0$$

$\therefore$ Locus of (h, k) is

$$\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}-1\right)\left(\frac{x^2}{a^4}+\frac{y^2}{b^4}\right)+\frac{c^2}{a^2b^2}\left(\frac{x^2}{a^2}+\frac{y^2}{b^2}\right)=0$$

---

8. The three vertices are $(a\cos\alpha, b\sin\alpha)$, etc and area of triangle is $\frac{1}{2}\{x_1(y_2-y_3)+...\}$

$$= -\frac{1}{2}\ \{a\cos\alpha\ (b\sin\beta - b\sin\gamma) + ...\}$$

$$= -\frac{1}{2}ab\ \{\sin(\alpha-\beta)+\sin(\beta-\gamma)+\sin(\gamma-\alpha)\}$$

$$= -\frac{1}{2}ab\ \left\{2\sin\frac{(\alpha-\beta)}{2}\cos\frac{\alpha-\beta}{2} + 2\sin\frac{\beta-\alpha}{2}\cos\frac{\alpha+\beta-2\gamma}{2}\right\}$$

$$= -\frac{1}{2}ab\cdot 2\sin\frac{\alpha-\beta}{2}\left(\cos\frac{\alpha-\beta}{2} - \cos\frac{\alpha+\beta-2\gamma}{2}\right)$$

$$= -ab\cdot\sin\frac{\alpha-\beta}{2}\cdot 2\sin\frac{\alpha-\gamma}{2}\sin\frac{\beta-\gamma}{2}$$

$$= 2ab\sin\frac{\alpha-\beta}{2}\sin\frac{\beta-\gamma}{2}\sin\frac{\gamma-\alpha}{2}$$

---

9. The equation of the co-axial system of which two members are given by

$$(x^2+y^2+5x+y+4) - \lambda(x^2+y^2+10x-4y-1) = 0$$

or $x^2+y^2+\frac{5-10\lambda}{1-\lambda}x+\frac{1+4\lambda}{1-\lambda}y+\frac{4+\lambda}{1-\lambda}=0,\ \lambda \neq 1$

The centre of this circle is

$$\left(-\frac{5(1-2\lambda)}{2(1-\lambda)}, -\frac{1+4\lambda}{2(1-\lambda)}\right)$$

Radius is given by

$$r^2 = \frac{(5-10\lambda)^2+(1+4\lambda)^2-4(1-\lambda)(4+\lambda)}{4(1-\lambda)^2}$$

$$= \frac{120\lambda^2-80\lambda+10}{4(1-\lambda)^2}$$

Radius will vanish if $120\lambda^2 - 80\lambda + 10 = 0$

$\Rightarrow\ 12\lambda^2 - 8\lambda + 1 = 0 \Rightarrow (6\lambda - 1)(2\lambda - 1) = 0$

$\lambda = 1/6$ and $1/2$

Substituting these values of λ we get the limiting points as $(0, -3)$ and $(-2, -1)$.

---

10. Any normal $y = mx - 2am - am^3$ meets the axis of the parabola i.e. x – axis where $y = 0$, in the point $(2a + am^2, 0)$

Since the abscissas are in A.P.

$$2(2a+am_2^2) = (2a+am_1^2)+(2a+am_3^2)$$

$$\Rightarrow\ 2(2+m_2^2) = 2+m_1^2+2+m_3^2$$

$$\Rightarrow\ 2m_2^2 = m_1^2+m_3^2$$

$$\Rightarrow\ 3m_2^2 = m_1^2+m_2^2+m_3^2 = (m_1+m_2+m_3)^2 - 2(m_1m_2+m_2m_3+m_3m_1)$$

$$\Rightarrow\ 3m_2^2 = 0 - 2\frac{2a-h}{a}\quad \Rightarrow m_2^2 = \frac{2(h-2a)}{3a}\ \ ...(1)$$

Again

$$2m_2^2 = (m_1+m_3)^2 - 2m_1m_3$$

$$\Rightarrow\ 2m_2^2 = (-m_2)^2 + 2\frac{k}{am_2}$$

$$\Rightarrow\ m_2^2 = \frac{2k}{am_2}\quad \Rightarrow\ m_2^3 = \frac{2k}{a}\qquad ...(2)$$

from (1) and (2)

$$8\frac{(h-2a)^3}{27a^3} = \frac{4k^2}{a^2}$$

$\therefore$ Locus is $27ay^2 = 2(x-2a)^3$. ■

19th Mathematical Challenge for I.I.T. MAINS

This section is designed to give IIT JEE aspirants a thorough grinding & exposure to variety of possible twists and turns of problems in mathematics that would be very helpful in facing IIT JEE. Each and every problem is well thought of in order to strengthen the concepts and we hope that this section would prove a rich resource for practicing challenging problems and enhancing the preparation level of IIT JEE aspirants.
The detailed solutions to these problems will be published in the next issue alongwith a new set of such problems.

1. Show that the conics through the intersection of two rectangular hyperbolas are also rectangular hyperbolas. If A, B, C and D be the four points of intersection of these two rectangular hyperbolas, then find the orthocentre of the triangle ABC.

2. Find the area of a right angle triangle if it is known that the radius of circle inscribed in the triangle is r and that of the circumscribed circle is R.

3. Q is any point on the line $x = a$. If A is the point $(a, 0)$ and QR, the bisector of the angle OQA, meets OX in R, then prove that the locus of the foot of the perpendicular from R to OQ has the equation
$$(x-2a)(x^2+y^2)+a^2x=0$$

4. Show that the equation $z^4+2z^3+3z^2+4z+5=0$ with $(Z \in C)$ have no purely real as well as purely imaginary root.

5. Prove that $\int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{\ell nx}{x}dx = \ell na \int_0^\infty f\left(\frac{a}{x}+\frac{x}{a}\right)\frac{dx}{x}$

6. A straight line moves so that the product of the perpendiculars on it from two fixed points is a constant. Prove that the locus of the foot of the perpendiculars from each of these points upon the straight line is a circle, the same for each.

7. Prove the identity :
$\int_0^x e^{zx-z^2}dz = e^{\frac{x^2}{4}}\int_0^x e^{-z^2/4}dz$, deriving for the function $f(x)=\int_0^x e^{zx-z^2}dz$ a differential equation and solving it.

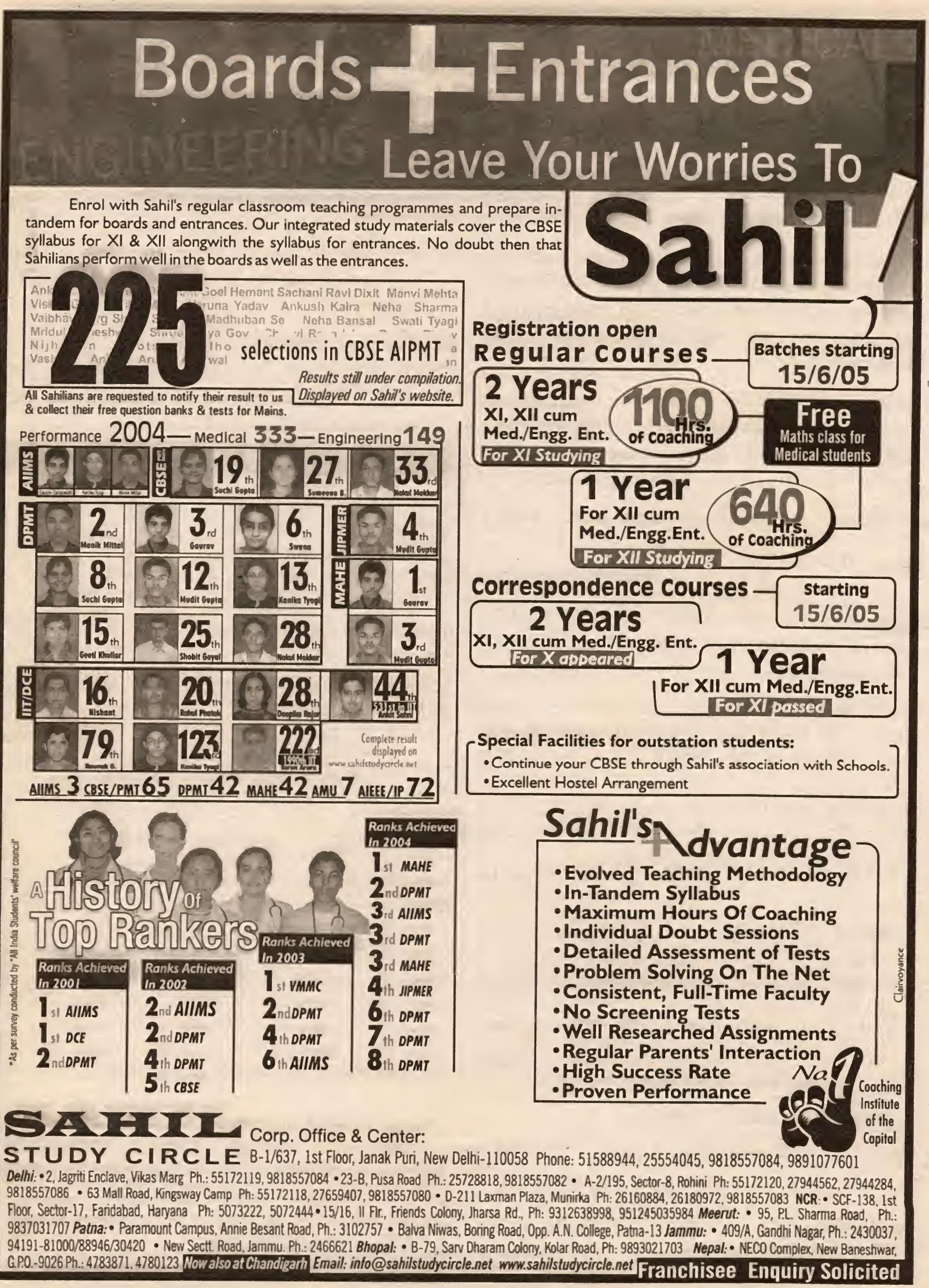
8. Let α, β be the roots of a quadratic equation, such that $\alpha\beta = 4$ and $\frac{\alpha}{\alpha-1}+\frac{\beta}{\beta-1}=\frac{a^2-7}{a^2-4}$. Find the set of values of a for which $\alpha, \beta \in (1, 4)$

9. Investigate the function $f(x) = x^{5/3} - 5x^{2/3}$ for points of extremum and find the values of k such that the equation $x^{5/3} - 5x^{2/3} = k$ has exactly one positive root.

10. Let $A = \{1, 2, 3, ..., 100\}$. If X is a subset of A containing exactly 50 elements then show that $\sum_{p\in x} p_{\min} = {}^{101}C_{51}$

Solution of Mathematical Challenges 18 and 19 will be published in next issue.

By : **Shailendra Maheshwari**, Career point, Kota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Mathematics Olympia for IIT-JEE (MAINS)

By : Er. Akhlak Ahmad, ABC Classes, Gorakhpur

1. From a point (1, 1, 21), a ball is dropped onto the plane $x + y + z = 3$, where x, y-plane is horizontal and z-axis is along the vertical. Find the co-ordinates of the point where the ball hits the plane the second time.

2. If the curve $y = ax^{1/2} + bx$ passes through the point (1, 2) and lies above the x-axis for $0 \le x \le 9$ and the area enclosed by the curve, the x-axis and the line $x = 4$ is 8 sq. units. Determine a and b.

3. If $p(\sin\alpha - \cos\alpha \tan\theta)\sec\theta = \tan\theta \cdot \sec(\alpha - \theta)$ then prove that $\theta = \frac{1}{2}\cot^{-1}\left(\frac{q + p\cos 2\alpha}{p\sin 2\alpha}\right)$.

4. $ABCD$ is a rectangle with $2AD = AB = 2r$. Find the area of the shaded region if arc(APB) and arc(DQC) are semicircles.

5. Prove that the curves $2|z|^2 - 4a\ \text{Re}(z) + c^2 = 0$ and $2|z|^2 - 4b\ \text{Im}(z) + c^2 = 0$ touch each other if a^2, c^2, b^2 are in H.P., where $a, b, c \in R - \{0\}$, $|a| \ge \frac{|c|}{\sqrt{2}}$ and $|b| \ge \frac{|c|}{\sqrt{2}}$.

6. If $2f(x) + f(-x) = \frac{1}{x}\sin\left(x - \frac{1}{x}\right)$, then find the value of $\int_{1/e}^{e} f(x)\,dx$.

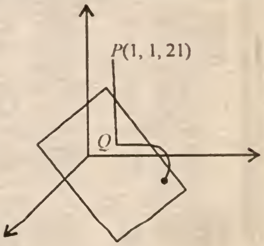
7. If y_1 and y_2 are the solution of differential equation $\frac{dy}{dx} + Py = Q$, where P and Q are function of x alone and $y_2 = y_1 z$, then prove that $z = 1 + c\cdot e^{-\int \frac{Q}{y_1}dx}$, where c is an arbitrary constant.

8. If f is an increasing function such that $f''(x) > 0\ \forall\ x \in (0,1)$; $f(0) = 0$; $f(1) = 1$, prove that $f(x).\ f^{-1}(x) \le x^2\ \forall\, x \in (0, 1)$.

9. If M, N be two square matrices of order $\lambda \times \lambda$ whose all the elements are positive integers, show that $\text{tr}(MN^2) \ge \lambda^3$.

10. Find the coordinates of the points in the rectangle $\{(x, y): |x| \le 10, |y| \le 3\}$ which lie on the curve $y^2 = x + \sin x$ and at which the tangents to the curve are parallel to the x-axis.

SOLUTION



1. Since it falls along the vertical, the x-y coordinates of the ball will not change before it strikes the plane.
$\Rightarrow$ If Q be the point where the ball meets the plane 1st time, then $Q = (1, 1, 1)$
Speed of the balls just before striking the plane is

$$\sqrt{2\times 10\times 20} = 20 \text{ m/s}.$$

Now let θ be the angle between PQ and normal to the plane

$$\Rightarrow \cos\theta = \frac{1}{\sqrt{3}} \Rightarrow \cos 2\theta = -\frac{1}{3},\ \sin 2\theta = \frac{2\sqrt{2}}{3}$$

Now component of velocity in the direction of z-axis after it strikes the plane

$$= -20\sin\left(2\theta - \frac{\pi}{2}\right) = -\frac{20}{3} \text{ m/s}$$

Hence in t time the z-coordinate of ball becomes

$$1 - \frac{20}{3}t - \frac{1}{2}\times 10t^2 = 1 - \frac{20}{3}t - 5t^2$$

The component of velocity in x-y plane is

$$20\cos\left(2\theta - \frac{\pi}{2}\right) = 20\sin 2\theta = \frac{20\times 2\sqrt{2}}{3} = \frac{40\sqrt{2}}{3}$$

Using symmetry, the component along the x-axis $= 40/3$ and the component along the y-axis $= 40/3$.
Hence x and y coordinates of the ball after t time

$= 1 + \frac{40}{3}t$

$\Rightarrow$ after t time the coordiante of the ball will become

$= \left(1 + \frac{40}{3}t,\ 1+\frac{40}{3}t,\ 1-\frac{20}{3}t-5t^2\right)$

It lies on the plane $\frac{80}{3}t-\frac{20}{3}t-5t^2 = 0$

$\Rightarrow 20t - 5t^2 = 0 \Rightarrow t = 4$

$\Rightarrow$ coordinate of the point where the ball strikes the plane the second time $=\left[\frac{163}{3}, \frac{163}{3}, \frac{-317}{3}\right]$.

---

2. Since the curve $y = ax^{1/2} + bx$ passes through the point (1, 2). $\therefore 2 = a + b$ ...(1)

by observation the curve also passes through (0, 0). Therefore the area enclosed by curve, x-axis and $x = 4$ is given by

$$A = \int_0^4 (ax^{1/2} + bx)dx = 8 \Rightarrow \frac{2a}{3}\cdot 8+\frac{b}{2}\cdot 16 = 8$$

$$\Rightarrow \frac{2a}{3}+b = 1 \qquad ...(2)$$

Solving (1) and (2), we get $a = 3$, $b = -1$.

---

3. $\frac{p}{q} = \frac{\sin\theta\cdot\cos\theta}{\sin(\alpha-\theta)\cos(\alpha-\theta)} = \frac{\sin 2\theta}{\sin(2\alpha-2\theta)}$

$1+\frac{q}{p} = \frac{2\sin\alpha\cos(2\theta-\alpha)}{\sin 2\theta}$

$\frac{(p+q)}{2p\sin\alpha} = \cot 2\theta\cos\alpha+\sin\alpha$

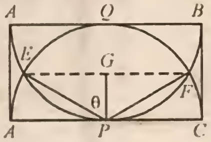
$\cot 2\theta = \left[\frac{(p+q)}{2p\sin\alpha}-\sin\alpha\right]\times\frac{1}{\cos\alpha}$

$\theta = \frac{1}{2}\cot^{-1}\left(\frac{q+p\cos 2\alpha}{p\sin 2\alpha}\right)$.

---

4. The required area is double of the area $QEFQ$.

In ΔEPG,



$\cos\theta = \frac{r}{2r} = \frac{1}{2} \Rightarrow \theta = \frac{\pi}{3}$.

Since area $QEFQ$ = area of segment $PEQFP$ – area of triangle PEF

$= \frac{1}{2}r^2\times\frac{2\pi}{3} - \frac{r}{2}\times\frac{r}{2}\tan 30°$

$\Rightarrow$ Area $QEFQ = \frac{1}{3}\pi r^2 - \frac{\sqrt{3}}{4}r^2$

$\Rightarrow$ Shaded area is

$$2r^2\left[\frac{\pi}{3}-\frac{\sqrt{3}}{4}\right] = \frac{r^2}{6}[4\pi-3\sqrt{3}] \qquad ...(1)$$

---

5. The two curves are circles and their equations can be rewritten as $2|z|^2 - 4a\text{Re}(z) + c^2 = 0$

$\Rightarrow |z|^2 - 2a\text{Re}(z) + \frac{c^2}{2} = 0$

$\Rightarrow \bar{z}\cdot z - a(z+\bar{z})+a^2 = a^2-\frac{c^2}{2} \Rightarrow |z-a| = \sqrt{a^2-\frac{c^2}{2}}$

Similarly for second curve $|z-ib| = \sqrt{b^2-\frac{c^2}{2}}$,

The curves will touch each other if $c_1c_2 = r_1 \pm r_2$

$\Rightarrow \sqrt{a^2+b^2} = \left|\sqrt{a^2-\frac{c^2}{2}} \pm \sqrt{b^2-\frac{c^2}{2}}\right|$

$\Rightarrow 2a^2b^2 = c^2(a^2+b^2) \Rightarrow c^2 = \frac{2a^2b^2}{a^2+b^2}$

$\Rightarrow a^2, c^2, b^2$ are in H.P.

---

6. Since $2f(x)+f(-x) = \frac{1}{x}\sin\left(x-\frac{1}{x}\right)$

$\therefore 2f(-x)+f(x) = \frac{1}{x}\sin\left(x-\frac{1}{x}\right)$

$2f(x)+f(-x) = 2f(-x)+f(x) \Rightarrow f(x) = f(-x)$

$\therefore 3f(x) = \frac{1}{x}\sin\left(x-\frac{1}{x}\right)$

Hence $I = \int_{1/e}^{e} f(x)dx = \frac{1}{3}\int_{1/e}^{e}\frac{1}{x}\sin\left(x-\frac{1}{x}\right)dx$

Now, put $x = \frac{1}{t}$, $dx = -\frac{1}{t^2}dt$

$\therefore t = \frac{1}{3}\int_e^{1/e} t\sin\left(\frac{1}{t}-t\right)\left(-\frac{1}{t^2}\right)dt = \frac{1}{3}\int_e^{1/e}\frac{1}{t}\sin\left(t-\frac{1}{t}\right)dt$

$= -\frac{1}{3}\int_{1/e}^{e}\frac{1}{t}\sin\left(t-\frac{1}{t}\right)dt = -I \Rightarrow 2I = 0 \Rightarrow I = 0$.

---

7. We have been given,

$$\frac{dy_1}{dx}+Py_1 = Q,\ \frac{dy_2}{dx}+Py_2 = Q$$

Now $y_2 = y_1z \Rightarrow \frac{dy_2}{dx} = y_1\frac{dz}{dx}+z\frac{dy_1}{dx}$

$\Rightarrow y_1\frac{dz}{dx}+z\frac{dy_1}{dx}+Py_1z = Q$

$\Rightarrow y_1\frac{dz}{dx}+z\left(\frac{dy_1}{dx}+Py_1\right) = Q$

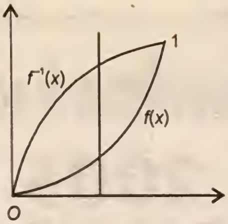
$\Rightarrow y_1\cdot\frac{dz}{dx}+zQ = Q \Rightarrow y_1\cdot\frac{dz}{dx} = Q(1-z)$

$\Rightarrow \frac{dz}{z-1} = -\frac{Q}{y_1}dx \Rightarrow \ln|z-1| = \int -\frac{Q}{y_1}dx+\lambda$

λ being constant of integration

$\Rightarrow z = 1+c\cdot e^{\int -\frac{Q}{y_1}dx}$

8. The function $f(x)$ is increasing and $f(0) = 0$



$\Rightarrow f(x) > 0 \ \forall \ x \in (0, 1)$

Now since

$f''(x) > 0 \ \forall \ x \in (0, 1)$

$\Rightarrow f(x)$ is concave upward.

Simultaneously, $f(1) = 1$,

therefore $f(x) < x \ \forall \ x \in (0, 1)$

Now, $f^{-1}(x)$ is symmetric about $y = x$

$\Rightarrow f^{-1}(x) > x \ \forall \ x \in (0, 1)$

For any $x \in (0, 1)$, Applying AM $\geq$ GM for $f(x)$ and $f^{-1}(x)$,

$$\frac{f(x)+f^{-1}(x)}{2} \geq (f(x)f^{-1}(x))^{1/2}$$

Also, $\dfrac{f(x)+f^{-1}(x)}{2} = x$ (symmetry)

$\Rightarrow f(x)f^{-1}(x) \leq x^2 \ \forall \ x \in (0, 1)$.

---

9. Let m_{ij} and n_{ij} be the elements of the matrices M and N respectively.

Thus, $tr(M) = \sum\limits_{i=j} m_{ij}$ and $tr(N) = \sum\limits_{i=j} n_{ij}$

As all the elements are positive integers, $m_{ij}, n_{ij} \geq 1$

$\Rightarrow \sum\limits_{i=j} m_{ij} \geq \lambda$ and $\sum\limits_{i=j} n_{ij} \geq \lambda$

$\Rightarrow tr(M) \geq \lambda$ and $tr(N) \geq \lambda$

Hence, $\text{tr}(MN^2) = \text{tr}(M)(\text{tr}(N))^2 \geq \lambda^3$.

---

10. $y^2 = x + \sin x \Rightarrow 2y\dfrac{dy}{dx} = 1 - \cos x$

$$\Rightarrow \frac{dy}{dx} = \frac{1-\cos x}{2y} = \frac{1-\cos x}{2\sqrt{x+\sin x}} \qquad ...(1)$$

when tangent at (x, y) is parallel to x-axis,

$\dfrac{dy}{dx} = 0 \Rightarrow 1 - \cos x = 0 \Rightarrow \cos x = 1$

$\Rightarrow x = 0, \pm 2\pi, \pm 4\pi, \pm 6\pi, ...$

but $-10 \leq x \leq 10 \Rightarrow x = 0, \pm 2\pi$

when $x = 0$, $y^2 = 0 + 0 \Rightarrow y = 0$

when $x = 2\pi$, $y^2 = 2\pi + \sin 2\pi \Rightarrow y = \pm\sqrt{2\pi}$

both of these satisfy $-3 \leq y \leq 3$

when $x = -2\pi$, $y^2 = -2\pi + \sin(-2\pi) \Rightarrow y^2 = -2\pi$

which is not possible for real y

Now, the slope is undefined at (0, 0), hence it is rejected

Thus the coordinates of the required points are

$(2\pi, \sqrt{2\pi})$ and $(2\pi, -\sqrt{2\pi})$. ■

Sea Shell Spirals

The chambered nautilus is a sea creature that belongs in the same class as the octopus. Unlike the octopus, it has a hard shell that's divided into chambers. As the nautilus matures and grows, it periodically seals off the shell behind it and creates a new, larger living chamber. The shells of adults may have as many as 30 such chambers.

This growth process yields an elegant spiral structure, visible when the shell is sliced to reveal the individual chambers. Many accounts describe this pattern as a logarithmic (or equiangular) spiral and link it to a number known as the golden ratio.

A logarithmic spiral follows the rule that, for a given rotation angle (such as one revolution), the distance from the pole (spiral origin) is multiplied by a fixed amount.

When this fixed amount is the golden ratio, $1+\sqrt{5}/2$, or 1.6180339887. . . , you get a particular type of logarithmic spiral. Such a logarithmic spiral can be inscribed in a rectangle whose sides have lengths defined by the golden ratio.

Does the spiral of a chambered nautilus shell actually fit such a model?

In 1999, it was found that the spirals of these shells could be inscribed within rectangles with sides in the ratio of about 1.33$#151; not 1.618 . . . , as they would be if a spiral based on the golden ratio matched the shell shape.

Roughly speaking, the spiral of the chambered nautilus triples in radius with each full turn whereas a golden-ratio spiral grows by a factor of about 6.85 per full turn. It was observed that shell spirals are logarithmic spirals, many people automatically assume that, because the golden ratio can be used to draw a logarithmic spiral, all shell spirals are related to the golden ratio, when, in fact, they are not.

Nonetheless, many accounts still insist that a cross section of nautilus shell shows a growth pattern of chambers governed by the golden ratio.

"One of the amazing things about such misconceptions is that it is so widespread, even by mathematicians who should know better". "It is a prime example of why geometry needs to be taught more widely and not only geometry, but the visual appreciation of shape and proportion."

And it's always useful to check things out in the real world.

OO

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Deepak Merugu	Deeptanshu Shukla	Mohd. H Ansari	Chaitanya G. Gokhale	Girish K Sahani	Kartik	Santosh Arya	Nitin Agarwal	Abhishek Ghosh

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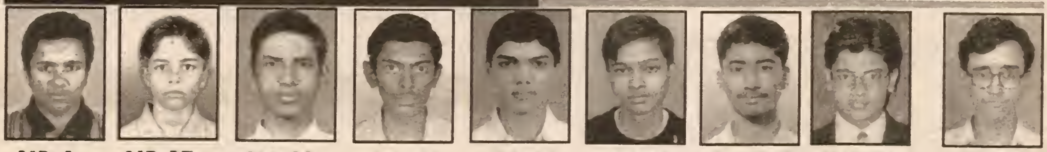

AIR-4	AIR-17	AIR-22	AIR-28	AIR-34	AIR-62	AIR-75	AIR-84	AIR-96
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Challenging PROBLEMS

By : B.L. Sharma, Jaipur

1. If $x^5 - x^3 + x = a$, prove that $x^6 \geq 2a - 1$.

2. The equation $x^2 + 19x + 92 - m^2 = 0$, where m is an integer, has integer roots. Find them.

3. The sum of certain number of consecutive positive integers is 1000. Find these integers.

4. Prove that

$$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$$

and find the seven roots of the equation

$$64x^7 - 112x^5 + 56x^3 - 7x - 1 = 0.$$

5. Find an expression for $\sum_{r=1}^{n} r^4$ as a polynomial of degree five in n.

6. Prove that $\cos^8\theta = \frac{1}{128}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]$

Evaluate $\int_{\pi/4}^{\pi/2} (\cos^8\theta + \sin^8\theta)d\theta$.

7. Prove that $(x + y + z)(x + y\omega + \omega^2 z)(x + \omega^2 y + \omega z) = x^3 + y^3 + z^3 - 3xyz$,

where ω is a complex root of unity. Hence or otherwise, solve the following problems.

(i) Prove that the product

$$(x^3 + y^3 + z^3 - 3xyz)(a^3 + b^3 + c^3 - 3abc)$$

is expressible in the form $A^3 + B^3 + C^3 - 3ABC$ where $A = ax + by + cz$, $B = ay + bz + cx$, $C = az + bx + cy$.

(ii) Solve the equation: $x^3 - 9x + 12 = 0$.

8. **(a)** If $(1 + px + x^2)^n = 1 + a_1 x + a_2 x^2 + ... + a_{2n}x^{2n}$, prove that

$1 + 3a_1 + 5a_2 + ... + (4n + 1)a_{2n} = (2n + 1)(2 + p)^n$.

(b) Find the coefficient of x^r in $(1 + x + x^2 + x^3)^n$.

9. **(a)** Prove that the inequality

$$|\sin a + \sin 2a + ... + \sin na| \leq \frac{1}{|\sin a/2|}.$$

is valid for any number a ($a \neq 2\pi n$, $n \in z$).

(b) Compare the numbers

$$\log_{1/3}\left(\frac{1}{80}\right) \text{ and } \log_{1/2}\left(\frac{1}{15+\sqrt{2}}\right).$$

10. **(a)** Show that

$$\hat{i} \times (\vec{r} \times \hat{i}) + \hat{j} \times (\vec{r} \times \hat{j}) + \hat{k} \times (\vec{r} \times \hat{k}) = 2\vec{r}.$$

where $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$.

(b) (i) Prove that if three variable points z_1, z_2, z_3 are such that $z_3 = \lambda z_2 + (1 - \lambda)z_1$, where λ is a complex constant, then the triangle with vertices z_1, z_2 and z_3 is similar to the triangle with vertices at the points 0, 1 and λ.

(ii) ABC is a triangle. On the sides BC, CA, AB, triangles BCA', CAB', ABC' are described similar to a given triangle DEF. Prove that the centroids of the ΔABC and $\Delta A'B'C'$ are coincident. If ΔDEF is equilateral, then the centroids of the triangles BCA', CAB', ABC' form an equilateral triangle.

SOLUTIONS

1. $a = x(x^4 - x^2 + 1) = \dfrac{x(x^6+1)}{x^2+1}$ ... (i)

From (i) if $x \leq 0 \Rightarrow a \leq 0$

If $x > 0 \Rightarrow a > 0$

$$x^6 + 1 = a\left(\frac{x^2+1}{x}\right) \quad \text{[from (i)]}$$

$$= a\left(x + \frac{1}{x}\right)$$

$x > 0 \Rightarrow x + \frac{1}{x} \geq 2$ (A.M. $\geq$ G.M.)

$\therefore$ $x^6 + 1 \geq 2a$ or, $x^6 \geq 2a - 1$.

2. $x^2 + 19x + 92 - m^2 = 0$

$$\therefore \quad x = \frac{-19 \pm \sqrt{4m^2 - 7}}{2}$$

Since the roots are integers

$\Rightarrow$ $4m^2 - 7 = n^2$ or, $4m^2 - n^2 = 7$

$\Rightarrow$ $(2m - n)(2m + n) = 7$

$\Rightarrow$ $2m - n = \pm 7$, $2m + n = \pm 1$

or, $2m - n = \pm 1$, $2m + n = \pm 7$

$\Rightarrow$ $2m = \pm 4$ or, $m = \pm 2$.

$\therefore \quad x=\dfrac{-19\pm\sqrt{16-7}}{2}=\dfrac{-19\pm 3}{2} \quad \therefore x=-8, \quad x=-11.$

3. Now, $m + (m+1) + ... + (m+k) = 1000$

$$\frac{k+1}{2}[2m+(k)]=1000 \quad \text{or,} \quad (2m+k)(k+1)=2000$$

Also, $(2m + k) - (k + 1) = 2m - 1$ (odd number)

$\Rightarrow \; 2m + k > k + 1$.

If $2m + k$ is even then $k + 1$ is odd. If $k + 1$ is even then $2m + k$ is odd.

Equality has the following solutions.

(i) $2m + k = 2000, k + 1 = 1 \Rightarrow m = 1000, k = 0$

(ii) $2m + k = 400, k + 1 = 5 \Rightarrow m = 148, k = 4$

(iii) $2m + k = 80, k + 1 = 25 \Rightarrow m = 28, k = 24$

(iv) $2m + k = 125, k + 1 = 16 \Rightarrow m = 55, k = 15$.

4. $\cos 7\theta + i\sin 7\theta = (\cos\theta + i\sin\theta)^7$

$= \cos^7\theta + {}^7C_1(\cos\theta)^6 (i\sin\theta) + {}^7C_2(\cos\theta)^5 (i\sin\theta)^2 + {}^7C_3(\cos\theta)^4 (i\sin\theta)^3 + {}^7C_4(\cos\theta)^3 (i\sin\theta)^4 + {}^7C_5(\cos\theta)^2 (i\sin\theta)^5 + {}^7C_6 \cos\theta (i\sin\theta)^6 + {}^7C_7(i\sin\theta)^7$

Separating the real and imaginary parts, we get

$\cos 7\theta = 64\cos^7\theta - 112\cos^5\theta + 56\cos^3\theta - 7\cos\theta$

(ii) Put $\cos 7\theta = 1$, and $x = \cos\theta$, we have

$64x^7 - 112x^5 + 56x^3 - 7x - 1 = 0$

Now, $\cos 7\theta = 1 = \cos 2n\pi + i\sin 2n\pi$

$$\theta=\left(\cos\frac{2n\pi}{7}+i\sin\frac{2n\pi}{7}\right), \quad n = 0, 1, 2, ..., 6.$$

The roots are $x = \cos\theta$, $\theta=\left(\cos\dfrac{2n\pi}{7}+i\sin\dfrac{2n\pi}{7}\right)$

$\theta = 0, 1, 2, ... 6.$

5. Assume $\displaystyle\sum_{r=1}^{n} r^4 = A+Bn+Cn^2+Dn^3+En^4+Fn^5$

Changing n to $n + 1$, we have

$$\sum_{r=1}^{n+1} r^4 = A+B(n+1)+C(n+1)^2+D(n+1)^3 + E(n+1)^4+F(n+1)^5$$

By subtraction, $(n+1)^4 = B+C(2n+1)+D(3n^2+3n+1) + E(4n^3 + 6n^2 + 4n + 1) + F(5n^4 + 10n^3 + 10n^2 + 5n + 1)$

Equating the coefficient of n^4 on both sides, we have

$1 = 5F$ or $F = 1/5$.

Equating the coefficient of n^3 on both sides, we have

$4 = 4E + 10F = 4E + 2$ or $E = 1/2$.

$D = 1/3, C = 0, B = -1/30$

When $n = 1, A = 0$

$$\sum_{r=1}^{n} r^4 = \frac{1}{5}n^5+\frac{1}{2}n^4+\frac{1}{3}n^3-\frac{1}{30}n.$$

6. $z = \cos\theta + i\sin\theta, \quad \dfrac{1}{z}=\cos\theta - i\sin\theta$

$2\cos\theta = z+\dfrac{1}{z}, \; 2\cos 2\theta = z^2+\dfrac{1}{z^2}, \; .....2\cos n\theta = z^n+\dfrac{1}{z^n}$

$$(2\cos\theta)^8=\left(z+\frac{1}{z}\right)^8 = z^8+8z^6+28z^4+56z^2+70 +\frac{56}{z^2}+\frac{28}{z^4}+\frac{8}{z^6}+\frac{1}{z^8}$$

$$=\left(z^8+\frac{1}{z^8}\right)+8\left(z^6+\frac{1}{z^6}\right)+28\left(z^4+\frac{1}{z^4}\right)+56\left(z^2+\frac{1}{z^2}\right)+35$$

$= 2\cos 8\theta + 16\cos 6\theta + 56\cos 4\theta + 112\cos 2\theta + 70$

$$\therefore \; (\cos\theta)^8=\frac{1}{128}[\cos 8\theta+8\cos 6\theta+28\cos 4\theta +56\cos 2\theta+35]$$

$$\int_{\pi/4}^{\pi/2}(\cos\theta)^8 d\theta=\frac{1}{128}\int_{\pi/4}^{\pi/2}(\cos 8\theta+8\cos 6\theta+28\cos 4\theta + 56\cos 2\theta + 35)d\theta$$

$= 35\pi/256.$

7. (i) $x^3 + y^3 + z^3 - 3xyz = (x + y + z)(x^2 + y^2 + z^2 - xy - yz - zx)$

Also $\omega + \omega^2 = -1$ and $\omega^3 = 1$.

$x^2 + y^2 + z^2 - xy - yz - zx = x^2 + y^2 + z^2 + (\omega + \omega^2)yz + xy(\omega + \omega^2) + zx(\omega + \omega^2)$

$= x(x + \omega y + \omega^2 z) + \omega^2 y(x + \omega y + \omega^2 z) + \omega z(x + \omega y + \omega^2 z)$

$= (x + \omega^2 y + \omega z)(x + \omega y + \omega^2 z)$

Hence the result.

$A^3 + B^3 + C^3 - 3ABC = (A + B + C)(A + \omega B + \omega^2 C)$

$(A+\omega^2 B+\omega C) = (ax+by+cz+ay+bz+cx+az+by+cy)$

$(ax + by + cz + a\omega y + b\omega z + c\omega x + a\omega^2 z + b\omega^2 x + \omega^2 cy)$

$(ax + by + cz + a\omega^2 y + b\omega^2 z + c\omega^2 x + a\omega z + b\omega x + \omega cy)$

$= (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)(x + y + z)$

$(x + y\omega + z\omega^2)(x + y\omega^2 + z\omega)$

Hence the result.

(ii) $x^3 - 9x + 12 = 0$

$(x)^3 + (3^{1/3})^3 + (3^{2/3})^3 - 9x = 0$

$(x + 3^{1/3} + 3^{2/3})(x + \omega x^{1/3} + \omega^2 3^{2/3})$

$(x + \omega^2 3^{1/3} + \omega 3^{2/3}) = 0$

$\therefore \; x = -(3^{1/3} + 3^{2/3}), -(\omega 3^{1/3} + \omega^2 3^{2/3}),$

$-(\omega^2 3^{1/3} + \omega x^{2/3})$

8. **(a)** $(1 + px + x^2)^n = 1 + a_1x + a_2x^2 + ... + a_{2n}x^{2n}$

Replace x by x^2

$\therefore \; (1 + px^2 + x^4)^n = 1 + a_1x^2 + a_2x^4 + ... + a_{2n}x^{4n}$

Multiply by x on both sides,

$x(1 + px^2 + x^4)^n = x + a_1x^3 + a_2x^5 + ... + a_{2n}x^{4n+1}$

Differentiating both sides with respect to x,

$(1 + px^2 + x^4)^n + nx(4x^3 + 2px)^n = 1 + 3a_1x^2 + 5a_2x^4 + ... + a_{2n}(4n + 1)x^{4n}$

Put $x = 1$ and we get the result.

(b) $(1 + x + x^2 + x^3)^n = (1 + x^2)^n (1 + x)^n$

$$= \left[1+\sum_{s=1}^{n} {}^nC_s\, x^s\right]\left[1+\sum_{t=1}^{n} {}^nC_t\, x^{2t}\right] = 1+\sum a_r\, x^r.$$

If $r = 2m$, then $a_r = {}^nC_m + {}^nC_{m-1}\, {}^nC_2 + {}^nC_{m-2}\, {}^nC_4 + \ldots,$ and if $r = 2m + 1$

$$a_r = {}^nC_m + {}^nC_{m-1}\, {}^nC_3 + {}^nC_{m-2}\, {}^nC_5 + \ldots$$

9. **(a)** We can easily show that

$$\sin a + \sin 2a + \ldots + \sin na = \frac{\sin\frac{na}{2}\sin\frac{(n+1)a}{2}}{\sin a/2}$$

$$|\sin a + \sin 2a + \ldots + \sin na| = \frac{\left|\sin\frac{na}{2}\right|\left|\sin\frac{(n+1)a}{2}\right|}{|\sin a/2|}$$

$$= \frac{1}{|\sin a/2|}, \text{ provided } \sin a/2 \neq 0 \;\; i.e. \;\; a \neq 2n\pi.$$

(b) $\log_{1/3}\left(\frac{1}{80}\right) < \log_{1/3}\left(\frac{1}{81}\right) = 4$ and

$$15+\sqrt{2} > 16 \text{ and } \log_{1/2}\left(\frac{1}{15+\sqrt{2}}\right) > \log_{1/2}\left(\frac{1}{16}\right) = 4$$

It follows that

$$\log_{1/3}\left(\frac{1}{80}\right) < \log_{1/2}\left(\frac{1}{15+\sqrt{2}}\right).$$

10. **(a)** Using the relation

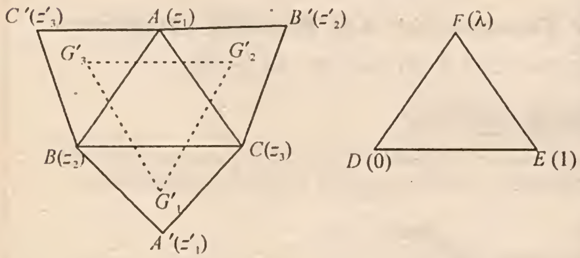
$$\vec{a}\times(\vec{b}\times\vec{c}) = (\vec{a}\cdot\vec{c})\vec{b} - (\vec{a}\cdot\vec{c})\vec{b}$$

$$\hat{i}\times(\vec{r}\times\hat{i}) = (\hat{i}\cdot\hat{i})\vec{r} - (\hat{i}\cdot\vec{r})\hat{i} = \vec{r} - x\hat{i}, \text{ similarly}$$

$$\hat{j}\times(\vec{r}\times\hat{j}) = \vec{r} - y\hat{j} \;;\quad \hat{k}\times(\vec{r}\times\hat{k}) = \vec{r} - z\hat{k}$$

$\therefore$ L.H.S. $= 3\vec{r} - (x\hat{i} + y\hat{j} + z\hat{k}) = 2\vec{r}$.

(b) (i) $z_3 = \lambda z_2 + (1 - \lambda)z_1,\ z_3 - z_1 = \lambda(z_2 - z_1)$



or, $\frac{|z_3 - z_1|}{|z_2 - z_1|} = \frac{|\lambda|}{1}$ $\quad$ *i.e.* $\frac{AC}{AB} = \frac{DF}{DE}$

Also, $z_3 - z_2 = (1 - \lambda)(z_1 - z_2)$

$$\frac{|z_3 - z_2|}{|z_1 - z_2|} = \frac{|1-\lambda|}{1} \quad i.e. \quad \frac{BC}{AB} = \frac{FE}{DE}$$

$\Rightarrow$ Two triangles are similar.

(ii) $z_1' = (1 - \lambda)z_3 + \lambda z_2$
$z_2' = (1 - \lambda)z_1 + \lambda z_3$
$z_3' = (1 - \lambda)z_2 + \lambda z_1$

Centroid of $\Delta A'B'C' = \frac{z_1' + z_2' + z_3'}{3}$

$$= \frac{(1-\lambda)z_3 + \lambda z_2 + (1-\lambda)z_1 + \lambda z_3 + (1-\lambda)z_2 + \lambda z_1}{3}$$

$$= \frac{z_1 + z_2 + z_3}{3}, \text{ hence the result.}$$

(iii) In case ΔDEF is equilateral, then
$\lambda^2 - \lambda + 1 = 0$ (using $z_1^2 + z_2^2 + z_3^2 - z_1z_2 - z_1z_3 - z_2z_3 = 0$)
$\Rightarrow\ \lambda = -\omega$. Then

$$G_1' = \frac{(1+\omega)z_3 - \omega z_2 + z_2 + z_3}{3} = \frac{(2+\omega)z_3 + z_2(1-\omega)}{3}$$

$$G_2' = \frac{(2+\omega)z_1 + z_3(1-\omega)}{3} \;;\quad G_3' = \frac{(2+\omega)z_2 + z_1(1-\omega)}{3}$$

Then $\Delta G_1'G_2'G_3'$ is equilateral if we prove that
$G_3' = (1 + \omega)G_1' - \omega G_2'$

$$= \frac{(1+\omega)[(2+\omega)z_3 + z_2(1-\omega)] - \omega[(2+\omega)z_1 + z_3(1-\omega)]}{3}$$

$$= \frac{z_2(2+\omega) + z_1(1-\omega)}{3}$$

Hence the result. ■

Challenging PROBLEMS

By : B.L. Sharma, Jaipur

1. If ω is a root other than 1 of the equation $\omega^7 = 1$ prove that other roots are $\omega^2, \omega^3, \omega^4, \omega^5, \omega^6$.
If $\alpha = \omega + \omega^6$, $\beta = \omega^2 + \omega^5$, $\gamma = \omega^3 + \omega^4$, prove that the equation with roots α, β, γ is $z^3 + z^2 - 2z - 1 = 0$.
Hence or otherwise, find the values of

(i) $\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}$

(ii) $\cos\frac{2\pi}{7}\cos\frac{4\pi}{7}\cos\frac{6\pi}{7}$

2. By considering the product $(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta)$, find the necessary and sufficient condition that two roots of the equation $x^3 + px^2 + qx + r = 0$ $(r \neq 0)$ should be equal in magnitude but opposite in sign.

3. Solve the simultaneous equations
$$x + y + z = 2;\ x^2 + y^2 + z^2 = 30;\ x^3 + y^3 + z^3 = 16.$$

4. $ABCD$ is a plane quadrilateral whose sides CD, BA intersect at O. If P, Q are the mid-points of the diagonals AC, BD, prove that the area of ΔOPQ is one-quarter of the area of the quadrilateral $ABCD$.

5. ABC is a triangle and points L, M, N are taken on BC, CA, AB respectively such that
$$\frac{BL}{LC} = \lambda,\ \frac{CM}{MA} = \mu,\ \frac{AN}{NB} = \nu$$
Prove that $\Delta LMN = \dfrac{(1+\lambda\mu\nu)\Delta ABC}{(1+\lambda)(1+\mu)(1+\nu)}$ and deduce that if LMN is a straight line, $\lambda\mu\nu = -1$.

6. **(a)** Using vector product, solve the following three linear simultaneous equations.
$$3x + 5y + 8z = -1$$
$$2x + 3y + 4z = 3$$
$$4x + 2y + 5z = -2$$
(b) Using the method used in 6(a) find a necessary and sufficient condition for the existence of a non-trivial solution to the set of three homogeneous linear equations.
$$a_1x + b_1y + c_1z = 0,$$
$$a_2x + b_2y + c_2z = 0,$$
$$a_3x + b_3y + c_3z = 0.$$

7. If a and b are positive integers, prove that the probability that $\frac{1}{2}(a^2 + b^2)$ is a positive integer is 9/25.

8. If p is a prime and r is any integer less than $p - 1$, prove that the sum of the products of the numbers $1, 2, 3, \ldots, p - 1$ taken r together is divisible by p.

9. **(a)** Show that if m is prime to n, the equations $x^m = 1$ and $x^n = 1$, have no root common except 1.
(b) If $n = pqr$, where p, q, r are primes, show that the roots of $x^n = 1$ are the n terms of the product
$$(1 + \alpha + \alpha^2 + \ldots + \alpha^{p-1})(1 + \beta + \beta^2 + \ldots + \beta^{q-1})(1 + \gamma + \gamma^2 + \ldots + \gamma^{r-1})$$
where α is a root of $x^p = 1$, β of $x^q = 1$, γ of $x^r = 1$.

10. **(a)** Prove that the derivative of
$$f(x) = e^x(x^2 - 6x + 12) - (x^2 + 6x + 12)$$
is never negative for any real value of x.
(b) $a_0, a_1, a_2, \ldots, a_{2n}$ are given constants and
$$P_r(x) = a_0x^r + ra_1x^{r-1} + \frac{1}{2}r(r-1)a_2x^{r-2} + \ldots + a_r$$
where $0 \le r \le 2n$. Prove that $P'_r(x) = r\,P_{r-1}(x)$ and that
$$\sum_{r=0}^{2n} (-1)^r\,{}^{2n}C_r\,P_r(x)\,P_{2n-r}(x) \text{ is a constant.}$$

SOLUTIONS

1. $x^7 = 1,\ x = \cos\frac{2n\pi}{7} + i\sin\frac{2n\pi}{7},\ n = 0, 1, 2 \ldots, 6$

$$\omega = \cos\frac{2\pi}{7} + i\sin\frac{2\pi}{7},\ \omega^2 = \cos\frac{4\pi}{7} + i\sin\frac{4\pi}{7}$$
$$\omega^3 = \cos\frac{6\pi}{7} + i\sin\frac{6\pi}{7},\ \omega^4 = \cos\frac{6\pi}{7} - i\sin\frac{6\pi}{7}$$
$$\omega^5 = \cos\frac{4\pi}{7} - i\sin\frac{4\pi}{7},\ \omega^6 = \cos\frac{2\pi}{7} - i\sin\frac{2\pi}{7}$$

$\alpha = 2\cos\frac{2\pi}{7},\ \beta = 2\cos\frac{4\pi}{7},\ \gamma = 2\cos\frac{6\pi}{7}$. Note $\omega^7 = 1$,

$1 + \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = 0$

$\alpha + \beta + \gamma = \omega + \omega^2 + \omega^3 + \omega^4 + \omega^5 + \omega^6 = -1$

$\alpha\beta + \beta\gamma + \alpha\gamma = (\omega + \omega^6)(\omega^2 + \omega^5) + (\omega + \omega^6)(\omega^3 + \omega^4) + (\omega^2 + \omega^5)(\omega^3 + \omega^4) = -2$

$\alpha\beta\gamma = (\omega + \omega^6)(\omega^2 + \omega^5)(\omega^3 + \omega^4) = 1$. Thus, α, β, γ are the roots of the equation

$z^3 + z^2 - 2z - 1 = 0$

Sum of the roots $= \alpha + \beta + \gamma$

$$= 2\left(\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7}\right) = -1$$

$$\therefore \quad \cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

$$\cos\frac{2\pi}{7}\ \cos\frac{4\pi}{7}\ \cos\frac{6\pi}{7} = \frac{1}{8}.$$

---

2. $\alpha + \beta + \gamma = -p$

$\beta + \gamma = -p - \alpha$ or, $y = -p - x$

$\therefore\ x = -p - y.$

$(-p - y)^3 + p(-p - y)^2 + q(-p - y) + r = 0$

$-(p^3 + 3p^2y + 3py^2 + y^3) + p(p^2 + y^2 + 2py) - qp - qy + r = 0$

or, $y^3 + y^2(2p + 3p^2) + y(q - 2p^2) + r - pq = 0$

has roots $(\beta + \gamma)$, $(\gamma + \alpha)$ and $(\alpha + \beta)$. If the original equation has two roots equal but opposite in sign then

$(\beta + \gamma)(\gamma + \alpha)(\alpha + \beta) = 0.$

$\therefore\ r = pq.$

Sufficient : Let $r = pq$.

$\Rightarrow\ \alpha\beta\gamma = (\alpha + \beta + \gamma)(\alpha\beta + \beta\gamma + \gamma\alpha)$

$\alpha\beta\gamma = \alpha^2(\beta + \gamma) + \beta^2(\alpha + \gamma) + \gamma^2(\alpha + \beta) + 3\alpha\beta\gamma$

or, $\alpha^2(\beta + \gamma) + \beta^2(\alpha + \gamma) + \gamma^2(\alpha + \beta) + 2\alpha\beta\gamma = 0$

$\therefore\ (\alpha + \beta)(\beta + \gamma)(\alpha + \gamma) = 0$. Hence the result.

---

3. Let x, y, z be the roots of the equation

$$t^3 - at^2 + bt - c = 0$$

Thus, $a = \sum x = 2$

$$b = \sum xy = \frac{1}{2}\left\{\left(\sum x\right)^2 - \sum (x)^2\right\} = -13$$

To find c, we use the identity

$x^3 + y^3 + z^3 - 3abc = (x + y + z)(x^2 + y^2 + z^2 - bc - ca - ab)$

$\therefore\ c = 10.$

Thus x, y, z are the roots of the equation

$t^3 - 2t^2 - 13t - 10 = 0.$

By observation, $t = -1$ is a root.

$\therefore\ (t + 1)(t^2 - 3t - 10) = (t + 1)(t + 2)(t - 5) = 0.$

$\therefore\ x = -1, y = -2, z = 5.$

---

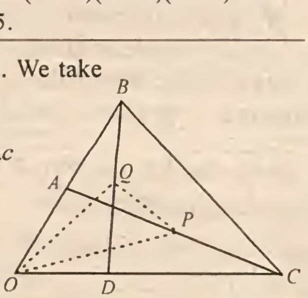
4. Take O as the origin. We take

$\overrightarrow{OA} = a,\ \overrightarrow{OC} = c,$

$\overrightarrow{OB} = b = \lambda a,\ \overrightarrow{OD} = \mu c$

$\overrightarrow{OP} = p = \frac{1}{2}(a + b)$

$\overrightarrow{OQ} = q = \frac{1}{2}(\lambda a + \mu c)$



The vector area of $OPQ = \frac{1}{2}(p \times q)$

$$= \frac{1}{8}(a + c) \times (\lambda a + \mu c) = \frac{1}{8}(\mu - \lambda)(a \times c)$$

But since $ABCD$ is a plane quadrilateral, its area is the magnitude of the sum of vector areas of ΔABC and ΔBDC. Thus,

$$ABCD = \frac{1}{2}[d \times b + b \times a + a \times d] + \frac{1}{2}[d \times c + c \times b + b \times d]$$

$$= \frac{1}{2}[b \times a + a \times d + d \times c + c \times b]$$

$$= \frac{1}{2}[a \times (d - b) - c \times (d - b)]$$

$$= \frac{1}{2}[(a - c) \times (d - b)] = \frac{1}{2}(a - c) \times (\mu c - \lambda a)$$

$$= \frac{1}{2}(\mu - \lambda)(a \times c).$$ Hence the result.

---

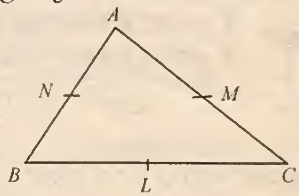
5. Take O as the origin.

$\overrightarrow{OA} = a,\ \overrightarrow{OB} = b,\ \overrightarrow{OC} = c$

Now, $\overrightarrow{OL} = \dfrac{b + \lambda a}{1 + \lambda},$

$\overrightarrow{OM} = \dfrac{c + \mu a}{1 + \mu},$

$\overrightarrow{ON} = \dfrac{a + \nu b}{1 + \nu}$



Vector area of ΔLMN

$$= \frac{1}{2}\left[\frac{(b + \lambda c) \times (c + \mu a)}{(1 + \lambda)(1 + \mu)} + \frac{(c + \mu a) \times (a + \nu b)}{(1 + \mu)(1 + \nu)} + \frac{(a + \nu b) \times (b + \lambda c)}{(1 + \nu)(1 + \lambda)}\right]$$

$$= \frac{1}{2}[(b \times c) + (c \times a) + (a \times b)]\frac{1 + \lambda\mu\nu}{(1 + \lambda)(1 + \mu)(1 + \nu)}.$$

If LML is a straight line then area of $\Delta LMN = 0$.

$\therefore\ 1 + \lambda\mu\nu = 0.$

---

6. **(a)** The three equations represent three planes P_1, P_2, P_3; their solution consists in finding the point Q common to each plane (assuming such a point exists). From equations P_1 and P_2, we obtain $3P_1 + P_2$, the equation of a plane through the line of intersection of P_1, P_2 containing the origin. Thus,

$11x + 18y + 28z = 0 \quad : \quad P_4$

is a plane containing O and Q. Similarly, $(2P_1 - P_3)$

$2x + 8y + 11z = 0 \quad : \quad P_5$

is a plane containing O and Q.

Thus, $\overrightarrow{OQ}$ is perpendicular to both $11\hat{i} + 18\hat{j} + 28\hat{k}$ and $2\hat{i} + 8\hat{j} + 11\hat{k}$.

$\therefore \quad \overrightarrow{OQ} = \lambda(11\hat{i}+18\hat{j}+28\hat{k})\times(2\hat{i}+8\hat{j}+11\hat{k})$

$\qquad = -13\lambda(2\hat{i}+5\hat{j}-4\hat{k})$

and Q has co-ordinates $(-26\lambda, -65\lambda, 52\lambda)$ where λ is some scalar. Write,

$-13\lambda = \mu$, Q has co-ordinates $(2\mu, 5\mu, -4\mu)$.

Substituting in P_1, we find $\mu = 1$. Hence the solution

$$x = 2,\ y = 5,\ z = -4.$$

(b) As in 6(a),

P_4 is $(a_1 - a_2)x + (b_1 - b_2)y + (c_1 - c_2)z = 0$

P_5 is $(a_1 - a_3)x + (b_1 - b_3)y + (c_1 - c_3)z = 0$

$\overrightarrow{OQ}$ is perpendicular to both

$$(a_1-a_2)\hat{i}+(b_1-b_2)\hat{j}+(c_1-c_2)\hat{k}$$

and $(a_1-a_3)\hat{i}+(b_1-b_3)\hat{j}+(c_1-c_3)\hat{k}$

$$\overrightarrow{OQ} = \lambda[\{(a_1-a_2)\hat{i}+(b_1-b_2)\hat{j}+(c_1-c_2)\hat{k}\} \times \{(a_1-a_3)\hat{i}+(b_1-b_3)\hat{j}+(c_1-c_3)\hat{k}\}]$$

$$= \lambda[\hat{i}\left((b_1-b_2)(c_1-c_3)-(c_1-c_2)(b_1-b_3)\right) + \hat{j}\left((c_1-c_2)(a_1-a_3)-(a_1-a_2)(c_1-c_3)\right) + \hat{k}\left((a_1-a_2)(b_1-b_2)-(b_1-b_2)(a_1-a_3)\right)]$$

$$= \lambda[\hat{i}\,\{(b_2c_3-c_2b_3)+(b_1c_2-c_1b_2)+(c_1b_3-b_1c_3)\} + \hat{j}\,\{(c_2a_3-a_2c_3)+(c_1a_2-a_1c_2)+(a_1c_3-c_1a_3)\} + \hat{k}\,\{(a_2b_3-b_2a_3)+(a_1b_2-b_1a_2)+(b_1a_3-a_1b_3)\}]$$

Q has co-ordinates

$(\lambda(b_2c_3 - c_2b_3) + \lambda(b_1c_2 - c_1b_2) + \lambda(c_1b_3 - b_1c_3),$
$\lambda(c_2a_3 - a_2c_3) + \lambda(c_1a_2 - a_1c_2) + \lambda(a_1c_3 - c_1a_3),$
$\lambda(a_2b_3 - b_2a_3) + \lambda(a_1b_2 - b_1a_2) + \lambda(b_1a_3 - a_1b_3))$

Q lies on P_1

$\therefore \quad a_1(b_2c_3-c_2b_3)\lambda + a_1(b_1c_2-c_1b_2)\lambda + \lambda a_1(c_1b_3-b_1c_3) + b_1(c_2a_3-a_2c_3)\lambda + b_1(c_1a_2-a_1c_2)\lambda + \lambda b_1(a_1c_3-c_1a_3) + a_1(a_2b_3-b_2a_3)\lambda + c_1(a_1b_2-b_1a_2)\lambda + \lambda c_1(b_1a_3-a_1b_3) = 0 \quad \text{... (i)}$

We denote $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = \Delta$

$A_1, B_1, C_1, A_2, B_2, C_2, A_3, B_3, C_3$ are co-factors of a_y, b_y, c_y, $y = 1, 2, 3$. Thus,

$\lambda\Delta + \lambda(a_1A_3 + b_1B_3 + c_1C_3) + \lambda(a_1A_2 + b_1B_2 + c_1C_2) = 0$

or, $\lambda\Delta = 0$, λ can not be zero. $\therefore \ \Delta = 0$.

It is left as an exercise for the students to prove that it is sufficient.

---

7. We take 1, 2, 3, ... 5 N. Now we divide this set in the following manner.

$\{5, 10, ..., 5N\} = A(0)$ $\quad \{1, 6, ... 5N - 4\} = A(1)$

$\{2, 7, ..., 5N - 3\} = A(2)$ $\quad \{3, 8, ... 5N - 2\} = A(3)$

$\{4, 9, ... 5N - 1\} = A(4)$

Sample space $= {}^{5n}C_2$

(i) If $a, b \in A(0)$, then $\dfrac{a^2+b^2}{5}$ is a +ve integer.

Number of favourable cases $= {}^nC_2$

(ii) If $a \in A(1), b \in A(2)$, then $\dfrac{a^2+b^2}{5}$ is a +ve integer.

$\therefore$ Number of favourable cases $= n^2$.

(iii) If $a \in A(1)$, $b \in A(3)$, then $\dfrac{a^2+b^2}{5}$ is an integer.

$\therefore$ Number of favourable cases $= n^2$.

(iv) If $a \in A(2)$, $b \in A(4)$, then $\dfrac{a^2+b^2}{5}$ is an integer.

$\therefore$ Number of favourable cases $= n^2$.

(v) If $a \in A(3)$, $b \in A(4)$, then $\dfrac{a^2+b^2}{5}$ is an integer.

$\therefore$ Number of favourable cases $= n^2$

Total favourable cases $= {}^nC_2 + 4n^2$

$$\text{Probability} = \lim_{n\to\infty} P_n = \lim_{n\to\infty} \frac{{}^nC_2 + 4n^2}{\frac{5n(5n-1)}{2}} = \frac{9}{25}.$$

---

8. Let $f(x) = (x + 1)(x + 2) \ldots (x + p - 1)$; we denote α_r the sum of the products of $1, 2, 3, ..., p - 1$ taken r together.

$f(x) = x^{p-1} + \alpha_1 x^{p-2} + \alpha_2 x^{p-3} + \ldots + \alpha_{p-1} \quad \text{... (i)}$

Now we have, $(x + p)f(x) = (x + 1)f(x + 1) \quad \text{... (ii)}$

i.e. $(x + p)(x^{p-1} + \alpha_1 x^{p-2} + \alpha_2 x^{p-3} + \ldots + \alpha_{p-1})$

$= (x + 1)^p + \alpha_1(x + 1)^{p-1} + \alpha_2(x + 1)^{p-2} + \ldots + \alpha_{p-1}(x + 1) \quad \text{... (iii)}$

Equating the coefficient of $x^{p-2}, x^{p-3}, ..., x$, we have

$p\alpha_1 = {}^pC_2 + {}^{p-1}C_1\,\alpha_1$

$p\alpha_2 = {}^pC_3 + {}^{p-1}C_2\,\alpha_1 + {}^{p-1}C_1\alpha_2$

....

....

$p\alpha_{p-2} = {}^pC_{p-1} + {}^{p-1}C_{p-2}\,\alpha_1 + \ldots + {}^2C_1\,\alpha_{p-2}$

Now ${}^pC_r$ is divisible by p, $r < p$; and ${}^{p-1}C_r$, ${}^{p-2}C_r$ etc. are all prime to p.

If follows in succession that

$\alpha_1, \alpha_2, \alpha_3, ..., \alpha_{p-2}$ are all divisible by p.

---

9. **(a)** Let α be a common root, then

$\alpha^{pm} = 1$ and $\alpha^{qn} = 1$, where p, q are any positive integers. Therefore $\alpha^{pm-qn} = 1$, since m is prime to n, p and q can be found so that $pm - nq = \pm 1 \quad \Rightarrow \quad \alpha = 1$.

(b) Take the case of three factors p, q, r; similar reason apply for all cases.

Any term of the product, for instance $\alpha^a \beta^b \gamma^c$ is a root.

$(\alpha^a)^n = (\alpha^p)^{aqr} = 1,$

and similarly, $(\beta^p)^n = 1$, and $(\gamma^c)^n = 1$.
If any two terms are equal, for instance, if
$\alpha^a \beta^b \gamma^c = \alpha^{a'} \beta^{b'} \gamma^{c'}$, then
$\beta^{b-b'} \gamma^{c-c'} = \alpha^{a'-a}$ which is impossible for $\beta^{b-b'} \gamma^{c-c'}$ is a root of $x^{qr} - 1 = 0$ and $\alpha^{a-a'}$ is a root of $x^{p-1} = 0$, since p is prime to qr, these equations have no common root except 1.

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10. (a) Let $F'(x) = f(x)$. Then
$F(x) = e^x(x^2 - 4x + 6) - (2x + 6)$ and $f(0) = 0$
$f'(x) = e^x(x^2 - 2x + 2) - 2$ and $f'(0) = 0$
$f''(x) = e^x x^2$ and is positive when $x \neq 0$.
When $x > 0$,
$f'(0) = 0, f''(x) > 0$ together gives $f'(x) > 0$;
$f(0) = 0, f'(x) > 0$ together gives $f(x) > 0$;
$F(0) = 0$, $F'(x) = f(x) > 0$ together gives $F(x) > 0$.
When $x < 0$
$f'(0) = 0, f''(x) > 0$ together gives $f'(x) < 0$;
$f(0) = 0, f'(x) < 0$ together gives $f(x) > 0$;
$F(0) = 0$, $F'(x) = f(x) > 0$ together gives $F(x) < 0$.
Thus $f(x)$ is never negative when x is real and $F(x)$ has the sign of x when $x \neq 0$.

(b) $P_r'(x) = ra_0 x^{r-1} + ra_1(r-1)x^{r-2}$

$$+ \frac{1}{2} r(r-1)(r-2)a_2 x^{r-3} + \dots + {}^rC_{r-1}\, a_{r-1}$$

$$= r\left[a_0 x^{r-1} + a_1(r-1)x^{r-2} + \frac{1}{2}a_2(r-1)(r-2)x^{r-3} + \dots + a_{r-1}\right] = r\, P_{r-1}(x) \qquad \dots \text{(i)}$$

Note $P_{-1} \equiv 0$
Differentiating,

$$F'(x) = \sum_{r=0}^{2n} (-1)^r\, {}^{2n}C_r\, r\, P_{r-1}(x)\, P_{2n-r}(x)$$

$$+ \sum_{r=0}^{2n} (-1)^r\, {}^{2n}C_r\, P_r(x)(2n-r)P_{2n-r-1}(x) \qquad \dots \text{(ii)}$$

$$= \sum_{r=0}^{2n} (-!)^r \left\{\frac{2n!}{(2n-r)!\,(r-1)!} P_{r-1}P_{2n-2} + \frac{2n!}{(2n-r-1)!\,r!} P_r\, P_{2n-r-1}\right\}$$

$$= \sum_{r=0}^{2n} (-1)^r \{a_r + b_r\} \text{ where}$$

$$a_r = \frac{2n!}{(2n-r)!\,(r-1)!} P_{r-1}\, P_{2n-r} \qquad \dots \text{(A)}$$

and $$b_r = \frac{2n!}{(2n-r-1)!\,r!} P_r\, P_{2n-r-1} \qquad \dots \text{(B)}$$

$$\therefore \quad b_r = a_{r+1} \qquad \dots \text{(iii)}$$

We put $r = 0, 1, 2, \dots, 2n$ and using the relation (ii), we get $F'(x) = 0$. Hence the result. ■

TIPS *for solving* MCQ'S

- The multiple choice question, consists of two parts:
  1. The stem - the statement or question.
  2. The choices - also known as the distracters. There are usually 3 to 5 options, that will complete the stem statement or question.

You are to select the correct choice, the option that completes the thought expressed in the stem. There is a 20% chance that you will guess the correct choice if there are 5 choices listed. Although multiple choice questions are most often used to test your memory of details, facts, and relationships, they are also used to test your comprehension and your ability to solve problems. Reasoning ability is a very important skill for doing well on multiple choice tests.

Read the stem as if it were an independent, free-standing statement. Anticipate the phrase that would complete the thought expressed, then compare each answer choice to your anticipated answer. It is important to read each choice, even if the first choice matches the answer you expected, because there may be a better answer listed.

- Another evaluation technique is to read the stem together with each answer choice as if it were a true-false statement. If the answer makes the statement a false one, cross it out. Check all the choices that complete the stem as a true statement. Try to suspend judgment about the choices you think are true until you have read all the choices.
- Beware of words like not, but, except. Mark these words because they specify the direction and limits of the answer.
- Also watch out for words like always, never, and only. These must be interpreted as meaning all of the time, not just 99% of the time. These choices are frequently incorrect because there are few statements that have no exceptions (but there are a few).
- If there are two or more options that could be the correct answer, compare them to each other to determine the differences between them, and then relate these differences with the stem to deduce which of the choices is the better one. (Hint: select the option that gives the most complete information.)
- If there is an encompassing answer choice, for example "all of the above," and you are able to determine that there are at least two correct choices, select the encompassing choice.
- Use hints from questions you know to answer questions you do not.
- If you do not find an answer, try to relate each answer to the stem to evaluate which one logically completes the thought.
- Make educated guesses—eliminate options any way you can.

INVERSE TRIGONOMETRIC FUNCTIONS

(Concept & Analysis)

By : **Prof. S.S. Dahiya**, Director Academics, Doon Ace Education Pvt. Ltd.

$$\sin\left(\frac{\pi}{6}\right)=\frac{1}{2}\Leftrightarrow \sin^{-1}\left(\frac{1}{2}\right)=\frac{\pi}{6}$$

Inverse Trigonometric function	Domain	Range
$y=f(x)=\sin^{-1}(x)$	$-1\le x\le 1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}$
$y=g(x)=\cos^{-1}(x)$	$-1\le x\le 1$	$0\le y\le\pi$
$y=\phi(x)=\tan^{-1}(x)$	$x\in$ Real	$-\frac{\pi}{2}< y<\frac{\pi}{2}$
$y=\Psi(x)=\cot^{-1}(x)$	$x\in$ Real	$0<y<\pi$
$y=G(x)=\sec^{-1}(x)$	$x\le -1$ or $x\ge 1$	$0\le y\le\pi,\ y\ne\frac{\pi}{2}$
$y=F(x)=\text{cosec}^{-1}(x)$	$x\le -1$ or $x\ge 1$	$-\frac{\pi}{2}\le y\le\frac{\pi}{2}, y\ne 0$

1. $\text{cosec}^{-1}(k)=\sin^{-1}\left(\frac{1}{k}\right)$
2. $\sec^{-1}(t)=\cos^{-1}\left(\frac{1}{t}\right)$
3. $\cot^{-1}(q)=\begin{cases}\tan^{-1}\left(\frac{1}{q}\right), \text{ if } q>0\\ \tan^{-1}\left(\frac{1}{q}\right)+\pi \text{ if } q<0\end{cases}$
4. $\sin^{-1}(x)+\cos^{-1}(x)=\frac{\pi}{2},\ -1\le x\le 1$
5. $\tan^{-1}(k)+\cot^{-1}(k)=\frac{\pi}{2}$
6. $\sec^{-1}(t)+\text{cosec}^{-1}(t)=\frac{\pi}{2}, t\le -1 \text{ or } t\ge +1$
7. $\sin^{-1}(-x)=-\sin^{-1}(x)$
8. $\tan^{-1}(-k)=-\tan^{-1}(k)$
9. $\cos^{-1}(-\lambda)=\pi-\cos^{-1}(\lambda)$

10. Determinor for $\sin^{-1}(x)$ is π (explain the rearson) If odd number of determinors are used than sign change gives answer, if even number of determinors are used then there is no sign change.

Example : $\sin^{-1}(\sin 10)=\theta$ then find θ

Because $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$ whereas 10 radians $\approx$ 570 degrees 10 radians = 3π + 30° approx or $10-3\pi$ lies within the required range, three determinors are used, hence $\theta=-(10-3\pi)=3\pi-10$.

11. Determinor for $\tan^{-1}(k)$ is π and there is no criteria of sign change (explain the reason)
$\tan^{-1}(\tan 10)=\alpha =$, find α

Because $-\frac{\pi}{2}\le\alpha\le\frac{\pi}{2}$, $10-3\pi$ is approx. 30°
$\therefore\ \alpha=10-3\pi$

12. Determinor for $\cos^{-1}(t)$ is 2π (explain the reason)
In first step adjust complete rounds, if angle left $\in[0,\pi]$ then angle left is answer, if angle left $\in(\pi, 2\pi)$ then 2π – angle left gives answer $\cos^{-1}(\cos 10)=\theta$, find θ.
10 radians = 2π + 210° approx.
$10-2\pi$ = 210° approx. $\in(\pi, 2\pi)$
$\therefore\ \theta=2\pi-(10-2\pi)=4\pi-10$,
$\cos^{-1}(\cos 15)=\alpha$, find α
15 radians = $2\pi+2\pi$ + 135° approx.
$15-4\pi$ = 135° approx $\in[0,\pi]$ hence $\theta=15-4\pi$

13. Find x if $\frac{2\pi}{3}=\tan^{-1}(x)$, because $\frac{-\pi}{2}<\tan^{-1}(x)<\frac{\pi}{2}$ there $\tan^{-1}(x)=\frac{2\pi}{3}$ is not possible, hence no value of x.

14. $\theta=\tan^{-1}(x)+\tan^{-1}(y)+\tan^{-1}(z)$ and $\tan(\theta)=0$ then find all possible values of θ

Because $\frac{-3\pi}{2}<\tan^{-1}(x)+\tan^{-1}(y)+\tan^{-1}(z)<\frac{3\pi}{2}$
$\therefore\ \theta\in\{-\pi, 0, \pi\}$.

5. r, x, y, z are real numbers such that $r^2 = x^2 + y^2 + z^2$ and $\tan^{-1}\left(\frac{rx}{yz}\right) + \tan^{-1}\left(\frac{ry}{zx}\right) + \tan^{1}\left(\frac{rz}{xy}\right) = \theta$ then find all possible values of θ.

$$\tan^{-1}\left(\frac{rx}{yz}\right) + \tan^{-1}\left(\frac{ry}{zx}\right) = \tan^{-1}\left(\frac{\frac{rx}{yz} + \frac{ry}{zx}}{1 - \frac{r^2}{z^2}}\right) = \tan^{-1}\left(\frac{-rz}{xy}\right)$$

$$\therefore \tan\theta = \tan\left(\tan^{-1}\left(\frac{rz}{xy}\right) + \tan^{-1}\left(\frac{-rz}{xy}\right)\right) = 0$$

$\therefore\ \theta \in \{-\pi, \pi\}$, here $\theta \neq 0$ because r, x, y, z cannot be zero and either all of $\frac{rx}{yz}, \frac{ry}{zx}, \frac{rz}{xy}$ are positive or all are negative.

6. $\sin^{-1}\left(x\sqrt{1-y^2} + y\sqrt{1-x^2}\right) =$

$$\begin{cases} -\sin^{-1}(x) - \sin^{-1}(y) + \pi, & x > 0, y > 0, x^2 + y^2 > 1 \\ \sin^{-1}(x) + \sin^{-1}(y) & \text{if } x^2 + y^2 \le 1 \text{ or } xy \le 0 \\ -\sin^{-1}(x) - \sin^{-1}(y) - \pi & \text{if } x < 0, y < 0, x^2 + y^2 > 1 \end{cases}$$

7. If $\sin^{-1}(\pm x) + \sin^{-1}(\pm y) = \pm\frac{\pi}{2}$ then $x^2 + y^2 = 1$.

8. If $x \le 0, y \le 0$ or $x \ge 0, y \ge 0$ then $\cos^{-1}\left(xy + \sqrt{1-x^2}\sqrt{1-y^2}\right) =$

$$\begin{cases} \cos^{-1}(x) - \cos^{-1}(y) & \text{when } x \le y \\ \cos^{-1}(y) - \cos^{-1}(x) & \text{when } y \le x \end{cases}$$

9. If $x \le 0, y \le 0$ or $x \ge 0, y \ge 0$ then

$\tan^{-1}(x) + \tan^{-1}(y) =$

$$\begin{cases} \tan^{-1}\left(\frac{x+y}{1-xy}\right) - \pi, & \text{when } x < 0, y < 0, xy > 1 \\ \tan^{-1}\left(\frac{x+y}{1-xy}\right), & \text{when } 0 \le xy < 1 \\ \tan^{-1}\left(\frac{x+y}{1-xy}\right) + \pi, & \text{when } x > 0, y > 0, xy > 1 \end{cases}$$

20. (A): $\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \begin{cases} -2\tan^{-1}(x) - \pi, & \text{when } x \le -1 \\ 2\tan^{-1}(x), & \text{when } -1 \le x \le 1 \\ -2\tan^{-1}(x) + \pi, & \text{when } x \ge 1 \end{cases}$

(B) $\tan^{-1}\left(\frac{2x}{1-x^2}\right) = \begin{cases} 2\tan^{-1}(x) + \pi, & x < -1 \\ 2\tan^{-1}(x), & -1 < x < 1 \\ 2\tan^{-1}(x) - \pi, & x > 1 \end{cases}$

(C) $\cos^{-1}\left(\frac{1-x^2}{1+x^2}\right) = \begin{cases} 2\tan^{-1}(x), & x \ge 0 \\ -2\tan^{-1}(x), & x \le 0 \end{cases}$

(D) For domain $0 \le x < 1$,

$$\sin^{-1}\left(\frac{2x}{1+x^2}\right) = \tan^{-1}\left(\frac{2x}{1-x^2}\right) = \cos^{-1}\left(\frac{1-x^2}{1+x^2}\right)$$
$$= 2\tan^{-1}(x).$$

21.(A) $\sin^{-1}\left(2x\sqrt{1-x^2}\right) = \begin{cases} -2\sin^{-1}(x) - \pi, & -1 \le x \le -1/\sqrt{2} \\ 2\sin^{-1}(x), & -\frac{1}{\sqrt{2}} \le x \le \frac{1}{\sqrt{2}} \\ -2\sin^{-1}(x) + \pi, & \frac{1}{\sqrt{2}} \le x \le 1 \end{cases}$

(B) $\cos^{-1}(2x^2 - 1) = \begin{cases} 2\cos^{-1}(x), & x \ge 0 \\ 2\pi - 2\cos^{-1}(x), & x \le 0 \end{cases}$

22. In inverse trigonometric function, start with angle θ, after simplification if angle decreases i.e. becomes $\frac{1}{2}\theta, \frac{1}{3}\theta$ etc then lies within the domain, if angle increases i.e. becomes 2θ, 3θ etc. then is liable to go out of domain and to bring it within the domain using determinor, different branches of the formulae are formed.

Example : $\cos(4\theta) = 8\cos^4(\theta) - 8\cos^2(\theta) + 1$;

Put $\cos\theta = x$ $\therefore$ $\theta = \cos^{-1}(x), -1 \le x \le 1$

$$\cos^{-1}(8x^4 - 8x^2 + 1) = \begin{cases} 4\cos^{-1}(x), & \frac{1}{\sqrt{2}} \le x \le 1 \\ 2\pi - 4\cos^{-1}(x), & 0 \le x \le \frac{1}{\sqrt{2}} \\ 4\cos^{-1}(x) - 2\pi, & -\frac{1}{\sqrt{2}} \le x \le 0 \\ 4\pi - 4\cos^{-1}(x), & -1 \le x \le -\frac{1}{\sqrt{2}} \end{cases}$$

23. Express $2\tan^{-1}(x)$ in terms of $\tan^{-1}\left(\frac{2x}{1-x^2}\right)$.

Hint use 20(B).

$$2\tan^{-1}(x) = \begin{cases} \tan^{-1}\left(\frac{2x}{1-x^2}\right) - \pi, & x < -1 \\ -\frac{\pi}{2}, & x = -1 \\ \tan^{-1}\left(\frac{2x}{1-x^2}\right), & -1 < x < 1 \\ +\frac{\pi}{2}, & x = 1 \\ \tan^{-1}\left(\frac{2x}{1-x^2}\right) + \pi, & x > 1 \end{cases}$$

24. Convert $\sin^{-1}(x) - \sin^{-1}(y) = \pi/2$ in the form $y = f(x)$

$\sin^{-1}(x)+\sin^{-1}(-y)=\frac{\pi}{2}$ $\therefore\ 0 \le x \le 1,$

$0 \le -y \le 1,\ (x^2)+(-y^2)=1$

Therefore, $0 \le x \le 1,\ -1 \le y \le 0.$ Hence $y = -\sqrt{1-x^2}$.

25. Solve for x the equation

$$\tan^{-1}\left(\frac{2x}{1-x^2}\right)+\cot^{-1}\left(\frac{1-x^2}{2x}\right) = \frac{2\pi}{3}$$

Answer : $x=-\sqrt{3},\ x=-(2-\sqrt{3}),\ x=\frac{1}{\sqrt{3}},\ x=2+\sqrt{3}.$

26. In inverse trigonometric functions wrong questions have been given in various books and other sources, for example prove that

$$\frac{1}{2}\cos^{-1}\left(\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}\right) = \tan^{-1}\left(\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right)$$

The question is wrong because for $\theta=\frac{2\pi}{3},\ \phi=\frac{4\pi}{3}$

Left hand side is positive whereas right hand side is negative correct version of the question is

Prove that $\frac{1}{2}\cos^{-1}\left(\frac{\cos\theta+\cos\phi}{1+\cos\theta\cos\phi}\right)=\tan^{-1}\left|\tan\frac{\theta}{2}\tan\frac{\phi}{2}\right|$

Further $\tan^{-1}\left(\frac{a_1x-y}{a_1y+x}\right)+\tan^{-1}\left(\frac{a_2-a_1}{1+a_2a_1}\right)+\ldots$

$$+\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)+\tan^{-1}\left(\frac{1}{a_n}\right) = \tan^{-1}\left(\frac{x}{y}\right)$$

is wrong, the correct version of the question is

$$\tan^{-1}\left(\frac{a_1x-y}{a_1y+x}\right)+\tan^{-1}\left(\frac{a_2-a_1}{1+a_2a_1}\right)+\ldots.+$$

$$\tan^{-1}\left(\frac{a_n-a_{n-1}}{1+a_na_{n-1}}\right)+\tan^{-1}\left(\frac{1}{a_n}\right) = \tan^{-1}\left(\frac{x}{y}\right)+\lambda\pi$$

where $\lambda \in$ integer and $-n \le \lambda \le n$.

27. (i) Express $\sin^{-1}(3x-4x^3)$ in terms of $3\sin^{-1}(x)$, $-1 \le x \le 1$

(ii) Express $\cos^{-1}(4x^3 - 3x)$ in terms of $3\cos^{-1}(x), -1 \le x \le 1$

(iii) Express $\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ in terms of $3\tan^{-1}(x)$

$x \in R - \left\{\pm\frac{1}{\sqrt{3}}\right\}$

28. (i) $\sqrt{1-x^2}+\sqrt{1-y^2} = a(x-y)$

where $x \in [-1, 1], y \in [-1, 1]$

convert in inverse trigonometric function.

(ii) Express $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ in terms of $\tan^{-1}(x)$

$x \in R - \{0\}$.

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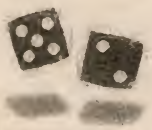


PROBABILITY THEORY
Mathematics of Chance

Probability, as the concept is most commonly understood, is a mathematical expression of the relationship between a particular outcome of an event and the total number of possible outcomes. In its commonest form, like a gambling game, probability becomes the ratio of the chances of winning — expressed in a number of ways: 50-50, a chance of 1 out of 2, 50 per cent chance, a .5 probability, and so on.

Probability is therefore concerned with the analysis of random phenomena. The outcome of a random event cannot be determined before it occurs. The actual outcome may be any one of several possible outcomes, and this is considered to be determined by chance.

Chance and Risk : The concepts of chance, fortune and luck are as old as the first dice games. Archaeologists have found evidence of games of chance in prehistoric excavations, indicating that gaming and gambling had been a major pastime for different races since the dawn of civilization.

For centuries human beings speculated about probabilities in connection with legal questions of evidence and contracts and, at times, insurance schemes. The Babylonians had several forms of maritime insurance. The Romans had annuities, i.e., exchanges of a lump sum in return for regular payments over a long time — the risk being that the person taking out the annuity would not live to collect the lump sum.

Given the penchant of the people of Greek, Egyptian, Chinese and Indian dynasties for gambling, one would expect the mathematics of chance to be one of the earliest to have developed. Surprisingly, it wasn't until the 17th century that an accurate mathematics of probability was developed by French mathematicians Pierre de Fermat and Blaise Pascal.

Mathematics Olympia for IIT-JEE (MAINS)

By : Er. Akhlak Ahmad, ABC Classes, Gorakhpur

1. Use integral calculus to find the sum of the series $\frac{1}{4!}+\frac{4!}{8!}+\frac{8!}{12!}+\ldots\infty$.

2. Find the minimum odd value of a (> 1) such that $\int_{10}^{19}\frac{\sin x}{1+x^a}dx<\frac{1}{9}$.

3. Find the area of the region bounded by the curve $2^{|x|}|y|+2^{|x|-1}\le 1$, with in the square formed by the lines $|x|\le 1/2$, $|y|\le 1/2$.

4. Let $f(x+2y)=f(x)(f(y))^2$ for all x and y. If $f'(0)=\ln 2$, then prove that

$$f(x)+f(2x)+f(3x)+\ldots+f(nx)=\frac{f(x)(f(nx)-1)}{f(x)-1}$$

(consider $f(x)$ is non-negative function).

5. If $H(x_0)=0$ for some $x=x_0$ and $\frac{d}{dx}H(x)>2cxH(x)\ \forall\ x\ge x_0$, where $c>0$, then prove that $H(x)$ cannot be zero for any $x>x_0$.

6. Find the domain of the function

$$f(x)=\frac{5}{\left[\frac{x-1}{2}\right]}-3^{\sin^{-1}x^2}+\frac{(7x+1)!}{\sqrt{x+1}},$$

(where [·] represents the greatest integer function.)

7. Integrate $\int\frac{dy}{y^2(1+y^2)^3}$.

8. A tangent to the ellipse $\frac{x^2}{4}+\frac{y^2}{1}=1$ cuts the circle $x^2+y^2=4$ at points A and B, C is any point on the circle $x^2+y^2=4$ such that A, B and C are to the same side of x-axis. Find the maximum area of the ΔABC.

9. Let $f(x)=\begin{cases}x^3-x^2+10x-5, & x\le 1\\ -2x+\log_2(b^2-2), & x>1\end{cases}$

Find all possible real values of b such that $f(x)$ has the greatest value at $x=1$.

10. A curve with equation of the form $y=ax^4+bx^3+cx+d$ has zero gradient at the point (0, 1) and also touches the x-axis at the point (–1, 0). Then find the values of x for which the curve has negative gradients.

SOLUTION

1. $S=\frac{1}{4!}+\frac{4!}{8!}+\frac{8!}{12!}+\ldots\infty$

$$=\sum_{k=0}^{\infty}\frac{1}{(4k+1)(4k+2)(4k+3)(4k+4)}$$

$$=\sum_{k=0}^{\infty}\left(\frac{1}{6(4k+1)}-\frac{1}{2(4k+2)}+\frac{1}{2(4k+3)}-\frac{1}{6(4k+4)}\right)$$

Now consider the following definite integrals,

$$\int_0^1 x^{4k}dx=\frac{1}{4k+1};\ \int_0^1 x^{4k+1}dx=\frac{1}{4k+2}$$

$$\int_0^1 x^{4k+2}dx=\frac{1}{4k+3};\ \int_0^1 x^{4k+3}dx=\frac{1}{4k+4}$$

$$\Rightarrow S=\sum_{k=0}^{\infty}\int_0^1\left(\frac{1}{6}x^{4k}-\frac{1}{2}x^{4k+1}+\frac{1}{2}x^{4k+2}-\frac{1}{6}x^{4k+3}\right)dx$$

$$=\int_0^1\sum_{k=0}^{\infty}\left(\frac{1}{6}x^{4k}-\frac{1}{2}x^{4k+1}+\frac{1}{2}x^{4k+2}-\frac{1}{6}x^{4k+3}\right)dx$$

$$=\frac{1}{6}\int_0^1\sum_{k=0}^{\infty}x^{4k}(1-3x+3x^2-x^3)\,dx=\frac{1}{6}\int_0^1\frac{(1-x)^3}{1-x^4}dx$$

$$=\frac{1}{6}\int_0^1\frac{(1-x)^2}{(1+x)(1+x^2)}dx=\frac{1}{6}\int_0^1\left(\frac{2}{1+x}-\frac{x+1}{1+x^2}\right)dx$$

Integrating, we get

$$S=\left[\frac{1}{3}\ln(1+x)-\frac{1}{12}\ln(1+x^2)-\frac{1}{6}\tan^{-1}x\right]_0^1$$

$$=\left(\frac{1}{3}-\frac{1}{12}\right)\ln 2-\frac{1}{6}\tan^{-1}(1)=\frac{1}{4}\ln 2-\frac{\pi}{24}.$$

2. $I=\int_{10}^{19}\frac{\sin x}{1+x^a}dx<\int_{10}^{19}\frac{1}{1+x^a}dx<\int_{10}^{19}\frac{1}{1+10^a}dx$

or $I<\frac{9}{1+10^a}$ or, $I<\frac{9}{1+10^a}<\frac{1}{9}$

$$x^2 - 2x\left(y\frac{dy}{dx} + x\right) + y^2 = 0$$

$$x^2 + y^2 - 2xy\frac{dy}{dx} - 2x^2 = 0 \text{ or } x^2 - y^2 + 2xy\frac{dy}{dx} = 0$$

54. (d) : $9x^2 + 4y^2 = 36$

$$\frac{x^2}{4} + \frac{y^2}{9} = 1 \text{ (equation of ellipse)}$$

Remember area enclosed by ellipse is πab.

i.e. $\pi\, 2\cdot 3 = 6\pi$

55. (b) : $(2x - y + 1)\, dx + (2y - x + 1)\, dy = 0$

$$\frac{dy}{dx} = \frac{2x - y + 1}{x - 2y - 1} \quad \text{put } x = x + h;\ y = y + k$$

$$\therefore \frac{dy}{dx} = \frac{2x - y + 2h - k + 1}{x - 2y + h - 2k - 1}$$

s.$+ 2h - k + 1 = 0 \quad \Rightarrow h = -1$

$$h - 2k - 1 = 0 \qquad k = -1 \qquad \therefore \frac{dy}{dx} = \frac{2x - y}{x - 2y}$$

put $y = vx \quad \therefore \frac{dy}{dx} = V + X\frac{dv}{dx}$

$$\Rightarrow V + X\frac{dv}{dx} = \frac{2x - vx}{x - 2vx} = \frac{2 - v}{1 - 2v}$$

$$X\frac{dv}{dx} = \frac{2 - v - 2v^2}{1 - 2v} = \frac{2v^2 - 2v + 2}{1 - 2v} \quad \therefore \frac{dx}{x} = \frac{(1 - 2v)dv}{2v^2 - 2v + 2}$$

put $v^2 - v + 1 = t \ \Rightarrow (2v - 1)\, dv = dt$

$$\therefore \frac{dx}{x} = \frac{-dt}{2t} \quad \therefore \log x = \log t^{-1/2} + \log c$$

$\therefore x = t^{-1/2} c \ \Rightarrow x = (v^2 - v + 1)\cdot C$

$x^2 = (v^2 - v + 1) = \text{constt.}$

$$(x+1)^2\left(\frac{(y+1)^2}{(x+1)^2} - \frac{(y+1)}{x+1} + 1\right) = \text{constt.}$$

$(y + 1)^2 - (y + 1)(x + 1) + (x + 1)^2 = c$

$y^2 + x^2 - xy + 2y + 2x - x - y + 1 - 1 + 1 = c$

$y^2 + x^2 - xy + x + y = c$

56. (d) : $y = \tan^{-1}\dfrac{\sqrt{1+x^2} - \sqrt{1-x^2}}{\sqrt{1+x^2} + \sqrt{1-x^2}}$

put $x^2 = \cos 2\theta$ or $\theta = \frac{1}{2}\cos^{-1} x^2$

$$y = \tan^{-1}\frac{\sqrt{1 + \cos 2\theta} - \sqrt{1 - \cos 2\theta}}{\sqrt{1 + \cos 2\theta} + \sqrt{1 - \cos 2\theta}}$$

$$= \tan^{-1}\left(\frac{\cos\theta - \sin\theta}{\cos\theta + \sin\theta}\right) = \tan^{-1}\left(\frac{1 - \tan\theta}{1 + \tan\theta}\right)$$

$$= \tan^{-1}\tan\left(\frac{\pi}{4} - \theta\right) = \frac{\pi}{4} - \theta = \frac{\pi}{4} - \frac{1}{2}\cos^{-1} x^2$$

$$\therefore \frac{dy}{dx} = +\frac{1}{2}\frac{1}{\sqrt{1 - (x^2)^2}}\ 2x = \frac{+x}{\sqrt{1 - x^4}}$$

57. (d) : $x = \sin t;\ y = \cos pt$

$$\frac{dx}{dt} = \cos t\ ;\ \frac{dy}{dt} = -p\sin pt\ ;\ \frac{dy}{dx} = \frac{-p\sin pt}{\cos t}$$

$$\frac{d^2y}{dx^2} = \frac{-\cos t\ p^2\cos pt(dt/dx) - p\sin pt \sin t\,(dt/dx)}{\cos^2 t}$$

$$\Rightarrow (1 - x^2)\frac{d^2y}{dx^2} - \frac{xdy}{dx} + p^2 y = 0$$

58 (a)

59 (c) : Solving equations $x^2 + y^2 = 5$ and $y^2 = 4x$ we get $x^2 + 4x - 5 = 0$ *i.e.* $x = 1, -5$

for $x = 1$; $y^2 = 4$; $\Rightarrow y = \pm 2$ for $x = -5$; $y^2 = -20$ (imaginary values) $\therefore$ points are $(1, 2)\ (1, -2)$

m_1 for $x^2 + y^2 = 5$ at $(1, 2)$

$$\frac{dy}{dx} = -\frac{x}{y}\bigg|_{(1,2)} = -\frac{1}{2} \text{ Similarly, } m_2 \text{ for } y^2 = 4x \text{ at } (1, 2) \text{ is } 1$$

$$\therefore \tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{-\frac{1}{2} - 1}{1 - \frac{1}{2}}\right| = 3$$

60. (b) : Volume $= v = \dfrac{4}{3}\pi r^3$, $\quad \dfrac{dv}{dt} = \dfrac{4}{3}\pi\cdot 3r^2\dfrac{dr}{dt}$

$$\Rightarrow \frac{dv}{dt} = 4\pi r^2\cdot\frac{dr}{dt} \quad \text{at } r = 7 \text{ cm}$$

$$35 \text{ cc/min} = 4\pi(7)^2\frac{dr}{dt} \quad \Rightarrow \frac{dr}{dt} = \frac{35}{4\pi(7)^2}$$

S.A. $= 4\pi r^2$

$$\frac{ds}{dt} = 8\pi r\frac{dr}{dt} = \frac{8\pi\cdot 7\cdot 35}{4\pi(7)^2} = 10 \text{ cm}^2/\text{min}$$

■

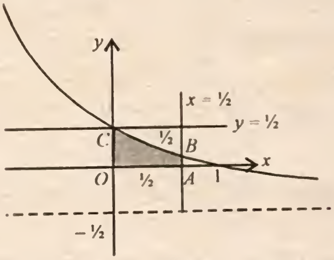
Answers for SUDOKU Challenge

5	1	8	4	9	3	6	2	7
2	6	3	7	1	8	4	9	5
7	4	9	2	6	5	1	3	8
6	8	1	9	7	2	3	5	4
3	9	5	6	8	4	7	1	2
4	7	2	3	5	1	8	6	9
1	5	6	8	2	7	9	4	3
9	3	7	5	4	6	2	8	1
8	2	4	1	3	9	5	7	6

or. $1 + 10^a > 81 \Rightarrow a = 2, 3, 4, 5, ...$
$\therefore$ Minimum odd value of a is 3.

3. $2^{|x|} \cdot |y| + 2^{|x|-1} \le 1$ ...(1)

Clearly, this region is symmetrical about x and y axis



Let, $x \ge 0$, $y \ge 0$, equation (1) gives,
$2^x \cdot y + 2^{x-1} \le 1$

$\Rightarrow y \le \dfrac{1-2^{x-1}}{2^x}$

$= 2^{-x} - \dfrac{1}{2}$

Clearly, bounded region in the first quadrant is $OABC$. The required area is 4 times the area of the region $OABC$

Required area $= 4\int_0^{1/2}\left(2^{-x}-\dfrac{1}{2}\right)dx = 4\left(-\dfrac{2^{-x}}{\ln 2}-\dfrac{x}{2}\right)_0^{1/2}$

$= \dfrac{4}{\ln 2}(1-2^{-1/2}) - 1$

4. $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$= \lim\limits_{h\to 0}\dfrac{f(x)f^2(h/2)-f(x)}{h} = f(x)\lim\limits_{h\to 0}\dfrac{f^2(h/2)-1}{h}$

$= f(x)\lim\limits_{h\to 0}\dfrac{(f(h/2)-1)}{(2\times(h/2))}\cdot\left(f\left(\dfrac{h}{2}\right)+1\right)$

$= f(x)f'(0) = f(x)\ln 2$ [since, $f(0) = 1$]

$\Rightarrow \dfrac{f'(x)}{f(x)} = \ln(2) \Rightarrow f(x) = 2^x + c$

Since, $f(0) = 1 \Rightarrow c = 0 \Rightarrow f(x) = 2^x$

$f(x)+f(2x)+...+f(nx) = 2^x + 2^{2x} + ...2^{nx}$

$= \dfrac{2^x(2^{nx}-1)}{2^x-1} = \dfrac{f(x)(f(nx)-1)}{f(x)-1}.$

5. Given that $\dfrac{d}{dx}H(x) > 2cxH(x)$

$\Rightarrow \dfrac{d}{dx}H(x) - 2cxH(x) > 0 \Rightarrow \dfrac{d}{dx}(H(x)e^{-cx^2}) > 0$

$\Rightarrow H(x)e^{-cx^2}$ is an increasing function.
But $H(x_0) = 0$ and e^{-cx^2} is always positive
$\Rightarrow H(x) > 0$ for all $x > x_0$
$\Rightarrow H(x)$ cannot be zero for any $x > x_0$.

6. $\left[\dfrac{x-1}{2}\right] = 0$ for $0 \le \dfrac{x-1}{2} < 1$ for $1 \le x < 3$

$\Rightarrow \dfrac{5}{\left[\dfrac{x-1}{2}\right]}$ is defined for all $x \in R-[1, 3)$

$\sin^{-1}x^2$ is defined for $-1 \le x \le 1$
$\Rightarrow 3^{\sin^{-1}x^2}$ is defined for $x \in [-1, 1]$
$(7x + 1)!$ is defined for $7x + 1 \ge 0$ with

$7x + 1 \in N \cup \{0\} \Rightarrow x \in \left\{-\dfrac{1}{7}, 0, \dfrac{1}{7}, 1, \dfrac{2}{7}, ...\right\}$

$\dfrac{1}{\sqrt{x+1}}$ is defined for $x \in (-1, \infty)$

taking intersection of all these domains
$D_1 = R - [1, 3)$

$D_2 = N \cup \{0\} = x \in \left\{-\dfrac{1}{7}, 0, \dfrac{1}{7}, \dfrac{2}{7}, ...\right\}$

$D_3 = (-1, \infty)$, Domain $= D_1 \cap D_2 \cap D_3 = \left\{-\dfrac{1}{7}\right\}$.

7. Put $y = \tan\theta$

$\Rightarrow \int\dfrac{dy}{y^2(1+y^2)^3} = \int\dfrac{\cos^6\theta}{\sin^2\theta}d\theta = \int\dfrac{(1-\sin^2\theta)^3\,d\theta}{\sin^2\theta}$

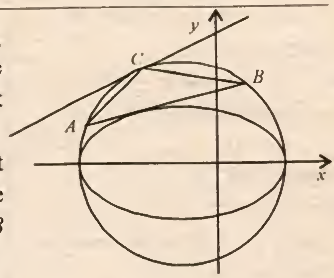
$= \int\dfrac{(1-3\sin^2\theta+3\sin^4\theta-\sin^6\theta)d\theta}{\sin^2\theta}$

$= \int(\mathrm{cosec}^2\theta-3+3\sin^2\theta-\sin^4\theta)d\theta$

$= -\cot\theta-3\theta+\dfrac{3}{2}\int(1-\cos 2\theta)d\theta-\dfrac{1}{4}\int(1-\cos 2\theta)^2\,d\theta$

$= -\dfrac{1}{y}-\dfrac{15}{8}\tan^{-1}y-\dfrac{1}{2}\sin(2\tan^{-1}y)-\dfrac{1}{32}\sin(4\tan^{-1}y)+c.$

8. For a fixed line AB, area of ΔACB will be fixed only when tangent at C is parallel to AB.



Let equations of tangent AB and tangent to the circle at C parallel to AB are $y = mx+\sqrt{4m^2+1}$

and $y = mx+2\sqrt{1+m^2}$ since, $y = mx+\sqrt{4m^2+1}$ cuts the circle $x^2 + y^2 = 4$ at A and B then :

$x^2+(mx+\sqrt{4m^2+1})^2 = 4$

$\Rightarrow x_1+x_2 = \dfrac{-2m\sqrt{4m^2+1}}{1+m^2}$ and $x_1x_2 = \dfrac{4m^2-3}{1+m^2}$

$\Rightarrow (x_1-x_2)^2 = \dfrac{4m^2(4m^2+1)}{(1+m^2)^2}-\dfrac{4(4m^2-3)}{1+m^2}$

$$= \frac{16m^4 + 4m^2 - 16m^2 - 16m^4 + 12 + 12m^2}{(1+m^2)^2} = \frac{12}{(1+m^2)^2}$$

$$\Rightarrow (y_1 - y_2) = m(x_1 - x_2)$$

$$\Rightarrow AB = \sqrt{1+m^2} \cdot \frac{\sqrt{12}}{(1+m^2)} = \frac{\sqrt{12}}{\sqrt{1+m^2}}$$

$\Rightarrow$ Altitude of triangle ABC = Distance between tangents

$$= \left| \frac{2\sqrt{1+m^2} - \sqrt{4m^2+1}}{\sqrt{1+m^2}} \right|$$

Area of $\Delta ABC = \frac{1}{2} \cdot \frac{\sqrt{12}}{\sqrt{1+m^2}} \cdot \frac{(2\sqrt{1+m^2} - \sqrt{4m^2+1})}{\sqrt{1+m^2}}$

$$A = \sqrt{3}\left(\frac{2}{\sqrt{1+m^2}} - \frac{\sqrt{4m^2+1}}{1+m^2} \right)$$

$$\frac{dA}{dm} = \sqrt{3} \cdot \left(\frac{-m}{(1+m^2)^{3/2}} - \frac{2m - 4m^3}{(1+m^2)^2\sqrt{4m^2+1}} \right)$$

$$= -\sqrt{3}m \left(\frac{\sqrt{1+m^2}\sqrt{4m^2+1} + 2 - 4m^2}{(1+m^2)^2\sqrt{4m^2+1}} \right) = 0$$

$\Rightarrow m = 0$, At $m = 0$, $\frac{dA}{dm}$ changes sign from positive to negative. So there is maximum.

Maximum area $= \sqrt{3}$.

9. For $x \le 1$;

$f'(x) = 3x^2 - 2x + 10$

Discriminant of

$f'(x) = 0 = -56 < 0$

and coefficient of

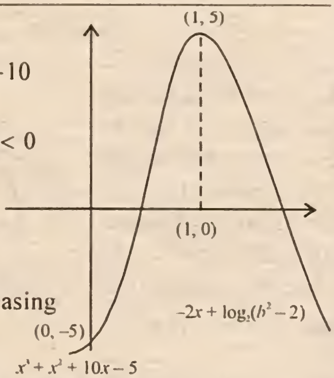
$x^2 > 0$

Hence $f'(x) > 0$

for all $x \le 1$

Hence $f(x)$ is an increasing function for $x \le 1$

For $x > 1$; $f'(x) = -2$



$\Rightarrow f(x)$ is a decreasing function for $x > 1$

$f(x)$ will have the greatest value at $x = 1$ if

$$\lim_{x \to 1^+} f(x) \le f(1) \Rightarrow \lim_{h \to 0} f(1+h) \le f(1)$$

$$\Rightarrow -2 + \log_2(b^2 - 2) \le 5 \Rightarrow \log_2(b^2 - 2) \le 7$$

$$\Rightarrow b^2 - 2 \le 2^7 \Rightarrow b^2 \le 130$$

Again $b^2 - 2 > 0$ for $\log_2(b^2 - 2)$ to be defined

$\Rightarrow 2 < b^2 < 130$

$\therefore b \in \left[-\sqrt{130}, -2\right) \cup \left(\sqrt{2}, \sqrt{130}\right]$

10. $y = ax^4 + bx^3 + cx + d$ ...(1)

y touches x-axis at $(-1, 0)$

so $(-1, 0)$ lies on it and $dy/dx = 0$

so, $0 = a - b - c + d$ ...(2)

From (1), $\frac{dy}{dx} = 4ax^3 + 3bx^2 + c$ ...(3)

Hence $\left(\frac{dy}{dx}\right)_{(-1,\,0)} = 0 \Rightarrow -4a + 3b + c = 0$ ...(4)

Also $\left(\frac{dy}{dx}\right)_{(0,\,1)} = c = 0$ (since curve touches (0, 1))

$\Rightarrow (0, 1)$ also lies on it hence $d = 1$

Putting values of c and d in (2) and (4), solving for a and b we get $a = 3$, $b = 4$

Therefore (3) becomes $\frac{dy}{dx} = 12x^3 + 12x^2$

Now $\frac{dy}{dx} < 0 \Rightarrow 12x^3 + 12x^2 < 0$

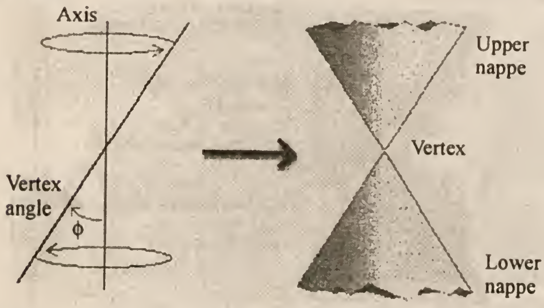
$\Rightarrow 12x^2(x+1) < 0 \Rightarrow x < -1.$ ■

Introduction to CONICS

Conic sections are the curves which result from the intersection of a plane with a cone. These curves were studied and used by the ancient Greeks, and were written about extensively by both Euclid and Appolonius. They remain important today, partly for their many and diverse applications.

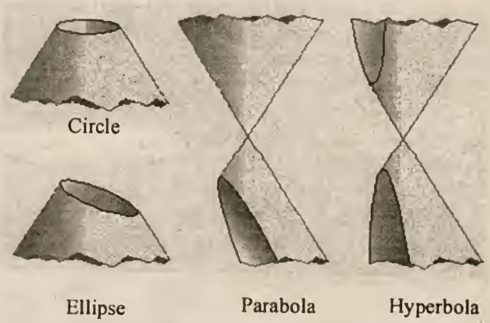
Although to most people the word "cone" conjures up an image of a solid figure with a round base and a pointed top, to a mathematician a cone is a surface, one which is obtained in a very precise way.

Imagine a vertical line, and a second line intersecting it at some angle ϕ (phi). We will call the vertical line the axis, and the second line the *generator*. The angle ϕ between them is called the *vertex angle*. Now imagine grasping the axis between thumb and forefinger on either side of its point of intersection with the generator, and twirling it. The generator will sweep out a surface, as shown in the diagram. It is this surface which we call a cone.



Notice that a cone has an upper half and a lower half (called the *nappes*), and that these are joined at a single point, called the vertex. Notice also that the nappes extend indefinitely far both upwards and downwards. A cone is thus completely determined by its vertex angle.

Now, in intersecting a flat plane with a cone, we have three choices, depending on the angle the plane makes to the vertical axis of the cone. First, we may choose our plane to have a greater angle to the vertical than does the generator of the cone, in which case the plane must cut right through one of the nappes. This results in a closed curve called an ellipse. Second, our plane may have exactly the same angle to the vertical axis as the generator of the cone, so that it is parallel to the side of the cone. The resulting open curve is called a parabola. Finally, the plane may have a smaller angle to the vertical axis (that is, the plane is steeper than the generator), in which case the plane will cut both nappes of the cone. The resulting curve is called a hyperbola, and has two disjoint "branches."

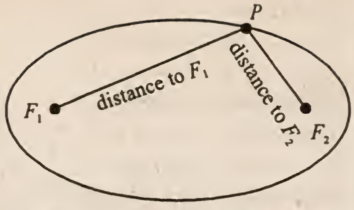


Notice that if the plane is actually perpendicular to the axis (that is, it is horizontal) then we get a circle – showing that a circle is really a special kind of ellipse. Also, if the intersecting plane passes through the vertex then we get the so-called *degenerate* conics; a single point in the case of an ellipse, a line in the case of a parabola, and two intersecting lines in the case of a hyperbola. Although intuitively and visually appealing, these definitions for the conic sections tell us little about their properties and uses. Consequently, one should master their "plane geometry" definitions as well. It is from these definitions that their algebraic representations may be derived, as well as their many important properties,such as the reflection properties. We will now look at each conic section in detail.

Ellipse

The set of all points in the plane, the sum of whose

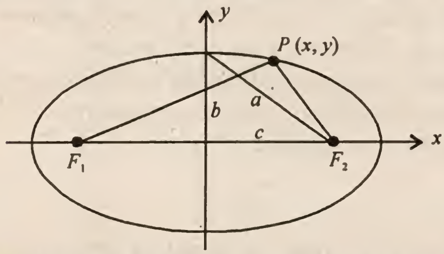
distances from two fixed points, called the foci, is a constant. Sometimes this definition is given in terms of "a locus of points" or even "the locus of a point" satisfying this condition – it all means the same thing.



Distance to F_1 + distance to F_2 = a constant

For reasons that will become apparent, we will denote the sum of these distances by $2a$.

We see from the definition that an ellipse has two axes of symmetry, the larger of which we call the major axis and the smaller the minor axis. The two points at the ends of the ellipse (on the major axis) are called the vertices. It happens that the length of the major axis is $2a$, the sum of the distances from any point on the ellipse to its foci. If we call the length of the minor axis $2b$ and the distance between the foci $2c$, then the Pythagorean Theorem yields the relationship $b^2 + c^2 = a^2$.



By imposing coordinate axes in this convenient manner, we see that the vertices are at the x intercepts, at a and $-a$, and that the y-intercepts are at b and $-b$. Let the variable point P on the ellipse be given the coordinates (x, y). We may then apply the distance formula for the distances from P to F_1 and from P to F_2 to express our geometrical definition of the ellipse in the language of algebra:

$$2a = PF_1 + PF_2 = \sqrt{(x+c)^2 + y^2} + \sqrt{(x-c)^2 + y^2}$$

Substituting $a^2 - b^2$ for c^2 and using a little algebra, we can then derive the standard equation for an ellipse centered at the origin,

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

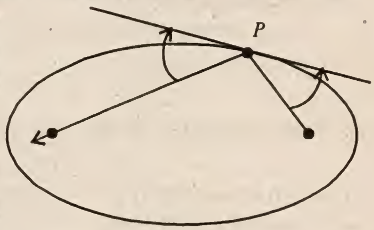
where a and b are the lengths of the semimajor and semiminor axes, respectively. (If the major axis of the ellipse is vertical, exchange a and b in the equation.)

The points $(a, 0)$ and $(-a, 0)$ are called the vertices of the ellipse. If the ellipse is translated up/down or left/right, so that its center is at (h, k), then the equation takes the form

$$\frac{(x-h)^2}{a^2} + \frac{(y-k)^2}{b^2} = 1$$

If $a = b$, we have the special case of an ellipse whose foci coincide at the center – that is, a circle of radius a.

The ellipse has the following remarkable reflection property. Let P be any point on the ellipse, and construct the line segments joining P to the foci. Then these lines make equal angles to the tangent line at P.

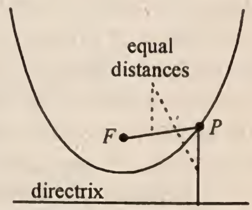


Consequently, any ray emanating from one focus will always reflect off of the inside of the ellipse in such a way as to go straight to the other focus. Architects have exploited this property in many famous buildings. The "whisper chamber" in the United States Capitol is one; stand at one focus and whisper, and anyone at the other focus can hear you with perfect clarity, even though they are much too far away from you to hear a whisper normally. The Mormon Tabernacle in Salt Lake City was also designed as an ellipse (indeed, it is the top half of an ellipsoid), to provide a perfect acoustical environment for choral and organ music.

Ellipses occur in nature as well, and are critical to understanding the motion of planets and other bodies moving in space. In Physics Kepler's laws are also depend upon this principle.

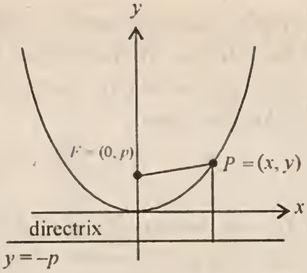
Parabola

The set of all points in the plane whose distances from a fixed point, called the focus, and a fixed line, called the directrix, are always equal.



The point directly between – and hence closest to – the focus and the directrix is called the *vertex* of the parabola. To derive the equation of a parabola in rectangular coordinates, we again choose a convenient location for the axes, placing the origin at the vertex so that the y-axis is the axis of symmetry. We denote the distance

from the vertex to the focus by p, so that the directrix is then the line $y = -p$.



Using the distance formula for the distance from P to F, and noting that the distance from P to the directrix is evidently $y + p$, and setting these distances equal, we obtain

$$\sqrt{x^2 + (y-p)^2} = y + p$$

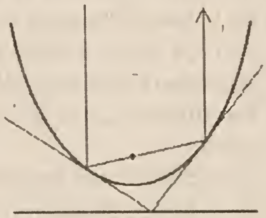
A direct application of ordinary algebra reduces this to

$$x^2 = 4py$$

This then is the equation of a parabola opening upwards, with its vertex at the origin. If we introduce a negative sign, we get a parabola opening downwards. If we interchange the roles of x and y, we get a parabola opening to the right (or to the left if there is a negative). We may translate the parabola up/down or back/forth, putting the vertex at the point (h, k) if we write our equation as

$$(x - h)^2 = 4p(y - k).$$

The reflection property of parabolas is very important because it has so many practical uses. Let P be any point on the parabola. Construct the line segment joining P to the focus, and a ray through P that is parallel to the axis of symmetry. The line segment and ray will always make equal angles to the tangent line at P. Consequently, any ray emanating from the focus will reflect off of the parabola so as to point directly outwards, parallel to the axis. This property is made use of in the design of flashlights, headlights, and spotlights, for instance. Conversely, any ray entering the parabola that is parallel to the axis will be reflected to the focus. This property is exploited in the design of radio and satellite receiving dishes, and solar collectors.

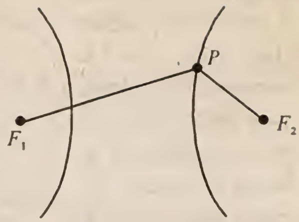


The reflection property of parabolas is related to the curious property that the tangent lines at the endpoints of any chord through the focus (as shown above) intersect on the directrix, and always do so in a right angle.

Parabolas are also important in the study of ballistics, the movement of a body under the force of gravity.

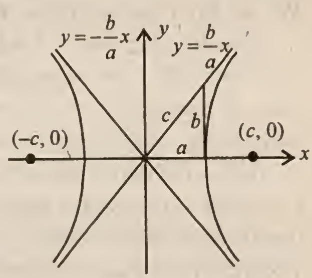
Hyperbola

The set of all points in the plane, the difference of whose distances from two fixed points, called the foci, remains constant.



$PF_1 - PF_2 =$ a constant

Mimicking our procedure with ellipses, we will choose the constant $2a$ to represent the difference of these distances, that is, $PF_1 - PF_2 = 2a$. We will call the two points of the hyperbola which lie on the line connecting the foci the vertices, and we then see that the distance between the vertices must be $2a$. Also, we will call the distance between the foci $2c$. Finally, we will define the constant b by $b^2 = c^2 - a^2$. (We may do this since evidently $c > a$.) Placing co-ordinate axes at the center as before, we obtain this picture.



Applying the distance formulas and substituting for c as we did in the previous cases, we can derive the standard formula of a hyperbola:

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

We note that solving this equation for y yields

$$y = \pm\frac{b}{a}\sqrt{x^2 - a^2}$$

and letting x become arbitrarily large causes this expression to become arbitrarily close to

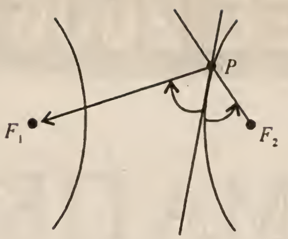
$$y = \pm\frac{b}{a}x.$$

Thus we see that the crisscrossing lines in the diagram above are asymptotes for the hyperbola, that is, the curve becomes indefinitely close to these lines as the absolute value of x grows without bound.

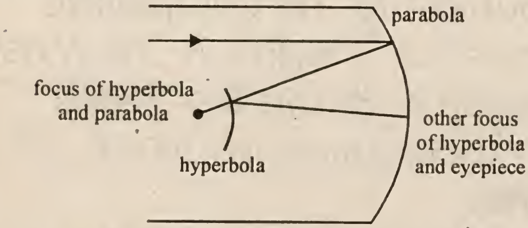
As before, if the principal axis of the hyperbola is vertical instead of horizontal, we switch the roles of a and b. We may also translate the hyperbola up/down and back/forth, placing the center at (h, k) by modifying our equation thusly:

$$\frac{(x-h)^2}{a^2} - \frac{(y-k)^2}{b^2} = 1$$

The reflection property of the hyperbola is of great importance in optics. Let P be any point on one branch of the hyperbola. Then the line segments joining P to each of the foci form an angle which is bisected by the tangent line at P.



Consequently, any ray approaching one of the foci from a convex side of the hyperbola is reflected to the opposite focus. An example of an application of this principle is the Cassegrain reflecting telescope:

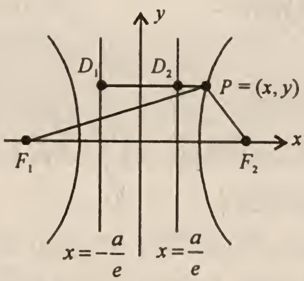


A concave parabolic mirror forms the back of the telescope, and this shares a focus with a convex hyperbolic mirror, the other focus of which is at the eyepiece.

Eccentricity

The unifying idea among these curves is that they are all conics, that is, conic sections. We have seen the geometric realization of this unifying notion, but how can it be expressed algebraically? The key notion is that of eccentricity.

To define the eccentricity of a conic, we must first observe a feature of the ellipse and the hyperbola that we neglected before, namely, that each of these curves has a directrix, just as the parabola does. Indeed, the ellilpse and hyperbola each have two directrices.



Now let P be a point on the conic curve, and consider its distance to a focus, and its distance tc the corresponding directrix. The curve's eccentricity is the ratio of these distances.

$$\text{Eccentricity} = \frac{PF_1}{PD_1} = \frac{PF_2}{PD_2}.$$

We will denote the eccentricity by the letter e. It can be shown geometrically that e is always equal to the ratio of c and a as these constants were defined in each case. That is, we always have $e = c/a$. It can also be shown that the directrices of an ellipse or hyperbola with principle axes horizontal are always the vertical lines given by

$$x = \pm\frac{a}{e}$$

as shown in the diagrams above.

Now recall that in a parabola the distance from a point to the focus, and from the same point to the directrix, are always the same. Consequently, a parabola always has eccentricity $e = 1$. An ellipse, on the other hand, always has $e < 1$, and for a hyperbola $e > 1$. (A circle is the special case of an ellipse with $e = 0$.) In summary, we have

$e < 1 \leftrightarrow$ ellipse
$e = 1 \leftrightarrow$ parabola
$e > 1 \leftrightarrow$ hyperbola

The names of these curves are related to their eccentricities. "Ellipse" comes from a Greek word meaning "deficiency" or "something left out," and is related to the English words "ellipsis" and "elliptical." The word "hyperbola," on the other hand, comes from the Greek word for "excess," and is related to the English word "hyperbole." Finally, "parabola" means something like "just right," and is related to the words "compare" and "parable."

What this discussion shows is that we may consider that there is only one general kind of curve, called a conic, with special cases called ellipse, parabola, and hyperbola depending on the conic's eccentricity. Algebraically, we may now consider conics in complete generality. To do so, consider a second degree polynomial in two variables, x and y.

$$Ax^2 + Bxy + Cy^2 + Dx + Ey + F = 0$$

The 'xy' term can be eliminated by a rotation of axes. The algebraic techniques for doing so can be found in any text on calculus with analytic geometry. By then completing the square with respect to both x and y, one will obtain one of the standard equations given above, for either an ellipse or a hyperbola. If only one of x and y appears as a square in the original conic equation, then the standard equation of a parabola may be obtained.

The study of conic sections is one of the most beautiful topics in classical mathematics. Every student of mathematics should take the time to master conic sections thoroughly, not only for the esthetic appeal of the subject, and not only because their applications are so varied and important, but also because they show – in a deep and clear way – the fundamental unification of geometry and algebra in the field of analytic geometry. ■

Mathematics Olympia for IIT-JEE (MAINS)

By : Er. Akhlak Ahmad, ABC Classes, Gorakhpur

1. If $\sqrt{2}\cos A=\cos B+\cos^3 B, \sqrt{2}\sin A=\sin B-\sin^3 B$, prove that $\sin(A-B) = \pm\frac{1}{3}$.

2. If $f(x)=(x-a)(x-b)-\left(\frac{a+b}{2}\right)$ and $f(x) = 0$ has both non-negative roots, then prove that $f(x) \geq -\frac{(a+b)^2}{4}$.

3. Find the fifth degree polynomial which leaves remainder 1 when divided by $(x-1)^3$ and remainder -1 when divided by $(x+1)^3$.

4. solve the equation $x^3 - [x] = 5$, where $[x]$ denotes the integral part of the number x.

5. If the quadratic equation
$$4^{\sec^2\alpha}x^2+2x+\left(\beta^2-\beta+\frac{1}{2}\right) = 0$$
have real roots, then find all the possible values of $\cos\alpha + \cos^{-1}\beta$.

6. A circle $x^2+y^2+4x-2\sqrt{2}y+c = 0$ is the director of circle S_1 and S_1 is the director circle of circle S_2 and so on. If the sum of radii of all these circles is 2, then find the value of c.

7. Let $S(n)$ denotes the sum of the first n terms of an A.P. Then find $S = \lim_{n\to\infty}\sum_{r=-n}^{n}\frac{f(r)}{n}$, where $f(n) = \frac{S(3n)}{S(2n)-S(n)}$.

8. Three numbers in A.P. are removed from first n consecutive natural numbers and average of remaining numbers is found to be 43/4. Find n as well as removed numbers if one of the removed number is perfect square.

ANSWERS

1. $\sin(A-B) = \sin A\cos B - \cos A\sin B$
substituting values of $\cos A$ and $\sin A$ from the given equations, we get

$$= \frac{\sin B\cos^2 B\cos B-(\cos B+\cos^3 B)\sin B}{\sqrt{2}}$$
$$= \frac{-\sin B\cos B}{\sqrt{2}}$$

Squaring and adding given equations
$(\cos B + \cos^3 B)^2 + (\sin B - \sin^3 B)^2 = 2$
$\Rightarrow \cos^2 B+\cos^6 B+2\cos^4 B + \sin^2 B + \sin^6 B - 2\sin^4 B = 2$
$\Rightarrow 1+2(\cos^2 B+\sin^2 B)+(\sin^4 B+\cos^4 B-\sin^2 B\cos^2 B) = 2$
$\Rightarrow 1 + 2\cos^2 B - 2\sin^2 B + 1 - 3\sin^2 B\ \cos^2 B = 2$
$\Rightarrow 3\cos^2 2B + 8\cos 2B - 3 = 0$
$$\Rightarrow \cos 2B = \frac{1}{3} \Rightarrow \sin 2B = \pm\frac{2\sqrt{2}}{3}$$
$$\Rightarrow \sin(A-B) = -\frac{\sin 2B}{2\sqrt{2}} = \pm\frac{1}{3}.$$

2. Given that $f(x) = (x-a)(x-b) - \left(\frac{a+b}{2}\right)$

Sum of the roots of the equation $f(x) = 0$, will be positive
$\Rightarrow (a + b) > 0$

The product of the roots of the equation will be greater than and equal to zero
$$\Rightarrow ab-\left(\frac{a+b}{2}\right) \geq 0$$
Now $f(x)$ will be minimum, when $f'(x) = 0$
$$\Rightarrow x = \frac{a+b}{2}$$
$$\Rightarrow (f(x))_{\min} = \left(\frac{a+b}{2}\right)^2 - \frac{(a+b)^2}{2}+ab-\left(\frac{a+b}{2}\right)$$
$$= \frac{-(a+b)^2}{4}+ab-\left(\frac{a+b}{2}\right)$$
$$= \frac{-(a+b)^2-2(a+b)+4ab}{4}$$
$$\Rightarrow f(x) \geq \frac{-(a+b)^2-2(a+b)+4ab}{4}$$
$$\geq \frac{-(a+b)^2-4ab+4ab}{4} = \frac{-(a+b)^2}{4}.$$

3. Let $f(x)$ be the required polynomial. Then $f'(x)$ is divisible by $(x-1)^2$ and $(x+1)^2$.

Hence $f'(x) = A(x^2-1)^2 = A(x^4-2x^2+1)$

$$\Rightarrow f(x) = A\left(\frac{x^5}{5}-\frac{2x^3}{3}+x\right)+B$$

But $f(1) = 1$ and $f(-1) = -1$ (by remainder theorem)

Therefore $f(x) = \frac{1}{8}(3x^5-10x^3+15x)$.

---

4. $x = (6)^{1/3}$

$\because$ $x = [x] + f$ where $0 \le f < 1$

$\therefore$ given equation becomes $x^3 - (x-f) = 5$

i.e. $x^3 - x = 5 - f \Rightarrow 4 < x^3 - x \le 5$

Now, $x^3 - x$ is negative for $x \in (-\infty, -1) \cup (0, 1)$

So, possible values of x lie in the interval

$[-1, 0] \cup [1, \infty)$

for $-1 \le x \le 0$, we have $x^3 - x < 1 < 4$;

for $x = 1$, we have $x^3 - x = 0 < 4$

further for $x \ge 2$ we have

$x^3 - x = x(x^2-1) \ge 2(4-1) = 6 > 5$;

Therefore, $1 < x < 2 \Rightarrow [x] = 1$

Now the original equation can be written as $x^3 - 1 = 5$, hence $x^3 = 6$, i.e. $x = (6)^{1/3}$

---

5. The quadratic equation

$4^{\sec^2\alpha}x^2+2x+\left(\beta^2-\beta+\frac{1}{2}\right) = 0$ have real roots

$\Rightarrow$ discriminant $= 4 - 4.\ 4^{\sec^2\alpha}\left(\beta^2-\beta+\frac{1}{2}\right) \ge 0$

$$\Rightarrow 4^{\sec^2\alpha}\left(\beta^2-\beta+\frac{1}{2}\right) \le 1 \qquad ...(1)$$

But $4^{\sec^2\alpha} \ge 4,\ \beta^2-\beta+\frac{1}{2} = \left(\beta-\frac{1}{2}\right)^2+\frac{1}{4} \ge \frac{1}{4}$

i.e. equation will be satisfied only when $4^{\sec^2\alpha} = 4$ and

$\beta^2-\beta+\frac{1}{2} = \frac{1}{4}$ $\Rightarrow \sec^2\alpha = 1$ and $\left(\beta-\frac{1}{2}\right)^2 = 0$

$\Rightarrow \cos^2\alpha = 1$ and $\beta = \frac{1}{2} \Rightarrow \alpha = n\pi$ and $\beta = \frac{1}{2}$

$$\cos\alpha+\cos^{-1}\beta = \cos n\pi+\cos^{-1}\frac{1}{2}$$

$= 1+\frac{\pi}{3}$ when n is even integer

$= -1+\frac{\pi}{3}$ when n is odd integer

i.e. values of $\cos\alpha+\cos^{-1}\beta$ is $\frac{\pi}{3}-1,\ \frac{\pi}{3}+1$.

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6. Radius of given circle

$$= \sqrt{4+2-c} = \sqrt{6-c} = a \text{ (say)}$$

Radius of circle $S_1 = \frac{a}{\sqrt{2}}$

Radius of circle $S_2 = \frac{a}{2}$ so on $a+\frac{a}{\sqrt{2}}+\frac{a}{2}+...+\infty = 2$

$$a\left[\frac{1}{1-1/\sqrt{2}}\right] = 2 \Rightarrow a\left(\frac{\sqrt{2}}{\sqrt{2}-1}\right) = 2$$

$$a = 2-\sqrt{2} \Rightarrow \sqrt{6-c} = 2-\sqrt{2}$$

$$\Rightarrow 6-c = 4+2-4\sqrt{2} \Rightarrow c = 4\sqrt{2}.$$

---

7. Let the first term of A.P. be a and its common difference be d.

Given that $S(3n) = \frac{3n}{2}[2a+(3n-1)d]$

$$S(2n) = \frac{2n}{2}[2a+(2n-1)d]$$

$$S(n) = \frac{n}{2}[2a+(n-1)d]$$

$$\Rightarrow S(2n)-S(n) = \frac{n}{2}[2\{2a+(2n-1)d\}-2a-(n-1)d]$$

$$= \frac{n}{2}[2a+(3n-1)d] = \frac{S(3n)}{3} \Rightarrow \frac{S(3n)}{S(2n)-S(n)} = 3 = f(n)$$

Now $S = \lim_{n\to\infty}\frac{1}{n}\sum_{r=-n}^{n} f(r) = \lim_{n\to\infty} 3\left(\frac{2n+1}{n}\right) = 6.$

---

8. Three numbers in A.P. with least and greatest sum are $(1, 2, 3)$ and $(n-2, n-1, n)$ respectively.

sum of removed numbers $= \frac{43}{4}\times(n-3)$

$$\Rightarrow \frac{n(n+1)}{2}-3(n-1) \le \frac{43}{4}(n-3) \le \frac{n(n+1)}{2}-6$$

$$\Rightarrow \frac{(n-3)(n-2)}{2} \le \frac{43}{4}(n-3) \le \frac{(n-3)(n+4)}{2}$$

$\Rightarrow 18 \le n \le 23$

So, possible numbers are 18, 19, 20, 21, 22, 23.

As $\frac{43}{4}(n-3)$ is a natural number

$\Rightarrow$ $(n-3)$ should be divisible by 4

so, only possibilities are 19 and 23

Also, $\frac{n(n+1)}{2}-\frac{43}{4}(n-3)$ is sum of removed number

$\Rightarrow$ this should be divisible by 3 $\Rightarrow$ 19 is only possibility

so, sum of removed number = 190 – 172 = 18

$\Rightarrow$ middle term of removed A.P. = 18/3 = 6

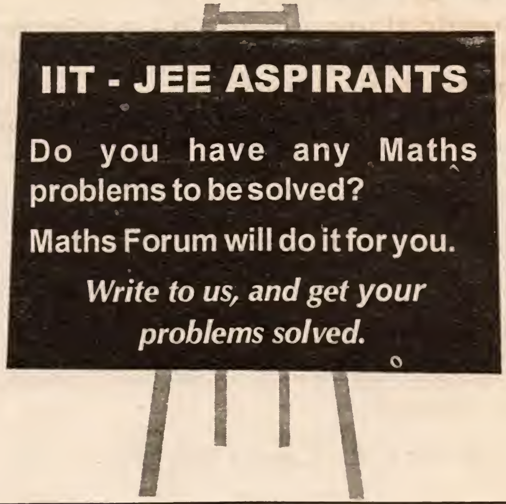
so, removed numbers can be

(1, 6, 11), (4, 6, 8), (3, 6, 9).

■

MATHS FORUM

by Prof. S.S. Dahiya



1. For $x > 1$ prove that $\dfrac{\ln x}{(x^3-1)} < \dfrac{(x+1)}{3(x^3+x)}$.

2. Prove that $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$
Give geometry of the result and deduce that
$$\left|\alpha+\sqrt{\alpha^2-\beta^2}\right|+\left|\alpha-\sqrt{\alpha^2-\beta^2}\right| = |\alpha+\beta|+|\alpha-\beta|$$

3. Find area of triangle whose angular points are t_1, t_2, t_3 on parabola $y^2 = 4ax$.

4. Find locus of mid-points of normal chords of parabola $y^2 = 4ax$. ***Gagandeep Singh, Aligarh***

5. Find all natural numbers n such that $n!$ ends with exactly 26 zeroes.

6. Show that there is no natural number n for which $n!$ ends with exactly 29 zeroes.

7. Find the largest two digit prime number which divides ${}^{200}C_{100}$

8. In ΔABC, prove that value of
$\tan A \tan B + \tan B \tan C + \tan C \tan A$
cannot lie between 0 and 9.
Arvind Kumar, Dibrugarh

9. Find natural numbers x and y, if $x + y = 52$ and LCM of x and y is 168. ***Alok Kumar, Patna***

10. Three integers are taken at a time from $\{1, 2, 3, \ldots\ldots\ldots, n\}$ and are multiplied, S_n denotes sum of all such products. For example $\{1, 2, 3\}$, $S_3 = 1 \times 2 \times 3 = 6$
$\{1, 2, 3, 4\}$, $S_4 = 1 \times 2 \times 3 + 1 \times 2 \times 4 + 1 \times 3 \times 4 + 2 \times 3 \times 4 = 6 + 8 + 12 + 24 = 50$
Establish a rule to find S_{n+1} and hence find S_{10}.
K.V.R. Murthy, Vishakhapatnam

11. Find the area bounded by $x = t^2$, $y = 2t$, x-axis for $t = 1$ to $t = 2$

12. Find area of region generated by $x = a \sin t$, $y = a \sin 2t$

13. Find area bounded by curve $r = 2 \sin \theta$

14. Convert $r = 2 \sin\theta$ in cartesian form
Nitesh Bhatia, Ajmer

ANSWERS

1. For $x > 1$ prove that $\dfrac{\ln x}{x^3-1} < \dfrac{(x+1)}{3(x^3+x)}$

Given that $x > 1$, therefore $x^3 > 1$ or $x^3 - 1$ is positive

then rewrite $\dfrac{\ln x}{(x^3-1)} < \dfrac{(x+1)}{3(x^3+x)}$

as $3 \ln x < \dfrac{(x^3-1)(x+1)}{(x^3+x)}$

$\therefore$ To prove $\dfrac{(x^3-1)(x+1)}{(x^3+x)} > 3 \ln x$ for $x > 1$

or $y = \dfrac{(x^3-1)(x+1)}{(x^3+x)} - 3\ln x > 0$ for $x > 1$

Let $y = f(x) = \dfrac{(x^3-1)(x+1)}{(x^3+x)} - 3\ln x$

$$= x + 1 - \frac{1}{x} - \frac{2}{(x^2+1)} - 3\ln x$$

$$f'(x) = 1 + \frac{1}{x^2} + \frac{4x}{(x^2+1)^2} - \frac{3}{x}$$

$$f'(x) = \left(1-\frac{1}{x}\right)+\left(\frac{1}{x^2}-\frac{1}{x}\right)+\frac{4x}{(x^2+1)^2}-\frac{1}{x}$$

$$f'(x) = \frac{(x-1)^2}{x^2} - \frac{(x-1)^2(x+1)^2}{x(x^2+1)^2} = \frac{(x-1)^4(x^2+x+1)}{x^2(x^2+1)^2}$$

$f'(x) > 0$ for $x > 1$ $\therefore$ $f(x) \uparrow$ for $x > 1$

$\because$ $f(1) = 0$, $f(x) \uparrow$ for $x > 1$ $\therefore$ $f(x) > f(1)$

or $f(x) > 0$ for $x > 1$

2. $$|z_1 + z_2|^2 = (z_1 + z_2)\ (\bar{z}_1 + \bar{z}_2)$$
$$= |z_1|^2 + |z_2|^2 + z_1\bar{z}_2 + z_2\bar{z}_1$$

$$|z_1 - z_2|^2 = (z_1 - z_2)\ (\bar{z}_1 - \bar{z}_2)$$
$$= |z_1|^2 + |z_2|^2 - z_1\bar{z}_2 - z_2\bar{z}_1$$

Add : $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

Geometrical representation : In a parallelogram sum of squares of lengths of diagonals is equal to sum of squares of lengths of four sides

Now using $|w|^2 = |w^2|$, put $z_1 = \sqrt{\dfrac{\alpha+\beta}{2}}$, $z_2 = \sqrt{\dfrac{\alpha-\beta}{2}}$

in $|z_1 + z_2|^2 + |z_1 - z_2|^2 = 2|z_1|^2 + 2|z_2|^2$

or $\left|(z_1 + z_2)^2\right| + \left|(z_1 - z_2)^2\right| = \left|2z_1^2\right| + \left|2z_2^2\right|$

We get

$$\left|\alpha + \sqrt{\alpha^2 - \beta^2}\right| + \left|\alpha - \sqrt{\alpha^2 - \beta^2}\right| = |\alpha + \beta| + |\alpha - \beta|.$$

---

3. Three points

$A(at_1^2, 2at_1)$, $B(at_2^2, 2at_2)$, $C(at_3^2, 2at_3)$

$$\text{Area } \Delta ABC = \frac{1}{2}\begin{vmatrix} at_1^2 & 2at_1 & 1 \\ at_2^2 & 2at_2 & 1 \\ at_3^2 & 2at_3 & 1 \end{vmatrix} = a^2\begin{vmatrix} t_1^2 & t_1 & 1 \\ t_2^2 & t_2 & 1 \\ t_3^2 & t_3 & 1 \end{vmatrix}$$

$\therefore$ Area $\Delta ABC = a^2(t_1 - t_2)(t_2 - t_3)(t_3 - t_1)$.

---

4. To parabola $y^2 = 4ax$, normal at $A(at_1^2,\ 2at_1)$ meets the parabola at $B(at_3^2,\ 2at_3)$ then $t_3 = -t_1 - \dfrac{2}{t_1}$.

Eliminate t_1, t_3

$$2x' = a\{(t_1 + t_3)^2 - 2t_1t_3\},\ 2y' = 2a(t_1 + t_3)$$

Put $t_1 + t_3 = \dfrac{y'}{a}$, $t_3 = -t_1 - \dfrac{2}{t_1}$, $\dfrac{y'}{a} = t_1 + t_3 = \dfrac{-2}{t_1}$

$\therefore t_1 = -\dfrac{2a}{y'}$,

$$2x' = a\left\{\frac{y'^2}{a^2} + 2t_1^2 + 4\right\}$$

$$2x' = a\left\{\frac{y'^2}{a^2} + \frac{8a^2}{y'^2} + 4\right\}$$

$\therefore$ locus is $2axy^2 = y^4 + 4a^2y^2 + 8a^4$

---

5. $\lfloor 100 = 5^{24}$ (Number not divisible by 5)

$\therefore$ $\lfloor 100$ ends with 24 zeroes

For ending with 26 zeroes, we need $(5 \times 2) \times (5 \times 2)$ multiply of 5 more than 100 are 105, 110, 115, ...

$\therefore$ $\lfloor 100 = 5^{26}$, (number not divisible by 5)

$\therefore$ $\lfloor x$ ends with 26 zeroes when $x = 110, 111, 112, 113, 114$

---

6. Show that there is no natural number x for which $\lfloor x$ ends with exactly 29 zeroes.

$\lfloor 115 = 5^{27} \times$ (Number not multiple of 5),

$\lfloor 120 = 5^{28} \times$ (No. not multiple of 5)

$\lfloor 125 = \lfloor 120 \times (121 \times 122 \times 123 \times 124 \times 5^3)$

$= 5^{31} \times$ (No. not multiple of 5)

$\therefore$ there is no value of n for which $\lfloor n$ ends with exactly 29 zeroes.

---

7. Find the largest two digit prime number which divides ${}^{200}C_{100}$

$${}^{200}C_{100} = \frac{\lfloor 200}{\lfloor 100\ \lfloor 100}$$

In $\lfloor 200$ there are exactly two numbers which are multiple of 67, 68, 69, ..., 99, 100 whereas in $\lfloor 100$ there is one number multiple of 67, 68, ..., 100 (number itself)

$\therefore$ These numbers 67, 68, ..., 97, 98, 99, 100 gets cancelled from numerator & Denominator in value of ${}^{200}C_{100}$.

In $\lfloor 200$ there are exactly three numbers which are multiple of 51, 52, ..., 66 where as in $\lfloor 100$ there is one number multiple of 51, 52, ..., 66 (number itself)

$$\therefore \frac{\lfloor 200}{\lfloor 100\ \lfloor 100} = \frac{(61)^3(\text{No. not divisible by } 61)}{(61 \times \text{No. not divisible by } 61)^2}$$

$= 61 \times$ integer *Answer* 61

---

8. To prove $\tan A \tan B + \tan B \tan C + \tan C \tan A \geq 9$

or $\tan A \tan B + \tan B \tan C + \tan C \tan A < 0$

In any ΔABC, $A + B + C = \pi$

$\tan A + \tan B + \tan C = \tan A \tan B \tan C \neq 0$

Case I : Triangle is acute angled, $\tan A > 0$, $\tan B > 0$, $\tan C > 0$

Using $(x_1 + x_2 + x_3)\left(\dfrac{1}{x_1} + \dfrac{1}{x_2} + \dfrac{1}{x_3}\right) \geq (3)^2$

for x_1, x_2, x_3 as positive

$$(\tan A + \tan B + \tan C)\left(\frac{1}{\tan A} + \frac{1}{\tan B} + \frac{1}{\tan C}\right) \geq 9$$

$\therefore$ $\tan A \tan B + \tan B \tan C + \tan C \tan A \geq 9$

Case II : Triangle is obtuse angled

$\therefore$ Two angle acute (less then $\pi/2$) one angle obtuse (more than $\pi/2$)

In any Δ, $\cot A + \cot B + \cot C = \dfrac{2R(a^2+b^2+c^2)}{2abc} > 0$

$$\frac{1}{\tan A}+\frac{1}{\tan B}+\frac{1}{\tan C}>0$$

$$\frac{\tan A\tan B+\tan B\tan C+\tan C\tan A}{\tan A\tan B\tan C}>0$$

Because $\tan A \tan B \tan C < 0$

$\therefore$ $\tan A \tan B + \tan B \tan C + \tan C \tan A < 0$

Value of $\tan A \tan B + \tan B \tan C + \tan C \tan A$ on number line is

and hence does not lie between 0 and 9

---

9. H.C.F of $x, y, x+y$, L.C.M. of x and y is common H.C.F of 52 and 168 is 4. Let $x = 4m$, $y = 4n$ where m and n are coprime (No common factor)

$\therefore$ $4m + 4n = 52$, L.C.M. $= \dfrac{(4m)(4n)}{4} = 168$

$\therefore$ $m + n = 13$, $mn = 42$; on solving $m = 7, n = 6$

$\therefore$ $x = 4m = 28$, $y = 4n = 24$

The two numbers are 28, 24

---

10. $$S_{n+1} = (n+1)\sum_{i<j}(i\ j)+S_n$$

$$= \frac{1}{2}\left\{\left(\frac{n(n+1)}{2}\right)^2-\frac{n(n+1)\ (2n+1)}{6}\right\}+S_n$$

$$S_{n+1} = \frac{(n-1)n(n+1)^2(3n+2)}{24}+S_n;$$

Here $S_1 = 0$, $S_2 = 0$

For $n = 2$, $S_3 = \dfrac{1\times 2\times (3)^2\times 8}{24} + S_2 = 6 + 0 = 6$

For $n = 3$,

$$S_4 = \frac{2\times 3\times (4)^2\times 11}{24}+S_3 = 44+6 = 50 \text{ and so on.}$$

Second Method

$$S_{n+1} = \sum_{k=1}^{n}\frac{(k-1)k(k+1)^2(3k+2)}{24}$$

---

11. Area upto x-axis $= \displaystyle\int_a^b y\,dx = \int_a^b (2t)\cdot\frac{dx}{dt}\,dt$

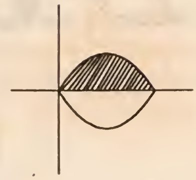
$$= \int_1^2 (2t)\,2t\,dt = \left.\frac{4t^3}{3}\right]_1^2 = \frac{4(8-1)}{3} = \frac{28}{3} \text{ units.}$$

---

12. $x = a\sin t$, $y = a\sin 2t$

$$\frac{dx}{dt} = a\cos t,\quad \frac{dy}{dt} = 2a\cos 2t,\quad \frac{dy}{dx} = \frac{2\cos 2t}{\cos t}$$

t	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
x	0	$a/\sqrt{2}$	a	$a/\sqrt{2}$	0
y	0	a	0	$-a$	0
$\frac{dy}{dx}$	2	0	∞	0	-2



$$\text{Area} = \int_0^\pi y\frac{dy}{dx}\,dt = \int_0^\pi (a\sin 2t)a\cos dt$$

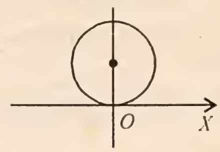
Put $\cos t = z$, $-\sin t\ dt = dz$

$$= \int_{-1}^{1} 2a^2z^2dz = 4a^2\int_0^1 z^2dz = \frac{4a^2}{3}.$$

---

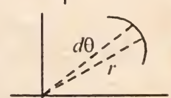
13. Curve $r = 2\sin\theta$

θ	0	$\pi/4$	$\pi/2$	$3\pi/4$	π
r	0	$\sqrt{2}$	2	$\sqrt{2}$	0



$$\text{Area} = \int_0^\pi \frac{1}{2}r^2\cdot d\theta = \int_0^\pi 2\sin^2\theta\cdot d\theta$$

$$= 4\int_0^{\pi/2}\sin^2\theta\,d\theta = \pi.$$



Area of sector $= \frac{1}{2}r^2\,d\theta$

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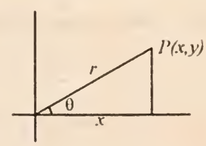
14. To convert Polar curve eq. to cartesean eq.

$r = \sqrt{x^2+y^2}$

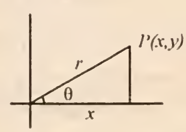


$x = r\cos\theta$ $\quad\therefore\ \cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$

$y = r\sin\theta$ $\quad\therefore\ \sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$

$\therefore$ put $r = \sqrt{x^2+y^2}$; $\quad \sin\theta = \dfrac{y}{\sqrt{x^2+y^2}}$,

$\cos\theta = \dfrac{x}{\sqrt{x^2+y^2}}$, $\quad \tan\theta = \dfrac{y}{x}$



$$\sqrt{x^2+y^2} = 2\frac{y}{\sqrt{x^2+y^2}}$$

or $x^2 + y^2 = 2y$ or $x^2 + y^2 - 2y = 0$

■

Extra edge at IIT-JEE

By : Zafar Ahmed, e-mail : zai-alpha@hotmail.com

The problems appearing in this section are meant for those students who are very well prepared for IIT-JEE and looking for new types of problems. The standard of these problems may be only slighly more than that of JEE and this endows them with an Extra Edge at IIT-JEE. Those who are not so well prepared may look into these problems/solutions at a later stage.

1. If $2a + 10b + 29c = 0$, prove that the equation $ax^3 + bx + c = 0$ will have at least one root in (0, 1).

2. Find the image of $x^2 - y^2 = 1$ under inversion ($\omega = 1/z$) in Argand plane ($\omega = u + iv$). Also find the area enclosed by the transformed curve.

3. Determine whether S_{2m} and S_{2m+1} can be found in closed forms, if $S_n = \sum_{k=0}^{n} (-1)^k \sin^n \frac{\pi k}{n}$.

4. Given that $\lim_{n\to\infty} \sum_{r=0}^{n} f(r/n)\frac{1}{n} = \int_0^1 f(x)dx + \frac{1}{2n}[f(1) - f(0)]$

   is a more accurate formula to convert such summations into integrations. Hence evaluate

$$\lim_{n\to\infty}\left(\frac{2^z + 4^z + 6^z + \ldots + (2n)^z}{1^z + 3^z + 5^z + \ldots + (2n-1)^z}\right)^n, \text{ if } z > 0.$$

5. For positive numbers a and b prove that $\frac{a+b}{2} - \sqrt{ab} \geq \frac{(a-b)^2(a+3b)(b+3a)}{8(a+b)(a^2+6ab+b^2)}$.

6. Prove that $\int_0^{\pi/2} \log(\sin x)\log(\tan x)dx < 2$.

7. Find all values of k, if straight lines $x + 2y = 3$, $2x + 3y = 5$, $k^2x + ky = -1$ represent a triangle which is not right-angled.

8. The triplet of vectors $\vec{u}_1, \vec{u}_2, \vec{u}_3$ are linearly independent non-orthogonal vectors. If the triplet $\vec{v}_1, \vec{v}_2, \vec{v}_3$ are orthogonal and of the type $\vec{v}_1 = \vec{u}_1$, $\vec{v}_2 = \vec{u}_2 + \alpha\vec{v}_1$, $\vec{v}_3 = \vec{u}_3 + \beta\vec{v}_1 + \gamma\vec{v}_2$. Find α, β, γ. Hence or otherwise find a triplet of normalized orthogonal vectors for $-i + j + k$, $i - j + k$, $i + j - k$.

9. If A is a non-singular square matrix whose elements are in general complex, prove that $\det(I + AA^T) = \det(I + A^TA)$ and that $\det(I + AA^*)$ is real. Here A^T and A^* denote the transpose and the complex-conjugate of the matrix A, respectively.

10. Out of the six planes $ax + z = f$, $x + cy + bz = g$, $x + dy + bz = h$, $ax + y + z = a^2$, $dy + cz = 0$, $fx + gy + hz = 0$, the first three planes are known to meet in a line. Prove that the next three planes will not meet in a point and if $adf \neq 0$ the last three planes will make a prism.

BRAIN TWISTERS for IIT MAINS

By : Er. S.K. Mishra, Career Point, Gorakhpur

1. Two lines L_1 and L_2 are drawn from point (α, α) making an angle $\pi/4$ with the lines $L_3 \equiv x + y - f(\alpha) = 0$ and $L_4 \equiv x + f(\alpha) + y = 0$. L_1 intersects L_3 and L_4 at A and B and L_2 intersect L_3 and L_4 at C and D respectively ($|2\alpha| > |f(\alpha)|$). If area of trapezium $ABCD$ is independent of α, find $f(\alpha)$.

2. Let $\alpha + \beta = 1$, $2\alpha^2 + 2\beta^2 = 1$ and $f(x)$ be a continuous function such that $f(x) = 2 - f(x+2)$ for all $x \in [0, 2]$ and $p = \int_0^1 f(x)dx - 4$, $q = \frac{\alpha}{\beta}$. Then find the least positive integral values of a for which the equation $ax^2 - bx + c = 0$ has both roots lie between p and q, where $a, b, c \in N$.

3. If $f(x)$ is a polynomial with integral coefficient such that $f(\alpha_1) = f(\alpha_2) = f(\alpha_3) = f(\alpha_4) = f(\alpha_5) = m$ where $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ are all different integers and m is a prime number. Prove that there exist no integer β such that $f(\beta) = M$, where M is another prime number such that $M - m = 5$.

4. A function $f : (0, \infty) \to R$ satisfy the equation $f(xy) = 2f(x) - f\left(\frac{x}{y}\right)$. If f is differentiable on R and $f(1) = 0, f'(1) = 1$, then show that

(i) $f(y) = -f\left(\frac{1}{y}\right)$ (ii) $f'(x) = \frac{1}{x}$.

And hence determines $f(x)$.

5. Let $f(x)$ is the periodic function such that $\int_0^x (f(t))^3 dt = \frac{1}{x^2}\left[\int_0^x f(t)dt\right]^3 \ \forall\ x \in R - \{0\}$. Find the function $f(x)$ if $f(1) = 1$.

6. Find the value of t so that volume contained inside the plane $\sqrt{1-t^2}\,x + tz = 2\sqrt{1-t^2}$; $z = 0$; $x = 2 + \frac{t\sqrt{4t^2 - 5t + 2}}{12(1-t^2)^{1/4}}$ and $|y| = 2$ is maximum.

7. A variable point C is on the circle $x^2 + y^2 - 1 = 0$ lying on XY plane. From point C perpendicular CN is drawn to the line $\frac{x-2}{1} = \frac{y-2}{1} = -\frac{z}{1}$. Find minimum length of perpendicular.

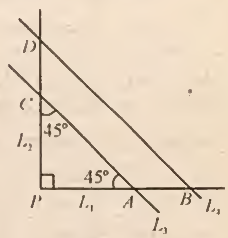
8. Show that $\lim\limits_{n \to \infty} \prod\limits_{r=1}^{n} \left[\phi\left(a + \frac{br}{n}\right)\right]^{1/n} = e^{\lambda}$, where $\lambda = \frac{1}{6}\int_a^{a+b} \log_e \phi(x)\,dx$.

9. Consider A is any point and O being origin. The circle on OA as diameter is drawn. Points B and C are taken on the circle to lie on the same side of OA such that $\angle AOB = \angle B = \angle C = \phi$. If A, B, C are z_1, z_2, z_3 such that $2\sqrt{3}z_2^2 = (2+\sqrt{3})z_1 \cdot z_3$ then find angle ϕ.

10. If $f(x) = (x-a)(x-b) - \left(\frac{a+b}{2}\right)$ and $f(x) = 0$ has both non negative roots, then prove that $f(x) \geq -\frac{(a+b)^2}{4}$.

SOLUTION

1. Distance between parallel lines AC and BD is



$$\frac{2|f(\alpha)|}{\sqrt{2}} = \sqrt{2}|f(\alpha)| ;$$

$$AC = 2\left(\sqrt{2}\alpha - \frac{|f(\alpha)|}{\sqrt{2}}\right)$$

And $BD = 2\left(\sqrt{2}\alpha + \frac{|f(\alpha)|}{\sqrt{2}}\right)$;

So $AC + BD = 4\sqrt{2}\alpha$

Again area of trapezium $ABDC = \frac{1}{2} \cdot \sqrt{2}f(\alpha) \cdot 4\sqrt{2}\alpha$

$$= 4\alpha \cdot f(\alpha)$$

If area is independent of α, then $f(\alpha) = p/\alpha$ where $p \in R$.

2. As given we have $\alpha + \beta = 1$ ...(1)

$$2\alpha^2 + 2\beta^2 = 1 \qquad ...(2)$$

From above two equations $\alpha = \beta = 1/2$

$$\Rightarrow \frac{\alpha}{\beta} = 1 = q \qquad ...(3)$$

and given $f(x)+f(x+2) = 2 \ \forall \ x \in [0, 2]$ ...(4)

Thus $p = \int_0^4 f(x)dx - 4 = \int_0^2 f(x)dx + \int_2^4 f(x)dx - 4$

$= \int_0^2 f(x)dx + \int_0^2 f(t+2)dt - 4$ [suppose $x = t + 2$]

$= \int_0^2 f(x)dx + \int_0^2 [2-f(x)]dx - 4 = 2\int_0^2 dx - 4 = 0$

Then $p = 0$; $q = 1$; Let the roots of equation $ax^2 - bx + c = 0$ be α and β

Let $f(x) = ax^2 - bx + c$;

Now $f(x) = a(x - \alpha)(x - \beta)$ ...(5)

Since equation $f(x) = 0$ has both roots between 0 and 1

$$\therefore \ f(0)\cdot f(1) > 0 \qquad ...(6)$$

But $f(0) \cdot f(1) = c(a - b + c) =$ an integer ...(7)

$\therefore$ Least value of $f(0) \cdot f(1) = 1$ ...(8)

Now from equation (5)

$$f(0) \cdot f(1) = a\alpha\ \beta a(1 - \alpha)(1 - \beta)$$
$$= a^2(\alpha\beta)(1 - \alpha)(1 - \beta) \qquad ...(9)$$

$\alpha(1 - \alpha)$ has greatest value 1/4 at $\alpha = \frac{1}{2}$ and $\beta(1 - \beta)$ has greatest value $\frac{1}{4}$ at $\beta = \frac{1}{2}$.

But $\alpha \neq \beta$. Then from equation (8) greatest value of

$$f(0)\cdot f(1) < \frac{a^2}{16} \qquad ...(10)$$

From equation (8) and (10) :

$$1 < \frac{a^2}{16} \Rightarrow a^2 - 16 > 0;\ a < -4 \text{ or } a > 4$$

$a \in N \Rightarrow$ least value of $a = 5$.

3. Let if possible there exists $\beta \in N$ such that $f(\beta) = M$. Since difference of two prime numbers can be odd only when one prime number is 2.

$\Rightarrow M = 7$ and $m = 2 \Rightarrow f(\beta) = 7$;

i.e. $f(x) - 7 = (x - \beta)g(x) \Rightarrow -5 = (\alpha_1 - \beta) \cdot g(\alpha_1)$

$\Rightarrow$ values of $(\alpha_1 - \beta)$, $(\alpha_2 - \beta)$, $(\alpha_3 - \beta)$, $(\alpha_4 - \beta)$ and $(\alpha_5 - \beta)$ can be 5, –5, 1 or –1

$\Rightarrow$ any two of $\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5$ must be equal hence contradiction.

4.(i) $f(xy) = 2f(x) - f\left(\frac{x}{y}\right)$

Since $f(1) = 0$; putting $x = 1$ in above functional equation

$$f(y) = -f\left(\frac{1}{y}\right) \qquad ...(1)$$

(ii) $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f\left\{x\left(1+\frac{h}{x}\right)\right\} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2f\left(1+\frac{h}{x}\right) - f\left(\frac{1}{x}+\frac{h}{x^2}\right) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{1}{x}\left[\frac{2\cdot f\left(1+\frac{h}{x}\right) - f(1)}{h/x}\right] - \frac{1}{x^2}\lim_{h\to 0}\frac{f\left(\frac{1}{x^2}+\frac{h}{x^2}\right)+f(x)}{h/x^2}$$

$$= \frac{2}{x}f'(1) - \frac{1}{x^2}\lim_{h\to 0}\frac{f\left(\frac{1}{x}+\frac{h}{x^2}\right) - f\left(\frac{1}{x}\right)}{h/x^2} = \frac{2}{x} - \frac{1}{x^2}f'\left(\frac{1}{x}\right)$$

Differentiating (1) w.r.t. y, we have

$$f'(y) = f'\left(\frac{1}{y}\right)\cdot\frac{1}{y^2}$$

using this result we have

$$f'(x) = \frac{2}{x} - f'(x) = 0 \text{ i.e. } f'(x) = \frac{1}{x}$$

On integration $f(x) = \log_e x + k$;

$f(1) = 0$ gives $k = 0 \Rightarrow f(x) = \log_e x$.

5. Let antiderivative of $\int_0^x f(t)dt$ is $g(x)$

$$\Rightarrow f(x) = g'(x) \Rightarrow \int_0^x (g'(x))^3 dx = \frac{1}{x^2}(g(x))^3;$$

Differentiation w.r.t. x

$$(g'(x))^3 = \frac{3x^2(g(x))^2 g'(x) - (g(x))^3\cdot 2x}{x^4}$$

$$x^3\cdot(g'(x))^3 = 3x(g(x))^2 g'(x) - 2(g(x))^3$$

$$\left\{\frac{xg'(x)}{g(x)}\right\}^3 = 3\left\{\frac{xg'(x)}{g(x)}\right\} - 2$$

Let $p = \left\{\frac{xg'(x)}{g(x)}\right\} \Rightarrow p^3 - 3p + 2 = 0,\ p = 1$ or $p = -2$

$$x\frac{g'(x)}{g(x)} = 1$$

$g(x) = cx$ (on integration) $\therefore f(x) = c$, $(f(x) = g'(x))$

But when $p = -2$ function is not periodic.

Hence $f(x) = c$ is the required function

Now as $f(1) = 1 \Rightarrow c = 1$. Thus $f(x) = 1$.

6. All the given planes are at right angles to xz plane, so a cross section parallel to xz plane will be same every where. Thus volume will be maximum when area of triangular cross section is maximum.

If we cut it by $y = 0$, the triangle so obtained is ABC

where $A = t\sqrt{\dfrac{4t^2-5t+2}{12(3-t^2)^{1/4}}}$;

Let $\angle CAB = \theta$ = angle between first plane and XY plane, then $\cos\theta = t$

Area of $\Delta ABC = \Delta = \frac{1}{2}(AB)^2\tan\theta$

$$\Delta = \frac{1}{2}\frac{t^2(4t^2-5t+2)}{12(1-t^2)^{1/2}}\times\frac{\sqrt{2-t^2}}{t} = \frac{1}{2}\frac{4t^3-5t^2+2t}{12}$$

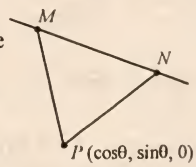
For maxima or minima

$$\frac{d\Delta}{dt} = 0 \Rightarrow 12t^2-10t+2 = 0$$

$$\Rightarrow t = \frac{1}{2} \text{ or } \frac{1}{3};\ \frac{d\Delta^2}{dt^2} = 24t-10$$

Δ is max at $t = 1/3$. $\left[\because \dfrac{d^2\Delta}{dt^2} < 0 \text{ for } t = \dfrac{1}{3}\right]$

7. Let the point P $(\cos\theta, \sin\theta, 0)$ and a fixed point M on the given line be $(2, 2, 0)$



MN = projection of PM on the line

$$= (\cos\theta-2)\cdot\frac{1}{\sqrt{3}} + \frac{(\sin\theta-2)}{\sqrt{3}}$$

$$(PM)^2 = (\cos\theta-2)^2+(\sin\theta-2)^2 - \frac{1}{\sqrt{3}}\left[(\cos\theta-2)+\frac{1}{\sqrt{3}}(\sin\theta-2)\right]^2$$

$$(PN)^2 = \frac{2}{3}[5-2\cos\theta-2\sin\theta-\cos\theta\cdot\sin\theta];$$

Let $x = 5 - 2\cos\theta - 2\sin\theta - \cos\theta \cdot \sin\theta$

$$\frac{dx}{d\theta} = 2\sin\theta-2\cos\theta+\sin^2\theta-\cos^2\theta$$

$$= (\sin\theta-\cos\theta)(\sin\theta+\cos\theta+2) = 0$$

$$\Rightarrow \tan\theta = 1 \Rightarrow \theta = \frac{\pi}{4}, \frac{5\pi}{4}$$

$X_{\min}$ occurs when $\theta = \frac{\pi}{4}$ as $\frac{dx}{d\theta}$ changes sign from negative to positive.

$$\Rightarrow X_{\min} = 5-\frac{2}{\sqrt{2}}-\frac{2}{\sqrt{2}}-\frac{1}{2} = \frac{9}{2}-2\sqrt{2}$$

$$\Rightarrow PN_{\min} = \sqrt{\frac{2}{3}\left(\frac{9}{2}-2\sqrt{2}\right)} = \sqrt{3-\frac{4\sqrt{2}}{3}}.$$

8. $S = \lim\limits_{n\to\infty}\left[\prod\limits_{r=1}^{n}\phi\left(a+\frac{br}{n}\right)\right]^{1/n}$

$$= \lim_{n\to\infty}\left[\phi\left(a+\frac{b}{n}\right)\cdot\phi\left(a+\frac{2b}{n}\right)....\phi\left(a+\frac{nb}{n}\right)\right]$$

Taking log of both sides with respect to base e

$$\log_e S = \lim_{n\to\infty}\frac{1}{n}\left[\log_e\phi\left(a+\frac{b}{n}\right)+...\log_e\phi\left(a+\frac{nb}{n}\right)\right]$$

$$= \int_0^1\log_e\phi(a+bx)dx = \frac{1}{b}\int_a^{a+b}\log_e\phi(z)dz$$

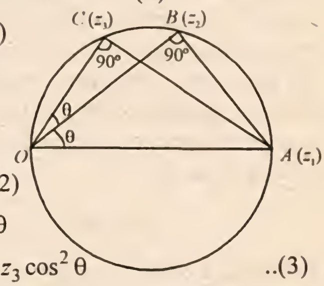
(by $a + bx = z$)

9. By rotation formula we have

$$z_2 = \frac{OB}{OA}z_1\cdot e^{i\theta} = z_1\cos\theta\cdot e^{i\theta} \quad ....(1)$$

$$z_3 = \frac{OC}{OA}z_1\cdot e^{2i\theta} = z_1\cos 2\theta\cdot e^{2i\theta} \quad ....(2)$$

(considering ΔOCA)



Now $z_2^2 = z_1^2\cos^2\theta\cdot e^{2i\theta}$

$$\Rightarrow z_2^2 = \frac{z_1^2 z_3\cos^2\theta}{z_1\cos 2\theta}$$

(from equation 2)

$$\Rightarrow z_2^2\cos 2\theta = z_1 z_3\cos^2\theta$$

$$\Rightarrow 2\sqrt{3}z_2^2\cos 2\theta = 2\sqrt{3}\,z_1 z_3\cos^2\theta \quad ..(3)$$

It is given that $2\sqrt{3}z_2^2 = (2+\sqrt{3})z_1 z_3$; using this result in equation 3, $(2+\sqrt{3})z_1 z_3\cos 2\theta = 2\sqrt{3}\,z_1 z_3\cos^2\theta$

$$\Rightarrow (2+\sqrt{3})(2\cos^2\theta-1) = 2\sqrt{3}\cos^2\theta$$

$$\Rightarrow \cos^2\theta = \frac{2+\sqrt{3}}{4} = \left(\frac{\sqrt{3}+1}{2\sqrt{2}}\right)^2;\ \theta = 15°;\ (\theta \text{ is acute}).$$

10. Given that $f(x) = (x-a)(x-b)-\left(\frac{a+b}{2}\right)$

sum of roots of the equation $f(x) = 0$, will be positive $\Rightarrow (a + b) > 0$; the product of the root's of the equation will be greater than and equal to zero

$$\Rightarrow ab-\left(\frac{a+b}{2}\right) \ge 0$$

Now $f(x)$ will be minimum, when $f'(x) = 0$

$$\Rightarrow x = \frac{a+b}{2}$$

$$\Rightarrow (f(x))_{\min} = \left(\frac{a+b}{2}\right)^2-\left(\frac{a+b}{2}\right)^2+ab-\left(\frac{a+b}{2}\right)$$

$$= -\frac{(a+b)^2}{4}+ab-\left(\frac{a+b}{2}\right) = -\frac{(a+b)^2-2(a+b)+4ab}{4}$$

$$\Rightarrow f(x) \ge \frac{-(a+b)^2-2(a+b)+4ab}{4}$$

$$\Rightarrow f(x) \ge \frac{-(a+b)^2-4ab+4ab}{4} \Rightarrow f(x) \ge \frac{-(a+b)^2}{4}.$$

■

वैदिक Vedic Mathematics

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Contd. from July issue

By - Ramkrishna B.S. Khandeparkar, Dept. of Mathematics,
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CUBE ROOT USING LOGARITHM TABLE:

Using logarithm table we can find cube root of a number in a single step. But this step entails having to find the value of the antilogarithm of a number. Since the logarithm table or antilogarithm are listing each number only in four or five digits, we can find the cube root with precision of at most 4 or 5 digits using logarithm table. These table also refer to logarithms where the base is TEN. To find cube root of a, we have

$$\log_{10} a^{1/3} = \frac{1}{3}\log_{10} a$$

Example 1 : Now, Let $a = 6$, then $\log_{10} 6^{1/3}$

$$= \frac{1}{3}\log_{10} 6 = \frac{1}{3}(0.7781) \quad \text{[4 digits of precision]}$$
$$= 0.2594.$$

Now, to get value of a, we need to find antilogarithm of 0.2594 which gives 1.8172.

$$\sqrt[3]{6} = 1.8172.$$

CUBE ROOTS USING NEWTON – RAPHSON METHOD

Newton – Raphson method is much better when compared with other methods, specially when we aim at accuracy to as many decimals as we need. According to this method, the cube root of a number a is found by beginning with a guess, say x_0, of the cube root. To find cube root of 6, let us take

$$x_0 = \frac{3}{2}$$

This guess x_0 is based on simple reasoning that cube of 2 is 8 and therefore 2 is too high and cube of 1 is 1 which means 1 is too low. We have the formula,

$$x_1 = \frac{1}{3}\left[2x_0 + \frac{a}{x_0^2}\right]$$

Example 2 : Let $a = 6$ or we find $\sqrt[3]{6}$ using Newton's method.

$$x_1 = \frac{1}{3}\left\{2\times\frac{3}{2} + \frac{6}{(9/4)}\right\} = \frac{1}{3}\left\{3 + \frac{24}{9}\right\} = \frac{51}{27} = 1.888...$$

We can repeat the process, and successively obtain better and better approximation

by taking $x_0 = \dfrac{51}{27}$ gives $\sqrt[3]{6} = 1.8198$

The same thing can also be achieved by solving equation $x^3 = 6$

Let $f(x) = x^3 - 6,\ f'(x) = 3x^2$

Let x_0 be the first guess at the root function then final approximation can be obtained using

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)}$$

$$x_0 = \frac{3}{2} \text{ then } x_1 = \frac{3}{2} - \frac{(3/2)^3 - 6}{3(3/2)^2} = 1.888....$$

Taking $x_0 = 1.8888$, a better approximation to cube root can be obtained.

Example 3 : To find cube root of 89

Let $x_0 = 4.5$ then

$$x_1 = \frac{1}{3}\left\{2\left(\frac{9}{2}\right) + \frac{89}{(9/2)^2}\right\} = 3 + \frac{356}{243} = 4.4650$$

Alternatively, we solve $x^3 = 89$

Let $f(x) = x^3 - 89, f'(x) = 3x^2$

$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 4.5 - \frac{(4.5)^3 - 89}{3(4.5)^2}$$

$$= 4.5 - \frac{2.125}{60.75} = 4.5 - 0.03498 = 4.4650$$

Babylonian method :

The algorithm given below is named as modified Babylonian algorithm. I am, rather doubtful, whether this method was known to them. Recently the view that Vedic Mathematics is a derivative of old

Mathematics Olympia for IIT-JEE (MAINS) 2006

By : Er. Akhlak Ahmad, ABC Classes, Gorakhpur

1. If n is a positive integer, prove that $\left|\operatorname{Im}\left(z^n\right)\right| \le n\left|\operatorname{Im}(z)\right||z|^{n-1}$.

2. If $1, \alpha_1, \alpha_2, \ldots\ldots, \alpha_{n-1}$ are the n, nth roots of unity, then prove that $\frac{1}{2-\alpha_1}+\frac{1}{2-\alpha_2}+\ldots+\frac{1}{2-\alpha_{n-1}}$ $=\frac{(n-2)2^{n-1}+1}{2^n-1}$.

3. Find a quadratic equation whose roots x_1 and x_2 satisfy the condition
$$x_1^2+x_2^2=5,\ 3(x_1^5+x_2^5)=11\left(x_1^3+x_2^3\right),$$
(Assume that x_1, x_2 are real).

4. If $I_n=\int_0^{\pi/4}\tan^n x\,dx$, show that $\frac{1}{I_2+I_4}$, $\frac{1}{I_3+I_5}, \frac{1}{I_4+I_6}, \frac{1}{I_5+I_7}, \ldots\ldots$ form an A.P. Find its common difference.

5. Show that $\left[\left(\sqrt{3}+1\right)^{2n}\right]+1$ is divisible by 2^{n+1} for all $n \in N$, where [.] denotes the greatest integer function.

6. If $x > 0$, $y > 0$, $z > 0$, prove that $x^{\log_e y-\log_e z}+y^{\log_e z-\log_e x}+z^{\log_e x-\log_e y} \ge 3$.

7. If a, b, c are all positive and $a < b < c$, show that $\frac{a^2}{c}<\frac{a^2+b^2+c^2}{a+b+c}<\frac{c^2}{a}$.

8. Consider the cartesian plane R^2, and let X denote the subset of points for which both co-ordinates are integers. A coin of diameters 1/2 is tossed randomly onto the plane. Find the probability p that the coin covers a point of X.

9. If $x^2y = 2x - y$ and $|x|<1$, then prove that $y+\frac{y^3}{3}+\frac{y^5}{5}+\ldots\ldots\infty=2\left[x+\frac{x^3}{3}+\frac{x^5}{5}+\ldots\ldots\infty\right]$.

10. Find the matrices of transformations T_1T_2 and T_2T_1, when T_1 is rotation through angle 60° and T_2 is the reflection in the y-axis. Also verify that $T_1T_2 \ne T_2T_1$.

SOLUTION

1. We have $\left|\frac{\operatorname{Im}\left(z^n\right)}{\operatorname{Im} z}\right|=\left|\frac{z^n-(\bar{z})^n}{z-\bar{z}}\right|$

$$=\left|z^{n-1}+z^{n-2}\bar{z}+z^{n-3}(\bar{z})^2+\ldots\ldots+(\bar{z})^{n-1}\right|$$
$$\le|z|^{n-1}+|z|^{n-2}|\bar{z}|+|z|^{n-3}|\bar{z}|^2+\ldots\ldots+|\bar{z}|^{n-1}$$
$$=|z|^{n-1}+|z|^{n-1}+|z|^{n-1}+\ldots\ldots+|z|^{n-1}=n|z|^{n-1}$$
$$\left|\frac{\operatorname{Im}\left(z^n\right)}{\operatorname{Im}(z)}\right| \le n|z|^{n-1},\quad \left|\operatorname{Im}\left(z^n\right)\right| \le n|\operatorname{Im} z||z|^{n-1}$$

2. $\because\quad x^n-1=(x-1)(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_{n-1})$

Taking logarithm both sides
$\ln(x^n-1) = \ln(x-1) + \ln(x-\alpha_1) + \ln(x-\alpha_2) + \ldots + \ln(x-\alpha_{n-1})$

Differentiating both sides w.r.t. x then
$$\frac{nx^{n-1}}{x^n-1}=\frac{1}{(x-1)}+\frac{1}{(x-\alpha_1)}+\frac{1}{(x-\alpha_2)}+\ldots+\frac{1}{(x-\alpha_{n-1})}$$
Putting $x = 2$ then we get
$$\frac{n(2)^{n-1}}{2^n-1}-1=\frac{1}{2-\alpha_1}+\frac{1}{2-\alpha_2}+\ldots+\frac{1}{2-\alpha_{n-1}}$$

3. We have $3\left(x_1^5+x_2^5\right)=11\left(x_1^3+x_2^3\right)$
$$\Rightarrow\quad \frac{x_1^5+x_2^5}{x_1^3+x_2^3}=\frac{11}{3}$$

$$\Rightarrow \quad \frac{\left(x_1^2+x_2^2\right)\left(x_1^3+x_2^3\right)-x_1^2x_2^2\left(x_1+x_2\right)}{\left(x_1^3+x_2^3\right)}=\frac{11}{3}$$

$$\Rightarrow \quad \left(x_1^2+x_2^2\right)-\frac{x_1^2x_2^2\left(x_1+x_2\right)}{\left(x_1+x_2\right)\left(x_1^2+x_2^2-x_1x_2\right)}=\frac{11}{13}$$

$$\Rightarrow \quad 5-\frac{x_1^2x_2^2}{5-x_1x_2}=\frac{11}{3} \Rightarrow \frac{4}{3}=\frac{x_1^2x_2^2}{5-x_1x_2}$$

$$\Rightarrow \quad 3x_1^2x_2^2+4x_1x_2-20=0$$

$$3x_1^2x_2^2+10x_1x_2-6x_1x_2-20=0$$

$$\Rightarrow \quad (x_1x_2-2)(3x_1x_2+10)=0 \quad \therefore \quad x_1x_2=2,-\frac{10}{3}$$

We have $(x_1+x_2)^2=x_1^2+x_2^2+2x_1x_2$

$= 5 + 2x_1x_2$

$\therefore$ $(x_1 + x_2)^2 = 5 + 4$ (if $x_1 x_2 = 2$)

$= 9$

$\therefore$ $x_1 + x_2 = \pm 3$

$(x_1 + x_2)^2 = 5 + 2(-10/3)$ (if $x_1x_2 = -10/3$)

$=\frac{-5}{3}$

which is not possible x_1, x_2 are real

Thus required quadratic equations are $x^2 \pm 3x+2=0$

We have $I_n+I_{n+2}=\int_0^{\pi/4}\left(\tan^n x+\tan^{n+2}x\right)dx$

$$=\int_0^{\pi/4}\tan^n x\left(1+\tan^2 x\right)dx$$

$$=\int_0^{\pi/4}\tan^n x\cdot\sec^2 x\,dx=\left[\frac{\tan^{n+1}x}{n+1}\right]_0^{\pi/4}=\frac{1}{n+1}$$

$$\frac{1}{I_n+I_{n+2}}=n+1$$

Putting $n = 2, 3, 4, 5, \ldots\ldots$

$$\frac{1}{I_2+I_4}=3,\ \frac{1}{I_3+I_5}=4,\ \frac{1}{I_4+I_6}=5,\ \frac{1}{I_5+I_7}=6,\ \ldots.$$

Hence $\frac{1}{I_2+I_4},\frac{1}{I_3+I_5},\frac{1}{I_4+I_6},\frac{1}{I_5+I_7},\ldots\ldots$ are in A.P. with common difference is 1.

Let $x=\left(\sqrt{3}+1\right)^{2n}=[x]+f$ ...(1)

where $0\le f<1$

Now let $\left(\sqrt{3}-1\right)^{2n}=f'$ ...(2)

where $0 < f' < 1$

Adding (1) and (2) we get

$$\Rightarrow \quad [x]+f+f'=\left(\sqrt{3}+1\right)^{2n}+\left(\sqrt{3}-1\right)^{2n}$$

$$=\left(4+2\sqrt{3}\right)^n+\left(4-2\sqrt{3}\right)^n=2^n\left\{\left(2+\sqrt{3}\right)^n+\left(2-\sqrt{3}\right)^n\right\}$$

$$=2^n\cdot 2\left\{{}^nC_0(2)^n+{}^nC_2(2)^{n-2}\left(\sqrt{3}\right)^2+{}^nC_4(2)^{n-4}\left(\sqrt{3}\right)^4+\ldots\right\}$$

$\Rightarrow$ $[x]+f+f'=2^{n+1}k.$ ...(3) (where k is an Integer)

Hence $f + f'$ is an integer.

$\Rightarrow$ *i.e.*, $f+f'=1$ $\{\because 0<f+f'<2\}$

from (3) $[x] + 1 = 2^{n+1} k$

$$\left[\left(\sqrt{3}+1\right)^{2n}\right]+1=2^{n+1}k \qquad \text{[from (1)]}$$

this shows that $\left[\left(\sqrt{3}+1\right)^{2n}\right]+1$ divisible by 2^{n+1} for all $n\in N$.

6. Let $\log_e x = a$, $\log_e y = b$ and $\log_e z = c$

$\therefore$ $x = e^a$, $y = e^b$, and $z = e^c$

then given inequality becomes

$$e^{a(b-c)}+e^{b(c-a)}+e^{c(a-b)}\ge 3$$

Since A.M. $\ge$ G.M.

$$\therefore \quad \frac{e^{a(b-c)}+e^{b(c-a)}+e^{c(a-b)}}{3}\ge\left[e^{a(b-c)}\cdot e^{b(c-a)}\cdot e^{c(a-b)}\right]^{1/3}$$

or $e^{a(b-c)}+e^{b(c-a)}+e^{c(a-b)}\ge 3\left[e^0\right]^{1/3}$

or $e^{a(b-c)}+e^{b(c-a)}+e^{c(a-b)}\ge 3$

Hence, $x^{\log_e y-\log_e z}+y^{\log_e z-\log_e x}+z^{\log_e x-\log_e y}\ge 3.$

7. Since $a < b < c$

we have $a + a + a < a + b + c < c + c + c$

or $3a < a + b + c < 3c$ ...(1)

and $3a^2 < a^2 + b^2 + c^2 < 3c^2$ ...(2)

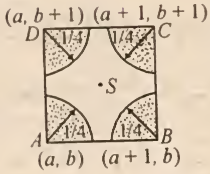
from (1) $\frac{1}{3a}>\frac{1}{a+b+c}>\frac{1}{3c}$

or $\frac{1}{3c}<\frac{1}{a+b+c}<\frac{1}{3a}$ ...(3)

Multiplying corresponding sides of (2) and (3), we get $3a^2.\frac{1}{3c}<\frac{a^2+b^2+c^2}{a+b+c}<3c^2.\frac{1}{3a}$

Hence, $\frac{a^2}{c}<\frac{a^2+b^2+c^2}{a+b+c}<\frac{c^2}{a}.$

8. Let S denote the set of points inside a square with corners (a, b), $(a, b+1)$, $(a+1, b+1)$, $(a+1, b) \in X$. Let P denote the set of points in S with distance less than $\frac{1}{4}$ from any corner point. (Observe that the area of P is equal to the area inside a circle of radius $\frac{1}{4}$).



Thus a coin, whose centre falls in S, will cover a point of X if an only if its centre falls in a point of P.

Hence, $p = \dfrac{\text{area of } P}{\text{area of } S} = \dfrac{\pi\left(\frac{1}{4}\right)^2}{1} = \dfrac{\pi}{16}$

9. Given $x^2y = 2x - y$, $y(1 + x^2) = 2x$

$\therefore \quad y = \dfrac{2x}{1+x^2}$

Now L.H.S. $= y + \dfrac{y^3}{3} + \dfrac{y^5}{5} + \ldots\infty = \dfrac{1}{2}\log_e\left(\dfrac{1+y}{1-y}\right)$

$$= \frac{1}{2}\log_e\left(\frac{1+\frac{2x}{1+x^2}}{1-\frac{2x}{1+x^2}}\right) = \frac{1}{2}\log_e\left(\frac{1+x^2+2x}{1+x^2-2x}\right)$$

$$= \frac{1}{2}\log_e\left(\frac{(1+x)^2}{(1-x)^2}\right) = \frac{1}{2}\cdot 2\log\left(\frac{1+x}{1-x}\right)$$

$$= \log_e\left(\frac{1+x}{1-x}\right) \qquad \text{given } |x| < 1$$

$$= 2\left(x + \frac{x^3}{3} + \frac{x^5}{5} + \ldots\infty\right) = \text{R.H.S.}$$

10. $T_1 = \begin{bmatrix} \cos 60^\circ & -\sin 60^\circ \\ \sin 60^\circ & \cos 60^\circ \end{bmatrix} = \begin{bmatrix} 1/2 & -\sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \text{ and } T_2 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$\therefore \quad T_1T_2 = \frac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} \times \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -1+0 & 0-\sqrt{3} \\ -\sqrt{3}+0 & 0+1 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} -1 & -\sqrt{3} \\ -\sqrt{3} & 1 \end{bmatrix}$$

$$= \begin{bmatrix} -1/2 & -\sqrt{3}/2 \\ -\sqrt{3}/2 & 1/2 \end{bmatrix} \qquad \ldots(1)$$

and $T_2T_1 = \begin{bmatrix} -1 & 0 \\ 0 & 1 \end{bmatrix} \times \dfrac{1}{2}\begin{bmatrix} 1 & -\sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix}$

$$= \frac{1}{2}\begin{bmatrix} -1+0 & \sqrt{3}+0 \\ 0+\sqrt{3} & 0+1 \end{bmatrix}$$

$$= \frac{1}{2}\begin{bmatrix} -1 & \sqrt{3} \\ \sqrt{3} & 1 \end{bmatrix} = \begin{bmatrix} -1/2 & \sqrt{3}/2 \\ \sqrt{3}/2 & 1/2 \end{bmatrix} \qquad \ldots(2)$$

It is clear from (1) and (2), $T_1T_2 \neq T_2T_1$

10 BEST PROBLEMS

ALGEBRA

By Er. Tapas Kumar Yogi, Lucknow

1. Find all numbers 'a' such that the equation $|x+|x|+a|+|x-|x|-a|=2$ has exactly three solutions.

2. Let the sequence of integers a_1, a_2, a_3, ... be defined by $a_1 = 2$, $a_2 = 3$, $a_3 = 7$, $a_4 = 43$,.... and in general $a_{n+1} = 1 + a_1 a_2 a_3 \dots a_n$ for all $n \geq 1$. If $S_n = \frac{1}{a_1} + \frac{1}{a_2} + \dots + \frac{1}{a_n}$ and $P_n = \frac{1}{a_1 a_2 \cdot a_3 \dots a_n}$ are the sum and the product of the reciprocals of the first n numbers in the sequence, then compute $S_{2005} + P_{2005}$.

3. Let $f_1, f_2, f_3 \dots$ be a sequence of integers satisfying $f_{n-1} + f_n = 2n \ \forall \ n \geq 2$. If $f_1 = 100$, Find f_{1000}.

4. Find all positive integers n such that $n^2 + 25n + 19$ is a perfect square.

5. The cubic polynomical $p(x) = x^3 - 4x^2 + 2$ has three distinct real roots say α, β and r. Find $\alpha^4 + \beta^4 + r^4$.

6. If x is any real number and $[x]$ denote the largest integer that does not exceed x. Suppose a & b are positive numbers such that $\frac{1}{a} + \frac{1}{b} = 1$. If m and n are positive integers such that $[ma] = [nb]$, show that ma & nb are integers.

7. For any positive integer n, let $S(n)$ denote the sum of its digits. Show that the equation $n + S(n) = 1{,}000{,}000$ has no solution. Also solve the equation $n + S(n) = 1{,}000{,}000{,}000$.

8. Let $F_1 = 1$, $F_2 = 1$, $F_3 = 2$, $F_4 = 3$, $F_5 = 5$ and in general $F_n = F_{n-1} + F_{n-2} \ \forall \ n \geq 3$. (This is the famous Fibonacci sequence). Show that $\frac{F_n}{F_{n-1}} < 1.7 \ \forall \ n \geq 4$.

9. Let f be a function which assigns to each positive integer 'n', a positive integer $f(n)$. We suppose that $f(ab) = f(a) \cdot f(b) - f(a) - f(b) + 2$ for all positive integers a, b and that $f(c\,!) = c\,! + 1 \ \forall \ c \geq 10^{10}$. Show that $f(n) = n + 1$ for all n.

10. Some of the people attending a party hate each other, but no one at the party hates more than three other guests. Prove that it is possible for all of the people at the party to assemble in two large rooms so that in each room, no individual hates more than one other person in that room. Assume that the hatred relation is symmetric, which means that if A hates B, then B also hates A.

SOLUTION

1. $|x+|x|+a|+|x-|x|-a|=2$ changing x to $-x$ in the LHS of the above equation, we have

$|-x+|-x|+a|+|-x-|-x|-a|$

$=|-x+|x|+a|+|-x-|x|-a|$

$=|x-|x|-a|+|x+|x|+a|=2$ from above.

Hence, if x is a solution then $-x$ is also a solution. Thus, in order to have an odd number of solutions, it is clear that $x = 0$ must be one of these.

substituting $x = 0$, we obtain

$|a|+|-a|=2 \Rightarrow 2|a|=2 \Rightarrow |a| = \pm 1$

If $a = 1$ and $x \geq 0$ then

$|x+x+1|+|x-x-1|=2;\ |2x+1|+1=2$

$2x + 2 = 2; x = 0$. But $-x$ is a solution. Iff x is, so there is only one solution in this case on the other hand if $a = -1$ and $x \geq 0$ then $|x+x-1|+|x-x+1|=2$ or, $x = 0, 1$. Thus in this case we have the three solutions $x = 0, 1, -1$.

$\therefore\ a = -1$ is the required value of a.

2. $a_{n+1} = 1 + a_1a_2a_3...a_n$ ...(1)

$S_n = \frac{1}{a_1}+\frac{1}{a_2}+...\frac{1}{a_n}$ ...(2)

$P_n = \frac{1}{a_1a_2...a_n}$ ...(3)

Let $t_n = S_n + P_n$

Now, $S_{n+1} = \frac{1}{a_{n+1}} + S_n...$ (by (2)) ...(4)

and $P_{n+1} = \frac{1}{a_{n+1}}\cdot P_n$ (by (3))

Now, $P_n - P_{n+1} = P_n - \frac{P_n}{a_{n+1}} = P_n\left[\frac{a_{n+1}-1}{a_{n+1}}\right]$

$= \frac{P_n \times (a_1a_2a_3...a_n)}{a_{n+1}}$ By (1)

$= \frac{P_n}{a_{n+1}} \times \frac{1}{P_n}$ By (3)

$= \frac{1}{a_{n+1}}$ ...(5)

Hence, from (4) and (5)

$S_{n+1} - S_n = P_n - P_{n+1} = \frac{1}{a_{n+1}}$

or, $S_{n+1} + P_{n+1} = S_n + P_n$ or, $t_{n+1} = t_n$.

$\therefore\ t_{2005} = t_{2004} = ...= t_2 = t_1 = S_1 + P_1 = 1$

3. Consider $f_{n-1} + f_n = 2n\ \forall\ n \geq 2$ ...(1)

then $f_{n-1} + f_{n-2} = 2(n-1)\ \forall\ n \geq 3$ ...(2)

substracting (2) from (1) we get

$f_n - f_{n-2} = 2\ \forall\ n \geq 3$

i.e. $f_n = 2 + f_{n-2}\ \forall\ n \geq 3$

so, $f_n = 2 + f_{n-2} = 2 + (2 + f_{n-4}) = 4 + f_{n-4}$

similarly, $f_n = 8 + f_{n-8} = \begin{cases}(n-2)+f_2 & \text{if } n \in \text{even}\\(n-1)+f_1 & \text{if } n \in \text{odd}\end{cases}$

and $f_2 + f_1 = 4$ from (1)

So, $f_2 = 4 - f_1 = 4 - 100 = -96$

Hence, $f_n = \begin{cases}(n-2)-96 & \text{if } n \in \text{even}\\(n-1)+100 & \text{if } n \in \text{odd}\end{cases}$

i.e. $f_n = \begin{cases}(n-98) & \text{if } n \in \text{even}\\n+99 & \text{if } n \in \text{odd}\end{cases}$

So, $f_{1000} = 1000 - 98 = 902$.

4. Now, $(n + 4)^2 = n^2 + 8n + 16 < n^2 + 25n + 19$ and $(n + 13)^2 = n^2 + 26n + 169 > n^2 + 25n + 19$

Hence, $n^2 + 25n + 19 = (n + k)^2 = n^2 + 2nk + k^2$ for some integer k, $5 \leq k \leq 12$.

So, from the above eqn. we have

$25n + 19 = 2nk + k^2$ or, $n = \frac{k^2-19}{25-2k}$ ...(1)

by considering the eight possible integers k between 5 and 12, we see that k can only equal 8, 11 or 12.

when $k = 8$, (1) gives $n = 5$

and $k = 11$ gives $n = 34$; $k = 12$ gives $n = 125$.

since, $n^2 + 25n + 19 = (n + k)^2$ for all such pairs (n, k), we conclude that

$n = 5, 34$ and 125 are the only possibilities.

5. $x^3 - 4x^2 + 2 = 0$ has α, β, γ as its roots so,by standard results of theory of equations, we have,

$\alpha + \beta + \gamma = 4$

$\alpha\beta + \beta\gamma + \alpha\gamma = 0$ and $\alpha\beta\gamma = -2$

Let $S_k = \alpha^k + \beta^k + \gamma^k$ for every integer $k \geq 0$ ...(1)

Now, $S_0 = \alpha^0 + \beta^0 + \gamma^0 = 3$

$S_1 = \alpha + \beta + \gamma = 4$ from above

$S_2 = \alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= (4)^2 - 2(0) = 16$

As α is a root of the equation, so $\alpha^3 - 4\alpha^2 + 2 = 0$

Multiplying by α^{k-3}, $\alpha^k - 4\alpha^{k-1} + 2\alpha^{k-3} = 0$

similarly, $\beta^k - 4\beta^{k-1} + 2\beta^{k-3} = 0$

and $\gamma^k - 4\gamma^{k-1} - 2\gamma^{k-3} = 0$

Adding and in view of (1); $S_k - 4S_{k-1} + 2S_{k-3} = 0$

So, $S_k = 4S_{k-1} - 2S_{k-3}$

Hence, $S_3 = 4S_2 - 2S_0 = 58$
and $S_4 = 4S_3 - 2S_1 = 224$
Obviously, we can continue this process to compute power sums with larger and larger exponents.

6. Let k be the integer with $k = [ma] = [nb]$.
then, $k \le m < k + 1$
or, $\frac{k}{b} \le m < \frac{k+1}{b}$ similarly, $\frac{k}{b} \le n < \frac{k+1}{b}$
Adding the above inequalities and using
$\frac{1}{a} + \frac{1}{b} = 1$, we have $k \le m + n < k + 1$
But, since $(m + n)$ is an integer (by given data), so we must have $k = m + n$ and hence all the $\le$ inequalities above must be equalities. In particular ma and nb are both equal to the integer k.

7. Consider the equation $n + S(n) = 1{,}000{,}000$.
Here $S(n) > 0$, we have $n \le 999{,}999$. Thus n has at the max. 6 digits and hence $S(n) \le 6 \times 9 = 54$. It follows that $n \ge 1{,}000{,}000 - 54 = 999{,}946$. So, $n = 999, 9xy$ where $x, y,$ are digits (i.e. between 0 & 9), with $x \ge 4$.
Now, $n = 999{,}900 + (10x) + (y)$ and
$S_{(n)} = 36 + x + y$ so the original equation becomes
$999{,}900 + 10x + y + 36 + x + y = 1{,}000{,}000$
i.e., $11x + 2y = 64$.
Obviously $x \le 5$, because if $x = 5$ then $2y = 9$ and if $x = 4$ then $2y = 20$. In either case, y is not a digit. So, the eqn. has no solution.
We solve the eqn. $n + S(n) = 1{,}000{,}000{,}000$ in a similar way. Here n has atmost 9 digits. So, $S(n) \le 9 \times 9 = 81$ and $n \ge 999{,}999{,}19$.
Thus $n = 999{,}999, 9xy = 999{,}999{,}900 + 10x + y$, with $x \ge 1$ and $S(n) = 63 + x + y$. So, the given eqn. and a bit of simplification becomes $11x + 2y = 37$; clearly x is not even and ≤ 3. If $x = 1$ then $2y = 26$ and y is not a digit. Hence $x = 3$, $2y = 4$, $y = 2$. so, $n = 999{,}999{,}932$.

8. We use mathematical induction on n to show that
$\frac{F_n}{F_{n-1}} < 1.7 \ \forall\ n \ge 4$
To start with, we have $\frac{F_4}{F_3} = \frac{3}{2} = 1.5 < 1.7$
and $\frac{F_5}{F_4} = \frac{5}{3} = 1.67 < 1.7$ and therefore we need consider only cases where $n \ge 6$. Let us assume that we have already proved.
$\frac{F_k}{F_{k-1}} < 1.7$ when $4 \le k \le n - 1$ and so in particular, we have (by taking $k = n - 1$ and then $k = n - 2$), $\frac{F_{n-1}}{F_{n-2}} < 1.7$
and $\frac{F_{n-2}}{F_{n-3}} < 1.7$
Hence, $F_n = F_{n-1} + F_{n-2} < 1.7\, F_{n-2} + 1.7\, F_{n-3}$
i.e., $F_n < 1.7\,(F_{n-2} + F_{n-3})$ or, $F_n < 1.7\, F_{n-1}$
we conclude that $\frac{F_n}{F_{n-1}} < 1.7$ and the result follows by induction.

9. Let us say that a positive integer n is "good" if $f(x) = n + 1$. Our aim, therefore, is to prove that all positive integers are good. First, we show that if n and m are positive integers and both n and nm are good, then m is also good. We have
$nm + 1 = f(nm) = f(n) \cdot f(m) - f(n) - f(m) + 2$
$= (n+1) f(m) - (n + 1) - f(m) + 2$
solving, $f(m) = m + 1$ and thus m is good, as claimed. If $c > 10^{10}$ then $c - 1 > 10^{10}$ and we know by assumption that both $(c - 1)!$ and $c\,!$ are good. But $c\,! = c \times (c-1)!$ and so if we set $n = (c - 1)$ and $m = c$, our previous result shows that c is good. Now, consider the set A of all positive integers that are not good, and assume that A is not the empty set. We have seen that no member of A can exceed 10^{10} and so A contains some largest number k. Then $(k + 1)$ is good and so is $k\,(k+1)$. If therefore follows by our earlier observation that k is also good. But this is not true since we chose k in the set A. This contradiction proves that our assumption that A is non empty must be wrong and so, A is empty. In other words, all positive integers are good.

10. Start by randomly dividing the guests in the two rooms. Measure the intra-room hatred by counting the total number N of pairs of people who are in the same room and who hate each other. If there is someone who hates more people in his own room than in the other room, choose one such person and move him to the other room. Note that this move causes the quantity N to decrease. We continue, for as long as we can move one person at a time from one room to the other, reducing N with each move. At each stage, N is an integer and since it can never be negative, we cannot continue to reduce it forever. Eventually we must arrive at a situation where no move will reduce N. When that happens, no one will hate more people in his own room than in the other room. Since each person hates at the max. three people in total, we see that when N has been reduced as far as possible, no one will hate as many as two people in his own room. ■

dividend, Q stands for quotient and R stands for remainder. Also to begin with we take an example of an exact cube and find its square root by general method and in the next article switch over to finding cube root of any number using this method.

Example 9. Find the cube root of exact cube 258474853. The given cube has 258 ′ 474 ′ 853 three groups and therefore cube root has three digits, say, xyz.
The largest perfect cube less than 258 is $216 = 6^3$. We write, $D = 3 \times 6^2 = 108$, $Q = 6$ and $R = 42$. Also we have $x = 6$

258 ′ 4 7 4 853
108 42
6 :

Next, $D = 424$, $424 \div 108$ giving $Q = 3$ and $R = 100$. We get $y = 3$

258 ′ 4 7 4853
108 42 100
6 : 3

Next, $D = 1007$, $3xy^2 = 3 \times 6 \times 3^2 = 162$
$1007 - 162 = 845$
$845 \div 108$ giving $Q = 7$ and $R = 89$. We get $z = 7$.
Now, note the following steps which are very much important when the number is not a perfect cube.

258 ′ 4 7 4 853
108 42 100 89
6 : 3 7

Next, $D = 984$, $y^3 + 6xyz = 27 + 756 = 783$.
$894 - 783 = 111$
$111 \div 108$ giving $Q = 0$ and $R = 111$.

258 ′ 4 7 4 8 53
108 42 100 89 111
6 : 3 7

Next, $D = 1118$, $3xz^2 + 3zy^2$
$= 882 + 189 = 1071$.
$1118 - 1071 = 47$
$47 \div 108$ giving $Q = 0$ and $R = 47$.

258 ′ 4 7 4 8 5 3
108 42 100 89 111 47
6 : 3 7

Next, $D = 475$, $3yz^2 = 441$.
$475 - 441 = 34$
$34 \div 108$ giving $Q = 0$ and $R = 34$.

258 ′ 4 7 4 8 5 3
108 42 100 89 111 47 34
6 : 3 7

Next, $D = 343$, $z^3 = 343$.
$343 - 343 = 0$
giving $Q = 0$ and $R = 0$.
Thus, the process terminates and we get required cube root 637.

to be continued ...

❖ ❖ ❖

Challenging PROBLEMS

By : Deptt. of Mathematics, Momentum, Jabalpur

1. Let $f(x)$ is polynomial having n real roots such that $f(-1) = f(1)$. Each root α of f is real and satisfies $0 < \alpha < 1$. Prove that the product of the roots doesn't exceed $1/4^n$.

2. Two runners A and B start at the origin and run along the positive x-axis, with b running 3 times as fast as A. An observer, standing one unit above the origin, keeps A and B in view. What is the maximum angle of sight θ between the observer's view of A and B?

3. If $f(x) = \int_1^x \frac{dt}{2+t^4}$, prove that $3f(2) < 1$.

4. Find the value of the positive integer n for which the quadratic equation $\sum_{k=1}^{n} (x+k-1)(x+k) = 10n$, has solution α and $\alpha + 1$ for some α.

5. If $\int_x^{x^2 y} f(t)dt$ is independent of x and $f(2) = 2$, find the value of $\int_1^x f(x)dx$.

6. Show that an equilateral triangle is a triangle of maximum area for a given perimeter and a triangle of minimum perimeter for a given area.

7. A boat starts off from one side of a river at a point A and heads toward a point B on the other side of the river directly across from A the river has a uniform width of c meter and its current downstream is a constant a m/s. At each moment, the boat is headed toward B with a speed through the water of b m/s. Under what conditions on a, b and c will the boat ever each the opposite bank?

8. Let $f(x)$ be a non-negative continuous function such that $f'(x) - f(x) \le 0, \forall\ x \ge 0$ and $f(0) = 0$. Find the value of $f(1)$.

ANSWERS

1. Let $f(x) = (x - \alpha_1)(x - \alpha_2) \ldots\ldots (x - \alpha_n)$

$f(-1) = f(1)$

$\Rightarrow (-1-\alpha_1)(-1-\alpha_2)\ldots(-1-\alpha_n) = (1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_n)$

$\Rightarrow (-1)^n(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_n) = (1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_n)$ ....(1)

$\because$ each root $\alpha_1, \alpha_2 \ldots \alpha_n$ is between 0 and 1.

$\therefore$ right side of equation (1) will be > 0

and if right side (+ve), then left side will also be (+ve)

$\Rightarrow (-1)^n(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_n) > 0$

$\Rightarrow (-1)^n = +\text{ve} = 1$

so, from equation (1)

$(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_n) = (1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_n)$

Now, $(1+\alpha_1)(1-\alpha_1) = 1 - \alpha_1^2 < 1$

$(1+\alpha_2)(1-\alpha_2) = 1 - \alpha_2^2 < 1$

$\Rightarrow [(1+\alpha_1)(1+\alpha_2)\ldots(1-\alpha_n)][(1-\alpha_1)(1-\alpha_2)\ldots(1-\alpha_n)] < 1$

$\Rightarrow (1+\alpha_1)(1+\alpha_2)\ldots(1-\alpha_n) < 1$ ...(2)

from AM - GM inequality,

$$\frac{1+\alpha_1}{2} \ge [(\alpha_1)(1)]^{1/2}$$

$$\frac{1+\alpha_2}{2} \ge [(1)(\alpha_2)]^{1/2}$$

$$\vdots$$

$$\frac{1+\alpha_n}{2} \ge [(1)(\alpha_n)]^{1/2}$$

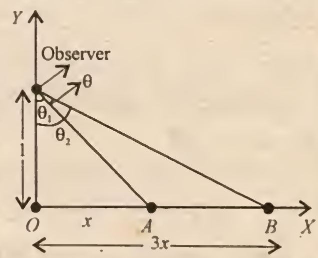
$\Rightarrow (1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_n) \ge 2^n(\alpha_1\alpha_2\ldots\alpha_n)$

$\Rightarrow \alpha_1\alpha_2\ldots\alpha_n \le \frac{1}{2^{2n}}(1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_n)$

$\Rightarrow (\alpha_1\alpha_2\ldots\alpha_n) \le \frac{1}{4^n}$

$(\because (1+\alpha_1)(1+\alpha_2)\ldots(1+\alpha_n) < 1)$

2. Let distance of A from the origin is x. Then B is $3x$ units from origin. Let θ_1 be the angle between y-axis and the line of sight of A, and θ_2 be the corresponding angle from B.



Then $\theta = \theta_2 - \theta_1$ ....(1)

From figure, $\tan\theta_1 = x$ and $\tan\theta_2 = 3x$

from equation (1),

$\tan\theta = \tan(\theta_2 - \theta_1) = \frac{\tan\theta_2 - \tan\theta_1}{1 + \tan\theta_2 \tan\theta_1}$

$$\Rightarrow \tan\theta = \frac{3x-x}{1+(3x)(x)} = \frac{2x}{1+3x^2}$$

Since, θ is beween 0 and $\pi/2$,
so, maximizing θ means to maximize tanθ

$$\Rightarrow \text{we have to maximize } \frac{2x}{1+3x^2}$$

$$\text{Let } y = \frac{2x}{1+3x^2} \Rightarrow \frac{dy}{dx} = \frac{2(1-3x^2)}{(1+3x^2)^2}$$

$$\frac{dy}{dx} = \theta \Rightarrow x = \frac{1}{\sqrt{3}} \quad (x \text{ cannot be} < 0)$$

$$\frac{d^2y}{dx^2} = \frac{-24x}{(1+3x^2)^3} < 0 \text{ for } x = \frac{1}{\sqrt{3}}$$

$$\Rightarrow y \text{ is maximum for } x = \frac{1}{\sqrt{3}}$$

$$\Rightarrow \max(\tan\theta) = \frac{2(1/\sqrt{3})}{1+3(1/\sqrt{3})^2} = \frac{1}{\sqrt{3}} \Rightarrow \theta_{max} = 30°.$$

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3. Given, $f(x) = \int_1^x \frac{dt}{2+t^4}$

Differentiating w.r.t. x, $f'(x) = \frac{(1)}{2+x^4} - 0$

$$\Rightarrow f'(x) = \frac{1}{2+x^4}$$

$\because f'(x)$ is continuous and differentiable

$\therefore$ Now applying Lagrange's Mean value theorem in the interval [1, 2]

$$f'(c) = \frac{f(2)-f(1)}{2-1} \Rightarrow \frac{1}{2+c^4} = \frac{f(2)-f(1)}{1}$$

$$f(1) = \int_1^1 \frac{dt}{2+x^4} = 0 \Rightarrow f(2) = \frac{1}{2+c^4}$$

$$\because 1 < c < 2 \Rightarrow 3 < 1+c^4 < 18$$

$$\Rightarrow f(2) < \frac{1}{3} \Rightarrow 3f(2) < 1.$$

---

4. $\sum_{k=1}^{n} (x+k-1)(x+k) = 10n$

$$\Rightarrow x(x+1)+(x+1)(x+2)+...+(x+n-1)(x+n) = 10n$$

$$\Rightarrow nx^2 + x\{1+3+5+...+(2n-1)\} + \{2+6+12+...+n(n-1)\} - 10n = 0$$

$$\Rightarrow nx^2 + xn^2 + s - 10n = 0 \quad ...(1)$$

where, $s = 2 + 6 + 12 + ... + n(n-1)$

$$S = \sum_{r=2}^{n} r(r-1) = \sum_{i=1}^{n-1} i(i+1) = \sum_{i=1}^{n-1} i^2 + \sum_{i=1}^{n-1} i$$

$$= \frac{1}{6}(n-1)(n-1+1)(2n-2+1) + (n-1)\frac{(n-1+1)}{2}$$

$$= \frac{1}{6}(n-1)n(2n-1) + (n-1)\frac{n}{2}$$

$$S = \frac{n(n-1)}{2}\left(\frac{2n-1}{3}+1\right) = \frac{n(n-1)(2n+2)}{6}$$

Putting the value of S in equation (1)

$$nx^2 + xn^2 + \frac{n(n-1)(2n+2)}{6} - 10n = 0$$

$$nx^2 + n^2x + \frac{n(n^2-1)}{3} - 10n = 0$$

$\therefore$ equation has roots α and (α + 1).

$\Rightarrow$ (Difference of roots)^2 = 1

$$\Rightarrow \left(-\frac{n^2}{n}\right)^2 - \frac{4}{n}\left[\frac{n(n^2-1)}{3} - 10n\right] = 1$$

$$\Rightarrow n^2 - 4\left[\frac{n^2-1}{3} - 10\right] = 1$$

$$\Rightarrow 3n^2 - 4n^2 + 124 = 3 \Rightarrow n^2 = 121 \Rightarrow n = 11.$$

---

5. $\because \int_x^{x^2y} f(t)\,dt$ is independent of x,

$$\Rightarrow \frac{d}{dx}\int_x^{x^2y} f(t)dt = 0 \Rightarrow 2xy\, f(x^2y) - (1)\, f(x) = 0$$

$$\Rightarrow 2xy\, f(x^2y) = f(x)$$

on putting $y = \frac{1}{x^2} \Rightarrow \frac{2}{x}f(1) = f(x)$ ...(1)

$$\text{Now } \int_1^x f(x)dx = 2f(1)\int_1^x \frac{dx}{x}$$

$$\Rightarrow \int_1^x f(x)dx = 2f(1)\ln x$$

on putting $x = 2$ in equation (1)

$$\Rightarrow \frac{2}{2}f(1) = f(2) \Rightarrow f(1) = f(2) = 2$$

$$\Rightarrow \int_1^x f(x)dx = 2(2)\ln x \Rightarrow \int_1^x f(x)dx = 4\ln x.$$

---

6. Let a, b, c are sides of a triangle,

Then area, $A = \sqrt{s(s-a)(s-b)(s-c)}$

Using AM-GM inequality for three (+ve) numbers $(s-a)(s-b)$ and $(s-c)$,

$$\frac{(s-a)+(s-b)+(s-c)}{3} \ge [(s-a)(s-b)(s-c)]^{1/3}$$

$$\Rightarrow \frac{3s-(a+b+c)}{3} \ge [(s-a)(s-b)(s-c)]^{1/3}$$

$$\Rightarrow [(s-a)(s-b)(s-c)]^{1/3} \le \frac{s}{3}$$

$$\Rightarrow s(s-a)(s-b)(s-c) \le \frac{s^4}{27}$$

$$\Rightarrow A \le \frac{s^2}{3\sqrt{3}} \Rightarrow A \le \frac{p^2}{12\sqrt{3}} \quad \text{(where } p = 2s \text{ is perimeter)}$$

From above relation, $A_{\max} \equiv \dfrac{p^2}{12\sqrt{3}}$

and equality holds only iff,

$$(s-a) = (s-b) = (s-c) \Rightarrow a = b = c$$

triangle is equilateral

If $A \le \dfrac{p^2}{12\sqrt{3}}$

Then $p^2 \ge 12\sqrt{3}A \Rightarrow p \ge \sqrt{12\sqrt{3}(A)}$

So now perimeter will be maximum for given area A, when equality will hold and equality holds when

$$(s-a) = (s-b) = (s-c) \Rightarrow a = b = c$$

triangle is equilateral.

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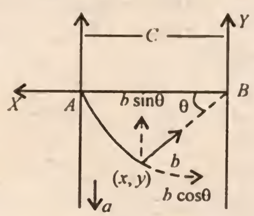
7. Let B be the origin.

The direction of axes are as shown.



Resultant velocity of boat in X direction is $b\cos\theta$ along negative X.

$$\therefore \frac{dx}{dt} = -b\cos\theta \quad \text{...(i)}$$

In Y direction resultant velocity is $b\sin\theta - a$

[θ is angle as shown]

$$\therefore \frac{dy}{dt} = b\sin\theta - a \quad \text{...(ii)}$$

Dividing (ii) by (i) $\dfrac{dy}{dx} = \dfrac{b\sin\theta - a}{-b\cos\theta}$ ...(iii)

If instantaneous co-ordinates of boat are (x, y) then from figure

$$\sin\theta = \frac{y}{\sqrt{x^2+y^2}} \text{ and } \cos\theta = \frac{x}{\sqrt{x^2+y^2}}$$

$\therefore$ Equation (iii) becomes $\dfrac{dy}{dx} = \dfrac{a\sqrt{x^2+y^2}+by}{bx}$

Now this is a homogenous differential equation of order one in x and y

Let $y = zx$

$$\Rightarrow z + x\frac{dz}{dx} = \frac{a\sqrt{1+z^2}+bz}{b} \Rightarrow x\frac{dz}{dx} = \frac{a}{b}\sqrt{1+z^2}$$

$$\Rightarrow x\frac{dz}{dx} = \frac{a}{b}\sqrt{1+z^2}$$

$$\int \frac{dz}{\sqrt{1+z^2}} = \frac{a}{b}\int \frac{dx}{x}$$

$\Rightarrow \ln\left|\sqrt{1+z^2}+z\right| = \dfrac{a}{b}\ln|x| + \ln d$ (where $\ln d$ is constant)

$$\Rightarrow \left|z+\sqrt{1+z^2}\right| = e^{\frac{a}{b}\ln|x| + \ln d}$$

$$\Rightarrow \left|z+\sqrt{1+z^2}\right| = d|x|^{a/b}$$

$$\Rightarrow \left|\frac{y+\sqrt{x^2+y^2}}{x}\right| = d|x|^{a/b}$$

$$\Rightarrow y + \sqrt{x^2+y^2} = dx^{\left(\frac{a}{b}+1\right)}$$

At $t = 0$, $x = c$ and $y = 0 \Rightarrow d = c^{-a/b}$

Thus, $c^{a/b}\left(\sqrt{x^2+y^2}+y\right) = x^{a/b+1}$

$$\Rightarrow y = \frac{1}{2c^{a/b}}\left(x^{a/b+1} - c^{2a/b}\,x^{1-a/b}\right).$$

Case I : If $1-\dfrac{a}{b} = 0$ (In this case, $b < a$, that the current is faster than speed of boat)

Then as $x \to 0$, $y \to -\infty$

(boat never reaches the opposite bank).

Case II : $1-\dfrac{a}{b} = 0$

(In this current is same as speed of boat)

Then as $x \to 0$, $y \to -\dfrac{1}{2}c$

(boat reaches at a distance below B equal half of width of river).

Case III : $1-\dfrac{a}{b} > 0$ (boat is faster than current)

Then as $x \to 0$, $y \to 0$

means boat reaches point B.

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8. Given $f'(x) - f(x) \le 0 \ \forall \ x \ge 0$

$\Rightarrow e^{-x}f'(x) - e^{-x}f(x) \le 0 \ \forall \ x \ge 0$

$\Rightarrow (e^{-x}f(x))' \le 0 \ \forall \ x \ge 0$

Let $g(x) = e^{-x}f(x) \Rightarrow g'(x) \le 0 \ \forall \ x \ge 0$

$\Rightarrow g(x)$ is decreasing function $\forall \ x \ge 0$

$\Rightarrow g(x) \le g(0) \ \forall \ x \ge 0$

$\Rightarrow g(x) < e^{-0}f(0) \ \forall \ x \ge 0$

$\Rightarrow g(x) \le f(0) \ \forall \ x \ge 0$

$\Rightarrow g(x) \le 0 \ \forall \ x \ge 0 \quad (\because f(0) = 0)$

$\Rightarrow g(1) \le 0 \Rightarrow e^{-1}f(1) \le 0 \Rightarrow f(1) \le 0 \quad (\because e^{-1} > 0)$

But it is given that $f(x)$ is non-negative $\Rightarrow f(1) = 0$. ■

SOLVED PAPER 2005

S.C.R.A.

– by Alok Kumar, B.Tech., IIT Kanpur

1. A tall electric pole is to be kept in a vertical position by a stretched wire from a point on the ground. The wire has to clear a wall of height a m and this wall is $8a$ m away from the pole. What is the least length of the wire that can be used for this purpose?
(a) $10a$ m (b) $12a$ m
(c) $5\sqrt{5}a$ m (d) $\frac{17}{2}\sqrt{5}a$ m

2. If $(2-3\lambda+\lambda^2)^k = A_0 + A_1\lambda + A_2\lambda^2 + ... + A_{2k}\lambda^{2k}$ and $A_0 + A_2 + A_4 + A_6 ... + A_{2k} = 648$. What is the value of k?
(a) 4 (b) 5 (c) 6 (d) 7

3. Five-letter words are to be constructed using the letters of the word 'Equation' so that each word contains exactly three vowels and two consonants. How many of them will have all the vowels together ?
(a) 3600 (b) 1800
(c) 1080 (d) None of these.

Directions : The following **five (5)** items consist of two statements : one labelled as the 'Assertion (A)' and the other as Reason (R)'. You are to examine these two statement carefully and select the answer to these items using the code given below :
code :
(a) Both A and R are individually true and R is the correct explanation of A.
(b) Both A and R are individually true but R is not the correct explanation of A.
(c) A is true but R is false.
(d) A is false but R is true.

4. **Assertion (A) :** $\int_0^{\infty} \frac{x\ln|x|}{(1+x^2)^2} = 0$

Reason (R) : $f(x) = \frac{x\ln|x|}{(1+x^2)^2}$ is an odd function.

5. Let $S_1 = p + (p+q) + (p+2q) + ... + \{p+(n-1)q\}$ and $S_2 = t + tr + tr^2 + ... + tr^{n-1}$ where $p, q, t, r > 0$.
Assertion (A) : If $p + (n-1)q = tr^{n-1}$, then $S_2 < S_1$.
Reason (R) : The geometric mean of any two unequal positive integers is always less than their arithmetic mean.

6. **Assertion (A) :** $\lim_{x\to1} = \frac{\sqrt{1-\cos(2x-2)}}{x-1}$ exists

Reason (R) : $\lim_{x\to a} f(x)$ exists if the left-hand limit is equal to right-hand limit.

7. **Assertion (A) :** $\left[\left(1+\frac{1}{10000}\right)^{10000}\right] = 2$,
where [.] is the greatest integer function.

Reason (R) : $2 < \left(1+\frac{1}{n}\right)^n < 2.5$ for all $n \in N$.

8. Let R_0 be set of all real numbers except zero. A binary operation $*$ on R_0 is defined by $a * b = |a|b$, where $|a|$ is the absolute value of a.
Assertion (A) : (R_0, $*$) is not a group.
Reason (R) : $*$ is not associative on R_0.

9. If $\frac{\sin\theta}{\theta} = \frac{5045}{5046}$, what is the approx. value of θ ?
(a) 1/30 (b) 1/29 (c) 1/28 (d) 1/27.

10. If $a * b = a \times b - 2$, and if $x * 3 = 7$; then what is the value of x^{-1}?
(a) 4 (b) –2 (c) 6 (d) 1.

11. What is the coefficient of $\cos^3\theta$ in the expansion of $\cos 7\theta$ in powers of $\cos\theta$?
(a) –112 (b) –56 (c) 112 (d) 56

12. What is the smallest positive number p for which $\cos(p \sin x) = \sin(p \cos x)$?
(a) $\pi/2$ (b) $\pi/(2\sqrt{2})$
(c) $1/(2\sqrt{2})$ (d) $\pi/\sqrt{2}$.

13. If $x = \frac{2}{1!} - \frac{4}{3!} + \frac{6}{5!} - \frac{8}{7!} + ... \infty$,
what is the value of x^2 ?
(a) sin 2 (b) 1 – sin2
(c) 1 + sin2 (d) sin2 – 1.

14. What is the value of
$\cos^6 5° - 15\cos^4 5° \sin^2 5° + 15\cos^2 5° \sin^4 5° - \sin^6 5°$?
(a) 1/2 (b) $1/\sqrt{2}$ (c) $\sqrt{3}/2$ (d) 1.

15. If $\cos(x-y) = -1$, what are the values of $(\sin x + \sin y)$ and $(\cos x + \cos y)$, respectively?
(a) 1 and 1 (b) 0 and 1
(c) 1 and 0 (d) 0 and 0.

Mathematics Olympia

By : Er. Akhlak Ahmad, ABC Classes, Gorakhpu

1. If line $x = y = z$ intersect the line $b^2x + (2 - 4a)y + z = 1$; $z^2x + (1 - 2b)y + z = -1$ then find all the possible values of a and b.
2. Prove that $\tan(-314°) > 61/59$.
3. If $a_n = \sum_{r=0}^{n} \frac{1}{{}^nC_r}$, then prove that
$$\sum_{r=0}^{n} \frac{r^2}{{}^nC_r} = (n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + a_n - (n+1)(n+6).$$
4. Let $f(x + 2y) = f(x)(f(y))^2$ for all x and y. If $f'(0) = \ln 2$, then prove that
$$f(x) + f(2x) + f(3x) + \ldots + f(nx) = \frac{f(x)\left(f(nx) - 1\right)}{f(x) - 1}$$
(consider $f(x)$ is non-negative function).
5. Let $2x + 3y = 6$ be a line meeting the coordinate axes at A and B respectively. A variable line $\frac{x}{a} + \frac{y}{b} = 1$ meets the axes at P and Q respectively in such a way that the lines BP and AQ always meet at right angle at R. Find the locus of the orthocentre of the ΔARB.
6. If the argument of $(z - a)(\bar{z} - b)$ is equal to that of $\left(\frac{(\sqrt{3}+i)(1+\sqrt{3}i)}{1+i}\right)$, where a, b are real numbers, then find the locus of z in the argand diagram. Find the values of a and b so that the locus becomes a circle with its centre at $(3 + i)/2$.
7. Evaluate $I = \int_1^4 \frac{\tan^{-1}[x^2]}{\tan^{-1}[x^2] + \tan^{-1}[25 + x^2 - 10x]}dx$ where [.] denotes the greatest integer function.
8. Solve the differential equation
$$(xy^2 - e^{1/x^3})dx - x^2ydy = 0.$$
9. An ellipse has the points $(1, -1)$ and $(2, -1)$ as its foci and $x + y - 5 = 0$ as one of its tangent. Find the coordinates of the point where this line touches the ellipse.
10. If any tangent to the curve $ax^2 + 2hxy + by^2 = 1$ $(h^2 \neq ab)$ makes with co-ordinate axes a triangle o constant area, then show that the curve is a rectangula hyperbola and also find the area of the triangle.

SOLUTIONS

1. $x = y = z = r$. If both lines intersect then intersectio of first line with both the planes is same.

i.e. $\frac{1}{b^2 + (2-4a) + 1} = -\frac{1}{a^2 + 1 - 2b + 1}$

i.e. $(b - 1)^2 + (a - 2)^2 = 0 \Rightarrow a = 2$ and $b = 1$.

2. $$\tan(-314°) = \tan 46° = \frac{1+\tan 1°}{1-\tan 1°}$$
$$= \frac{1+\tan\frac{\pi}{180°}}{1-\tan\frac{\pi}{180°}} > \frac{1+\frac{\pi}{180°}}{1-\frac{\pi}{180°}} > \frac{1+\frac{3}{180°}}{1-\frac{3}{180°}} = \frac{61}{59}.$$

3. $$\sum_{r=0}^{n} \frac{r^2}{{}^nC_r} = \sum_{r=0}^{n}\left[\frac{(r+1)^2}{{}^nC_r} - \frac{2r+1}{{}^nC_r}\right]$$
$$= \sum_{r=0}^{n}(r+1)\frac{n+1}{{}^{n+1}C_{r+1}} - \sum_{r=0}^{n}\frac{2r+1}{{}^nC_r}$$
$$= (n+1)\sum_{r=0}^{n}\frac{r+2}{{}^{n+1}C_{r+1}} - (n+1)\sum_{r=0}^{n}\frac{1}{{}^{n+1}C_{r+1}} - 2\sum_{r=0}^{n}\frac{r+1}{{}^nC_r} + \sum_{r=0}^{n}\frac{1}{{}^nC_r}$$
$$= (n+1)(n+2)\sum_{r=0}^{n}\frac{1}{{}^{n+2}C_{n+2}} - (n+1)\sum_{r=0}^{n}\frac{1}{{}^{n+1}C_{n+1}} - 2(n+1)\sum_{r=0}^{n}\frac{1}{{}^{n+1}C_{r+1}} + \sum_{r=0}^{n}\frac{1}{{}^nC_r} \quad ...(1$$
$$= (n+1)(n+2)\sum_{r=0}^{n}\frac{1}{{}^{n+2}C_{r+2}} = (n+1)\sum_{r=0}^{n}\frac{1}{{}^{n+1}C_{r+1}} - 2(n+1)\sum_{r=0}^{n}\frac{1}{{}^{n+1}C_{r+2}} + \sum_{r=0}^{n}\frac{1}{{}^nC}$$
$$= (n+1)(n+2)\left[a_{n+2} - \frac{1}{{}^{n+2}C_0} - \frac{1}{{}^{n+2}C_1}\right]$$

$$-3(n+1)\left[a_{n+1}-\frac{1}{{}^{n+1}C_0}\right]+a_n.$$

$= (n+1)(n+2)a_{n+2} - (n+1)(n+2) - (n+1)$

$-3(n+1)a_{n+1} - 3(n+1) + a_n$

$= (n+1)(n+2)a_{n+2} - 3(n+1)a_{n+1} + a_n - (n+1)(n+6)$

4. $f'(x) = \lim_{h\to 0} \frac{f(r+h)-f(x)}{h}$

$= \lim_{h\to 0} \frac{f(x)f^2(h/2)-f(x)}{h} = f(x)\lim_{h\to 0}\frac{f^2(h/2)-1}{h}$

$= f(x)\lim_{h\to 0} f(x)\frac{(f(h/2)-1)}{(2\times h/2)}\cdot\left(f\left(\frac{h}{2}\right)+1\right)$

$= f(x)f'(0)$ [since, $f(0) = 1$]

$= f(x)\ln 2 \Rightarrow \frac{f'(x)}{f(x)} = \ln 2 \Rightarrow f(x) = 2^x + c$

Since, $f(0) = 1 \Rightarrow c = 0 \Rightarrow f(x) = 2^x$

$f(x) + f(2x) + \ldots. f(nx) = 2^x + 2^{2x} + \ldots.. 2^{nx}$

$= \frac{2^x(2^{nx}-1)}{2^x-1} = \frac{f(x)(f(nx)-1)}{f(x)-1}.$

5. Since AB is fixed for all positions of R and $\angle ARB = 90°$, locus of R is a circle with AB as diameter. i.e. required locus is

$x(x-3) + y(y-2) = 0 \Rightarrow x^2 + y^2 - 3x - 2y = 0.$

6. $(z-a)(\bar{z}-b) = z\bar{z} - a\bar{z} - bz + ab$ $(\because z = x+iy)$

$= x^2 + y^2 - ax - bx + ab + i(a-b)y$

$\therefore$ argument of $(z-a)(\bar{z}-b)$

$= \tan^{-1}\left(\frac{y(a-b)}{x^2+y^2-ax-bx+ab}\right)$ ...(1)

Also, $\frac{(\sqrt{3}+i)(1+\sqrt{3}i)}{1+i} = \frac{\sqrt{3}-\sqrt{3}+i(1+3)}{1+i} = 2+2i.$

Argument of $2+2i = \tan^{-1}(2/2) = \pi/4$ ...(2)

From (1) and (2) $\left(\frac{y(a-b)}{x^2+y^2-(a+b)x+ab}\right) = 1$

$\Rightarrow x^2 + y^2 - (a+b)x - (a-b)y + ab = 0$

Thus the locus of z is a circle.

Given centre is $\frac{3+i}{2} \Rightarrow \frac{a+b}{2} = \frac{3}{2}$

and $\frac{a-b}{2} = \frac{1}{2} \Rightarrow a = 2, b = 1.$

7. Let $I = \int_1^4 \frac{\tan^{-1}[x^2]}{\tan^{-1}[x^2]+\tan^{-1}[25+x^2-10x]}dx$ ...(1)

Appling $\int_a^b f(x)dx = \int_a^b f(a+b-x)dx,$ we get

$I = \int_1^4 \frac{\tan[(5-x)^2]}{\tan^{-1}[(5-x)^2]+\tan^{-1}[x^2]}dx$ ...(2)

Adding (1) and (2), $2I = \int_1^4 dx \Rightarrow 2I = 3 \Rightarrow I = \frac{3}{2}.$

8. $xy(ydx - xdy) = e^{1/x^3}dx$

$\Rightarrow \frac{-xy}{x^2}(xdy - ydx) = \frac{e^{1/x^3}}{x^2}dx$

$\Rightarrow -xyd\left(\frac{y}{x}\right) = \frac{e^{1/x^3}}{x^2}dx \Rightarrow -\frac{y}{x}d\left(\frac{y}{x}\right) = \frac{e^{1/x^3}}{x^4}dx.$

Integrating, we get $\frac{y^2}{x^2} = \frac{2}{3}e^{1/x^3} + c.$

9. Given $S(1,-1)$ and $S'(2,-1)$ are focii.

Hence $2ae = SS' = \sqrt{(2-1)^2+(-1+1)^2} = 1$

$\Rightarrow ae = 1/2$ ...(1)

Also $P_1P_2 = b^2$, where P_1, P_2 are lengths of perpendiculars from S and S' to the tangent.

So, $\frac{5}{\sqrt{2}}\times\frac{4}{\sqrt{2}} = b^2$ or $b^2 = 10$ .....(2)

10. Let any point on the curve be (α, β), then the equation of tangent at this point will be,

$T = 0$, *i.e.* $a\alpha x + h\alpha y + h\beta x + h\beta y = 1$

This tangent will cut the co-ordinate axes at

$A\left(\frac{1}{a\alpha+h\beta}, 0\right)$ and $B\left(0, \frac{1}{h\alpha+b\beta}\right)$

Now it is given that the area of the triangle OAB = constant = k(say)

So, $\frac{1}{2}\left|\left(\frac{1}{a\alpha+h\beta}\right)\left(\frac{1}{h\alpha+b\beta}\right)\right| = k$

$\Rightarrow h(a\alpha^2+b\beta^2)+(ab+h^2)\alpha\beta = \pm 1/2k$

Now, as (α, β) is on the curve,

$a\alpha^2 + b\beta^2 = 1 - 2h\alpha\beta$

So, $h(1-2h\alpha\beta) + (ab+h^2)\alpha\beta = \pm 1/2k.$

$\Rightarrow (h^2-ab)\alpha\beta = h\pm\frac{1}{2k} \Rightarrow \alpha\beta = \frac{h\pm\frac{1}{2k}}{h^2-ab}$

As, (α, β) represents a general point on the curve, the curve is the rectangular hyperbola $xy = \frac{h\pm\frac{1}{2k}}{h^2-ab}$

Now, comparing the coefficients with given curve we get, $a = b = 0$ and $\frac{1}{2h} = \frac{h\pm\frac{1}{2k}}{h^2-ab} \Rightarrow k = \frac{1}{|h|}$

Practice Paper for IIT-JEE

ON LATEST IIT-JEE PATTERN

Q.A : Given an odd function f defined and integrable everywhere and periodic with period 2. Let

$$g(x)=\int_0^x f(t)dt$$

Answer the following questions–

1. $f(4)$ is equal to–
(a) 0 (b) 2 (c) 4 (d) none of these.

2. $g(4)$ is equal to–
(a) 0 (b) 2 (c) 4 (d) none of these.

3. $g(x+2)$ is equal to–
(a) $g(x)\,g(2)$ (b) $g(x)+g(2)$
(c) $g(x)-g(2)$ (d) none of these

4. If $f'(-2)=-2$, then what should be $f'(2)$?
(a) 0 (b) –2 (c) 2 (d) none of these.

5. If $g(x^2)=x^2(1+x)$, then roots of the equation $x^2-f(x^2)=0$ are
(a) $-\frac{1}{2},2$ (b) $\frac{1}{2},2$ (c) $\frac{1}{2},-2$ (d) $-\frac{1}{2},-2$

Q.B : Ramesh, a student of class XII found when the graph of a one-one function is reflected about the line $y=x$, the result is graph of the inverse function.He also found that image of any point on the graph of function taken about 45° line, lie on the graph of inverse function.
Answer the following questions–

6. Suppose $f(x)=(x+1)^2-1\ \forall x\geq -1$. If $g(x)$ is the function whose graph is the reflection of the graph of $f(x)$ with respect to the line $y=x$, then $g(x)$ equals–
(a) $\sqrt{x+1}-1,\ x\geq -1$ (b) $\sqrt{x-1}-1,\ x\geq 1$
(c) $\sqrt{x-1},\ x\geq 1$ (d) $-\sqrt{x+1}-1,\ x\geq -1$

7. What will be the solution set

$$S=\{x:f(x)=f^{-1}(x)\}\ \text{in part (ii) ?}$$

(a) Empty (b) $\{0,-1\}$
(c) $\{0,1,-1\}$
(d) $\left\{0,-1,\frac{-3+i\sqrt{3}}{2},\frac{-3-i\sqrt{3}}{2}\right\}$

8. let $e^{f(x)}=\ln x$. If $g(x)$ is the inverse function of $f(x)$, then $g'(1)$ equals–
(a) 0 (b) e (c) 1 (d) none of these

9. If (1,2) is a point on the graph of a function, What should be its image on the inverse function ?
(a) (1,-2) (b) (2,1) (c) (1,1) (d) none of these

10. Product of slopes of tangents drawn at point A having x co-ordinate x_1 ,of graph of function $f(x)$ and at point B having x co-ordinate x_2 of graph of inverse of function is (Point B is mirror image of point A)–
(a) 1 (b) –1 (c) 2 (d) –2

Q.C : Let Z is any complex number satisfying the equation $Z^2+pZ+q=0$; $q\in C$ and both the roots of the equation have unit modulus.
Answer the following questions–

11. Modulus of q is–
(a) 1/2 (b) 2 (c) 1 (d) 3

12. Which of the following is true?
(a) $|p|\leq 2$ (b) $|p|\leq 1$ (c) $|p|\geq 1$ (d) $|p|\geq 2$

13. If $\arg(p)=\alpha$ and $\arg(q)=\beta$, then realtion between α and β is–
(a) $\alpha=2\beta$ (b) $2\alpha=\beta$ (c) $\alpha=\frac{1}{\beta}$ (d) $\alpha=-\beta$

Q.D : Consider the function $f(x)=\max\left\{x^2,(x-1)^2,2x(1-x)\right\},0\leq x\leq 1$
Answer the following questions--

14. Which of the following is true ?
(a) $f(x)$ is differentiable for all x
(b) $f(x)$ is differentiable for all x except at one point
(c) $f(x)$ is differentiable for all x except at two points
(d) $f(x)$ is not differentiable at more than two points

15. Area bounded by $f(x)$ with x-axis between lines $x=0$ and $x=1$ is
(a) 17/27 (b) 19/27
(c) 11/27 (d) none of these.

16. What should be the answer in above question, if $f(x)$ is $\min\left\{x^2,(x-1)^2,2x(1-x)\right\},0\leq x\leq 1$
(a) 7/12 (b) 1/12 (c) 3/12 (d) 5/12

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8 BEST Challenging Problems

ALGEBRA

By : Tapas Kumar Yogi, Lucknow

1. Find the smallest positive integer m such that m is not a square and yet in the decimal expansion of $\sqrt{m}$, the decimal point is followed by atleast four consecutive zeros.

Soln. : Let k be the integral part of $\sqrt{m}$ then according to question

$$K < \sqrt{m} < K + 0.0001$$

so, $K^2 < m < K^2 + \dfrac{1}{(10000)^2} + \dfrac{K}{5000}$

since m is an integer, it follows that

$$\frac{K}{5000} + \frac{1}{(10000)^2} \geq 1 \text{ so, } K \geq 5000$$

Hence, the smallest m occurs when $k = 5000$ and $m = k^2 + 1 = (5000)^2 + 1$.

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2. Determine whether or not the number $\sqrt[3]{26-15\sqrt{3}} + \sqrt[3]{26+15\sqrt{3}}$ is an integer. Prove that your answer is correct.

Soln.: Let $a = \sqrt[3]{26-15\sqrt{3}}$ and $b = \sqrt[3]{26+15\sqrt{3}}$

Note that $a^3 + b^3 = (26-15\sqrt{3}) + (26+15\sqrt{3}) = 52$

and $ab = \sqrt[3]{(26-15\sqrt{3})(26+15\sqrt{3})} = 1$.

Let $x = a + b$

Now, $x^3 = a^3 + b^3 + 3ab(a + b)$ i.e., $x^3 = 52 + 3(1)(x)$

or, $x^3 - 3x - 52 = 0$ or $(x - 4)(x^2 + 4x + 13) = 0$

i.e., $(x-4)\left[(x+2)^2 + 9\right] = 0$ clearly $x = 4$, an integer.

---

3. Find all positive real numbers x, y and z that satisfy the following three equations :

$$x + \frac{4}{xy} = 3,\quad y + \frac{4}{yz} = 3 \text{ and } z + \frac{4}{zx} = 3.$$

Soln. : Without any loss of generality, we assume that x is the largest of the three unknowns.

In particular $x \geq y \rightarrow (1)$ so, $zx \geq zy$ as $z > 0$

Thus, $\dfrac{4}{zx} \leq \dfrac{4}{zy}$ and this gives $z = 3 - \dfrac{4}{zx} \geq 3 - \dfrac{4}{zy} = y$

i.e., $z \geq y \rightarrow (2)$

Again, $zx \geq xy$ as $x > 0$

So, $z = 3 - \dfrac{4}{zx} \geq 3 - \dfrac{4}{xy} = x$ i.e., $z \geq x \rightarrow (3)$

From (1), (2) and (3) and in view of assumption we have $x = z$ and thus the third equation gives, $x + \dfrac{4}{x^2} = 3$ i.e., $x^3 - 3x^2 + 4 = 0$ i.e., $(x+1)(x-2)(x-2) = 0$ or, $x = 2, -1$

So, $x = 2$ is the only positive solution.

Hence, $x = 2 = z$ and y also $= 2$.

---

4. Let a and b be positive numbers with $\dfrac{1}{a} + \dfrac{1}{b} = 1$ and let $[x]$ denote the largest integer that does not exceed the number x. If neither a nor b can be written as the ratio of two integers, show that each positive integer is either of the form $[ma]$ or $[mb]$ for some positive integer m.

Soln. : Let t be a positive integer and suppose, we assume the contradictory that $t \neq [ma]$ and $t \neq [mb]$ for any integer m. If ra is the largest integer multiple of a at most equal to t, then $ra \leq t$ and $(r + 1)\, a > t$. In fact $(r + 1)\, a > t + 1$ since otherwise we would have $t < (r + 1)\, a < t + 1$ and $t = [(r + 1)\, a]$. Further $ra \neq t$ and $(r + 1)\, a \neq t + 1$ since a is not the ratio of two integers. In other words, we have

$$ra < t < t + 1 < (r + 1)\, a$$

or, $$r < \frac{t}{a} < \frac{t+1}{a} < r + 1 \quad (a > 0) \qquad ...(1)$$

similarly there exists an integer S with

$$s < \frac{t}{b} < \frac{t+1}{a} < s + 1 \qquad ...(2)$$

Adding (1) and (2) we get,

$$r + s < t\left(\frac{1}{a} + \frac{1}{b}\right) < (t+1)\left(\frac{1}{a} + \frac{1}{b}\right) < r + s + 2$$

using given condition, $r + s < t < t + 1 < r + s + 2$; but there is only one integer strictly contained between $r + s$ and $r + s + 2$, so this is a contradiction. Hence, we must have $t = [ma]$ or $t = [mb]$ for some integer m.

5. Given a real number x, the floor of x, denoted by $\lfloor x \rfloor$ is defined to be the largest integer n such that $n \le x$. Similarly, the ceiling of x, denoted by $\lceil x \rceil$ is the smallest integer m such that $x \le m$. We also define average (x) to be the average of $\lfloor x \rfloor$ and $\lceil x \rceil$.

Prove that $\lfloor x+y \rfloor \le$ avg. (x) + avg. $(y) \le \lceil x+y \rceil$ for all real numbers x and y.

Soln. : If x is an integer then

$\lfloor x+y \rfloor = x + \lfloor y \rfloor$ ...(1) and $\lceil x+y \rceil = x + \lceil y \rceil$ ...(2)

Now, $\lfloor y \rfloor \le \text{avg.}(y) \le \lceil y \rceil$

Adding x, $x + \lfloor y \rfloor \le x + \text{avg.}(y) \le x + \lceil y \rceil$

or, $\lfloor x+y \rfloor \le \text{avg}(x) + \text{avg}(y) \le \lceil x+y \rceil$ in view of (1) and (2) and avg. $(x) = \dfrac{\lceil x \rceil + \lfloor x \rfloor}{2} = \dfrac{x+x}{2} = x$ if $x \in I$.

Hence the result, similarly the result hold if y is an integer. Finally, suppose that neither x nor y is an integer. Then for some integers a and b, we have

$$a < x < a+1$$
$$b < y < b+1$$
$$a + b < x + y < a + b + 2 \quad ...(3)$$

and $\lceil x \rceil = a+1, \lfloor x \rfloor = a$

so, avg.$(x) = a + \dfrac{1}{2}$ and avg. $(y) = b + \dfrac{1}{2}$

so, avg.(x) + avg.$(y) = a + b + 1$ ...(4)

Now (3) gives $\lceil x+y \rceil \ge a+b+1$ ...(5)

and $\lfloor x+y \rfloor \le a+b+1$ ...(6)

combining the results (4), (5) and (6), we have

$$\lfloor x+y \rfloor \le \text{avg.}(x) + \text{avg.}(y) \le \lceil x+y \rceil.$$

6. Do there exists primes p and q such that the quadratic equation $px^2 - qx + p = 0$ has a rational solution? If so, find all possibilities.

Soln. $px^2 - qx + p = 0$

Note that $f(0) = p(> 0)$ and sum of the roots is also positive.

So, it is clear that if x is a real root then $x > 0$. Let $x = \dfrac{m}{n}$ where m and n are positive integers with no common factors.

$$p\left(\frac{m}{n}\right)^2 - q\left(\frac{m}{n}\right) + p = 0$$

Simplifying a bit, $m(qn - pm) = Pn^2$

Thus, m divides pn^2 and since m and n have no proper factors in common, we see that m divides p. In other words, $m = 1$ or p and similarly $n = 1$ or p.

Thus, $x = m/n = 1$, p or $1/p$.

If $x = 1$ then the quadratic equation yields $q = 2p$, contradicting the fact that p and q are prime. And if $x = p$ or $\dfrac{1}{p}$ then quadratic equations yields $q = p^2 + 1$. Now if p is odd, then q is an even number ≥ 10, again cortradicting the fact that q is prime. Hence, p must be even and $p = 2$, $q = 5$ and the roots of $2x^2 - 5x + 2 = 0$ are 2 and $\dfrac{1}{2}$.

7. Let a, b, c and d be four distinct integers. Find the smallest possible value for

$$4(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2.$$

Soln. : Consider $E = (a-b)^2 + (a-c)^2 + (a-d)^2 + (b-c)^2 + (b-d)^2 + (c-d)^2$

Simplifying this we have,

$$E = 4(a^2 + b^2 + c^2 + d^2) - (a + b + c + d)^2$$

and so our task is to minimise E subject to the condition that a, b, c, d are distinct integers.

Without loss of generality, we can assume that

$$a > b > c > d$$

Now each of $(a-b)^2$, $(b-c)^2$, $(c-d)^2$ is at least $1^2 = 1$ and each of $(a-c)^2$, $(b-d)^2$ is at least $2^2 = 4$ and finally $(a-d)^2$ is at least $3^2 = 9$.

So, the smallest E could possibly be

$$1 + 1 + 1 + 4 + 4 + 9 = 20$$

Infact by taking $d = 1$, $c = 2$, $b = 3$, $a = 4$ $\therefore$ $E = 20$.

8. I want to buy a pair of fish for my aquarium. The salesman at the pet shop says that if he nets 2 fish at random from his big tank, the probability that both will be of the same sex is exactly 1/2.

Assuming that the salesman is telling the truth, prove that the number of fishes in the tank is a square.

Soln. : Given a set of n objects, there are exactly

$${}^nC_2 = \frac{n(n-1)}{2} \text{ pairs}$$

Now let m and f respectively be the numbers of male and female fish in the tank. The total number of pairs of fish is, therefore, $\dfrac{(m+f)(m+f-1)}{2}$. We know that exactly half of these are same-sex pairs and thus exactly half consist of one male and one female.

But there are m ways to choose the male and f ways to choose the female, giving mf male-female pairs.

So, $mf = \dfrac{1}{2} \times \dfrac{(m+f)(m+f-1)}{2}$

simplifying, $m + f = (m+f)^2 - 4mf = (m-f)^2$

Thus, the total no. of fish in the tank = $m + f$ is a square.

■

Contd. from page no. 11

(a) (7, –5) if $|PA - PB|$ is maximum

(b) $\left(\frac{1}{5}, \frac{1}{5}\right)$ if $|PA - PB|$ is maximum

(c) (7, –5) if $|PA - PB|$ is minimum

(d) $\left(\frac{1}{5}, \frac{1}{5}\right)$ if $|PA - PB|$ is minimum

45. The equation $axy + bx + cy + d = 0$ represents a pair of straight lines, then

(a) $bc = ad$ (b) the line are parallel

(c) the lines can not be parallel

(d) the lines are coincident

46. The function $f(x) = \cos^{-1}\left(\frac{2-|x|}{4}\right) + (\log(3-x))^{-1}$ is defined, if x lies in

(a) [–2, 6] (b) [–6, 2)

(c) (2, 3) (d) $(-2, 2) \cup (2, 3)$

47. If $f(x) = (x-1)/x$ for all real numbers except $x = 0$ and $g(u) = u^2 + 1$ for all $u \in R$ then $f[g(u)]$ is defined for

(a) all real numbers u (b) $u = -1$

(c) $u^2 = 1$ (d) $u = 0$

48. Let $g(t) = [t[1/t]]$ for $t > 0$. ([·] denotes the greatest integer function). Then $g(t)$ has

(a) Discontinuities at finite number of points

(b) Discontinuities at infinite number of points

(c) $g(1/2) = 1$ (d) $g(3/4) = 1$

49. Let $f(x) = \begin{cases} x^\alpha \sin^2 1/nx, & x \neq 0 \\ 0 & , x = 0 \end{cases}$ where $n \in I$, $n \neq 0$. If Rolle's theorem is applicable to $f(x)$ in the interval [0, 1], then

(a) $\alpha > 0$, greatest value of n is $\frac{1}{\pi}$

(b) $\alpha > 2$, greatest value of n is $\frac{1}{\pi}$

(c) $\alpha > 0$, least value of n is $-\frac{1}{\pi}$

(d) α cannot be < 1

50. If $f(x) = \begin{cases} x^2 + 2, & x < 0 \\ 3 & , x = 0 \\ x + 2, & x > 0 \end{cases}$, then

(a) $f'(x)$ has a maximum at $x = 0$

(b) $f'(x)$ is strictly decreasing on the left of 0

(c) $f'(x)$ is strictly increasing on the left of 0

(d) $f'(x)$ is strictly increasing on the right of 0

51. If $\lambda_1 > \lambda_2$ are roots of $x^2 + kx - 1 = 0$, then

(a) $\tan^{-1}\lambda_1 - \tan^{-1}\lambda_2 = -\frac{\pi}{2}$

(b) $\tan^{-1}\lambda_1 + \tan^{-1}\frac{1}{\lambda_1} = -\frac{\pi}{2}$

(c) $\tan^{-1}\lambda_2 + \tan^{-1}\frac{1}{\lambda_2} = -\frac{\pi}{2}$

(d) $\tan^{-1}\lambda_1 - \tan^{-1}\lambda_2 = \frac{\pi}{2}$

52. If $x + y = 60$, $x > 0$, $y > 0$, then the expression $x^2(30-y)^2$ has

(a) least value = 0 (b) greatest value = 15^4

(c) two extrema (d) no greatest value

53. If $y = ae^{-1/x} + b$ is a solution of $\frac{dy}{dx} = \frac{y}{x^2}$, then possible values of a and b are

(a) $a = 2, b = 0$ (b) $a = 5, b = 0$

(c) $a = -2, b = 0$ (d) $a = 1, b = 1$

54. Let $\vec{a} = x\hat{i} + x^2\hat{j} + 2\hat{k}$, $\vec{b} = -3\hat{i} + \hat{j} + \hat{k}$, $\vec{c} = (3x+11)\,\hat{i} + (x-9)\,\hat{j} - 3\,\hat{k}$ be three vectors. Then angle between $\vec{a}$ and $\vec{b}$ is acute and angle between $\vec{c}$ and $\vec{a}$ is obtuse, if x lies in

(a) $(-\infty, 1) \cup (2, 3)$ (b) $(-\infty, 1)$

(c) (2, 3) (d) none of these

55. If $\alpha + \beta + \gamma + \delta = 2\pi$, then

$$\cos^2\alpha + \cos^2\beta - \cos^2\gamma - \cos^2\delta =$$

(a) $2\sin(\beta+\gamma)\sin(\gamma+\alpha)\cos(\alpha+\beta)$

(b) $2\sin(\beta+\gamma)\cos(\gamma+\alpha)\sin(\alpha+\beta)$

(c) $2\cos(\beta+\gamma)\sin(\gamma+\alpha)\sin(\alpha+\beta)$

(d) $2\sin(\beta+\gamma)\cos(\gamma+\alpha)\cos(\alpha+\beta)$

ANSWER

1. (c)	2. (d)	3. (c)	4. (c)	5. (d)
6. (a)	7. (c)	8. (d)	9. (a)	10. (c)
11. (b)	12. (c)	13. (b)	14. (b)	15. (a)
16. (d)	17. (c)	18. (c)	19. (d)	20. (a)
21. (a)	22. (c)	23. (d)	24. (a)	25. (c)
26. (b)	27. (a)	28. (d)	29. (a)	30. (d)
31. (b)	32. (b)	33. (c)	34. (a)	35. (d)
36. (b)	37. (a)	38. (c)	39. (d)	40. (b)
41. (b,c,d)	42. (b,c,d)	43. (All)	44. (a)	45. (a,c)
46. (b,c,d)	47. (All)	48. (b,c)	49. (a,c,d)	50. (a,c)
51. (c,d)	52. (a,c)	53. (a,b,c)	54. (a,b,c)	55. (a)

PRACTICE PAPER for West Bengal JEE 2006

Exam on 22nd & 23rd April '06

Note : Actual paper will consist 100 questions. 1 mark will be awarded for each correct answer and –½ for each wrong answer. For detailed information see prospectus for JEE 2006 or visit at http://jexab.becs.ac.in

Time : 1 hr Marks : 1 for each correct,–½ for each wrong

1. $\sqrt{2+\sqrt{2+\sqrt{2+\ldots\infty}}}$ is equal to:

(a) –1 (b) $\sqrt{2}$ (c) 2 (d) 1/2.

2. l, m, n are pth, qth and rth term of G.P. are all positive, then $\begin{vmatrix} \log l & p & 1 \\ \log m & q & 1 \\ \log n & r & 1 \end{vmatrix}$ equals:

(a) –1 (b) 2 (c) 1 (d) 0.

3. Sum of the series $\frac{2}{3}+\frac{8}{9}+\frac{26}{27}+\frac{80}{81}+\ldots$ to n terms is:

(a) $n-\frac{1}{2}(3^n-1)$ (b) $n+\frac{1}{2}(3^n-1)$

(c) $n+\frac{1}{2}(1-3^{-n})$ (d) $n+\frac{1}{2}(3^{-n}-1)$.

4. In the expansion of $\left[x-\frac{3}{x^2}\right]^9$ the term independent of x is:

(a) non existent (b) ${}^9C_2$

(c) –2268 (d) 2268.

5. $f: R \to R$ is a function defined by $f(x) = 10x - 7$. If $g = f^{-1}$ then $g(x)$ is equal to:

(a) $\frac{1}{10x-7}$ (b) $\frac{1}{10x+7}$

(c) $\frac{x+y}{10}$ (d) $\frac{x-7}{10}$.

6. If the two pairs of line $x^2 - 2mxy - y^2 = 0$ and $x^2 - 2nxy - y^2 = 0$ are such that one of them represents the bisector of the angles between the other, then:

(a) $mn + 1 = 0$ (b) $mn - 1 = 0$

(c) $\frac{1}{m}+\frac{1}{n}=0$ (d) $\frac{1}{m}-\frac{1}{n}=0$.

7. The lines represented by the equation $Ax^2 + 2Bxy + Hy^2 = 0$ are perpendicular if :

(a) $A + H = 0$ (b) $B + H = 0$

(c) $AH = -1$ (d) $A + B = 0$.

8. The angle between a pair of tangents drawn from a point P to the circle $x^2+y^2+4x-6y+9\sin^2\alpha+13\cos^2\alpha=0$ is 2α. The equation of the locus of the point P is

(a) $x^2 + y^2 + 4x - 6y + 4 = 0$

(b) $x^2 + y^2 + 4x - 6y - 9 = 0$

(c) $x^2 + y^2 + 4x - 6y - 4 = 0$

(d) $x^2 + y^2 + 4x - 6y + 9 = 0$.

9. The equation $2x^2 + 4xy - ky^2 + 4x + 2y - 1 = 0$ represents a pair of lines. The value of k is :

(a) –5/3 (b) +5/3 (c) 1/3 (d) –1/3.

10. The angle between the lines $x^2 + 4xy + y^2 = 0$ is:

(a) 60° (b) 15° (c) 30° (d) 45°.

11. The radius of the circle $x^2 + y^2 + 4x - 6y - 12 = 0$ is:

(a) 9 (b) 5 (c) –3 (d) –6.

12. $\lim_{x\to 0}\frac{\cos(2x^3)-1}{\sin^6(2x)}$ is equal to:

(a) 1/16 (b) –1/16 (c) 1/32 (d) –1/32.

13. $\lim_{x\to 0}\frac{(1+x)^n-1}{x}$ is :

(a) 0 (b) –1

(c) n (d) non existent.

14. Equation of the hyperbola whose vertices are (±3, 0) and foci (±5, 0) is :

(a) $16x^2 - 9y^2 = 144$ (b) $9x^2 - 16y^2 = 144$

(c) $25x^2 - 9y^2 = 225$ (d) $9x^2 - 25y^2 = 81$.

15. The angle between the asymptotes of all hyperbola $27x^2 - 9y^2 = 24$ is:

(a) 30° (b) 120° (c) 60° (d) 240°.

16. What is the equation of the tangent to the parabola $y^2 = 8x$ and perpendicular to the line $x - 3y + 8 = 0$

(a) $9x + 3y + 2 = 0$ (b) $3x + y + 2 = 0$

(c) $3x - y - 1 = 0$ (d) $9x - 3y + 2 = 0$.

17. The equation of a chord of a contact of tangents

(a) 0 (b) $\pi/4$
(c) 1 (d) none of these

49. The total number of non-negative integer pairs (m, n) such that $2m^2 + 5mn + 3n^2 - 29m - 30n + 27 = 0$ are
(a) 7 (b) 4 (c) 3 (d) 2

50. 20 objects are arranged in a row. A subset of 4 elements is selected such that no two of the selected elements are consecutive in the given arrangement. Number of such subsets is
(a) 2380 (b) 2480 (c) 2580 (d) 2180

SECTION - (II)

True or False (Q. No. 51 to 55)

51. Let $X^2 + Y^2 + X + Y = 1$ where $X = y - 2x$, $Y = x/2 + y$ represent a conic. The eccentricity of the conic in (x, y) plane is zero

52. Using the sense of the infinite G. P that $(1-z)^{-1} = 1 + z + z^2 + z^3 + \ldots$ if $|z| < 1$, one can expand $f(x) = \dfrac{1}{(x-2)(x-3)}$ as $f(x) = c + \sum_{n=1}^{\infty} a_n x^n + \sum_{n=1}^{\infty} b_n x^{-n}$ for $2<x<3$. Here c, a_n, b_n are non-zero & independent of x.

53. $y(x) = a + \log bx + ce^{d+x-\log c}$ with a, b, c, d as non-zero parameters is the solution of a differential equation of order three.

54. Given a variable line $y = x + c$ and a set of n-points (x_i, y_i), $i = 1, 2, 3.....n$. If the sum of squares of the perpendicular distances from each point on the line is the least, then $c = \bar{y} - \bar{x}$. Here $\bar{x}, \bar{y}$ are the arithmetic means of x_i and y_i, respectively.

55. If x, y, z are distinct natural numbers such that $\dfrac{\tan A}{x} = \dfrac{\tan B}{y} = \dfrac{\tan C}{z}$ then the triangle ABC is not always possible.

SECTION - (III)

(May have one or more than one options correct)
(Q. No. 56 to 63)

56. Number of distinct neckless which can be formed using 36 identical diamonds and 3 identical pearls is
(a) 127 (b) $\dfrac{21!}{18!3!}$
(c) 1330 (d) ${}^{18}C_3 - {}^{6}C_3 - {}^{6}C_1$

57. Let $f(x)$ be cubic polynomial with two real zeros as 1 and 5. Which of the following polynomial equation(s) will have at least one real root in (1, 5)
(a) $f(x) = 4f'(x)$ (b) $f'(x) = f^2(x)$
(c) $xf'(x) = f(x)$ (d) $f'(x) = xf(x)$

58. The function $y = x^2 \int \dfrac{e^{-f(x)}}{x^4} dx$ with $f(x) = \int \dfrac{2-x^2}{x^2-2x} dx$ is one solution of
(a) $y'' - y = 0$ (b) $(1-x)y'' + xy' - y = 0$
(c) $(x^2 - 2x)y'' + (2 - x^2)y' = 2(x-1)$
(d) none of these

59. Let a function $f(x)$ be continuous, differentiable and monotonic in the domain [1, 9] such that $2f(x) + f(10-x) = 81/x$. The value(s) of N so that we necessarily have $a \in (1, 9)$ and satisfying $f(a) = N$ is (are)
(a) 50 (b) 0 (c) –20 (d) –25

60. Let $\Delta = 2fgh + abc - af^2 - bg^2 - ch^2$. The parametric condition(s) that the equation $ax^2 + by^2 + 2hxy + 2fy + 2gx + c = 0$ does not represent any figure (curve/lines/points) in x–y plane is(are)
(a) $\Delta = 0$ and $h^2 < ab$ (b) $h^2 < ab, a\Delta > 0$ and $b\Delta > 0$
(c) $h^2 < ab$ and $a\Delta > 0$ or $b\Delta > 0$
(d) $a\Delta > 0$ and $b\Delta > 0$

61. If AD and AD' are the internal and external bisectors of $\angle A$ of ΔABC $(b < c)$. The points B, D, C, D' are collinear, then
(a) $BD = \dfrac{ac}{c+b}$ (b) $BD' = \dfrac{ac}{c-b}$
(c) $DD' = \dfrac{2abc}{c^2-b^2}$ (d) $a\,AD > b\,BD$

62. Which of the following function(s) does (do) not have a maximum or a minimum
(a) $10 - (x-7)^{11}$ (b) $(x-3)^{14}$
(c) $x^9 + x^5 + 1$ (d) $x^{14} + x^{13} + 2x$

63. Number of ways in which x & y can be selected from the set of first 25 natural numbers such that $x^4 - y^4$ is divisible by 5 is
(a) ${}^{25}C_2 - 4 \cdot {}^{5}C_1 \cdot {}^{5}C_1$ (b) ${}^{25}C_2$
(c) $2^3({}^{5}C_1)^2$ (d) ${}^{25}C_2 - {}^{20}C_2$

ANSWERS

1. (a)	2. (d)	3. (d)	4. (d)	5. (c)	6. (b)	7. (d)
8. (b)	9. (F)	10. (T)	11. (F)	12. (F)	13. (T)	
14. (a,b,c,d)		15. (a,c)	16. (a,d)	17. (a,b,c)	18. (a,b,d)	
19. (c,d)	20. (a,c)	21. (a,c)	22. (a)	23. (a)	24. (a)	25. (b)
26. (a)	27. (b)	28. (b)	29. (a)	30. (T)	31. (F)	32. (T)
33. (F)	34. (T)	35. (a,b,c)	36. (a,b,c)	37. (a,b)	38. (b,c)	
39. (a,b,c)	40. (a,b)	41. (a,b,c)	42. (c)	43. (c)	44. (b)	45. (d)
46. (c)	47. (a)	48. (b)	49. (a)	50. (a)	51. (F)	52. (T)
53. (F)	54. (T)	55. (F)	56. (a,d)	57. (a,c,d)	58. (a,b)	
59. (a,b,c)	60. (a,b)	61. (a,b,c)	62. (a,c)	63. (a,c)		

For Paper II See Physics For You

CONCEPTS & ANALYSIS

How to Prepare for IIT-JEE

FUNCTIONS

By : Prof. S. S. Dahiya

Part (i)

While dealing with functions mainly consider

(i) Existence of a function (ii) Continuity of the function (iii) Differentiability of the function.

Consider $f(x) = (x)^n$ where $x \in R, n \in R$ this function is discontinuous everywhere for $x < 0$ and $n = \frac{p}{2q}$ where p & q are coprime integers. $f(x) = (x)^n$ is continuous at $x = 0$ only when $n = \frac{p}{2q-1}$ form where $p, q \in N$.

Because $(x)^{3/2}$ is discontinuous at $x = 0$ because left hand limit $= \underset{h \to 0}{Lt}\,(0-h)^{3/2}$ which is imaginary.

Similarly for $f(x) = \begin{cases} x^n.\sin\left(\frac{1}{x}\right), & x \neq 0 \\ 0 & , x = 0 \end{cases}$ is discontinuous at $x = 0$ for $n = \frac{p}{2q}$ where p & q are co-prime integers. For continuity of $f(x)$ at $x = 0$ n must be of the type $\frac{p}{2q-1}$ where $p, q \in N$

$$f'(x) = nx^{n-1}\sin\left(\frac{1}{x}\right) - x^{n-2}\cos\left(\frac{1}{x}\right)$$

Here $n - 2 > 0$ and n must be of type $2 + \frac{p}{2q-1}$

If $n > 2$ and $n = \frac{p}{2q}$ type then left hand derivative becomes imaginary.

Part (ii)

Consider $f(x) = (x)^{2n}$, and $g(x) = (x^2)^n$. If n is an integer then $f(x) = g(x)$. If $x > 0$ and $n \in R$ then $f(x) = g(x)$. Surprisingly when $x \in R$, $n \in R$ then these functions may be equal, may not be equal

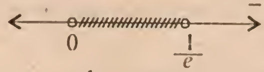
Example : $f(x) = (x)^{2n}$, $g(x) = g(x^2)^n$ For $x = -5$, $n = \frac{3}{4}$, value of $f(x)$ is imaginary; For $x = -5$, $n = \frac{3}{4}$, value of $g(x)$ is $(25)^{3/4}$

$\therefore$ $\underset{\substack{x \to 0 \\ n \to \infty}}{Lt}\,(x)^{2n}$ does not exist for $n \in R$

whereas $\underset{\substack{x \to 0 \\ n \to \infty}}{Lt}\,(x^2)^n$ is zero

Part (iii) $\underset{\substack{m \to \infty \\ n \to \infty}}{Lt}\left\{\cos^{2m}(n!\pi x)\right\}$. This limit does not exist for $x \in$ Irrational. When $x \in$ rational then $n!\pi x = 2k\pi$ where $k \in N$ for large values of n, here say $x = \frac{1}{17}$. Hence $\cos(n!\pi x) = 1$. For $x = \frac{1}{\sqrt{289+h}}$, we consider the case $h \to 0$ then $x \to \frac{1}{17}$ therefore $n!\pi x \to 2k\pi$ As a result $\cos(n!\pi x) \to 1$ then $\underset{\substack{m \to \infty \\ n \to \infty}}{Lt}\cos^{2m}(n!\pi x)$ is Indeterminant form 1^∞_-. Therefore no conclusion can be drawn for this limit when x is irrational

Part (iv) $f(x) = \log_{e^2 x}\left(\frac{2\ln x + 2}{-x}\right)$ For domain of function $f(x)$ $e^2x > 0$, $e^2x \neq 1$ and $\frac{2\ln(x)+2}{-x} > 0$



we get $x > 0, x \neq \frac{1}{e^2}$. Therefore domain is $x \in \left(0, \frac{1}{e^2}\right) U \left(\frac{1}{e^2}, \frac{1}{e}\right)$ $g(x) = 1 + \{x\}$ where $\{x\}$ denotes fractional part of x where $x \in R$ and $1 \le g(x) < 2$

Therefore $f(g(x))$ is not possible whereas $g(f(x))$ exists

Part (v) : Some functions are based on their particular properties, for example $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ where $x \in R$, $y \in R$ is property of linear function.

Similarly for property $f(x) + f\left(\frac{1}{x}\right) = f(x).f\left(\frac{1}{x}\right)$ and $f(x)$ is polynomial then $f(x) = x^n + 1$ or $f(x) = -x^n + 1$

Here the property can be changed by putting $\frac{1}{x} = y$ we get $f(x) + f(y) = f(x).f(y)$ and $xy = 1$; when $xy = 1$ then $f(xy) = f(1)$ We can rewrite the property as $f(x) + f(y) = f(x).f(y) + f(xy) - f(1)$. Therefore for property $f(x) + f(y) = f(x).f(y) + f(xy) - f(1)$ and $f(x)$ is polynomial, our function is $f(x) = x^n + 1$ or $f(x) = -x^n + 1$.

concept BOOSTERS

Limit, Continuity & Differentiability

Class XII

The response from the readers is overwhelming. To make our booster rockets more powerful, MTG has decided to create under the aegis of Mathematics Today – The I.I.T. Think Tank, to pool the experience, insight and knowledge of a panel of experts.

We shall be bridging the gap between the school syllabus and that of the Engineering entrance examinations both objective and subjective questions of various types will be given in order to prepare you to face any type of examination better – whether it is I.I.T. (JEE), DCE, AIEEE and at the same time CBSE also.

This is meant for the total development of our readers.

By : **MTG Editorial Board**

Evaluating Limits

1. In all the problems given below find the value of the limits.

(i) $\lim\limits_{x\to 4}\dfrac{(\cos\alpha)^x-(\sin\alpha)^x-\cos 2\alpha}{x-4}$, $\alpha\in(0,\ \pi/2)$

(ii) $\lim\limits_{x\to 0}\dfrac{(e^x-1)-(e^{x\cos x}-1)}{x+\sin x}$

(iii) $\lim\limits_{x\to 0}\dfrac{\tan([-\pi^2]x^2)-\tan([-\pi]^2)x^2}{\sin^2 x}$

(iv) $\lim\limits_{x\to 0}\dfrac{e-(1+x)^{1/x}}{\tan x}$

(v) $\lim\limits_{x\to 0}\left(\dfrac{1^x+2^x+3^x+\ldots+n^x}{n}\right)$

(vi) $\lim\limits_{n\to\infty} n^{-n^2}\left\{(n+1)\left(n+\dfrac{1}{2}\right)\left(n+\dfrac{1}{4}\right)\ldots\left(n+\dfrac{1}{2n-1}\right)\right\}$

2.(i) A square is inscribed in a circle of radius R, a circle is then inscribed in this square, then a square in this circle and so on for n times. Evaluate the limit of the sum of areas of all squares as $n\to\infty$.

(ii) Find $\lim\limits_{x\to 2}\dfrac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$

(iii) Calculate the values of α and β in order that

$$\lim_{x\to 0}\frac{x(1+\alpha\cos x)-\beta\sin x}{x^3}=1.$$

(iv) $\lim\limits_{\theta\to\pi/4}\dfrac{\sqrt{2}-\cos\theta-\sin\theta}{(40-\pi)^2}$

(v) Without using L'Hospital rule evaluate

$$\lim_{x\to 1}\frac{x^{k+1}-(k+1)x+k}{(x-1)^2}$$

Testing for continuity

3.(i) Let $f(x)=\dfrac{\cos^{-1}(1-\{x\}^2)\cdot\sin^{-1}(1-\{x\})}{\sqrt{2}(\{x\}-\{x\}^3)}$ for $x\neq 0$

$=\dfrac{\pi}{2}$ for $x=0$

Consider another function $g(x)$ defined by

$$g(x)=\begin{cases}f(x), & x\geq 0\\ 2\sqrt{2}\,f(x), & x<0\end{cases}$$

(ii) Let $f(x)=x^3-3x^2+6\ \forall\ x\in R$

$$g(x)=\begin{cases}\max\{f(t): x+1\leq t\leq x+2,\ -3\leq x<0\}\\ 1-x,\ \text{for } x\geq 0\end{cases}$$

Find the points where the function $g(x)$ is discontinuous in the interval $x\in[-3,1]$.

(iii) Let $[x]$ denote the greatest integer function and $f(x)$ be defined in a neighbourhood of 2 by $f(x)$ given by

$$f(x)=\begin{cases}\dfrac{\left(\exp\{(x+2)\ln 4\}^{\left(\frac{x+1}{9}\right)}\right)-16}{4^x-16}, & x<2\\[2ex] \Delta\,\dfrac{1-\cos(x-2)}{(x-2)\tan(x-2)}, & x>2\end{cases}$$

Find the values of Δ and $f(2)$ in order that $f(x)$ may be continuous at $x=2$.

(iv) It is given that the function

$$f(x)=\frac{2-(256-7x)^{1/8}}{(5x+32)^{1/5}-2},\ x\neq 0$$

is continuous everywhere, then what is the value of $f(0)$?

Mixed Problems on Continuity and Differentiability

4.(i) Let $f(x)=\begin{cases}x+2, & 0\leq x<2\\ 6-x, & x\geq 2\end{cases}$,

$$g(x) = \begin{cases} 1+\tan x, & 0 \le x < \pi/4 \\ 3-\cot x, & \pi/4 \le x < \pi \end{cases}$$

Find the composite function $fog(x)$ and test its continuity and derivability.

(ii) Given that f and g are differentiable functions, evaluate (using L'Hospitals rule)

$\lim_{x\to 2} \frac{f(x)g(4-x)-f(4-x)g(x)}{x-2}$. Assume $f(2) = 2$, $f'(2) = -3$, $g(2) = 4$ and $g'(2) = 1$.

5.(i) Let $g(x)$ be a polynomial of degree 1 and $f(x)$ be defined by

$$f(x) = \begin{cases} g(x), & x \le 0 \\ \left(\frac{1+x}{2+x}\right)^{1/x}, & x > 0 \end{cases}$$

Find the continuous function satisfying $f'(1) = f(-1)$.

(ii) Test for continuity the function

$$f(x) = \lim_{n\to\infty} \frac{(1+\sin x)^n + \ln x}{2+(1+\sin x)^n}.$$

(iii) If a function $f(x)$ satisfies

$$f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3} \quad \forall\ x, y \in R$$

and $f'(0) = 1$, then prove that $f(x)$ is continuous for all $x \in R$.

(iv) Let $f: R \to R$ be a function defined as

$$f(x) = \begin{cases} 1-|x|, & |x| \le 1 \\ 0, & |x| > 1 \end{cases} \text{ and}$$

$$g(x) = f(x-1) + f(x+1) \ \forall\ x \in R$$

Describe $g(x)$ in terms of x and test for its continuity and derivability.

6.(i) Let $f: [0, \infty) \to [1, \infty)$ be a one–one function satisfying $f(x)f(y)+2=f(x)+f(y)+f(xy)\ \forall\ x,\ y \ge 0$ and $f'(1)=2 \ne f(0)$ Evaluate $\int_0^1 f(x)dx$

(ii) If f be a polynomial function satisfying $x + f(x)\cdot f(y) = f(x) + f(y) + f(xy)\ \forall\ x,\ y \in R$ and if $f(2) = 5$, then find $f(f(2))$.

(iii) Define $f(x) = x^2 - 2x$, $x \in R$. Let $g(x)$ be defined by $g(x) = f(f(x) - 1) + f(5 - f(x))$. Show that $g(x) \ge 0\ \forall\ x \in R$. Also find the critical points of $g(x)$.

Application to function-based questions

7.(i) A function $f(x)$ is defined for all $x \in R$ and satisfies $f(x+y) = f(x) + 2y^2 + kxy\ \forall\ x, y \in R$ where k is a given constant. If $f(1) = 2$, $f(2) = 8$, find $f(x)$ and show that $f(x+y)\cdot f\left(\frac{1}{x+y}\right) = k,\ x+y \ne 0$.

(ii) Let f and g be real function such that

$$f(x+y) + f(x-y) = 2f(x)g(y)\ \forall\ x, y \in R.$$

If $f(x)$ is not identically zero and $|f(x)| \le 1\ \forall\ x \in R$, then prove that $|g(y)| \le 1\ \forall\ y \in R$.

(iii) If $e^{-xy}f(xy) = e^{-x}f(x) + e^{-y}f(y),\ \forall\ x, y \in R^+$, and $f'(1) = e$, determine $f(x)$.

Miscellaneous

8.(i) Find the values of a, b, c if the function $f(x) = a|\sin x| + be^{|x|} + c|x|^3$ is differentiable at $x = 0$.

(ii) Test for differentiability of the function

$$f(x) = (x^2-1)|x^2-3x+2| + \cos|x|$$

9.(i) Find $\lim_{x\to 0} \frac{\cot x \tan^{-1}(m\tan x) - m\cos^2(x/2)}{\sin^2(x/2)}$

(ii) Find $\lim_{x\to\infty}\left(\frac{a^{1/x}+b^{1/x}+c^{1/x}}{3}\right)^{3x}$.

10.(i) Let $f: R \to R$ be a function satisfying

$$f(x+2y) = f(x)e^{2y} + f(2y)e^x + x^2(1-e^{2y}) + 4y^2(1-e^x) + 4xy\ \forall\ x,\ y \in R.$$

Also $f'(0) = 1$. Find $f(x)$.

(ii) Let $f(x)$ be a continuous function in $[-1, 1]$ and satisfies $f(2x^2-1) = 2xf(x)\ \forall\ x \in [-1, 1]$. Show that $f(x)$ is identically zero $\forall\ x \in [-1, 1]$.

MULTIPLE CHOICE QUESTIONS

1. The value of $\lim_{x\to 0} \frac{\sin^{-1}x - \tan^{-1}x}{x^3}$ is

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$ (c) $-1/2$ (d) $1/2$

2. A function $f(x)$ satisfies $3f(x) + 4f\left(\frac{1}{x}\right) = \frac{1}{x} - 5$.

Then the value of $f(2)$ is

(a) $-3/14$ (b) $3/7$ (c) $3/14$ (d) $-3/7$.

3. Let $f(x) = f(x)\,g(x)\,h(x)$ when f, g, h are differentiable function. Assume $f'(x_0) = 13f(x_0)$, $f'(x_0) = 9f(x_0)$, $g'(x_0) = 4g(x_0)$ then $h'(x_0)/h(x_0)$, $(h(x_0) \ne 0)$ is

(a) 0 (b) 26 (c) -26 (d) 17.

4. About the function represented parametrically as $x = 2t - |t|$, $y = t^3 + t^2|t|$, which of the following statement is correct?

(a) f is differentiable everywhere, except $x = 0$ and $x = 2$

(b) f is differentiable everywhere except $x = 0$

(c) f is differentiable at $x = 0$ and $f'(0) = 0$

(d) f is differentiable at $x = 0$ but $f'(0) \neq 0$.

5. $f(x)$ is a real valued function not identically equal to zero such that $f(x + y^n) = f(x) + (f(y))^n$, $y \in R$ and n is natural number > 1 and $f'(0) \geq 0$, then which of the following statements is correct?

(a) $f(15) = -15,\ f'(20) = 1$

(b) $f(15) = 15,\ f'(20) = -1$

(c) $f(15) = -15,\ f'(20) = 1$

(d) $f(15) = 15,\ f'(20) = 1$

6. Let D be the domain and R the range of

$$f(x) = \left[\ln(\sin^{-1}\sqrt{x^2+3x+2}\right]$$

where $[\cdot]$ denotes the greatest integer function, then of the following which statements is correct?

(a) $D = \left[\frac{-3-\sqrt{5}}{2}, -2\right) \cup \left(-1, \frac{-3+\sqrt{5}}{2}\right]$,

R = real of non-positive integers

(b) $D = \left[-\frac{3-\sqrt{5}}{2}, -2\right) \cup \left[-1, \frac{-3+\sqrt{5}}{2}\right)$,

R = set of non-positive integers

(c) $D = \left(\frac{-3-\sqrt{5}}{2}, -2\right) \cup \left[-1, \frac{-3+\sqrt{5}}{2}\right)$,

R = set of non-negative integer

(d) $D = \left[-\frac{3-\sqrt{5}}{2}, -2\right) \cup \left(-1, \frac{-3+\sqrt{5}}{2}\right)$,

R = set of non-negative integers.

7. If $x + y = e^{x-y}$, then $\frac{d^2y}{dx^2}$ at $x = x_0$ is given by

(a) $\frac{4e^{y_0}}{e^{x_0}+e^{y_0}}$ (b) $\frac{4e^{x_0}}{e^{x_0}+e^{y_0}}$

(c) $\frac{4(x_0+y_0)}{(x_0-y_0+1)^3}$ (d) $\frac{4(x_0+y_0)}{(x_0+y_0+1)^2}$

8. The value of $\lim\limits_{x\to 0} \frac{27^x - 9^x - 3^x + 1}{\sqrt{5}-\sqrt{4+\cos x}}$ is

(a) $4\sqrt{5}(\ln 3)^2$ (b) $8\sqrt{5}(\ln 3)^2$

(c) $4\sqrt{5}(\ln 3)$ (d) $8\sqrt{5}(\ln 3)$

9. The value of a such that f is continuous at $x = 0$, where $f(x) = \frac{\sin 2x + a\sin x}{x^3}$, $x \neq 0$ is

(a) -2 and then $f(0) = -1$

(b) 2 and then $f(0) = -1$

(c) -2 and then $f(0) = 1$ (d) 2 and then $f(0) = 1$

10. The value of $\lim\limits_{n\to\infty} \cos\left(\frac{x}{2}\right)\cos\frac{x}{4}\cdot\cos\frac{x}{8}\ldots\cos\frac{x}{2^n}$ is

(a) $\frac{\sin x}{2x}$ (b) $\frac{x}{\sin x}$ (c) $\frac{\sin x}{x}$ (d) $\frac{2x}{\sin x}$

11. The values of a and b if the function given by

$$f(x) = \begin{cases} x^2+ax+1, & x \text{ rational} \\ ax^2+bx+1, & x \text{ irrational} \end{cases}$$

are respectively

(a) 1 and 2 (b) 1 and 1

(c) 2 and 1 (d) 1 and -1.

12. The value of $\lim\limits_{n\to\infty}(4^n+5^n)^{1/n}$ is given by

(a) $\sqrt{20}$ (b) 4 (c) 5 (d) 4.5.

13. Let $f(x) = \cos 2x \cot\left(\frac{\pi}{4}-x\right)$. Given that f is continuous at $x = \pi/4$, the value of $f(\pi/4)$ is

(a) -2 (b) 2 (c) 1 (d) -1.

14. Let f and g be two continuous and differentiable functions satisfying $f(x+y) = f(x) + f(y)\ \forall\ x, y \in R$. Also $f(x) = x^2g(x)$. Then $|f(15) - f(-15)|$ is

(a) -30 (b) 30

(c) 0 (d) cannot be determined.

15. The value of $\lim\limits_{x\to 1} \frac{x^{1/3}-x^{1/4}-2}{x^3-1}$ is

(a) 1/36 (b) $-1/36$ (c) $-1/12$ (d) 1/12.

SOLUTIONS

1. $\lim\limits_{x\to 4} \frac{(\cos\alpha)^x - (\sin\alpha)^x - \cos 2\alpha}{x-4}$

$= \lim\limits_{x\to 4} \frac{(\cos\alpha)^x - (\sin\alpha)^x - (\cos^2\alpha - \sin^2\alpha)(\cos^2\alpha+\sin^2\alpha)}{x-4}$

$= \lim\limits_{x\to 4} \frac{(\cos\alpha)^x - (\sin\alpha)^x - \cos^4\alpha + \sin^4\alpha}{x-4}$

$= \lim\limits_{x\to 4} \frac{(\cos\alpha)^x - \cos^4\alpha - (\sin\alpha)^x + \sin^4\alpha}{x-4}$

$= \lim\limits_{x\to 4} \frac{(\cos\alpha)^4\{(\cos\alpha)^{x-4}-1\} - \sin^4\alpha\{(\sin\alpha)^{x-4}-1\}}{x-4}$

$$= \cos^4\alpha \cdot \lim_{x\to 4}\frac{(\cos\alpha)^{x-4}-1}{x-4} - \sin^4\alpha \cdot \lim_{x\to 4}\frac{(\sin\alpha)^{x-4}-1}{x-4}$$

$$= \cos^4\alpha \cdot \lim_{h\to 0}\frac{a^h-1}{h} - \sin^4\alpha \cdot \lim_{h\to 0}\frac{b^h-1}{h} \quad ...(1)$$

where $a = \cos\alpha$ and $b = \sin\alpha$

Now $\lim_{h\to 0}\frac{a^h-1}{h} = \lim_{h\to 0}\frac{e^{h\log_e a}-1}{h} = \ln a$

Using this we have from (1)

the desired limit $= \cos^4\alpha \cdot \ln(\cos\alpha) - \sin^4\alpha \cdot \ln(\sin\alpha)$.

(ii) $\lim_{x\to 0}\frac{(e^x-1)-(e^{x\cos x}-1)}{x+\sin x}$

$$= \lim_{x\to 0}\frac{e^x-1}{x+\sin x} - \lim_{x\to 0}\frac{e^{x\cos x}-1}{x+\sin x}$$

$$= \lim_{x\to 0}\frac{e^x-1}{x\left(1+\frac{\sin x}{x}\right)} - \lim_{x\to 0}\frac{e^{x\cos x}-1}{x\cos x\left(\sec x+\frac{\sin x}{x\cos x}\right)}$$

$$= \lim_{x\to 0}\frac{e^x-1}{x}\cdot\frac{1}{\left(1+\frac{\sin x}{x}\right)} - \lim_{x\to 0}\frac{e^{x\cos x}-1}{x\cos x}\cdot\frac{1}{\sec x+\frac{\sin x}{x\cos x}}$$

$$= \left(\lim_{x\to 0}\frac{e^x-1}{x}\right)\left(\lim_{x\to 0}\frac{1}{1+(\sin x/x)}\right) - \left(\lim_{x\to 0}\frac{e^{x\cos x}-1}{x\cos x}\right)\cdot\left(\lim_{x\to 0}\frac{1}{\sec x+(\tan x/x)}\right)$$

$$= 1\cdot\frac{1}{1+1} - 1\cdot\frac{1}{1+1} = \frac{1}{2}-\frac{1}{2} = 0.$$

Remark : *Note now we have pulled x and x cosx from the numerator while evaluating limits $e^x - 1$ and $e^{x\cos x} - 1$, to make use of the fact that $\lim_{x\to 0}\frac{e^x-1}{x} = 1$. Such obervations help greatly when you want to simplify your caluclations and avoid the traps laid by L'Hospital rule.*

(iii) $\lim_{x\to 0}\frac{\tan([-\pi^2]x^2)-\tan([-\pi^2])x^2}{\sin^2 x}$

(Observe that $\pi = 3.14 \Rightarrow \pi^2 = 9.86$
$\Rightarrow -\pi^2 = -9.86 \;\therefore\; [-\pi^2] = -10$).

$$= \lim_{x\to 0}\frac{\tan(-10x^2)-(\tan(-10))x^2}{\sin^2 x}$$

$$= \lim_{x\to 0}\frac{-\tan(10x^2)+(\tan 10)x^2}{\sin^2 x}$$

$$= \lim_{x\to 0}\frac{(-\tan(10x^2)+(\tan 10)x^2)x^2}{(\sin^2 x)/x^2}$$

$$= \lim_{x\to 0}\frac{-10\cdot(\tan(10x^2)/10x^2)+\tan 10}{(\sin x/x)^2}$$

$$= \frac{-10\cdot 1+\tan 10}{1^2} = \tan 10 - 10.$$

(iv) $\lim_{x\to 0}\frac{e-(1+x)^{1/x}}{\tan x} = \lim_{x\to 0}\frac{e-e^{\ln(1+x)^{1/x}}}{(\tan x/x)x}$

$$= \lim_{x\to 0}\frac{e-e^{(1/x)\ln(1+x)}}{x} = \lim_{x\to 0}\frac{e-e^{\left(\frac{1}{x}\right)\left(x-\frac{x^2}{2}+\frac{x^3}{3}-\frac{x^4}{4}+...\right)}}{x}$$

$$= \lim_{x\to 0}\frac{e-e\cdot e^{-\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}....\right)}}{x}$$

$$= -e\cdot\lim_{x\to 0}\frac{e^{-\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}....\right)}-1}{x}$$

$$= -e\cdot\lim_{x\to 0}\frac{\left[e^{-\left(\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}....\right)}-1\right]\left[-\left(\frac{1}{2}-\frac{x}{3}-...\right)\right]}{\left(-\frac{x}{2}-\frac{x^2}{3}+\frac{x^3}{4}....\right)} = \frac{e}{2}.$$

(v) $\lim_{x\to 0}\left(\frac{1^x+2^x+3^x+...+n^x}{n}\right)^{\frac{a}{x}}$

$$= e^{\lim_{x\to 0}\left(\frac{1^x+2^x+3^x+...+n^x}{n}-1\right)\cdot\frac{a}{x}}$$

$$= e^{\lim_{x\to 0}\left(\frac{1^x+2^x+...+n^x-n}{n}\right)\cdot\frac{a}{x}}$$

$$= e^{\lim_{x\to 0}\left\{\frac{(1^x-1)+(2^x-1)+...+(n^x-1)}{x}\right\}\cdot\frac{a}{n}}$$

$$= e^{\left\{\lim_{x\to 0}\frac{1^x-1}{x}+\lim_{x\to 0}\frac{2^x-1}{x}+...+\lim_{x\to 0}\frac{n^x-1}{x}\right\}\cdot\frac{a}{n}}$$

$$= e^{(\ln 1+\ln 2+...+\ln n)\cdot\frac{a}{n}} = e^{\{\ln(n!)\}\frac{a}{n}}$$

$$= e^{\frac{a}{n}\ln(n!)} = e^{\ln(n!)^{a/n}} = (n!)^{a/n}$$

(vi) $\lim_{n\to\infty} n^{-n^2}\left\{(n+1)\left(n+\frac{1}{2}\right)....\left(n+\frac{1}{2^{n-1}}\right)\right\}^n$

$$= \lim_{n\to\infty}\left\{\frac{(n+1)\left(n+\frac{1}{2}\right)....\left(n+\frac{1}{2n-1}\right)}{n^n}\right\}^n$$

$$= \lim_{n\to\infty}\left\{\frac{n+1}{n}\cdot\frac{n+\frac{1}{2}}{n}...\frac{n+\frac{1}{2n-1}}{n}\right\}^n$$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^n\cdot\left(\frac{n+\frac{1}{2}}{n}\right)^n....\left(\frac{n+\frac{1}{2n-1}}{n}\right)^n\right]$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{2n}\right)^{\frac{2n}{2}}....\left(1+\frac{1}{2^{n-1}\cdot n}\right)^{\frac{2^{n-1}\cdot n}{2^{n-1}}}\right]$$

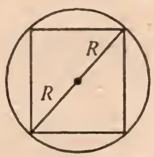
$$= \left\{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right\}\left\{\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{\frac{2n}{2}}\right\}....$$

$$\left\{\lim_{n\to\infty}\left(1+\frac{1}{2^{n-1}\cdot n}\right)^{\frac{2^{n-1}\cdot n}{2^{n-1}}}\right\}$$

$$= e\cdot e^{1/2}\cdot e^{1/4}....e^{1/2^{n-1}} = e^{1+\frac{1}{2}+\frac{1}{4}+....+} = e^{\frac{1}{1-(1/2)}}$$

$= e^2$.

2.(i) If in a circle of radius R, we inscribed a square, then its side is given by



$$= \frac{\text{diagonal}}{\sqrt{2}} = \frac{2R}{\sqrt{2}} = \sqrt{2}R = a \text{ (say)}$$

Let a_1 be the side of another square,

then $a_1\sqrt{2} = a \Rightarrow a_1 = \dfrac{a}{\sqrt{2}}$

Again $a_2\sqrt{2} = a_1 \Rightarrow a_2 = \dfrac{a_1}{\sqrt{2}} = \dfrac{a}{2}$

S_n = sum of square of all areas

$$= a^2 + a_1^2 + a_2^2 + ... + a_n^2$$

$$= a^2 + \frac{a^2}{2} + \frac{a^2}{4} + ... + \text{to } n \text{ terms}$$

$$= a^2\left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right) = 2a^2\left(1-\frac{1}{2^n}\right)$$

$$\lim_{n\to\infty} S_n = 2a^2 = 4R^2.$$

(ii) $$\lim_{x\to 2}\frac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$$

$$= \lim_{x\to 2}\frac{\sqrt{x+7}-\sqrt{9}+\sqrt{9}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-\sqrt[3]{8}+\sqrt[3]{8}-2\sqrt[3]{3x-5}}$$

$$= \lim_{x\to 2}\frac{\left\{(x+7)^{1/2}-9^{1/2}\right\}-3\left\{(2x-3)^{1/2}-1\right\}}{\left\{(x+6)^{1/3}-8^{1/3}\right\}-2\left\{(3x-5)^{1/3}-1\right\}}$$

$$= \lim_{x\to 2}\frac{\dfrac{(x+7)^{1/2}-9^{1/2}}{x+7-9}-3\left\{\dfrac{(2x-3)^{1/2}-1}{(2x-3)-1}\right\}\cdot 2}{\dfrac{(x+6)^{1/3}-8^{1/3}}{x+6-8}-2\left\{\dfrac{(3x-5)^{1/3}-1^{1/3}}{(3x-5)-1}\right\}\cdot 3}$$

$$= \frac{\frac{1}{2}\cdot(9)^{\frac{1}{2}-1}-6\cdot\frac{1}{2}\cdot(1)^{\frac{1}{2}-1}}{\frac{1}{3}(8)^{\frac{1}{3}-1}-6\cdot\frac{1}{3}\cdot(1)^{\frac{1}{3}-1}} = \frac{\frac{1}{2}\cdot\frac{1}{3}-3}{\frac{1}{3}\cdot\frac{1}{4}-2}$$

$$= -\frac{17}{6}\times\frac{12}{-23} = \frac{34}{23}.$$

(iii) $$\lim_{x\to 0}\frac{x(1+\alpha\cos x)-\beta\sin x}{x^3} = 1$$

i.e. $$\lim_{x\to 0}\frac{x\left[1+\alpha\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-...\right)\right]-\beta\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-...\right]}{x^3} = 1$$

i.e. $$\lim_{x\to 0}\frac{x(1+\alpha-\beta)+x^3\left(-\frac{\alpha}{2!}+\frac{\beta}{3!}\right)+x^5\left(\frac{\alpha}{4!}-\frac{\beta}{5!}\right)...}{x^3} = 1$$

$$\Rightarrow \lim_{x\to 0}\frac{1+\alpha-\beta}{x^2}+\lim_{x\to 0}\left(-\frac{\alpha}{2!}+\frac{\beta}{3!}\right)+\lim_{x\to 0}\left(\frac{\alpha}{4!}-\frac{\beta}{5!}\right)x^2... = 1$$

As the limit tends to a finite value, we have

$1+\alpha-\beta = 0$ and $-\dfrac{\alpha}{2!}+\dfrac{\beta}{3!} = 1$ i.e. $-3\alpha+\beta = 6$

Solving we get $\alpha = -5/2$, $\beta = -3/2$.

(iv) $$\lim_{\theta\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$$

$$= \sqrt{2}\lim_{\theta\to\pi/4}\frac{1-\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)}{(4\theta-\pi)^2}$$

$$= \sqrt{2}\lim_{\theta\to\pi/4}\frac{1-\cos(\theta-(\pi/4))}{16(\theta-(\pi/4))^2}$$

Set $\theta = \dfrac{\pi}{4}+h$, so that when $\theta\to\pi/4$, $h\to 0$

$$\therefore \text{ above limit} = \sqrt{2}\lim_{h\to 0}\frac{1-\cos h}{16h^2} = \frac{\sqrt{2}}{16}\cdot\lim_{h\to 0}\frac{1-\cos h}{h^2}$$

$$= \frac{1}{8\sqrt{2}}\cdot\lim_{h\to 0}\left(\frac{2\sin^2(h/2)}{h^2}\right) = \frac{1}{8\sqrt{2}}\cdot 2\cdot\lim_{h\to 0}\left(\frac{\sin(h/2)}{h/2}\right)^2\cdot\frac{1}{4}$$

$$= \frac{1}{16\sqrt{2}}\cdot 1 = \frac{1}{16\sqrt{2}}.$$

(v) $$\lim_{x\to 1}\frac{x^{k+1}-(k+1)x+k}{(x-1)^2} = \lim_{x\to 1}\frac{(x^{k+1}-x)-k(x-1)}{(x-1)^2}$$

$$= \lim_{x\to\infty}\frac{x(x^k-1)-k(x-1)}{(x-1)^2}$$

$$= \lim_{x\to 1}\frac{x(x-1)(x^{k-1}+x^{k-2}+...+x+1)-k(x-1)}{(x-1)^2}$$

$$= \lim_{x\to 1} \frac{(x-1)\{x(x^{k-1}+x^{k-2}+...+x+1)-k)}{(x-1)^2}$$

$$= \lim_{x\to 1} \frac{x^k + x^{k-1} + ... + x^2 + x - k}{x-1}$$

$$= \lim_{x\to 1} \frac{x^k + x^{k-1} + ... + x^2 + x - (1+1+... \text{ to } k \text{ times})}{x-1}$$

$$= \lim_{x\to 1} \frac{(x^k - 1)+(x^{k-1}-1)+...+(x^2-1)+(x-1)}{x-1}$$

$$= \lim_{x\to 1} \frac{x^k-1}{x-1} + \lim_{x\to 1} \frac{x^{k-1}-1}{x-1} + ... + \lim_{x\to 1} \frac{x-1}{x-1}$$

$$= k + (k-1) + (k-2) + ... + 2 + 1 = \frac{k(k+1)}{2}.$$

3.(i) For $x > 0$

Right hand limit $= \lim_{x\to 0^+} f(0) = \lim_{h\to 0^+} f(h)$

$$= \lim_{h\to 0} \frac{\cos^{-1}(1-\{h\}^2)\cdot \sin^{-1}(1-\{h\})}{\sqrt{2}\left(\{h\}-\{h\}^3\right)}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}(1-h^2)\cdot \sin^{-1}(1-h)}{\sqrt{2}(h-h^3)}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}(1-h^2)\cdot \sin^{-1}(1-h)}{\sqrt{2}\cdot h(1-h^2)}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}(1-h)^2\cdot \sin^{-1}(1-h)}{\sqrt{2}\cdot h(1-h)(1+h)}$$

$$= \lim_{h\to 0} \frac{\sin^{-1}(1-h)}{\sqrt{2}(1-h)(1+h)} \cdot \lim_{h\to 0} \frac{\cos^{-1}(1-h^2)}{h}$$

$$= \frac{\sin^{-1}1}{\sqrt{2}} \cdot \lim_{\theta\to 0} \frac{2\theta}{\sqrt{2}\sin\theta}$$

$$= \frac{\sin^{-1}1}{\sqrt{2}} \cdot \frac{2}{\sqrt{2}} \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = (\sin^{-1}1)1 = \frac{\pi}{2}$$

For $x < 0$

Left hand limit $= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0} \frac{\cos^{-1}(1-\{0-h\}^2)\cdot \sin^{-1}(1-\{0-h\})}{\sqrt{2}(\{0-h\}-\{0-h\}^3)}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}(1-\{1-h\}^2)\cdot \sin^{-1}(1-\{1-h\})}{\sqrt{2}(\{1-h\}-\{1-h\}^3)}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}(h(2-h))\cdot \sin^{-1}h}{\sqrt{2}(1-h)\cdot(2-h)\cdot h}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}h(2-h)}{\sqrt{2}\cdot(1-h)(2-h)} \cdot \lim_{h\to 0} \frac{\sin^{-1}h}{h}$$

$$= \frac{\cos^{-1}0}{2\sqrt{2}}\cdot 1 = \frac{\pi}{4\sqrt{2}} \quad \text{Also } f(0) = \frac{\pi}{2}$$

Thus $f(x)$ is discontinuous at $x = 0$.

Checking the continuity for $g(x)$

$g(0) = f(0) = \pi/2$

Right hand limit $= \lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} f(x) = \frac{\pi}{2}$

Left hand limit $= \lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} 2\sqrt{2}\, f(x)$

$$= 2\sqrt{2} \lim_{x\to 0-} f(x) = 2\sqrt{2}\cdot \frac{\pi}{4\sqrt{2}} = \frac{\pi}{2}$$

Thus $g(x)$ is continuous at $x = 0$.

(ii) $f(x) = x^3 - 3x^2 + 6$

$f'(x) = 3x^2 - 6x \Rightarrow x = 0, 2$

These are the critical points of the function $f(x)$

$f''(x) = 6x, f'''(x) = 6, \ f''(2) = 12 > 0$

$\therefore x = 2$ is a point of local minima and $x = 0$ is a point of local maxima.

Clearly $f(x)$ is increasing in $(-\infty, 0)$ and $(2, \infty)$ and decreasing in $(0, 2)$.

$x + 2 \le 0 \Rightarrow x \le -2 \Rightarrow g(x) = f(x+2), -3 \le x \le -2$

If $x + 1 < 0$ and $0 < x + 2 < 2$ i.e. $x < -1$ and $-2 < x < 0$

Thus $-2 < x < -1$, $g(x) = f(0)$

Now, for $0 \le x + 1, x + 2 \le 2 \Rightarrow -1 \le x \le 0$,

$g(x) = f(x+1)$

$$\Rightarrow g(x) = \begin{cases} f(x+2) & -3 \le x < -2 \\ f(0) & -2 \le x < -1 \\ f(x+1) & -1 \le x < 0 \\ 1-x & x \ge 0 \end{cases}$$

Thus $g(x)$ is continuous in the interval $[-3, 1]$.

(iii) For finding left hand limit, observe that $1 < x < 2$, so that $[x + 1] = 2$

$$\text{LHL} = \lim_{x\to 2^-} \frac{(4^{x+2})^{1/2}-16}{4^x-16} = \lim_{x\to 2^-} \frac{4(2^x-4)}{(2^x-4)(2^x+4)}$$

$$= \lim_{x\to 2^-} \left(\frac{4}{2^x+4}\right) = \frac{4}{8} = \frac{1}{2}$$

$$\text{RHL} = A \lim_{x\to 2^+} \frac{1-\cos(x-2)}{(x-2)\tan(x-2)} = A \lim_{h\to 0} \frac{1-\cos h}{h\tan h}$$

where $x = h + 2$, so that when $x \to 2^+, h \to 0$

$$= A \lim_{h\to 0} \frac{2\sin^2(h/2)}{h\tan h} = A \lim_{h\to 0} \frac{2\sin^2(h/2)}{h^2\cdot(\tan h/h)}$$

$$= A \lim_{h\to 0} \frac{\left(\frac{\sin(h/2)}{h/2}\right)^2 \cdot \frac{1}{4}\cdot 2}{\frac{\tan h}{h}} = A\cdot\frac{2}{4} = \frac{A}{2}$$

For continuity at $x = 2$, LHL = RHL = $f(2)$

$\Rightarrow \quad A = 1$ and $f(2) = 1/2$.

(iv) $f(0)$ = Right hand limit

$$= \lim_{x\to 0} f(x) = \lim_{h\to 0} f(0+h) = \lim_{h\to 0} \frac{2-(256-7h)^{1/8}}{(5h+32)^{1/5}-2}$$

$$= -\lim_{h\to 0} \frac{(256-7h)^{1/8}-(256)^{1/8}}{(5h+32)^{1/5}-(32)^{1/5}}$$

$$= -\lim_{h\to 0} \frac{\dfrac{(256-7h)^{1/8}-(256)^{1/8}}{(256-7h)-256}\cdot(-7h)}{\dfrac{(5h+32)^{1/5}-(32)^{1/5}}{(5h+32)-32}\cdot(5h)}$$

$$= \frac{7}{5}\cdot\frac{(1/8)(256)^{\frac{1}{8}-1}}{\left(\frac{1}{2}\right)\cdot(32)^{\frac{1}{5}-1}} = \frac{7}{5}\cdot\frac{\left(\frac{1}{8}\right)\cdot(256)^{-7/8}}{\left(\frac{1}{2}\right)\cdot(32)^{-4/5}}$$

$$= \frac{7}{5}\cdot\frac{1/8}{1/2}\cdot\frac{(2)^{-7}}{(2)^{-4}} = \frac{7}{8}\cdot\frac{1}{2^3} = \frac{7}{64}.$$

4.(i) $f(x) = \begin{cases} x+2, & 0\le x<2 \\ 6-x, & 0\ge 2\end{cases}$

$g(x) = \begin{cases} 1+\tan x, & 0\le x<\pi/4 \\ 3-\cot x, & \pi/4\le x<\pi\end{cases}$

For $0 \le x < \pi/4$, $g(x) = 1 + \tan x$

So $f(g(x)) = f(1+\tan x) = 1+\tan x+2$

Now $x \in [0, \pi/4) \Rightarrow 1 + \tan x \in [1, 2)$

and for $x\in\left[\frac{\pi}{4}, \pi\right)$, $g(x) = 3-\cot x$

Also for $x\in\left[\frac{\pi}{4}, \pi\right)$, $3-\cot x\in[2, \infty)$

So $f(g(x)) = f(3 - \cot x) = 6 - (3 - \cot x)$

Again denote by $h(x)$ the composition of f w.r.t. g

$h(x) = f(g(x)) = \begin{cases} 3+\tan x, & 0\le x<\pi/4 \\ 3+\cot x, & \pi/4\le x<\pi\end{cases}$

$f(x)$ is obviously continuous in $[0, \pi]$

$$h'+\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}+}(-\text{cosec}^2 x) = -2h'\left(\frac{\pi^-}{4}\right)$$

$$= \lim_{x\to\pi/4^-}(\sec^2 x) = 2$$

Thus $f(g(x))$ is differentiable everywhere in $[0, \pi)$ except at $x = \pi/4$.

(ii) $\displaystyle\lim_{x\to 2}\frac{f(x)g(4-x)-f(4-x)g(x)}{x-2}$

Let $h = x - 2$, so that when $x \to 2$, $h \to 0$

$$= \lim_{h\to 0}\frac{f(h+2)g(2-h)-f(2-h)\cdot g(2+h)}{h}$$

$$= \lim_{h\to 0}\frac{\begin{array}{c}f(2+h)g(2-h)-f(2)(2-h)+f(2)g(2-h)\\ -f(2-h)g(2+h)+f(2-h)g(2)-f(2-h)g(2)\end{array}}{h}$$

$$= \lim_{h\to 0}\frac{\begin{array}{c}g(2-h)[f(2+h)-f(2)]-f(2-h)(g(2+h)-g(2))\\ +g(2-h)f(2)-f(2-h)g(2)\end{array}}{h}$$

$$= g(2)\cdot f'(2)-f(2)\cdot g'(2)+\lim_{h\to 0}\frac{g(2-h)f(2)-f(2-h)\cdot g(2)}{h}$$

$$= 4\cdot(-3)-2\cdot 1+\lim_{h\to 0}\frac{f(2)g(2-h)-f(2)g(2)+f(2)g(2)-f(2-h)g(2)}{h}$$

$$= -12-2+f(2)\left\{\lim_{h\to 0}\frac{g(2-h)-g(2)}{h}\right\}-g(2)\left\{\lim_{h\to 0}\frac{f(2-h)-f(2)}{h}\right\}$$

$$= -14+f(2)\cdot(-g'(2))+g(2)f'(2)$$

$= -14 - 2\cdot(+1) + 4\cdot(-3) = -14 - 2 - 12 = -28.$

5.(i) Consider $g(x) = ax + b$, where a and b are constants to be determined.

$$f(x) = \begin{cases} ax+b, & x\le 0 \\ \left(\dfrac{1+x}{2+x}\right)^{1/x}, & x>0\end{cases}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) \Rightarrow \left(\frac{1}{2}\right)^{\infty} = b \Rightarrow b = 0$$

$$f(1) = \frac{2}{3}$$

Now $f(x) = \left(\dfrac{1+x}{2+x}\right)^{1/x}$ on taking logarithm yields

$$\ln f(x) = \frac{1}{x}\{\ln|x+1|-\ln|2+x|\}$$

$$\Rightarrow \frac{f'(x)}{f(x)} = -\frac{1}{x^2}\ln\frac{1+x}{2+x}+\frac{1}{x(x+1)(x+2)}$$

$$\Rightarrow \frac{f'(1)}{f(1)} = -1\ln\frac{2}{3}+\frac{1}{1\cdot 2\cdot 3} = \ln\frac{3}{2}+\frac{1}{6}$$

$$\Rightarrow f'(1) = \frac{2}{3}\ln\frac{3}{2}+\frac{2}{3}\times\frac{1}{6} = \frac{2}{3}\ln\frac{3}{2}+\frac{1}{9}$$

$$f(-1) = b-a$$

$$\Rightarrow b-a = \frac{2}{3}\ln\frac{3}{2}+\frac{1}{a} \Rightarrow 0-a = \frac{2}{3}\ln\frac{3}{2}+\frac{1}{9}$$

$\therefore\ a = -\frac{2}{3}\ln\frac{3}{2}-\frac{1}{9}$ Thus $f(x)=-\left(\frac{2}{3}\ln\frac{3}{2}+\frac{1}{9}\right)x.$

(ii) $f(x)$ is defined for $\forall\ x \in (0, \infty)$

Now, for $x = n+1$, $1+\sin x = 1$

$x\in((2n-1)\pi,\ 2n\pi),\ 0 \le 1+\sin x < 1$

For $x\in(2n\pi,\ (2n+1)\pi),\ 2\ge 1+\sin x > 1$

Thus $f(x)$ can be described as

$$f(x) = \begin{cases}\dfrac{1+\ln x}{3}, & x = n\pi\\ \dfrac{\ln x}{2}, & x\in((2n-1)\pi,\ 2n\pi)\\ 1, & x\in(2n\pi,\ (2n+1)\pi)\end{cases}$$

It is seen from the above that $f(x)$ is discontinuous at integral multiple of π.

(iii) Given relation is

$$f\left(\frac{x+2y}{3}\right)=\frac{f(x)+2f(y)}{3}\quad \forall\ x,\ y\in R$$

Replacing x by $3x$ and y by 0, we get

$$f(x) = \frac{f(3x)+2f(0)}{3}\Rightarrow f(3x)+2f(0) = 3f(x)$$

$$\Rightarrow\ f(3x) = 3f(x)-2f(0)$$

Now $f'(0)=\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f\left(\dfrac{3x+2\cdot(3h/2)}{3}\right)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(3x)+2f(3h/2)-3f(x)}{3h}$$

$$=\lim_{h\to 0}\frac{3f(x)-2f(0)+2f(3h/2)-3f(x)}{3h}$$

$$=\lim_{h\to 0}\frac{f(3h/2)-f(0)}{(3h/2)}=f'(0)=1$$

Thus $f'(x) = 1 \Rightarrow f(x) = x + c$, which being a linear function in x is always continuous.

(iv) Rewrite $f(x)$ as $f(x)=\begin{cases}0, & x\in(-\infty,-1)\\ 1+x, & x\in[-1,\ 0)\\ 1-x, & x\in(0,\ 7]\\ 0, & x\in(1,\infty)\end{cases}$

$$\Rightarrow f(x-1)=\begin{cases}0 & x-1\in(-\infty,-1)\\ 1+(x-1) & x-1\in[-1,\ 0)\\ 1-(x-1) & x-1\in(0,\ 1]\\ 0 & x-1\in(1,\ \infty)\end{cases}$$

or $\quad f(x-1)=\begin{cases}0, & x<0\\ x, & 0\le x<1\\ 2-x, & 1<x\le 2\\ 0, & x>2\end{cases}$

Also $f(x+1)=\begin{cases}0 & x+1\in(-\infty,-1)\\ 1+(x+1) & x+1\in[-1,0]\\ 1-(x+1) & x+1\in(0,1)\\ 0 & x+1\in(1,\ \infty)\end{cases}$

$$=\begin{cases}0, & x<-2\\ 2+x, & -2\le x\le -1\\ -x, & -1<x\le 0\\ 0, & x>0\end{cases}$$

Now, $g(x)=f(x-1)+f(x+1)=\begin{cases}0, & x<-2\\ 2+x, & -2\le x\le -1\\ -x, & -1<x\le 0\\ x, & 0<x\le 1\\ 2-x, & 1<x\le 2\\ 0, & x>2\end{cases}$

It is easy to check that $g(x)$ is continuous $\forall\ x\in R$ and non-differentiable at $x=-2,-1,0,1,2$ and differentiable else where.

6. (i) We have $f(x)f(y)+2=f(x)+f(y)+f(xy)$

Let $x = y = 1$, we get $f^2(1)+2 = 3f(1)\Rightarrow f(1) = 1, 2$

Setting $x = y = 0$, we obtain

$f^2(0)+2=3f(0)\Rightarrow f(0)=1, 2$

As f is $1-1$ and $f(0)\ne 2 \Rightarrow f(0) = 1$ and $f(1) = 2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)\cdot f\left(1+\frac{h}{x}\right)+2-f(x)-f\left(1+\frac{h}{x}\right)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(f(x)-1)f(1+(h/x))-2(f(x)-1)}{h}$$

$$=\lim_{h\to 0}\frac{(f(x)-1)(f(1+(h/x))-2)}{h}$$

$$=\lim_{h\to 0}\frac{(f(x)-1)(f(1+(h/x))-f(1))}{h}$$

$$=\lim_{h\to 0}\frac{(f(x)-1)(f(1+(h/x))-f(1))}{x-(h/x)}=\frac{f(x)-1}{x}\cdot f'(1)$$

$xf'(x) = 2(f(x)-1)$

Integrating both sides w.r.t x in between 0 and 1, we

obtain

$$\int_0^1 x f'(x)dx = 2\int_0^1 (f(x)-1)dx \Rightarrow xf(x)\Big|_0^1 - \int_0^1 f(x)dx$$

$$= 2\int_0^1 f(x)dx - 2 \Rightarrow \int_0^1 f(x)dx = \frac{4}{3}$$

(ii) $2 + f(x)f(y) = f(x) + f(y) + f(xy)$

$\Rightarrow\ 1 - f(x) - f(y) + f(x)f(y) = f(xy) - 1$

$\Rightarrow\ (1 - f(x))(2 - f(y)) = f(xy) - 1$

The above holds if and only if $f(x) = 1 + x^n$ for if $f(x) = a_n x^n + a_{n-1}x^{n-1} + ... + a_0$

Then consider $(1 - f(x))(1 - f(y)) = f(xy) - 1$

Compare constant term on either side

$1 - a_0 = a_0 - 1 \Rightarrow a_0 = 1$

Compare coefficient of $x^n y^n$

$\Rightarrow a_n^2 = a_n \Rightarrow a_n = 1$

($a_n \neq 0$, for then polynomial could not be of n)

Again compare coefficient of $x, x^2 ..... x^{n-1}$ on either side, we have $a_1 = a_2 = ... a_{n-1} = 0$

$\Rightarrow a_n = 1$ and $f(x) = 1 + x^n$

$f(2) = 3 \Rightarrow 1 + 2^n = 3 \Rightarrow 2^n = 4 = 2^2 \therefore n = 2$

Thus $f(x) = 1 + x^2$ and $f(f(2)) = f(5) = 1 + 5^2 == 26$

(iii) $g(x) = f(f(x)) - 1) + f(5 - f(x))$

$g'(x) = \{f'(f(x) - 1) - f'(5 - f(x))\} f'(x)$

Since $f(x)$ is differentiable everywhere $\Rightarrow g(x)$ exist for all $x \in R$ and then there is no point on which function is not differentiable.

$g(x)$ is thus continuous as well.

For critical points $g'(x) = 0$

$\Rightarrow f'(x)\ \{f'(f(x) - 1) - f'(5 - f(x))\} = 0$

Either $f'(x) = 0$ or $f(x) - 1 = 5 - f(x)$

$\Rightarrow x = 1$ or x is given by $f(x) = 3$

$f(x) = 3 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x = -1, 3$

Thus $x = -1, 1, 3$ are the critical points.

$g(x) = f(f(x) - 1) + f(5 - f(x))$

$= (f(x) - 1)^2 - 2(f(x) - 1) + (5 - f(x))^2 - 2(5 - f(x))$

$= f^2 - 2f + 1 - 2f + 2 + 25 - 10f + f^2 - 10 + 2f$

$= 2f^2 - 12f + 18 = 2(f^2 - 6f + 9) = 2\{f(x) - 3\}^2 \geq 0$

7.(i) $f(x + y) = f(x) + 2y^2 + kxy$

Setting $x = 1$, $y = 1$, we get $f(x) = f(1) + 2 + k$

$\Rightarrow 8 = 2 + 2 + k \Rightarrow k = 4$

Now $\dfrac{f(x+y) - f(x)}{y} = 2y + kx$

Let $y = h \Rightarrow \dfrac{f(x+h) - f(x)}{h} = 2h + kx$

Taking limits $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}(2h + kx)$

$\Rightarrow f'(x) = kx = 4x \Rightarrow f(x) = 4.\dfrac{x^2}{2} + C = 2x^2 + C$

$f(1) = 2 \Rightarrow C = 0$

Here $f(x) = 2x^2$ and

$f(x + y)\cdot f\left(\dfrac{1}{x+y}\right) = 2(x+y)^2\ \dfrac{2}{(x+y)^2} = 4 = k$

(ii) Let $\max f(x) = M$, where $0 < M \leq 1$ (Becauses $f(x)$ is not zero identically and $|f(x)| \leq 1\ \forall\ x \in R$)

Now $f(x + y) + f(x - y) = 2f(x)\cdot g(y)$

$\Rightarrow\ 2f(x)\,g(y) = f(x + y) + f(x - y)$

$\Rightarrow\ |2f(x)\,g(y)| = |f(x + y) + f(x - y)|$

$\Rightarrow\ 2|f(x)|\,|g(y)| \leq |f(x + y)| + |f(x - y)| \leq M + M$

$\Rightarrow\ 2|f(x)|\,|g(y)| \leq 2M \Rightarrow |g(y)| \leq \dfrac{M}{|f(x)|}$

$\Rightarrow\ |g(y)| \leq \dfrac{M}{M}$ *i.e.* $|g(y)| \leq 1$

(iii) $e^{-xy}f(xy) = e^{-x}f(x) + e^{-y}f(y)$ .... (1)

Let $x = y = 1$ in (1) to obtain

$e^{-1}f(1) = e^{-1}f(1) + e^{-1}f(1)$

$\Rightarrow f(1) = 2f(1) \therefore f(1) = 0$

By definition $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f\left(x\left(1+(h/x)\right)\right) - f(x-1)}{h}$$

$$= \lim_{h\to 0}\frac{e^{x+h}\{e^{-x}f(x) + e^{-1-\frac{h}{x}}f\left(1+(h/x)\right)\} - e^{x}(e^{-x}f(x) + e^{-1}f(1))}{h}$$

$$= \lim_{h\to 0}\frac{e^h f(x) + e^{x+h-1-\frac{h}{x}}f\left(1+(h/x)\right) - f(x) - e^{x-1}f(1)}{h}$$

$$= f(x)\cdot \lim_{h\to 0}\frac{e^h - 1}{h} + e^{x-1}\lim_{h\to 0}\frac{e^{h-\frac{h}{x}}f\left(1+(h/x)\right)}{x\cdot(h/x)}$$

$$= f(x)\cdot 1 + e^{x-1}\cdot\frac{f'(1)}{x} = f(x) + \frac{e^{x-1}\cdot e}{x} = f(x) + \frac{e^x}{x}$$

$$\Rightarrow f'(x) = f(x) + \frac{e^x}{x} \Rightarrow e^{-x}f'(x) - e^{-x}f(x) = \frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-x}f(x)\right) = \frac{1}{x}$$

on integrating, we have $e^{-x}f(x) = \ln x + k$

as $f(1) = 0 \Rightarrow k = 0$, $\therefore f(x) = \ln x\cdot e^x$

8(i) $f(x) = a|\sin x| + be^{|x|} + C|x|^3$

$$f'(0^+) = \lim_{h\to 0}\frac{f(0+h) - f(0)}{h}$$

$$= \lim_{h\to 0} \frac{a\sin h + be^h + ch^3 - b}{h}$$

$$= \lim_{h\to 0}\left[a\left(\frac{\sin h}{h}\right) + b\left(\frac{e^h - 1}{h}\right) + ch^2\right] = a + b$$

$$f'(0^-) = \lim_{h\to 0} \frac{f(0) - f(0-h)}{h}$$

$$= \lim_{h\to 0} \frac{(b - a\sin h + be^h + ch^3)}{h}$$

$$= \lim_{h\to 0} - a\left(\frac{\sin h}{h}\right) - b\left(\frac{e^h - 1}{h}\right) + ch^2 = -(a+b)$$

for f to be differentiable at $x = 0$,

we have $a + b = -(a + b)$

$\Rightarrow$ $a + b = 0$. Thus the values of a, b, c for f to be differentiable at $a, b \in R : a + b = 0$ and $C \in R$.

(ii) $f(x) = (x^2 - 1)\,|x^2 - 3x + 2| + \cos|x|$

$= (x^2 - 1)\,|(x - 1)(x - 2)| + \cos x$

Thus $f(x) = -(x^2 - 1)(x - 1)(x - 2) + \cos x$, $1 \le x \le 2$

$= (x^2 - 1)(x - 1)(x - 2) + \cos x$, $x \in (-\infty, 1) \cup (2, \infty)$

The only points where the function may fail to be differentiable is $x = 1, 2$.

For $x = 1$

$$f'(1^+) = \lim_{h\to 0} \frac{f(1+h) - f(1)}{h}$$

$$= \lim_{h\to 0} \frac{\left[\{1 - (1+h)^2\}(1+h-1)(1+h-2) + \cos(1+h)\right] - \cos 1}{h}$$

$$= \lim_{h\to 0} \frac{(h^2 + 2h)\cdot h(1-h)}{h} + \lim_{h\to 0} \frac{\cos(1+h) - \cos 1}{h}$$

$$= 0 + \lim_{h\to 0} \frac{-\sin(1+h)}{1} = -\sin 1$$

$$f'(1-) = \lim_{h\to 0} \frac{f(1) - f(1-h)}{h}$$

$$= \lim_{h\to 0} \frac{\cos 1 - \left[(1-h)^2 - 1(1-h-1)(1-h-2) + \cos(1-h)\right]}{h}$$

$$= \lim_{h\to 0} \frac{-(h^2 - 2h)h(1+h)}{h} + \lim_{h\to 0} \frac{\cos 1 - \cos(1-h)}{h}$$

$$= 0 + \lim_{h\to 0} \frac{-\sin(1-h)}{h} = 0 - \sin 1 = -\sin 1$$

Hence f is derivable at $x = 1$

For $x = 2$

$$f'(2^+) = \lim_{h\to 0} \frac{f(2+h) - f(2)}{h}$$

$$= \lim_{h\to 0} \frac{[\{(2+h)^2 - 1\}(2+h-1)(2+h-2) + \cos(2+h)] - \cos 2}{h}$$

$$= \lim_{h\to 0} \frac{(h^2 + 4h + 3)(1+h)h}{h} + \lim_{h\to 0} \frac{\cos(2+h) - \cos 2}{h}$$

$$= 3 + \lim_{h\to 0} \frac{-\sin(2+h)}{h} = 3 - \sin 2$$

$$f'(2^-) = \lim_{h\to 0} \frac{f(2) - f(2-h)}{h}$$

$$= \lim_{h\to 0} \frac{\cos 2 - [1 - \{(2-h)^2\}(2-h-1)(2-h-2) + \cos(2-h)]}{h}$$

$$= \lim_{h\to 0} \frac{-(h^2 - 4h + 3)h(1-h)}{h} + \lim_{h\to 0} \frac{\cos 2 - \cos(2-h)}{h}$$

$$= -3 + \lim_{h\to 0} \frac{-\sin(2-h)}{h} = -3 - \sin 2$$

As $f'(2^+1) \ne f'(2^-)$. thus $f(x)$ is not differentiable at $x = 2$. Then the only point where the function not differentiable is $x = 2$.

9. (i) : $$\lim_{x\to 0} \frac{\cot x \tan^{-1}(m\tan x) - m\cos^2(x/2)}{\sin^2(x/2)}$$

$$= \lim_{x\to 0} \frac{\cot x \tan^{-1}(m\tan x) - m\left(1 - \sin^2(x/2)\right)}{\sin^2(x/2)}$$

$$= \lim_{x\to 0} \frac{\cot x \tan^{-1}(m\tan x) - m}{\sin^2(x/2)} + m$$

$$= \lim_{x\to 0} \frac{\tan^{-1}(m\tan x) - m\tan x}{\tan x \sin^2(x/2)} + m$$

$$= \lim_{x\to 0} \frac{\tan^{-1}(m\tan x) - m\tan x}{x\cdot(x/2)^2} + m$$

$$= \lim_{x\to 0} \frac{\tan^{-1}(m\tan x) - m\tan x}{x^3/4} + m$$

$$= \lim_{t\to 0} \frac{t - \tan t}{t^3/4m^3} + m \quad (\text{set } m\tan x = \tan t)$$

$$= m + 4m^3 \lim_{t\to 0} \frac{t - \tan t}{t^3} = m + 4m^3 \lim_{t\to 0} \frac{1 - \sec^2 t}{3t^2}$$

(applying L'Hospital rule)

$$= m + 4m^3 \lim_{t\to 0} \frac{-2\sec^2 t \tan t}{6t} = m + 4m^3\left(-\frac{1}{3}\right) = m - \frac{4}{3}m^3$$

(ii) $$\lim_{x\to\infty} \left(\frac{a^{1/x} + b^{1/x} + c^{1/x}}{3}\right)^{3x}$$

$$= \lim_{t\to 0}\left(\frac{a^t + b^t + c^t}{3}\right)^{3/t} \quad \text{Set } t = \frac{1}{x}$$

$$= \lim_{t\to 0} e^{\frac{3}{t}\ln\left(\frac{a^t + b^t + c^t}{3}\right)}$$

Now,

$$\lim_{t\to 0} \frac{3}{t}\ln\left(\frac{a^t+b^t+c^t}{3}\right) = \lim_{t\to 0} \frac{3}{t}\ln\left(1+\frac{a^t+b^t+c^{t-3}}{3}\right)$$

$$= \lim_{t\to 0} \frac{a^t+b^t+c^t-3}{t}\cdot\frac{\ln\left(1+\dfrac{a^t+b^t+c^t-3}{3}\right)}{\dfrac{a^t+b^t+c^t-3}{3}}$$

$$= \lim_{t\to 0} \frac{(a^t-1)+(b^t-1)+(c^t-1)}{t}\cdot\lim_{t\to 0}\frac{\ln(1+u)}{u}$$

$$= \lim_{t\to 0}\left[\frac{(a^t-1)}{t}+\frac{(b^t-1)}{t}+\frac{(c^t-1)}{t}\right]\cdot 1$$

$$= \left(\lim_{t\to 0}\frac{a^t-1}{t}\right)+\left(\lim_{t\to 0}\frac{b^t-1}{t}\right)+\left(\lim_{t\to 0}\frac{c^t-1}{t}\right)$$

$$= \ln a + \ln b + \ln c = \ln(abc)$$

Hence the limit is $e^{\ln(abc)} = abc$.

10.(i) We have $f(x + 2y) = f(x)\, e^{2y} + f(2y)\,.\, e^x + x^2(1 - e^{2y}) + 4y^2(1 - e^x) + 4xy$

Set $x = y = 0$, we obtain $f(0) = 2f(0) \Rightarrow f(0) = 0$

By definition $f'(x) = \lim_{h\to 0}\dfrac{f(x+2h)-f(x)}{2h}$

$$= \lim_{h\to 0}\frac{f(x)e^{2h}+e^x f(2h)+x^2(1-e^{2h})+4h^2(1-e^x)+4xh-f(x)}{2h}$$

$$= \lim_{h\to 0}\frac{f(x)(e^{2h}-1)+e^x(f(2h)-f(0)-(e^{2h}-1)x^2+4xh-4h^2(1-e^x))}{2h}$$

$$= f(x)\lim_{h\to 0}\frac{e^{2h}-1}{2h}+e^x\lim_{h\to 0}\frac{f(2h)-f(0)}{2h}-x^2\lim_{h\to 0}\frac{e^{2h}-1}{2h}+2x$$

$= f(x) + e^x \cdot f'(0) - x^2 + 2x$

$\Rightarrow f'(x) = f(x) + e^x - x^2 + 2x$

$\Rightarrow e^{-x}f'(x) - e^{-x}f(x) = 1 + 2xe^{-x} - x^2e^{-x}$

$$\Rightarrow \frac{d}{dx}\left(e^{-x}f(x)\right)=\frac{d}{dx}\left(x^2e^{-x}+x\right)$$

$\Rightarrow e^{-x}f(x)=x+x^2e^{-x}+k$

As $f(0) = 0 \Rightarrow k = 0 \quad \therefore f(x) = xe^x + x^2$

(ii) $f(2x^2-1) = 2xf(x)\ \forall\ x \in [-1, 1]$

Replacing x by $-x$, we obtain

$f(2x^2 - 1) = -2x\,f(-x) \Rightarrow 2x\,f(-x) + f(2x^2 - 1) = 0$

$\Rightarrow 2x\,f(-x) + 2x\,f(x) = 0$

$\Rightarrow 2x\,(f(x) + f(-x)) = 0\ \ \forall\ x \in [-1, 1]$

$\therefore f(x) + f(-x) = 0$ Thus $f(x)$ is odd.

As $f(x)$ is odd and continuous in $[-1, 1] \Rightarrow f(0) = 0$

$$\Rightarrow f(x)=\frac{f(2x^2-1)}{2x}\Rightarrow 1\lim_{x\to 0}f(x)=\lim_{x\to 0}\frac{f(2x^2-1)}{2x}=0$$

Replacing $x = \cos\theta$ in the given functional equation, we obtain

$$f(\cos 2\theta) = 2\cos\theta\, f(\cos\theta) = 2\cos\theta\left(2\cos\frac{\theta}{2}f\left(\cos\frac{\theta}{2}\right)\right)$$

$$= 2^2\cos\theta\cos\frac{\theta}{2}f\left(\cos\frac{\theta}{2}\right)$$

$$= 2^2\cos\theta\ \cos\frac{\theta}{2}\left(2\cos\frac{\theta}{4}f\left(\cos\frac{\theta}{4}\right)\right)$$

$= \ldots\ldots\ldots\ldots\ldots\ldots$

$$= 2^{n+1}\cdot\cos\theta\cdot\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^{n-1}}\cdot\cos\frac{\theta}{2^n}f\left(\cos\frac{\theta}{2^n}\right)$$

$$= \sin 2\theta\frac{f\left(\cos\dfrac{\theta}{2^n}\right)}{\sin\dfrac{\theta}{2^n}}$$

Taking limit on both sides as $n \to \infty$, we get

$$\lim_{n\to\infty}f(\cos 2\theta)=\sin 2\theta\lim_{n\to\infty}\frac{f\left(\cos\dfrac{\theta}{2^n}\right)}{\sin\dfrac{\theta}{2^n}}$$

$$= \sin 2\theta\lim_{n\to\infty}\frac{f\left(2\cos^2\dfrac{\theta}{2^{n+1}}-1\right)}{2\sin\dfrac{\theta}{2^{n+1}}\cdot\cos\dfrac{\theta}{2^{n+1}}}$$

$$= \sin 2\theta\cdot\left[\lim_{n\to\infty}\frac{f\left(2\sin^2\dfrac{\theta}{2^{n+1}}-1\right)}{2\sin\dfrac{\theta}{2^{n+1}}\cdot\cos\dfrac{\theta}{2^{n+1}}}\right]=0$$

Thus $f(\cos 2\theta)=0\ \forall\ \theta\in R$

This gives $f(x) = 0\ \ \forall x\in[-1, 1]$

SOLUTIONS TO MCQS

1. (d) : $$\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}=\lim_{x\to 0}\frac{\dfrac{1}{\sqrt{1-x^2}}-\dfrac{1}{1+x^2}}{3x^2}$$

$$= \lim_{x\to 0}\frac{(1+x^2)-\sqrt{1-x^2}}{3x^2}\,\frac{1}{(1+x^2)\sqrt{1-x^2}}$$

$$= \lim_{x\to 0} \frac{(1+x^2)^2-(1-x^2)}{3x^2\left[(1+x^2)+\sqrt{1-x^2}\right]}\cdot 1$$

$$= \lim_{x\to 0} \frac{x^4+3x^2}{3x^2}\cdot\frac{1}{(1+x^2)+\sqrt{1-x^2}} = \lim_{x\to 0}\frac{x^2+3}{3}\cdot\frac{1}{2}=\frac{1}{2}$$

2. (c) : The functional relation is

$$3f(x) + 4f\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \qquad ....(1)$$

Changing x to $1/x$ in (1) we have

$$3f\left(\frac{1}{x}\right) + 4f(x) = x-5 \qquad ....(2)$$

Multiplying (1) by 3 and (2) by 4 and then by subtracting

$$9f(x)-16f(x)=3\left(\frac{1}{x}-5\right)-4(x-5)$$

$$\Rightarrow -7f(x) = -15+20+\frac{3}{x}-4x$$

$$\Rightarrow f(x)=\frac{5+(3/x)-4x}{-7} \quad \Rightarrow f(x)=\frac{4x-(3/x)-5}{7}$$

$$f(2) = \frac{8-(3/2)-5}{7} = \frac{16-3-10}{2\times 7} = \frac{3}{14}$$

3. (a) : $F(x) = f(x)\, g(x)\, h(x)$

Taking logarithm, we have

$\ln F(x) = \ln f(x) + \ln g(x) + \ln h(x)$

Differentiative w.r.t x we obtain

$$\frac{F'(x)}{F(x)} = \frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}+\frac{h'(x)}{h(x)}$$

$$\Rightarrow \frac{F'(x_0)}{F(x_0)} = \frac{f'(x_0)}{f(x_0)}+\frac{g'(x_0)}{g(x_0)}+\frac{h'(x_0)}{h(x_0)}$$

$$\Rightarrow 13 = 9+4+\frac{h'(x_0)}{h(x_0)} \quad \Rightarrow 0 = \frac{h'(x_0)}{h(x_0)} \quad \therefore h'(x_0)=0$$

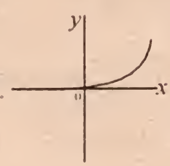
4. (c) : $x = 2t - |t|,\ y = t^3 + t^2|t|$

Function takes the form

$x = 3t,\ y = 0$, when $t < 0$

$x = t,\ y = 2t^3,\ t > 0$

Eliminating the parameter, we obtain



$y = 0 \quad x < 0$

$= 2x^3 \quad x \geq 0$

Differentiation w.r.t x gives

$$\frac{dy}{dx} = 0,\ x < 0$$

$$= 6x^2,\ x > 0$$

Hence the function is differentiable everywhere and $f'(0) = 0$

5. (d) : $f(x + yn) = f(x) + (f(y))^n$

Set $x = y = 0 \Rightarrow f(0) = f(0) + f(0) = f(0) = 0$

$$\text{Now } \lim_{y\to 0}\frac{f(x+yn)-f(x)}{y^n} = \lim_{y\to 0}\frac{(f(y))^n}{y^n}$$

$$\Rightarrow f'(x) = \left(\lim_{y\to 0}\frac{f(y)}{y}\right)^n = \left(\lim_{y\to 0}\frac{f(y+(0)-f(0)}{y}\right)^n$$

$$=(f'(0))^n = \text{constant}$$

$\therefore f(x) = ax + b,\ f'(0) = 1,\ f(0) = 0 \quad \therefore a = 1,\ b = 0$

Thus $f(x) = x \quad \therefore f(15) = 15,\ f'(20) = 1$

6. (a) : For domain $0 < x^2 + 3x + 2 \leq 1$

$x^2 + 3x + 2 > 0 \Rightarrow (x + 1)(x + 2) > 0$

$\Rightarrow x \in (-\infty, -2) \cup (-1, \infty)$

$x^2 + 3x + 2 \leq 1 \Rightarrow x^2 + 3x + 1 \leq 0$

$$\Rightarrow x \in \left[\frac{-3-\sqrt{5}}{2},\ \frac{-3+\sqrt{5}}{2}\right]$$

$$\text{Thus domain } D = \left[\frac{-3-\sqrt{5}}{2},\ -2\right)\cup\left(-1, \frac{-3+\sqrt{5}}{2}\right]$$

For range, $0 < \sin^{-1}\sqrt{x^2+3x+2} \leq \frac{\pi}{2}$

$$\Rightarrow -\infty < \ln\left(\sin^{-1}\sqrt{x^2+3x+2}\right) \leq \ln\frac{\pi}{2}$$

Thus $\ln\left(\sin^{-1}\sqrt{x^2+3x+2}\right)$ can take all non-positive integral values.

7. (b) : $x + y = e^{x\,y}$

Differentiating w.r.t x, we get

$$1+\frac{dy}{dx} = e^{x-y\left(1-\frac{dy}{dx}\right)} \Rightarrow y'+1 = e^{x-y(1-y')}$$

$$\Rightarrow y'+1 = (x+y)(1-y') \Rightarrow y'(x+y+1) = x+y-1$$

Differentiating again

$$y''(x+y+1)+y'(1+y') = 1+y'$$

$$\Rightarrow y''(x+y+1)+y'^2-1 = 0$$

$$\Rightarrow y''(x+y+1)+\left(\frac{x+y-1}{x+y+1}\right)^2-1=0$$

$$y''(x+y+1)+\frac{(x+y-1)^2-(x+y+1)^2}{(x+y+1)}=0$$

$$\Rightarrow y''(x+y+1)+\frac{(-4(x+y))}{(x+y+1)^2}=0 \Rightarrow y'' = \frac{4(x+y)}{(x+y+1)^3}$$

$$\frac{d^2y}{dx^2}\Big|x=x_0 = \frac{4(x_0+y_0)}{(x_0+y_0+1)^3} = \frac{4e^{x_0-y_0}}{1+e^{x_0-y_0}} = \frac{4e^{x_0}}{e^{x_0}+e^{y_0}}$$

8. (b) : $\lim_{x\to 0}\frac{27^x-9^x-3^x+1}{\sqrt{3}-\sqrt{4+\cos x}}=\lim_{x\to 0}\frac{(9^x-1)(3^x-1)}{\sqrt{3}-\sqrt{4+\cos x}}$

$= \lim_{x\to 0}\frac{(9^x-1)(3^x-1)(\sqrt{5}+\sqrt{4+\cos x})}{5-(4+\cos x)}$

$= \lim_{x\to 0}\frac{(9^x-1)(3^x-1)(\sqrt{5}+\sqrt{4+\cos x})}{1-\cos x}$

$=\left(\lim_{x\to 0}\frac{9^x-1}{x}\right)\cdot\left(\lim_{x\to 0}\frac{3^x-1}{x}\right)\lim_{x\to 0}\left(\frac{x^2}{2\sin^2\frac{x}{2}}\right)\cdot\left(\sqrt{5}+\sqrt{5}\right)$

$= \ln 9 \cdot \ln 3 \cdot 2 \cdot 2\sqrt{5} = 2\ln 3(\ln 3)\cdot 4\sqrt{5}$

$=8\sqrt{5}(\ln 3)^2$

9. (a) : $\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{\sin 2x + a\sin x}{x^3}$

$= \lim_{x\to 0}\frac{2\cos 2x + a\cos x}{3x^2}$

For the limit to exist i.e. to be finite, the numerator must tend to zero when denominator tends to zero.

$2 + a = 0 \Rightarrow a = -2$

Putting the value of a, we obtain

$\lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{2(\cos 2x-\cos x)}{3x^2}$

$= \lim_{x\to 0}\frac{2(-2\sin 2x+\sin x)}{6x}$

$= \lim_{x\to 0} -\frac{4}{3}\cdot\left(\frac{\sin 2x}{2x}\right)+\lim_{x\to 0}\frac{1}{3}\left(\frac{\sin x}{x}\right) = -\frac{4}{3}+\frac{1}{3} = -1.$

10. (c) : $\lim_{n\to\infty}\cos\frac{x}{2}\cos\frac{x}{4}\cos\frac{x}{8}\,.....\cos\frac{x}{2^n}$

$= \lim_{n\to\infty}\frac{\cos\frac{x}{2}\cos\frac{x}{4}\,.....\cos\frac{x}{2^n}\cdot\sin\frac{x}{2^n}}{\sin\frac{x}{2^n}}$

$= \lim_{n\to\infty}\frac{\cos\frac{x}{2}\cos\frac{x}{4}\,.....\frac{1}{2}\cdot\sin\frac{x}{2^{n-1}}}{\sin\frac{x}{2^n}}$

$= \lim_{n\to\infty}\frac{\frac{1}{4}\cos\frac{x}{2}\cos\frac{x}{4}.....\sin\frac{x}{2^{n-2}}}{\sin\frac{x}{2^n}}$

$= \lim_{n\to\infty}\frac{\frac{\sin x}{2^n}}{\sin\left(\frac{x}{2^n}\right)} = \lim_{n\to\infty}\frac{\frac{x}{2^n}}{\sin\left(\frac{x}{2^n}\right)}\cdot\frac{\sin x}{x}$

$=\frac{\sin x}{x}\lim_{t\to 0}\frac{t}{\sin t} = \frac{\sin x}{x}\cdot 1 = \frac{\sin x}{x}$

(we have repeatedly made use of the identity $\sin\theta = 2\sin\frac{\theta}{2}\cos\frac{\theta}{2}$)

11. (b) : f is continuous at $x = 1$

$\Rightarrow a + 2 = a + b + 1 \Rightarrow b = 1$

f is continuous at $x = 2 \Rightarrow 2a + 3 = 4a + 2b + 1$

$\Rightarrow 2a + 3 = 4a + 3 \Rightarrow 2a = 2 \therefore a = 1$

12. (c) : $\lim_{n\to\infty}\left(4^n+5^n\right)^{\frac{1}{n}} = \lim_{n\to\infty}\left\{5^n\left(\frac{4^n}{5^n}+1\right)\right\}^{\frac{1}{n}}$

$= \lim_{n\to\infty}\left\{5^n\left\{1+\left(\frac{4}{5}\right)^n\right\}\right\}^{\frac{1}{n}} = 5\lim_{n\to\infty}\left(1+\left(\frac{4}{5}\right)^n\right)^{\frac{1}{n}}$

$=5(1+0)=5$

13. (b) : Since f is continuous at $x = \pi/4$

$\therefore\ f(\pi/4) = \lim_{x\to\pi/4}\cos 2x\cot\left(\frac{\pi}{4}-x\right)$

$= \lim_{x\to\frac{\pi}{4}}\left(\frac{-2\sin 2x}{-\sec^2\left(\frac{\pi}{4}-x\right)}\right) = \frac{-2\cdot 1}{-1} = 2$

14. (c) : We have $f(x + y) = f(x) + f(y)$ ..... (A)

and $f(x) = x^2 g(x)$

As $f(x)$ is differentiable, by definition,

$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f(h)}{h}$ (using A)

$= \lim_{h\to 0}\frac{h^2 g(h)}{h} = \lim_{h\to 0} hg(h) = 0$

$\Rightarrow f'(x) = 0 \therefore f(x) = \text{constant} = c$ (says)

Thus $|f(15) - f(-15)| = |c - c| = 0$

15. (a) : $\lim_{x\to 1}\frac{x^{1/3}-x^{1/4}-2}{x^3-1} = \lim_{x\to 1}\frac{(x^{1/3}-1)-(x^{1/4}-1)}{x^3-1}$

$\lim_{x\to 1}\frac{(x^{1/3}-1)-(x^{1/4}-1)}{(x-1)(x^2+x+1)}$

$= \left(\lim_{x\to 1}\frac{x^{1/3}-1}{x-1} - \lim_{x\to 1}\frac{x^{1/4}-1}{x-1}\right)\frac{1}{(1^2+1+1)}$

$=\left[\left(\frac{1}{3}\right)(1)^{-2/3}-\frac{1}{4}(1)^{-3/4}\right]\frac{1}{3}$

$= \left(\frac{1}{3}-\frac{1}{4}\right)\frac{1}{3} = \frac{1}{12}\times\frac{1}{3} = \frac{1}{36}$

15. If $f(x) = \begin{vmatrix} \cos(x+\alpha) & \cos(x+\beta) & \cos(x+\gamma) \\ \sin(x+\alpha) & \sin(x+\beta) & \sin(x+\gamma) \\ \sin(\beta-\gamma) & \sin(\gamma-\alpha) & \sin(\alpha-\beta) \end{vmatrix}$

and $f(2) = 5$, then $\sum_{r=1}^{20} f(x)$ equals

(a) 5 (b) 10

(c) 100 (d) none of these

16. Given $f(x) = \log_{10}x$ and $g(x) = e^{\pi i x}$

If $\phi(x) = \begin{vmatrix} f(x)g(x) & [f(x)]^{g(x)} & 1 \\ f(x^2)g(x^2) & [f(x^2)]^{g(x^2)} & 0 \\ f(x^3)g(x^3) & [f(x^3)]^{g(x^3)} & 1 \end{vmatrix}$

(a) 1 (b) 2

(c) 0 (d) none of these

17. If $x > 0, y > 0, z > 0$ then the value of determinant

$\Delta = \begin{vmatrix} x\log 2 & 1 & 5+(\log 4)^x \\ y\log 3 & 2 & 10+(\log 9)^y \\ z\log 5 & 3 & 15+(\log 25)^z \end{vmatrix}$ is

(a) $2x + 3y + 5z$ (b) $x\log2^x + y\log3^y + z\log5^z$

(c) 1 (d) none of these

18. The roots of the equation

$$\begin{vmatrix} 3x^2 & x^2+x\cos\beta+\cos^2\beta & x^2+x\cos\beta+\cos^2\beta \\ x^2+x\cos x+\cos^2 x & 3\cos^2\beta & 1+(\sin 2\beta)/2 \\ x^2+x\sin\beta+\sin^2\beta & 1+(\sin 2\beta)/2 & 3\sin^2\beta \end{vmatrix} = 0$$

(a) $\sin\beta, \cos^2\beta$ (b) $\sin^2\beta, \cos^2\beta$

(c) $\sin^2\beta, \cos\beta$ (d) $\sin\beta, \cos\beta$

19. The value of determinant

$$\begin{vmatrix} 2 & a+b+c+d & ab+cd \\ a+b+c+d & 2(a+b)(c+d) & ab(c+d)+cd(a+b) \\ ab+cd & ab(c+d)+cd(a+b) & 2abcd \end{vmatrix} \text{ is}$$

(a) $2abc$ (b) ab/cd (c) $a+b+c+d$ (d) 0

20. If $\sin 2x = 1$, then $\begin{vmatrix} 0 & \cos x & -\sin x \\ \sin x & 0 & \cos x \\ \cos x & \sin x & 0 \end{vmatrix}^2$ equal

(a) 0 (b) 3/2

(c) 2/3 (d) none of these

21. The value of

$$\Delta = \begin{vmatrix} 1 & 1 & 1 \\ (e^{i\alpha}-e^{-i\alpha}) & (e^{i\alpha}-e^{-i\alpha})^2 & -(\sin\gamma-\cos\gamma)^2 \\ (e^{i\alpha}-e^{-i\alpha})^2 & (e^{i\alpha}-e^{-i\alpha})^2 & -(\sin\gamma+\cos\gamma)^2 \end{vmatrix} \text{ is equal}$$

(a) 0 (b) $e^{i\alpha}e^{\beta}$

(c) 1 (d) $(e^{i\alpha}/(e^{\beta}\sin\gamma))^2$

22. If maximum and minimum value of the determinant

$\begin{vmatrix} 1+\sin^2 x & \cos^2 x & \sin 2x \\ \sin^2 x & 1+\cos^2 x & \sin 2x \\ \sin^2 x & \cos^2 x & 1+\sin 2x \end{vmatrix}$ are α and β then

(a) $\alpha + \beta^{99} = 4$ (b) $\alpha^3 - \beta^3 = 26$

(c) $(\alpha^{2n} - \beta^{2n})$ is integer (d) None of these

23. If x, y, z are the integers in A.P., lying between 1 and 9 and $x51, y41, z31$ are three digit numbers, then the value of

$\begin{vmatrix} 5 & 4 & 3 \\ x51 & y41 & z31 \\ x & y & z \end{vmatrix}$ is

(a) $x + y + z$ (b) $x - y + z$

(c) 0 (d) none of these

24. If $f(x) = \begin{vmatrix} \sin x & \sec x & x^2-1 \\ \operatorname{cosec} x & x\sin x & \cos x \\ \tan x & x\tan x & x^2+1 \end{vmatrix}$, then $\int_{-\pi/3}^{+\pi/3} f(x)$ dx equals

(a) 1 (b) $\pi/3+1$ (c) 0 (d) $\pi/3-1$

ANSWERS

1. (a)	2. (d)	3. (d)	4. (c)	5. (b)
6. (a)	7. (b)	8. (c)	9. (c)	10. (a)
11. (c)	12. (b)	13. (b)	14. (a)	15. (c)
16. (c)	17. (c)	18. (d)	19. (d)	20. (a)
21. (a)	22. (a)	23. (c)	24. (c)	

10 SUBJECTIVE PROBLEMS — CO-ORDINATE GEOMETRY

1. Two straight lines PBC, PAD are drawn through $P(-1, -2)$ to intersect $5x + 12y - 13 = 0$ in A and B and $5x + 12y - 39 = 0$ at C and D respectively and such that $BC = 2$ units length and $AD = 26/5$ units length. Find the equation to PBC and PAD.

2. The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ is rotated through a right angle in its own plane about its centre which is fixed. Prove that the locus of the point of intersection of a tangent to the ellipse in its original position with the tangent at the same point of the curve in its new position is $(x^2 + y^2)(x^2 + y^2 - a^2 - b^2) = 2(a^2 - b^2)xy$

3. From any point P on the curve $b^4x + 2a^2y^2 = 0$ chords are drawn to the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$. Tangents are drawn at the extremities of the chord to hyperbola which intersect at Q. Prove that Q moves along a straight line for a given P. Also show that the straight line always touches a fixed parabola.

4. A parabola of latus rectum l, touch a fixed equal parabola, the axes of two parabolas being parallel. Prove that the locus of the vertex of the moving parabola is a parabola of latus rectum $2l$.

5. Find the equation of the circle touching the pair of lines $7x^2 - 18xy + 7y^2 = 0$ and the circle $x^2 + y^2 - 8x - 8y = 0$, and contained in the given circle.

6. Given the base of a triangle and the ratio of the tangents of half the base angles. Show that the vertex moves on a hyperbola whose foci are the extremities of the base.

7. A line intersects the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ at P and Q and the parabola $y^2 = 4d(x + a)$ at R and S. The line segment PQ subtends a right angle at the center of the ellipse. Find the locus of the point of intersection of the tangents to the parabola at R and S.

8. The distance between two parallel lines is 1. A point a lies between the lines at a distance 'a' from one of them. Find the length of the side of an equilateral triangle ABC, the vertex B of which lies on one of the parallel sides and the vertex C on the other.

9. Show that a circle with centre (π, e) cannot have more than one rational point on its circumference. Further if radius of the circle is a rational number then show that none of the point of the circumference is rational (a point is called rational if and only if both of its coordinates are rational numbers.)

10. Tangent is drawn at any point (x_1, y_1) on the parabola $y^2 = 4ax$. Now tangents are drawn from any point on this tangent to the circle $x^2 + y^2 = a^2$ such that all the chords of contact pass through a fixed point (x_2, y_2). Prove that $4\frac{x_1}{x_2}+\left(\frac{y_1}{y_2}\right)^2=0.$

SOLUTIONS

1. Consider line PBC

$C(-1 + r_2\cos\theta, -2 + r_2\sin\theta)$ on $5x + 12y - 39 = 0$

$B(-1 + r_1\cos\theta, -2 + r_1\sin\theta)$ on $5x + 12y - 13 = 0$

$\therefore\ 5(-1 + r_2\cos\theta) + 12(-2 + r_2\sin\theta) - 39 = 0$ ...(1)

$5(-1 + r_1\cos\theta) + 12(-2 + r_1\sin\theta) - 13 = 0$ ...(2)

(1) – (2)

$\Rightarrow\ 5(r_2 - r_1)\cos\theta + 12(r_2 - r_1)\sin\theta = 26$ ...(3)

but $r_2 - r_1 = 2$

$\therefore\ 10\cos\theta + 24\sin\theta = 26$

reducing these to $\tan\theta = t$

$\therefore\ (10 + 24t)^2 = 676(1 + t^2) \Rightarrow t = \frac{12}{5}$

$\therefore$ Equation of the line PBC is $y + 2 = \frac{12}{5}(x + 1)$

$\Rightarrow 12x - 5y + 2 = 0$

<u>In case of PAD</u> $\quad r_2 - r_1 = \frac{26}{5}$

And proceed as above we shall get

$\Rightarrow \theta = 0$ or $\tan^{-1}\left(\frac{-120}{119}\right)$

$\therefore$ Equation of PAD are $y + 2 = 0$ (or)

$120x + 199y + 358 = 0$

2. Let $(a\cos\theta, b\sin\theta)$ be any point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

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$\therefore$ $(-b\sin\theta,\ a\cos\theta)$ be the corresponding point after rotation

$\therefore$ Equation of the tangent at the two positions

$$\frac{x}{a}\cos\theta+\frac{y}{b}\sin\theta=1 \qquad ....(1)$$

$$\frac{-x}{b}\sin\theta+\frac{y}{a}\cos\theta=1 \qquad ....(2)$$

$\therefore$ point of intersection of (1) and (2)

$$(x,y)=\left(\frac{ab(b\cos\theta-a\sin\theta)}{a^2\sin^2\theta+b^2\cos^2\theta},\frac{ab(b\cos\theta+a\sin\theta)}{a^2\sin^2\theta+b^2\cos^2\theta}\right)$$

Now eliminate θ, we shall get

$$(x^2+y^2)(x^2+y^2-a^2-b^2)=2(a^2-b^2)xy$$

3. For a given P locus of Q is the polar of P w.r.t. $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$.

$P(\alpha,\beta)\Rightarrow$ equation of polar is $\frac{\alpha x}{a^2}-\frac{\beta y}{b^2}=1$

also $b^4\alpha+2a^2\beta^2=0 \quad b^2\alpha x-a^2\beta y=a^2b^2$

$$\Rightarrow b^2\alpha=\frac{-2a^2}{b^2}\beta^2,\ \frac{-2a^2\beta^2}{b^2}x-a^2\beta y=a^2b^2$$

which is st. line for a given (α,β).

$$y=-\frac{2\beta}{b^2}x-\frac{b^2}{\beta}\ \Rightarrow\ y=\frac{-2\beta}{b^2}x+\frac{2}{\frac{-2\beta}{b^2}}$$

$$\therefore m=\frac{-2\beta}{b^2},\ a=2$$

$y=mx+\frac{a}{m}\Rightarrow$ touches a parabola $y^2=4ax$

$\therefore y^2=8x$

4. Let given parabola be $y^2=lx$ ....(1)
and moving parabola be $(y-\beta)^2=-l(x-\alpha)$ .....(2)
Since two parabolas touches each other hence

$$(y-\beta)^2=-l\left(\frac{y^2}{l}-\alpha\right)$$

should have equal roots i.e. $D=0$

$\Rightarrow\ 4\beta^2-8(\beta^2-l\alpha)=0$

$\beta^2=2l\alpha$

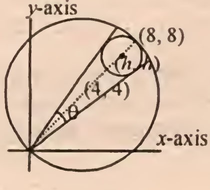
therefore required locus is $y^2=2lx$ which has latus rectum double that of given parabola.

5. $\tan\theta=\left|\frac{2\sqrt{h^2-ab}}{a+b}\right|=\frac{4\sqrt{2}}{7}$

$\tan\frac{\theta}{2}=\frac{1}{2\sqrt{2}}$

on solving

$\sin\frac{\theta}{2}=\frac{1}{3}=\frac{\sqrt{2}(8-h)}{\sqrt{2}h}$



Hence equation of circle is $(x-6)^2+(y-6)^2=8$.

6. BC = base of the triangle = a (constant)
A (vertex)

$$\frac{\tan B/2}{\tan C/2}=\frac{\sqrt{\frac{(s-c)(s-a)}{s(s-b)}}}{\sqrt{\frac{(s-a)(s-b)}{s(s-c)}}}=\frac{s-c}{s-b}=\frac{2s-2c}{2s-2b}$$

$$=\frac{a+b-c}{a-b+c}=k\quad\text{(let)}$$

By compodendo and dividendo

$$\Rightarrow\frac{k-1}{k+1}=\frac{b-c}{a}\Rightarrow b-c=a\left(\frac{k-1}{k+1}\right)=\text{constant}$$

$\Rightarrow CA-BA=$ constant

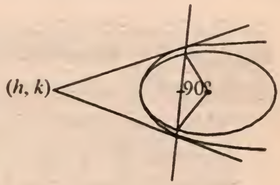
by the focal property, the locus of A is hyperbola, whose foci are B and C

7. Let the point of intersection of tangents at R and S to the parabola $y^2=4d(x+a)$ be (h,k) $\therefore$ RS is chord of contact of (h,k)

$\therefore$ Equation of RS is $Ky=2d(x+h)+4ad$

$Ky-2dx=2dh+4ad$

$$\frac{Ky-2dx}{2dh+4ad}=1$$



Homogenesing $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}-1.\frac{(Ky-2dx)^2}{(2dh+4ad)}=0$$

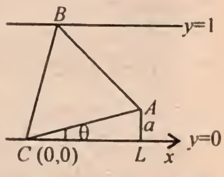
$\theta=90^\circ\Rightarrow$ coef.(x^2) + coef.$(y^2)=0$

$$\frac{1}{a^2}-\frac{4d^2}{(2dh+4ad)^2}+\frac{1}{b^2}-\frac{K^2}{(2dh+4ad)^2}=0$$

$$(4d^2+K^2)=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)(2dh+4ad)^2$$

locus of (h,k) is $4d^2+y^2=\left(\frac{1}{a^2}+\frac{1}{b^2}\right)(4ad+2dx)^2$

8. Without loss of generality, we can choose $y=0$ (i.e., the x-axis and $y=1$ as the two parallel lines) and the vertex C as the origin (0, 0) Choose the points A and B as shown in the figure and drop AL perpendicular to the x-axis. Let each side of the equilateral triangle ABC be r, and put $\angle ACL=\theta$, so that $\angle BCL=\theta+60^\circ$. Then $AL=a=AC\sin\theta=r\sin\theta$, i.e. $\sin\theta=a/r$. Furthermore, since B lies on $y=1$, $1=r\sin(\theta+60^\circ)$. That is $r(\sin\theta\cos60^\circ+\cos\theta\sin60^\circ)=1$

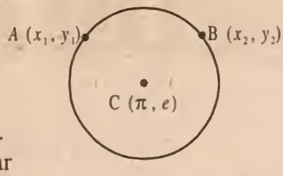


$$\Rightarrow r\left[\left(\frac{a}{r}\right)\left(\frac{1}{2}\right)+\sqrt{1-\frac{a^2}{r^2}}\left(\frac{\sqrt{3}}{2}\right)\right]=1$$

$$\Rightarrow a+\sqrt{3}\sqrt{r^2-a^2}=2 \Rightarrow (a-2)^2=3(r^2-a^2)$$

$$\Rightarrow 3r^2=4a^2-4a+4 \Rightarrow r=\frac{2}{3}\sqrt{3(a^2-a+1)}$$

9. Suppose there are two rational points $A(x_1, y_1)$ and $B(x_2, y_2)$ on the cirumference of a circle having Centre $C(\pi, e)$.



Equation of perpendicular bisector of AB is of the form $ax + by + c = 0$, where a, b and c are rational numbers which can never be satisfied by the point (π, e). Hence there is not more than one rational point on the circumference of a circle having centre (π, e). Further if radius 'r' of a circle having centre (π, e) is a rational number and its circumference has a rational point $A(x_1, y_1)$, then $r^2 = (\pi - x_1)^2 + (e - y_1)^2$, a contradiction as R.H.S. is irrational Hence such a circle cannot have even a single rational point on its circumference.

10. Point (x_1, y_1) can be written as $(at^2, 2at)$

Tangent at this point is $ty = x + at^2$

Any point on this tangent will be $\left(\alpha, \dfrac{\alpha+at^2}{t}\right)$

Equation of chord of contact of the point $\left(\alpha, \dfrac{\alpha+at^2}{t}\right)$ w.r.t circle $x^2 + y^2 = a^2$ is

$$\alpha x+\left(\frac{\alpha+at^2}{t}\right)y=a^2 \text{ or, } (aty-a^2)+\alpha\left(x+\frac{y}{t}\right)=0$$

which is a family of straight lines passing through point of intersection of $ty - a = 0$ and $x+\dfrac{y}{t}=0$

$\therefore$ the fixed point is $\left(-\dfrac{a}{t^2}, \dfrac{a}{t}\right) \therefore x_2=-\dfrac{a}{t^2}, y_2=\dfrac{a}{t}$

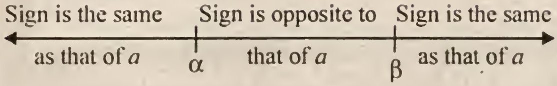
Also $x_1 = at^2$, $y_1 = 2at$

$$\Rightarrow \frac{x_1}{x_2}=-t^4, \frac{y_1}{y_2}=2t^2$$

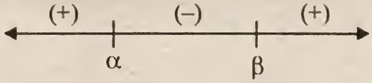
$$\Rightarrow \frac{x_1}{x_2}+\frac{1}{4}\left(\frac{y_1}{y_2}\right)^2=0 \qquad \Rightarrow 4\frac{x_1}{x_2}+\left(\frac{y_1}{y_2}\right)^2=0$$

■

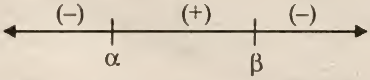
❐ If (α, β)are real and unequal ($\alpha < \beta$)i.e., D (discriminant) > 0, then



$\therefore$ if $a > 0$,

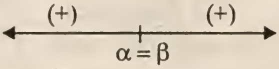


if $a < 0$,

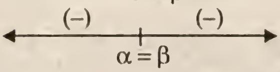


❐ If (α, β)are real and equal i.e., D (discriminant) = 0, then

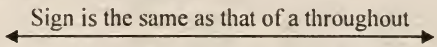
if $a > 0$,



if $a < 0$



❐ If (α, β) are imaginary (nonreal complex) i.e., D (discriminant) < 0, then



$\therefore$ if a > 0, the expression is always positive
if a < 0, the expression is always negative

Positive definiteness and negative definiteness of a quadratic polynomial

❐ $ax^2 + bc + c > 0$ holds for all $x \in R$, i.e., $ax^2 + bx + c$ is positive definite. if $D < 0$ and $a > 0$ where $D = b^2 - 4ac$

❐ $ax^2 + bx + c \geq 0$ holds for all $x \in R$, i.e., $ax^2 + bx + c$ is non-negative, if $D \leq 0$ and $a > 0$

❐ ax^2 bx ÷ $c < 0$ holds for all $x \in R$, i.e., $ax^2 + bx + c$ is negative definite, if $D < 0$ and $a < 0$.

Graphical representation

Graph of $y = f(x) = ax^2 + bx + c$:

The graph of $y = ax^2 + bx + c$ is called a parabola. The point at which its direction changes is called its turning point. We can observe following points in graph of $y = ax^2 + bx + c$

(Through which line of symmetry passes).

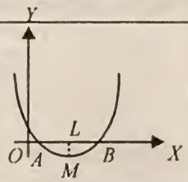
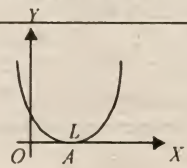
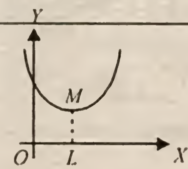
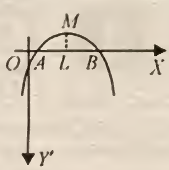
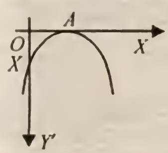
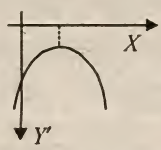
(i) The graph of the function is concave upwards when $a > 0$ and concave downwards when $a < 0$.

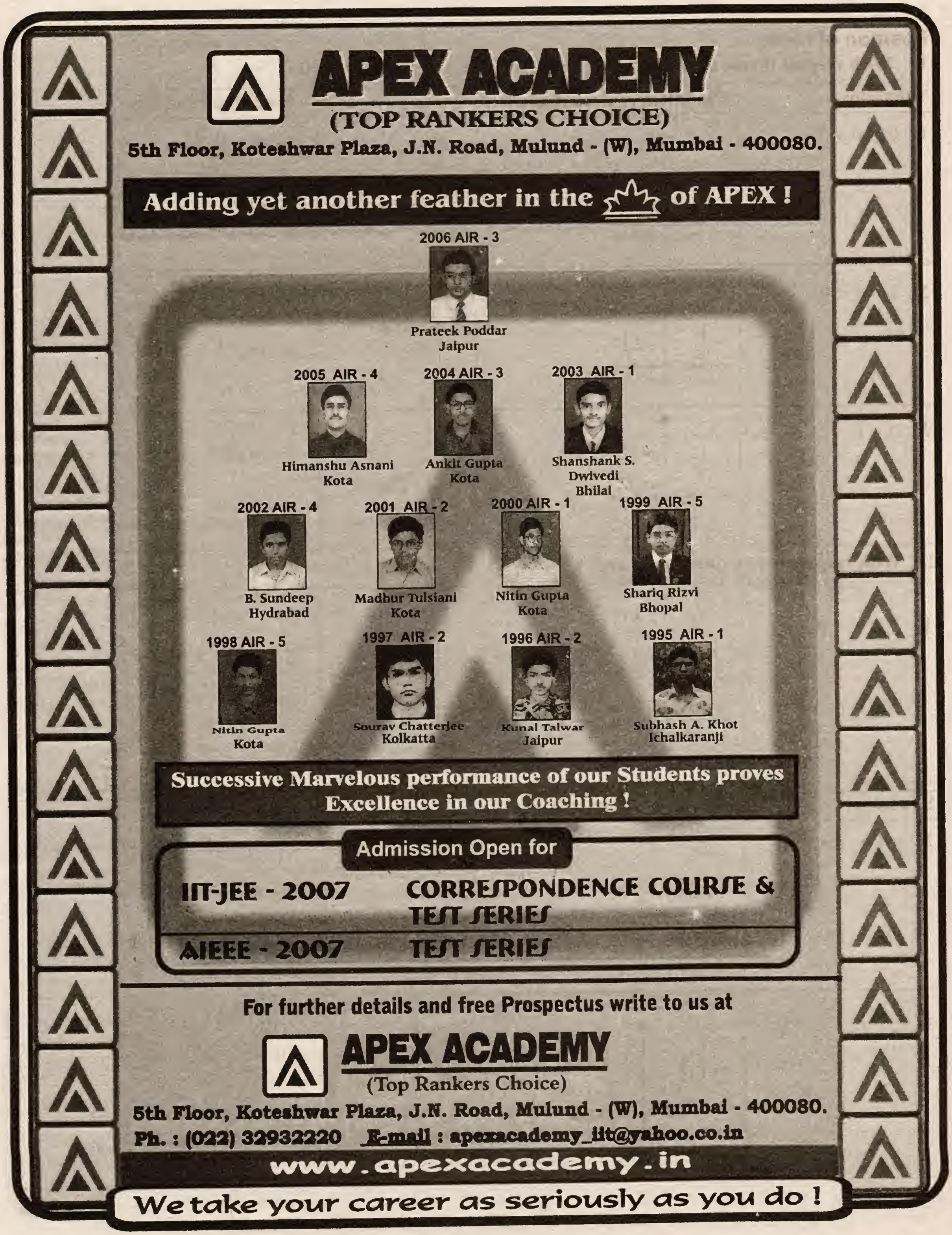
(ii) The character and values of the roots of a quadratic equation can also be determined by graphing, Thus

❐ If the graph cuts the x-axis, the roots of the equation will be real and unequal. Their values will be given by the abscissae of the points of intersection of the graph *ad* the x-axis.

❐ If the graph is tangent to the x-axis, the roots are real and equal.

❐ If the graph has no points in common with the x-axis, the roots of the equation are imaginary and cannot be determined from the graph.

	Case 1 : D > 0	**Case 2 : D = 0**	**Case 3 : D < 0**
When a is +ve i.e., $a > 0$	 Real roots OA and OB. Minimum value $LM = \frac{4ac - b^2}{4a}$ At $X = OL = \frac{-b}{2a}$	 Real and equal roots. Each root = OA	 Roots imaginary. Curve entirely above the X-axis with lowest point M at a height, $LM = \frac{4ac - b^2}{4a}$ above the X-axis
When a is +ve i.e., $a < 0$	 Real roots OA and OB. Minimum value $LM = \frac{4ac - b^2}{4a}$ At $x = OL = \frac{-b}{2a}$	 Real and Each root = OA	 Roots imaginary. The curve lies entirely below the X-axis and the vertex will be at a distance $= \frac{4ac - b^2}{4a}$ below OX.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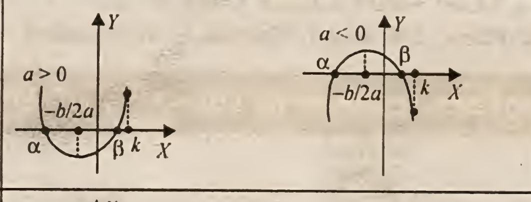
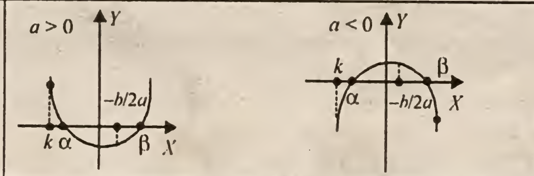
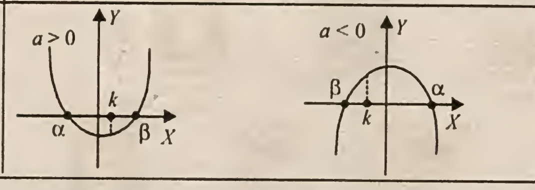
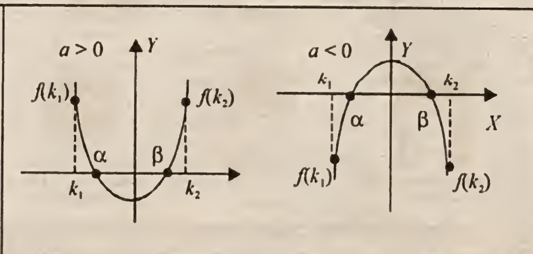
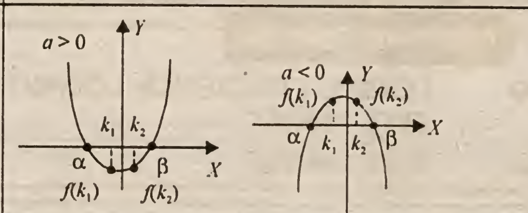
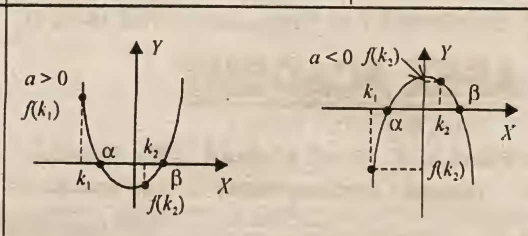
Position of roots

1. With respect to one quantity (k) : Let $ax^2 + bx + c = 0$ has roots α and β

Situations	Graphical representation	Required condition
1. $\alpha < \beta < k$		(i) $D \geq 0$ & (ii) $af(k) > 0$ & (iii) $k > \frac{-b}{2a}$
2. $k < \alpha < \beta$		(i) $D \geq 0$ & (ii) $af(k) > 0$ & (iii) $k < \frac{-b}{2a}$
3. $\alpha < k < \beta$		(i) $D \geq 0$ (ii) $af(k) < 0$

2. With respect to two quantity k_1 and k_2 :

Situations	Graphical representation	Required condition
1. $k_1 < \alpha < \beta < k_2$		(i) $D \geq 0$ (ii) $af(k_1) > 0$ (iii) $af(k_2) > 0$ (iv) $k_1 < \frac{-b}{2a} < k_2$

Situations	Graphical representation	Required condition
2. $\alpha < k_1 < k_2 < \beta$		(i) $D \geq 0$ (ii) $af(k_1) < 0$ (iii) $af(k_2) < 0$
3. $k_1 < \alpha < k_2 < \beta$		(i) $D \geq 0$ (ii) $f(k_1)\,f(k_2) < 0$

CONCEPTS & ANALYSIS

How to Prepare for IIT-JEE

FUNCTIONS

By : Prof. S. S. Dahiya

SECTION - I

Definition : A relation in which for some value of independent variable x there corresponds unique (single) value of dependent variable y then y is known to be function of x or we write $y = f(x)$.

Note that every function is a relation but every relation cannot be a function.

Domain : Set of feasible values of x or defined values of x is called domain.

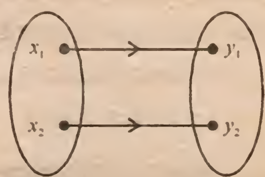
Co-domain : Set of defined values of y is called co-domain.

Range : Set of calculated values of y is called range.

One-One function (Injective)

if $x_1 \neq x_2$ then $y_1 \neq y_2$

function $y = f(x)$ is one-one



Many-One function :

$x_1 \neq x_2$ but $y_1 = y_2$ function $y = f(x)$ is many-one.

Onto function (Surjective) Range = Co-domain

Into function Range $\neq$ Co-domain

SECTION - II

(Characteristic functions)

Category:

1. Linear functions

(i) $f(\alpha x + \beta y) = \alpha\, f(x) + \beta\, f(y)$ where $\alpha + \beta = 1$ using calculus or section formula of co-ordinate geometry such functions are $f(x) = mx + c$ where $c \neq 0$ (m and c are arbitary constants)

(ii) $f(\alpha x + \beta y) = \alpha f(x) + \beta f(y)$ where $\alpha + \beta \neq 1$. This property gives $f(0) = 0$, hence $c = 0$, such functions are $f(x) = mx$ where m is arbitrary constant.

2. Exponential functions

(i) $f(x + y) = f(x) : f(y)$

(ii) $f(x-y) = \dfrac{f(x)}{f(y)}$

(iii) $f(x + y)\, f(x - y) = (f(x))^2$

Further $f(x) > 0$ for $x \in R$ and $f(0) = 1$

Using calculus such functions are $f(x) = e^{kx}$

3. Logarithmic functions : For $x > 0$, $y > 0$

(i) $f(xy) = f(x) + f(y)$

(ii) $f\left(\dfrac{x}{y}\right) = f(x) - f(y)$

(iii) $f(xy) + f\left(\dfrac{x}{y}\right) = 2f(x)$

Using calculus such functions are $f(x) = k \ln(x)$, where k is arbitrary constant

4. Equational functions : We can solve for u, two equations are $a_1u + b_1v = k_1$, $a_2u + b_2v = k_2$

Example : $5f(x) + 2f(6 - x) = 2x^2 + x + 5$ ... (i)

Replace x by $(6 - x)$, we get

$2f(x) + 5f(6 - x) = 2(6 - x)^2 + (6 - x) + 5$ ... (ii)

Solve equ. (i) and equ. (ii) for $f(x)$.

5. Analytic functions

First step: Put $x = 0$, $y = 0$ we get value of $f(0)$

Second step: Put $y = 0$ in the given eqn., we get $f(x)$

Example : $f(x + y) + \lambda xy = 6x^2 + f(y)$

When $x = 0, y = 0$ we get $0 = 0$;

$f(0)$ = any real value k

When $y = 0$, $f(x) = 6x^2 + k$.

6. Polynomial function with no intermediate term (Ghost function)

$f(x)$ is polynomial such that

$$f(x) + f\left(\frac{1}{x}\right) = f(x) \cdot f\left(\frac{1}{x}\right)$$

Assume $f(x) = a_0x^n + a_1x^{n-1} + \ldots. + a_n$, $a_0 \neq 0$

Using $f(x)+f\left(\frac{1}{x}\right)=f(x).f\left(\frac{1}{x}\right)$,

we get $a_1 = a_2 = ... a_{n-1} = 0$, $a_n = 1$, $a_0 = \pm 1$

Therefore $f(x) = 1 + x^n$ or $f(x) = 1 - x^n$.

7. Functions which do not belong to categories 1 to 6 can be formulated using differentiation and then by solving the resulting equation only when some feasible conditions are given.

Example : $f(x) f(y) + f(xy) = f(x) + f(y)$ where $f(1) = 0, f'(1) = -2$.

SECTION - III

One-One and Onto functions or Bijective functions or Invertible functions

A function is combination of mathematical operations, if effect of these operation is subsyded by some combination of operations then such combination of operations is known as Inverse function.

Example : $f(x) = 4x + 7$

Function is : value of x is multiplied by 4, then add 7

First work (Operation) $\longrightarrow$ Second work $\longrightarrow$

Inverse function is : Divide by 4 (Second work) $\longleftarrow$ Subtract 7 (First work) $\longleftarrow$

Hence $f^{-1}(x)=\dfrac{(x-7)}{4}$.

For finding inverse function, first delete Many-One part of the function, make inverse for one-one part.

Example (i) : $f(x) = |x - 2|, 0 \le x \le 7$

$$f(x)=\begin{cases} 2-x, & 0\le x<2, \quad 0<y\le 2 \\ x-2, & 2<x\le 4, \quad 0<y\le 2 \\ 0, & x=2 \\ x-2, & 4<x\le 7, \quad 2<y\le 5 \end{cases}$$

One-One part of the function is

$$f(x)=\begin{cases} 0, & x=2, \quad y=0 \\ x-2, & 4<x\le 7, 2<y\le 5 \end{cases}$$

Now replace x by y and y by x.

$$f^{-1}(x)=\begin{cases} 2, & x=0 \\ x+2, & 2<x\le 5 \end{cases}$$

Example (ii) : $f(x) = x^2, x \in R$

$$f^{-1}(x)=\begin{cases} x^2, & x>0, y>0 \\ x^2, & x<0, y>0 \\ 0, & x=0 \end{cases}$$

$f(x)$ is One-One only at one point *i.e.* at $x = 0$.

$f^{-1}(x)=0, x=0$.

Example (iii) : $f(x)=\dfrac{x^2-x+1}{x^2+x+1}, x\in R$

Note that $f(x)$ behaves One-One at three points only *i.e.* at $x = 0, 1, -1$.

Range is $\frac{1}{3}\le y\le 3$

$$f(x)=\left.\begin{cases} 1, & x=0 \\ \frac{1}{3}, & x=1 \\ 3, & x=-1 \end{cases}\right\} \text{One–One}$$

Many-One for $x \in R - \{-1, 0, 1\}$

$$f^{-1}(x)=\begin{cases} 0, & x=1 \\ 1, & x=\frac{1}{3} \\ -1, & x=3 \end{cases}$$

SECTION - IV

Composite Functions

Note that composite functions may exist, may not exist which purely depends upon the domain and range of the functions.

Without consideration the domain and range of the functions, we cannot think of composite functions.

If $y = f(x)$, domain D_1, range R_1.

Note that $f(f(x))$ is possible only when $D_1\cap R_1\neq\phi$, if $D_1\cap R_1=\phi$ then $f(f(x))$ does not exist.

Example (i) : $f(x) = 2x + 6, 0 \le x \le 4, 6 \le y \le 14$

D_1 is [0, 4], R_1 is [6, 14], $D_1\cap R_1\neq\phi$, $f(f(x))$ does not exist.

Example (ii) : $f(x) = 2x + 6, 0 \le x \le 8$,

$6 \le f(x) \le 22$

$f(f(x)) = 2f(x) + 6, 0 \le f(x) \le 8$

$f(f(x)) = 2(2x + 6) + 6, 6 \le 2x + 6 \le 8$

Because $0 \le f(x) < 6$ is not possible

$f(f(x)) = 4x + 18,\ 0 \le x \le 1$

$f(x) = 2x + 6,\ D_1 = [0, 8],\ R_1 = [6, 22]$

$f(f(x)) = 4x + 18,\ D_2 = [0, 1],\ R_2 = [18, 22]$

Note that $D_1 \cap R_1 \ne \phi$;

$D_2 \subseteq D_1,\ R_2 \subseteq R_1$

No conclusion for relationship of D_2 and R_1 or relationship of D_1 and R_2.

Example (ii) : $f(x) = 2x + 2,\ 0 \le x \le 4,\ 2 \le f(x) \le 10$

$g(x) = 8 - 4x,\ 0 \le x \le 2,\ 0 \le g(x) \le 8$

$\therefore\ f(g(x)) = 2\,g(x) + 2,\ 0 \le g(|x) \le 4$

$f(g(x)) = 2(8 - 4x) + 2,\ 0 \le 8 - 4x \le 4$

$f(g(x)) = 18 - 8x,\ 1 \le x \le 2$

$g(f(x)) = 8 - 4f(x),\ 0 \le f(x) \le 2$

$g(f(x)) = 8 - 4f(x),\ f(x) = 2$

Becuase $0 \le f(x) < 2$ is not possible

$g(f(x)) = 8 - 4(2x + 2),\ 2x + 2 = 2$

$g(f(x)) = -8x,\ x = 0$

$f(x)$, domain D_1, range R_1.

$g(x)$, Domain D_2, Range R_2

$f(g(x))$ is possible only when $D_1 \cap R_2 \ne \phi$

$g(f(x))$ is possible only when $D_2 \cap R_1 \ne \phi$

SECTION - V

Q1. $f(x + y) = f(x) \cdot f(y)$;

$f'(0) = 3, f(5) = 2.$

Prove that $f'(5) = 6$.

The question is wrong

$f(x + y) = f(x) \cdot f(y)$ ... (i)

$f'(0) = 3$ ... (ii)

$f(5) = 2$ ... (iii)

In mathematics there is no function which can satisfy conditions (i), (ii), (iii).

- Make function using conditons (i) and (ii), then condition (iii) does not obey the function
- Make function using conditions (i) and (iii) then condition (ii) does not obey the funciton
- Make function using conditions (ii) and (iii) then condition (i) does not obey the function.

Q2. a, b, c, d are real $f(x) = \dfrac{ax+b}{cx+d}, x \ne -\dfrac{d}{c}$

Find the condition for which $f^{-1}(x) = f(x)$.

Ans. $a + d = 0$.

Q3. a, b, c, d are real $f(x) = ax^3 + bx^2 + cx + d,\ a \ne 0$ find the condition for which $f(x)$ is One-One and Onto *i.e.* invertible.

Ans. $b^3 \le 3ac$.

Q4. Functions $f(x)$ and $f^{-1}(x)$ are symmetric about $y = a_0x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n$.

Find values of $a_0, a_1, a_2, ..., a_n$.

Ans. $a_0 = a_1 = a_2 = ... = a_{n-2} = 0,$

$a_{n-1} = 1,\ a_n = 0.$

Indian American sets a 'pi' record

An Indian American teenager has memorised 10,980 digits of 'pi', the ratio of circumference of a circle to its diameter, rewriting a 27-year-old North American record.

Fifteen-year-old Gaurav Raja, an 11th grader of Salem High School in Virginia, has recited two numbers per second for one hour, 14 minutes and 28 seconds to break the North American record of 10,625 digits.

The world record belongs to Hiroyuki Goto of Japan who has memorised 42,195 digits. This endeavour has earned Raja ninth position in this regard.

"I know what the numbers are, but it is not like I can see them. I just know what they are," Raja said.

The encouragement for his attempt came when his math and science teacher Linda Gooding asked her students to memorise 40 digits of the pi, the non-ending decimal more accurately expressed as a fraction 22/7.

Starting off with memorising about 250, Raja kept practising and realised that numbers came to him easily.

"I just don't know how he can do it," his father, Jogesh Raja said. The proud father said that even in kindergarten, the teachers said that their child was gifted in math.

Meanwhile, Raja has his next task cut out for himself — memorising every Nobel Prize winner. "I guess those are a bit more useful," he said.

Even after getting 1st Rank in AIIMS Satvik preferred pursuing engineering.

When I got 12th rank in IIT-JEE in my category at that time I was firm that I will take admission in IIT-Delhi. I wanted Electrical Engineering and I got it too.

text books, NCERT books and MTG Explorers.

MTG : In your words what are the components of an ideal preparation plan?
Satvik : Regular studies is the key component of an ideal preparation plan. Clear basic concepts first. Don't cram.A lot of practice of new questions is essential.

MTG : What role did the following play in your success ?
(a) parents (b) teachers (c) school.
Satvik : (a) Parents lent a great support and helped me throughout my preparation. They motivated me at each step.

(b) Teachers in my school were my ultimate guides. They gave constant encouragement and were available whenever I needed them.

(c) School laid foundation for success.

MTG : Your family background?
Satvik : My father is a businessman and mother is a senior manager.

MTG : What mistake you think you shouldn't have made?
Satvik : I was over confident in my preparation.

MTG : How have MTG magazines helped you in your preparation?
Satvik : I read Mathematics Today, Chemistry Today, Physics For You and Biology Today regularly. These magazines are a good source of knowledge. The best part is I got the solved papers of all the national level/state level on my table, month after month. I got superb practice and developed lot of confidence to face the real test. Challenging problems and olympiad section infused in me the winning spirit. I bought MTG Explorers to practice for AIIMS, CBSE-PMT, DPMT, IIT-JEE. These books are exceptionally good and the explanations given in the books has helped in achieving the merit.

MTG : Was this your first attempt?
Satvik : Yes, this is my first attempt.

MTG : Had you not been selected then what would have been your future plan?
Satvik : I would have done B.Sc.

MTG : What do you think is the secret of your success?
Satvik : Pure hard work, regular studies and the faculty of Aakash Institute.

MTG : What advice would you like to give our readers who are PET aspirants?
Satvik : There is no substitute to hardwork and self study. Coaching is only a guidance, the real effort has to come from within.

"Regular studies is the key component of an ideal preparation plan. Clear basic concepts first. Don't cram.A lot of practice of new questions is essential."

OO

10 Challenging Subjective Problems

Algebra

by Er. Tapas Kumar Yogi

1. Let A and B be two square matrices such that $A + B = AB$. Prove that A and B commute.

2. Let A, B, C be $n \times n$ matrices such that $A + B + C = AB + BC + CA$. Prove that the equalities $ABC = AC - CA$, $BCA = BA - AB$, $CAB = CB - BC$ are equivalent.

3. If A and B are different matrices satisfying $A^3 = B^3$ and $A^2B = B^2A$, find $\det(A^2 + B^2)$.

4. Find the solution to the system of equations

$$x + [y] + \{z\} = 1.1$$
$$[x] + \{y\} + z = 2.2$$
$$\{x\} + y + [z] = 3.3$$

where $[\cdot]$ denotes [illegible] integer function and $\{\cdot\}$ denotes fractional part function.

5. A graph is defined in polar co-ordinates by $r(\theta) = \cos\theta + \frac{1}{2}$. Find the smallest x-co-ordinate of any point on this graph.

6. Prove that for all $x \in R$, $x^4 + ax^3 + bx^2 + cx + 1 > 0$ if $a^2 + c^2 \le 4b$, $a, b, c \in R$.

7. Let A be an $n \times n$ matrix such that $A^n = \alpha A$ where α is a real number ($\neq \pm 1$). Prove that the matrix $A + I_n$ is invertible.

8. Let A, B, C be $n \times n$ real matrices that are pairwise commutative and $ABC = 0_n$. Prove that

$$\det(A^3 + B^3 + C^3) \cdot \det(A + B + C) \ge 0.$$

9. Let p and q be real numbers such that $x^2 + px + q \neq 0$ for every real number x. Prove that if n is an odd positive integer, then $x^2 + px + qI_n \neq 0_n$ for all real matrices X of order $n \times n$.

10. Prove that $4x - x^4 \le 3$, $x \in R$.

SOLUTIONS

1. $A + B = AB$ ... (i)

$\Rightarrow -A - B + AB = 0$

$I_n^2 - I_nA - I_nB + AB = I_n$

$\Rightarrow I_n(I_n - A) - B(I_n - A) = I_n$

or, $(I_n - A)(I_n - B) = I_n$

i.e., $I_n - A$ is invertible and its inverse is $I_n - B$.

So, $(I_n - B)(I_n - A) = I_n$

$\Rightarrow I_n^2 - I_nA - BI_n + BA = I_n$

or, $-A - B + BA = 0$

$\Rightarrow A + B = BA$ ... (ii)

So, vide (1), (2); $AB = BA$.

i.e., A and B commute.

2. Now, $A + B + C + ABC = AB + BC + CA + AC - CA$ ... (i)

(from the first equality statement)

and $(A - I_n)(B - I_n)(C - I_n)$

$= (A - I_n)(BC - BI_n - CI_n + I_n)$

$= ABC - AB - AC + A - BC + B + C - I_n$

$= -I_n$ ... [vide (i)]

$\Rightarrow$ $(A - I_n)$, $(B - I_n)$ and $(C - I_n)$ are invertible.

So, $(C - I_n)(A - I_n)(B - I_n) = -I_n$

Simplifying further and using (i), we have

$CAB = CB - BC$.

This shows that the first and third equalities are equivalent. Permuting the letters, we obtain the result.

3. We have

$$(A^2 + B^2)(A - B) = A^3 - A^2B + AB^2 - B^3 = 0_n$$

Since $A \neq B$, this shows that $A^2 + B^2$ has a zero divisor. Hence, it is not invertible. So, $\det.(A^2 + B^2) = 0$.

4. $x + [y] + z - [z] = 1.1$ ... (i)

$[x] + y - [y] + z = 2.2$ ... (ii)

$x - [x] + y + [z] = 3.3$ ... (iii)

Adding the first two equations and subtracting from the third, we have

$[x] + z - [z] = 0 \quad \Rightarrow \quad z \to$ integer

So, $z = [z] \Rightarrow [x] = 0 \Rightarrow 0 \le x < 1$

So, now (i) becomes $x + [y] = 1.1$ or $x = 1.1 - [y]$

$\Rightarrow 0 \le 1.1 - [y] < 1$ or, $[y] = 1$

So, $x = 1.1 - 1 = 0.1 \quad \Rightarrow \quad 1 \le y < 2$

From (ii), $y + [z] = 3.2$

$\Rightarrow \; y = 3.2 - [z] \quad$ or, $\quad 1 \le 3.2 - [z] < 2$

$\Rightarrow \; 1.2 < [z] \le 2.2 \quad$ or, $\quad [z] = 2$

$\Rightarrow \; z = 2 \quad$ or $\quad y = 1.2.$

So, $x = 0.1, y = 1.2, z = 2.$

5. For polar co-ordinates, we have

$x = r\cos\theta$

$$\text{So, } x = \left(\cos\theta + \frac{1}{2}\right)\cos\theta = \cos^2\theta + \frac{1}{2}\cos\theta$$

$$= \cos^2\theta + \frac{1}{2}\cos\theta + \frac{1}{16} - \frac{1}{16}$$

$$= \left(\cos\theta + \frac{1}{4}\right)^2 - \frac{1}{16} \text{ [completing the square]}$$

Hence the smallest x-co-ordinate is $-\frac{1}{16}$.

6. $x^4 + ax^3 + bx^2 + cx + 1$

$$= x^4 + 2\cdot\frac{ax}{2}\cdot x^2 + \frac{a^2x^2}{4} + bx^2 - \frac{a^2x^2}{4} + cx + 1$$

$$= \left(x^2 + \frac{ax}{2}\right)^2 + bx^2 - \frac{a^2x^2}{4} + 2\cdot\frac{cx}{2}\cdot 1 + \frac{c^2x^2}{4} + 1 - \frac{c^2x^2}{4}$$

$$= \left(x^2 + \frac{ax}{2}\right)^2 + bx^2 - \frac{a^2x^2}{4} + \left(1 + \frac{cx}{2}\right)^2 + \left(b - \frac{a^2}{4} - \frac{c^2}{4}\right)x^2$$

$$= \left(x^2 + \frac{ax}{2}\right)^2 + \left(1 + \frac{cx}{2}\right)^2 + (4b^2 - a^2 - c^2)\frac{x^2}{4} > 0$$

by using the given condition.

7. Let $B = A + I_n$

So, $A^n = \alpha A \quad \Rightarrow \quad (B - I_n)^n = \alpha(B - I_n)$

or, $B^n - {}^nC_1 \cdot B^{n-1} + {}^nC_2 B^{n-2} - \ldots (-1)^n I_n$

$= \alpha B - \alpha I_n$

[By using binomial expansion]

or, $B^n - {}^nC_1 \cdot B^{n-1} + {}^nC_2 B^{n-2} - \ldots (-1)^{n-1} B - \alpha B$

$= -\alpha I_n - (-1)^n I_n$

or, $B[B^{n-1} - {}^nC_1 B^{n-2} + {}^nC_2 B^{n-3} - \ldots (-1)^{n-1} I_n - \alpha I_n]$

$= I_n[(-1)^{n+1} - \alpha]$

i.e., B is invertible since, $(-1)^{n+1} - \alpha \ne 0$; $(\alpha \ne \pm 1)$.

8. Using the identity,

$a^3 + b^3 + c^3 = (a + b + c)(a + b\omega + c\omega^2)(a + b\omega^2 + c\omega)$

where $\omega \rightarrow$ complex cube root of unity.

We have, $A^3 + B^3 + C^3 = (A + B + C)(A + \omega B + \omega^2 C)$

$(A + \omega^2 B + \omega C)$

$= (A + B + C)(A + \omega B + \omega^2 C)\overline{(A + \omega B + \omega^2 C)}$

So, $\det(A^3 + B^3 + C^3) \cdot \det(A + B + C)$

$= \det(A + B + C)^2 \cdot \det(A + \omega B + \omega^2 C) \cdot$

$\det\overline{(A + \omega B + \omega^2 C)}$

$= (\det(A + B + C))^2 \cdot |\det(A + \omega B + \omega^2 C)|^2 \ge 0.$

9. Proof by contradiction.

Let $x^2 + px + qI_n = 0_n$

$$\text{Completing the square, } \left(x + \frac{p}{2}I_n\right)^2 = \frac{p^2 - 4q}{4}I_n$$

$$\text{or, } \left(\det\left(x + \frac{p}{2}I_n\right)\right)^2 = \left(\frac{p^2 - 4q}{4}\right)^n \quad \ldots \text{(i)}$$

Now given by $x^2 + px + q \ne 0$

$\Rightarrow \; x^2 + px + q > 0 \quad$ or $\; < 0$

and in any case $p^2 - 4q < 0$.

So, in the equation (i) above, LHS is ≥ 0

but RHS is < 0. This contradicts our assumption.

10. Consider $x^4 - 4x + 3 = x^4 - 2x^2 + 1 + 2x^2 - 4x + 2$

$= (x^2 - 1)^2 + 2(x^2 - 2x + 1)$

$= (x^2 - 1)^2 + 2(x - 1)^2 \ge 0$

or, $3 \ge 4x - x^4 \quad$ *i.e.* $\; 4x - x^4 \le 3.$

■

Puzzle ???

Suppose there are twin brothers, one which always tells the truth and one which always lies. (So in this case they know what is true and what is false, or as you put it, both are accurate in their knowledge). What one yes-no question could you ask to either one of the brothers to figure out which one he is?

Ans.: The question one could ask is, "If I were to ask your brother whether you always tells the truth, what would he say?" A reply of "no" means you are talking to the truth teller, a reply of "yes" means you are talking to the liar.

Another possible question is, "If I were to ask you whether you always tell the truth, what would you say?" In this case a reply of "yes" means you are talking to the truth teller and a reply of "no" means you are talking to the liar.

Both questions take advantage of the liar lying about what he or his brother would say, creating a double negative type situation.

Chinese Olympiad Problems

SECTION - 1

1. All diagonals of a convex *n-gon f*, where $n \geq 4$, have equal length. Which of the following statements is true ?

(a) F must be a quadrilateral
(b) F must be a pentagon
(c) F is either a quadrilateral or a pentagon
(d) F is either an equilateral or an equiangular polygon

2. What curve is defined by the polar equation

$$p = \frac{1}{1-\cos\theta+\sin\theta}$$

(a) circle (b) ellipse
(c) hyperbola (d) parabola

3. Suppose $\log_2[\log_{\frac{1}{2}}(\log_2 x)]$

$$= \log_3[\log_{\frac{1}{3}}(\log_3 y)] = \log_5[\log_{\frac{1}{5}}(\log_5 z)]$$

Which of the following statements is true ?

(a) $z < x < y$ (b) $x < y < z$
(c) $y < z < x$ (d) $z < y < x$

4. What is the area of the region enclosed by the curve defined by the equation

$$|x - 1| + |y - 1| = 1 ?$$

(a) 1 (b) 2
(c) π (d) 4

5. Suppose ϕ is any angle in the interval $\left(0, \frac{\pi}{2}\right)$. Which of the following statemetns is true ?

(a) $\sin \sin \phi < \cos \phi < \cos \cos \phi$
(b) $\sin \sin \phi > \cos \phi > \cos \cos \phi$
(c) $\sin \cos \phi > \cos \phi > \cos \sin \phi$
(d) $\sin \cos \phi < \cos \phi < \cos \sin \phi$

6. Let x_1 and x_2 be the real roots of the equation

$$x^2 - (k - 2)x + (k^2 + 3k + 5) = 0$$

What is the maximum value of $x_1^2 + x_2^2$?

(a) 19 (b) 18
(c) $\frac{50}{9}$ (d) non existent

7. Suppose $M = \{(x, y) \mid |x y| = 1, x > 0\}$ and $N = \{(x, y) | \arctan x + \arctan y = \pi\}$. Which of the following statements is true ?

(a) $M \cup N = \{(x, y) \mid |x y| = 1\}$
(b) $M \cup N = M$ (c) $M \cup N = N$
(d) $M \cup N = \{(x, y) \mid |x y| = 1$, x, y not negative simultaneously$\}$

8. Let a and b be distinct positive real numbers. Which of the expressions $P = \left(a+\frac{1}{a}\right)\left(b+\frac{1}{b}\right)$, $Q = \left(\sqrt{ab}+\frac{1}{\sqrt{ab}}\right)^2$ and $R = \left(\frac{a+b}{2}+\frac{2}{a+b}\right)^2$ is the largest?

(a) P (b) Q
(c) R (d) dependent on a and b

SECTION - 2

1. In the tetrahedron $SBAC$, $\angle ASB = \frac{\pi}{2}$, $\angle ASC = \alpha$ and $\angle BSC = \beta, 0 < \alpha, \beta < \frac{\pi}{2}$. Prove that the dihedral angle about the edge SC is given by

$$\pi - \arccos(\cot \alpha \cot \beta).$$

2. AB is the diameter of a semi-circle. T is a point on the extension of BA, with $AT < (1/4)AB$. The line l passes through T and is perpendicular to AB. Lines through two distinct points M and N on the semi-circle and perpendicular to l intersect it at P and Q respectively. If $MP = AM$ and $NQ = AN$, prove that $AM + AN = AB$.

3. ABC is an equilateral triangle of side length 4. D, E and F are points on BC, CA and AB respectively, with $AE = BF = CD = 1$.

QRS is the triangle formed by joining AD, BE and CF. P is a variable point in or on QRS. Consider the product of its distances from the three sides of ABC.

(a) Prove that this product is minimum when P coincides with Q, R or S.
(b) Determine the minimum value of this product.

SOLUTIONS

Section - 1

1. Suppose $A_1 A_2 A_3 A_4 A_5 .... A_n$ is a convex n-gon with $n \geq 6$. Then $A_1 A_4$, $A_1 A_4$, $A_2 A_4$ and $A_2 A_5$ are all diagonals. Let $A_1 A_4$ cut $A_2 A_4$ at B. Then

$A_1A_4 + A_2A_5 = A_1B + A_5B + A_2B + A_4B > A_1A_5 + A_2A_4$ by the triangle inequality. Hence it is not possible for these four diagonals to have equal length. A regular pentagon shows that F does not have to be a quadrilateral. An isosceles trapezoid shows that F does not have to be a pentagon, equilateral or equiangular.

2. Converting to Cartesian coordinates, we obtain

$$\sqrt{x^2+y^2} = x - y + 1$$

This may be rewritten as $(x+1)(y-1) = \frac{1}{2}$. Hence the curve is a hyperbola.

3. We have $x = 2^{1/2}$, $y = 3^{1/3}$ and $\overset{z}{x} = 5^{1/5}$. Now $z^{10} = 25 < 32 = x^{10}$ and $x^6 = 8 < 9 = y^6$. Hence $z < x < y$.

4. Depending on whether $x - 1$ and $y - 1$ are negative or otherwise, the given equation becomes one of $x + y = 1$, $x + y = 3$, $y = x + 1$ and $y = x - 1$. They enclose a square of area 2.

5. Note that $0 < \cos\phi < 1 < \frac{\pi}{2}$. For all x in $\left(0, \frac{\pi}{2}\right)$, we have $\sin x < x$. Hence $\sin\cos\phi < \cos\phi$. Also, $\cos x$ is a decreasing function on this interval. Hence $\cos\phi < \cos\sin\phi$. Take $\phi = \frac{\pi}{4}$. Then $0 < \frac{1}{\sqrt{2}} < \phi < \frac{\pi}{2}$

Hence $\cos\phi < \cos\frac{1}{\sqrt{2}} = \cos\cos\phi$

On the other hand, take $\phi = \arccos\frac{\pi}{4}$. Then

$$\cos\phi = \frac{\pi}{4} > \frac{1}{\sqrt{2}} = \cos\frac{\pi}{4} = \cos\cos\phi$$

It follows that neither $\cos\phi$ nor $\cos\cos\phi$ dominates the other on $(0, \pi/2)$.

6. Since the equation has real roots, $0 \le (k-2)^2 - 4(k^2 + 3k + 5) = (3k+4)(k+4)$ or $-4 \le k \le -4/3$.

By Vieta's Formulae

$$x_1^2 + x_2^2 = (k-2)^2 - 2(k^2 + 3k + 5) = -(k+5)^2 + 19$$

The allowable value of k closest to -5 is -4, which yields the maximum value. When $k = -4, x_1^2 + x_2^2 = 18$.

7. In N, we have $x = \tan(\pi - \arctan y) = -\tan(\text{arccot} y) = -1/y$ or $xy = -1$. Moreover, if $x < 0$ and $y > 0$, then $\arctan x < 0$ and $\arctan y < \pi/2$. This contradicts $\arctan x + \arctan y = \pi$. It follows that N lies entirely in the fourth quadrant. Now M consists of the curves $xy = 1$ and $xy = -1$ in the first and fourth quadrants. Hence $N \subset M$. Each of the other two sets contains points in the second quadrant, and cannot be equal to $M \cup N$.

8. We have $P = ab + \frac{1}{ab} + \frac{a}{b} + \frac{b}{a} > ab + \frac{1}{ab} + 2 = Q$ by the Arithmetic Mean Geometric Mean Inequality.

Take $a = 1$ and $b = 2$. Then $P = 5 > \frac{169}{36} = R$

On the other hand, take $a = 1$ and $b = 5$. Then

$$P = \frac{52}{5} < \frac{100}{9} = R$$

Hence whether P or R is greater is dependent on a and b.

Section - 2

1. We may assume that $\angle SCA = \pi/2 = \angle SCB$ and $SC = 1$. Then $CA = \tan\alpha$, $SA = \sec\alpha$, $CB = \tan\beta$ and $SB = \sec\beta$. Since $\angle ASB = \pi/2$, we have $AB^2 = \sec^2\alpha + \sec^2\beta$. Hence $\cos ACB = \frac{CA^2 + CB^2 - AB^2}{2CA \cdot CB}$

$$= \frac{\tan^2\alpha + \tan^2\beta - \sec^2\alpha - \sec^2\beta}{2\tan\alpha\tan\beta} = -\cot\alpha\cot\beta.$$

It follows that $\angle ACB = \arccos(-\cot\alpha\cot\beta)$
$= \pi - \arccos(\cot\alpha\cot\beta)$.

2. First solution :

In figure 1, consider the circles with centres M and N passing through A. They intersect at another point A' and are tangent to l at P and Q respectively. Let the line AA' intersect l at R. Then $RP^2 = RA \cdot RA' = RQ^2$. It follows that R is the midpoint of PQ.

Draw the line through R perpendicular to l, cutting MN at S. Then S is the midpoint of MN. Let O be the centre of the semicircle. Then OS is perpendicular to MN, and hence parallel to RA. It follows that $OARS$ is a parallelogram, so that $RS = AO$. We now have $AM + AN = MP + NQ = 2RS = 2AO = AB$.

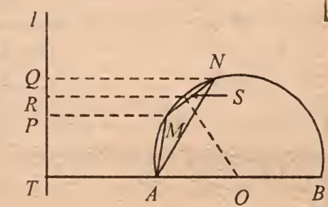


Figure 1

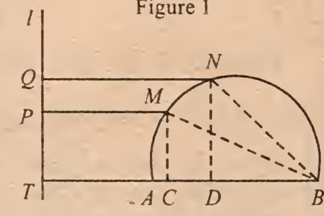


Figure 2

Second solution :

In figure 2, project M and N onto C and D on AB. Since triangles CAM and MAB are similar, we have $AM^2 = AC \cdot AB$. Similarly, $AN^2 = AD \cdot AB$. Hence $AB \cdot CD = AB(AD - AC) = AN^2 - AM^2$
$= (AN + AM)(NQ - MP) = (AN + AM)\,CD$
or $AM + AN = AB$.

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Olympiad Enrichment Series-II

useful for IIT-JEE 2008-09

This series is selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solution.

1. Several (at least two) nonzero numbers are written on a board. One may erase any two numbers, say a and b, and then write the numbers $a+\frac{b}{2}$ and $b-\frac{a}{2}$ instead. Prove that the set of numbers on the board, after any number of the preceding operations, cannot coincide with the initial set.

2. The polynomial $1-x+x^2-x^3+\ldots+x^{16}-x^{17}$ may be written in the form $a_0+a_1y+a_2y^2+\ldots+a_{16}y^{16}+a_{17}y^{17}$, where $y=x+1$ and a_is are constants. Find a_2.

3. Let a, b, and c be distinct nonzero real numbers such that $a+\frac{1}{b}=b+\frac{1}{c}=c+\frac{1}{a}$. Prove that $|abc|=1$.

4. Find polynomials $f(x)$, $g(x)$, and $h(x)$, if they exist, such that for all x,

$$|f(x)|-|g(x)|+h(x)=\begin{cases} -1 & \text{if} \quad x<-1 \\ 3x+2 & \text{if} \quad -1\le x\le 0. \\ -2x+2 & \text{if} \quad x>0 \end{cases}$$

5. Find all real numbers x for which

$$\frac{8^x+27^x}{12^x+18^x}=\frac{7}{6}.$$

SOLUTIONS

1. Let S be the sum of the squares of the numbers on the board. Note that S increases in the first operation and does not decrease in any successive operation, as

$$\left(a+\frac{b}{2}\right)^2+\left(b-\frac{a}{2}\right)^2=\frac{5}{4}(a^2+b^2)\ge a^2+b^2$$

with equality only if $a=b=0$

This completes the proof.

2. Alternative 1

Let $f(x)$ denote the given expression. Then

$xf(x) = x - x^2 + x^3 - .... - x^{18}$ and $(1 + x)f(x) = 1 - x^{18}$

$$\text{Hence } f(x) = f(y-1) = \frac{1-(y-1)^{18}}{1+(y-1)} = \frac{1-(y-1)^{18}}{y}$$

Therefore a_2 is equal to the coefficient of y^3 in the expansion of

$1 - (y - 1)^{18}$ *i.e.*, $a_2 = \binom{18}{3} = 816$

Alternative 2

Let $f(x)$ denote the given expression. Then

$f(x) = f(y - 1) = 1 - (y - 1) + (y - 1)^2 - ... - (y - 1)^{17}$

$\quad = 1 + (1 - y) + (1 - y)^2 + ... + (1 - y)^{17}$

$$\text{Thus } a_2 = \binom{2}{2} + \binom{3}{2} + ... + \binom{17}{2} = \binom{18}{3}$$

Here we used the formula $\binom{n}{k} + \binom{n}{k+1} = \binom{n+1}{k+1}$

and the fact that $\binom{2}{2} = \binom{3}{3} = 1.$

3. From the given conditions it follows that

$$a - b = \frac{b-c}{bc},\ b - c = \frac{c-a}{ca} \text{ and } c - a = \frac{a-b}{ab}$$

Multiplying the above equations gives $(abc)^2 = 1$, from which the desired result follows.

4. Alternative 1 Since $x = -1$ and $x = 0$ are the two critical values of the absolute functions, one can suppose that

$F(x) = a\,|\,x + 1\,| + b\,|\,x\,| + cx + d$

$$= \begin{cases} (c-a-b)x + d - a & \text{if } \quad x < -1 \\ (a+c-b)x + a + d & \text{if } \ -1 \le x \le 0 \\ (a+b+c)x + a + d & \text{if } \quad x > 0, \end{cases}$$

which implies that $a = 3/2$, $b = -5/2$, $c = -1$, and $d = 1/2$

Hence $f(x) = (3x + 3)/2$, $g(x) = 5x/2$, and $h(x) = -x + \frac{1}{2}$

Alternative 2

Note that if $r(x)$ and $s(x)$ are any two functions, then

$$\max(r, s) = \frac{r + s + |r - s|}{2}$$

Therefore, if $F(x)$ is the given function, we have

$F(x) = \max\{-3x - 3, 0\} - \max\{5x, 0\} + 3x + 2$

$\quad = (-3x - 3 + |3x + 3|)/2 - (5x + |5x|)/2 + 3x + 2$

$\quad = |(3x+3)/2| - |5x/2| - x + \frac{1}{2}.$

5. By setting $2^x = a$ and $3^x = b$, the equation becomes

$$\frac{a^3 + b^3}{a^2b + b^2a} = \frac{7}{6} \quad i.e., \quad \frac{a^2 - ab + b^2}{ab} = \frac{7}{6}$$

i.e., $6a^2 - 13ab + 6b^2 = 0$,

i.e., $(2a - 3b)(3a - 2b) = 0$

Therefore $2^{x+1} = 3^{x+1}$ or $2^{x-1} = 3^{x-1}$, which implies that $x = -1$ and $x = 1$

It is easy to check that both $x = -1$ and $x = 1$ satisfy the given equation.

(Apply $C_2 \to C_2 - C_1$, $C_3 \to C_3 - C_1$)

$$= \begin{vmatrix} 1 & 0 & 0 \\ 1 & x & 0 \\ 1 & 0 & y \end{vmatrix} = 1(xy - 0) = xy$$

Hence D is divisible by both x and y.

97. **(a)** : $A^2 = \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix} \begin{bmatrix} 5 & 5\alpha & \alpha \\ 0 & \alpha & 5\alpha \\ 0 & 0 & 5 \end{bmatrix}$

$$A^2 = \begin{bmatrix} 25 & 25\alpha + 5\alpha^2 & 5\alpha + 5\alpha + 25\alpha^2 \\ 0 & \alpha^2 & 25\alpha + 5\alpha^2 \\ 0 & 0 & 25 \end{bmatrix}$$

Given $|A^2| = 25$, $625\alpha^2 = 25 \Rightarrow |\alpha| = \frac{1}{5}$.

98. **(b)** : ${}^nC_4 a^{n-4}(-b)^4 = -\left({}^nC_5 a^{n-5}(-b)^5\right)$

$\Rightarrow \ \frac{a}{b} = \frac{n-4}{5}$.

99. **(a)** : Number of way $= {}^{12}C_4 \times {}^8C_4 \times {}^4C_4 = \frac{12!}{(4!)^3}$.

100. **(d)** : $\because \ {}^{20}C_0 + {}^{20}C_1 x + ... + {}^{20}C_{10}x^{10} + ... + {}^{20}C_{20}x^{20} = (1+x)^{20}$

After putting $x = -1$, we get

$${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + ... + {}^{20}C_{10} - {}^{20}C_{11} - {}^{20}C_{12} + ... + {}^{20}C_{20} = 0$$

$$2({}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + ... - {}^{20}C_9) + {}^{20}C_{10} = 0$$

$${}^{20}C_0 - {}^{20}C_1 + {}^{20}C_2 - {}^{20}C_3 + ... - {}^{20}C_9 + {}^{20}C_{10} = \frac{1}{2}\,{}^{20}C_{10}$$

Booming biz of coaching

Charmed by the power of being driven around in a beacon ambassador car or attracted to fat paycheques drawn by engineers and doctors, the growing demand by students for professional courses like engineering, medicine and civil service has led to private coaching industry flourishing like never before.

Educationists however, consider this as a grim reflection of the country's education system. "We have yet not done any survey about how big the coaching industry is but definitely it is mushrooming across the country because of the lack of proper education system." says a senior official in the education department, Federation of Indian Chamber of Commerce and Industry (FICCI).

The public perception now is that private tuition is a necessity to succeed in entrance tests and board exams, she adds. "The main problems that plague our education system today are lack of infrastructure and the environment to prepare students for competitive exams," she says. Coaching centres all over the country thrive on the ever-growing demand of students who are keen on getting a professional education. Some of these centres have grown to become big business houses by providing their services all around the country, and reaping profits.

The average private institute that trains students for IIT-JEE demands anything between Rs.40,000 and Rs.50,000. And this doesn't include the cost of study material. The total number of students who appeared for the IIT-JEE this year was 2,43,029 of which 7,209 are eligible to seek admission to 5,537 seats in IITs at Mumbai, Delhi, Guwahati, Kharagpur, Kanpur, Chennai and Roorkee. "My child is talented and he always got good marks in school but to perform well in competitive exams like the IIT he needs an extra effort. I think the coaching institute has done wonders to my child and I did not mind spending some extra money for that," says Anup Mishra, clerk at a bank. Some institutes also conduct motivational workshops apart from regular classes. As many as 2,10,318 candidates had registered for the preliminary examination of CBSE All India Pre-Medical 2007, out of which 17,135 has qualified for the final examination.

Stiff competition in competitive exams creates a platform for these institutes to attract students through advertisement in media or by distributing pamphlets. Medical coaching classes charge anything between 50-60 thousand for the eight month course and also provide installment facility. The coaching industry has become highly professional and corporate, many of them operated by IIT graduates. Services of retired IIT professors and even current IIT students are roped in for hefty compensation.

"Many of my seniors and colleagues take classes in coaching institutes and get a healthy remuneration. Even I have got calls from these institutes so many times but I rejected their offers," says Abhishek, a IIT student. Some educationists say that the phenomenal growth of such centres is a sad reflection of the quality of school and college education available in India. Vivekanand Upadhayay, a professor, says, "Growth of coaching industries shows the poor quality of our education system, Government spends only 3% of the GDP on education." ❖

Math Archives

10 Best Problems

Prof. Shyam Bhushan*

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE.

1. The line with direction cosines proportional to 2, 1, 2 meets each of the lines $x = y + 1 = z$ and $x + 1 = 2y = 2z$. The coordinates of each of the points of intersection are given by

(a) (3, 3, 3) (1, 1, 1) (b) (3, 2, 3) (1, 1, 1)
(c) (3, 2, 3) (1, 1, 2) (d) (2, 3, 3) (2, 1, 1)

2. Total number of common tangents of $y^2 = 4ax$ and $xy = c^2$, is equal to

(a) 1 (b) 2 (c) 3 (d) 4

3. Let $f : R \to R$ be a function defined by $f(x) = [x]^2 + [x + 1] - 3$ {where [] denotes greatest integer function}, then $f(x)$ is

(a) many-one into (b) many-one onto
(c) one-one into (d) one-one onto

4. If $2^{3n} - \alpha n - \beta$ is divisible by 49, then (α, β) is, $n \in N$

(a) (–7, –1) (b) (7, 1) (c) (49, 1) (d) (49, 7)

5. The length of projection of the line segment joining the points (1, –1,0) and (–1, 0, 1) to the plane $2x + y + 6z = 1$ is equal to

(a) $\sqrt{\frac{255}{41}}$ (b) $\sqrt{\frac{237}{41}}$ (c) $\sqrt{\frac{137}{41}}$ (d) $\sqrt{\frac{155}{41}}$

6. $\sum_{r=1}^{15} \frac{r2^r}{(r+2)!}$ is equal to

(a) $\frac{17!-2^{16}}{17!}$ (b) $\frac{18!-2^{17}}{18!}$
(c) $\frac{16!-2^{15}}{16!}$ (d) $\frac{15!-2^{14}}{15!}$

7. There are n locks and n matching keys. If all the locks and keys are to be perfectly matched, then maximum number of trials is equal to

(a) ${}^nC_2$ (b) ${}^{n-1}C_2$
(c) ${}^{n+1}C_2$ (d) none of these

8. Maximum length of the chord of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ such that eccentric angles of its extremities differ by $\frac{\pi}{2}$ is $(a > b)$

(a) $a\sqrt{2}$ (b) $b\sqrt{2}$
(c) $ab\sqrt{2}$ (d) none of these

9. The number of ordered pairs (m, n) $(m, n \in \{1, 2,...., 20\})$ such that $3^m + 7^n$ is a multiple of 10, is

(a) 100 (b) 200
(c) 4! × 4! (d) none of these

10. If the length of the tangent drawn from a variable point to one given circle is k $(\neq 1)$ times the length of the tangent from it to another circle, then the locus of the variable point is

(a) an ellipse (b) a parabola
(c) a circle (d) a hyperbola

SOLUTIONS

1. (b) : The first line $x = y + 1 = z = \mu$ gives general point as $(\mu, \mu - 1, \mu)$. Second line $x + 1 = 2y = 2z$ gives general point as $(1 - \lambda, \lambda/2, \lambda/2)$. The ratios of direction ratios of line joining points of intersection and direction cosines proportional to 2, 1, 2 are same

$$\frac{\mu-\lambda+1}{2}=\frac{\mu-1-\frac{\lambda}{2}}{1}=\frac{\mu-\frac{\lambda}{2}}{2}$$

$$\mu-\lambda+1=2\mu-2-\lambda=\mu-\frac{\lambda}{2}$$

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Solving, we get $\mu = 3$, $\lambda = 2$
Coordinates are (3, 2, 3) (1, 1, 1).

2. **(a) :** Any tangent of $xy = c^2$ is $\frac{x}{t} + yt = 2c$

i.e., $y = -\frac{x}{t^2} + \frac{2c}{t}$

Comparing it with $y = mx + \frac{a}{m}$, we get

$$\frac{1}{1} = \frac{mt^2}{-1} = \frac{at}{2cm} \Rightarrow m = -\frac{1}{t^2}, m = \frac{at}{2c}$$

$$\Rightarrow \frac{at}{2c} = -\frac{1}{t^2} \Rightarrow t = -\left(\frac{2c}{a}\right)^{1/3}$$

So, there is only one common tangent.

3. **(a) :** $f(x) = [x]^2 + [x] - 2 = ([x] + 2)([x] - 1)$
$f(x) = 0$ for $x \in [1, 2)$ and $x \in [-2, -1]$
So $f(x)$ is many one. As $f(x)$ will take only integral values, so it is into.

4. **(b) :** $8^n - \alpha n - \beta = (1 + 7)^n - \alpha n - \beta$
$= 1 + 7n + {}^nC_2 7^2 + {}^nC_3 7^3 + \ldots + {}^nC_n 7^n - \alpha n - \beta$
$= (1 - \beta) + (7 - \alpha)n + 49\,({}^nC_2 + {}^nC_3 7 + \ldots\ldots)$
$\therefore$ If $8^n - \alpha n - \beta$ is divisible by 49, then
$1 - \beta = 0$, $7 - \alpha = 0$ $\therefore \beta = 1, \alpha = 7$.

5. **(b) :** $A \equiv (1, -1, 0), B \equiv (-1, 0, 1)$
Direction ratios of segment AB are 2, −1, −1
If θ be the acute angle between segment AB and normal to plane $\cos\theta = \frac{|2\times2 - 1\times1 - 1\times6|}{\sqrt{4+1+36}\sqrt{4+1+1}} = \frac{3}{\sqrt{246}}$
Length of projection

$$= AB\sin\theta = \sqrt{6}\sqrt{1 - \frac{9}{246}} = \sqrt{\frac{237}{41}} \text{ units.}$$

6. **(a) :** $\frac{r2^r}{(r+2)!} = \frac{(r+2-2)2^r}{(r+2)!}$

$$= \frac{2^r}{(r+1)!} - \frac{2^{r+1}}{(r+2)!} = -\left(\frac{2^{r+1}}{(r+2)!} - \frac{2^r}{(r+1)!}\right)$$

$$\Rightarrow \sum_{r=1}^{15} \frac{r2^r}{(r+2)!} = -\left(\frac{2^{16}}{17!} - \frac{2}{2!}\right) = 1 - \frac{2^{16}}{17!}.$$

7. **(c) :** For the first key, maximum number of trials needed is n. For second key, it will be $(n - 1)$
In general, for r^{th} key, maximum number of trials needed is $(n - r + 1)$, then total number of trials needed

$$= n + (n-1) + (n-2) + \ldots\ldots + 1 = \frac{n(n+1)}{2} = {}^{n+1}C_2.$$

8. **(a) :** Let the extremities of the chord be
$P_1 \equiv (a\cos\theta, b\sin\theta)$ and $P_2 \equiv (-a\sin\theta, b\cos\theta)$
Now $P_1P_2^2 = a^2(\cos\theta + \sin\theta)^2 + b^2(\sin\theta - \cos\theta)^2$

$$\Rightarrow P_1P_2^2 = a^2 + b^2 + (a^2 - b^2)\sin 2\theta \le a^2 + b^2 + a^2 - b^2$$

$$\Rightarrow P_1P_2^2 = 2a^2 \Rightarrow P_1P_2 \le a\sqrt{2}.$$

9. **(a) :** The last digit of any power of 3 can be 3, 9, 7, 1. Similarly last digit of any power of 7 can be 7, 9, 3, 1
$\Rightarrow$ Total number of ways $= 5 \times 5 + 5 \times 5 + 5 \times 5 + 5 \times 5 = 100$.

10. **(c) :** $\sqrt{S_1} = k\sqrt{S_2} \Rightarrow S_1 = k^2 S_2$
So locus is
$x^2 + y^2 + 2g_1x + 2f_1y + c_1 = k^2(x^2 + y^2 + 2g_2x + 2f_2y + c_2)$
$\Rightarrow (x^2 + y^2)(1 - k^2) + 2x(g_1 - k^2g_2) + 2y(f_1 - k^2f_2) + c_1 - k^2c_2 = 0$
which is a circle. ■

47. In the binomial expansion of $(a + bx)^{-2}$, $|x| < \frac{a}{b}$; $a, b > 0$, is $\frac{1}{4} - 3x + ...$, then

(a) $a = 2, b = 12$ (b) $a = 12, b = 2$
(c) $a = 2, b = 8$ (d) none of these

48. The number of permutations of n distinct objects taken $r (< n - 3)$ at a time which exclude $3 (< n)$ particular objects is

(a) $3!\,P(n, r - 3)$ (b) $P(n, 3)\,P(n, r - 3)$
(c) $P(r, 3)\,P(n, r - 3)$ (d) $P(n - 3, r)$

49. If corresponding to every positive integer n, there are m consecutive positive integers, none of which is prime, then $m =$

(a) $n - 1$ (b) $n + 1$ (c) $\underline{|n}$ (d) n

50. A convex polygon of n sides has twice as many diagonals as the number of sides. The value of n is

(a) 5 (b) 6 (c) 7 (d) 8

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1. Ⓐ Ⓑ Ⓒ Ⓓ	11. Ⓐ Ⓑ Ⓒ Ⓓ	21. Ⓐ Ⓑ Ⓒ Ⓓ	31. Ⓐ Ⓑ Ⓒ Ⓓ	41. Ⓐ Ⓑ Ⓒ Ⓓ
2. Ⓐ Ⓑ Ⓒ Ⓓ	12. Ⓐ Ⓑ Ⓒ Ⓓ	22. Ⓐ Ⓑ Ⓒ Ⓓ	32. Ⓐ Ⓑ Ⓒ Ⓓ	42. Ⓐ Ⓑ Ⓒ Ⓓ
3. Ⓐ Ⓑ Ⓒ Ⓓ	13. Ⓐ Ⓑ Ⓒ Ⓓ	23. Ⓐ Ⓑ Ⓒ Ⓓ	33. Ⓐ Ⓑ Ⓒ Ⓓ	43. Ⓐ Ⓑ Ⓒ Ⓓ
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6. Ⓐ Ⓑ Ⓒ Ⓓ	16. Ⓐ Ⓑ Ⓒ Ⓓ	26. Ⓐ Ⓑ Ⓒ Ⓓ	36. Ⓐ Ⓑ Ⓒ Ⓓ	46. Ⓐ Ⓑ Ⓒ Ⓓ
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9. Ⓐ Ⓑ Ⓒ Ⓓ	19. Ⓐ Ⓑ Ⓒ Ⓓ	29. Ⓐ Ⓑ Ⓒ Ⓓ	39. Ⓐ Ⓑ Ⓒ Ⓓ	49. Ⓐ Ⓑ Ⓒ Ⓓ
10. Ⓐ Ⓑ Ⓒ Ⓓ	20. Ⓐ Ⓑ Ⓒ Ⓓ	30. Ⓐ Ⓑ Ⓒ Ⓓ	40. Ⓐ Ⓑ Ⓒ Ⓓ	50. Ⓐ Ⓑ Ⓒ Ⓓ

Answer sheet other than the one printed here will not be accepted. The answer sheet may be photocopied. Mark your Answers with blue/black ball point pen.

Math Archives

10 Best Problems

Prof. Shyam Bhushan*

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE.

1. Let

$$p=\left(1+\cos\frac{\pi}{10}\right)\left(1+\cos\frac{3\pi}{10}\right)\left(1+\cos\frac{7\pi}{10}\right)\left(1+\cos\frac{9\pi}{10}\right)$$

and

$$p=\left(1+\cos\frac{\pi}{8}\right)\left(1+\cos\frac{3\pi}{8}\right)\left(1+\cos\frac{5\pi}{8}\right)\left(1+\cos\frac{7\pi}{8}\right)$$

then

(a) $p = q$ (b) $2p = q$
(c) $p = 2q$ (d) $p + q = 1/4$

2. Let $2x - 3y = 0$ be a given line and $P(\sin\theta, 0)$ and $Q(0, \cos\theta)$ be two points. Then P and Q lie on the same side of the given line if θ lies in the

(a) 1st quadrant (b) 2nd quadrant
(c) 3rd quadrant (d) 4th quadrant

3. If $\sin^4 x\cos 3x=\sum_{k=0}^{n} a_k \cos kx$. Then the value of n and all the a_k's

(a) 5 (b) 6
(c) 7 (d) none of these

4. Let $f: R \to R$ be a function defined by,

$$f(x)=\frac{2x^2-x+5}{7x^2+2x+10}\ \text{the } f \text{ is}$$

(a) injective but not surjective
(b) surjective but not injective
(c) injective as well as surjective
(d) neither injective nor surjective

5. If α and β are two distinct roots of the equation $a\tan x + b\sec x = c$, then $\tan(\alpha + \beta)$ is equal to

(a) $\dfrac{ac}{a^2+c^2}$ (b) $\dfrac{2ac}{a^2+c^2}$

(c) $\dfrac{2ac}{a^2-c^2}$ (d) none of these

6. If $g(\theta)=\sin^2\left(\theta+\frac{\pi}{3}\right)+\cos\theta.\cos\left(\theta+\frac{\pi}{3}\right)$ and $f\left(\frac{5}{4}\right)=1$ then find $(fog)(x)$

7. If $\{x\}$ and $[x]$ denote the fractional and integral parts of a real number x, respectively then solve $2x + \{x + 1\} = 4[x + 1] - 6$

8. Find the domain of the following functions

(a) $\dfrac{\cos^{-1}x}{[x]}$ wher [.] denotes the greatest integer function

(b) $\dfrac{1}{x}+2^{\sin^{-1}x}+\dfrac{1}{\sqrt{x-2}}$

9. Evaluate $\displaystyle Lt_{x\to 0}\frac{1-\cos x\sqrt{\cos 2x}}{x^2}$ (without using L 'Hospital' Rule)

10. $$f(x)=\begin{cases}(1+|\sin x|)^{\frac{p}{|\sin x|}}; & -\frac{\pi}{6}<x<0\\ q & x=0\\ e^{\frac{\tan 3x}{\tan 5x}}; & 0<x<\frac{\pi}{6}\end{cases}$$

is continuous at $x=0$ find the values of p and q.

By : **Prof. Shyam Bhushan, Resonance, Kota.** Mobile: 09928390590

SOLUTIONS

1. **(b) :** $p=\left(\sin\frac{\pi}{10}\sin\frac{3\pi}{10}\right)^2=\frac{1}{16}$ and

$q=\left(\sin\frac{\pi}{8}\sin\frac{3\pi}{8}\right)^2=\frac{1}{8}$. Hence, $q = 2p$.

2. **(b) :** $L \equiv 2x - y; L(P) . L(Q) > 0$
$L(\sin\theta, 0) . L(0, \cos\theta) > 0$
$\sin\theta . \cos\theta < 0 \Rightarrow \sin 2\theta < 0 \therefore \pi/2 < \theta < \pi$.

3. **(c) :** $n = 7$; $a_1 = -3/16$, $a_3 = 3/8$, $a_5 = -1/4$, $a_7 = 1/16$ & $a_k = a_2 = a_4 = a_6 = 0$.

4. **(d)**

5. **(c) :** $(a^2 - b^2)\tan^2 x - 2ac\tan x + (c^2 - b^2) = 0$

$\therefore \tan\alpha + \tan\beta = \frac{2ac}{a^2-b^2}$

$\tan\alpha\tan\beta=\frac{c^2-b^2}{a^2-b^2}$ Hence, $\tan(\alpha+\beta)=\frac{2ac}{a^2-c^2}$.

6. $g(\theta)=\sin^2\theta+\sin^2\left(\theta+\frac{\pi}{3}\right)+\cos\theta.\cos\left(\theta+\frac{\pi}{3}\right)$

$=\frac{1}{2}\left[1-\cos 2\theta+1-\cos\left(2\theta+\frac{2\pi}{3}\right)+\cos\left(2\theta+\frac{\pi}{3}\right)+\cos\frac{\pi}{3}\right]$

$=\frac{1}{2}\left[\frac{5}{2}-2\cos\left(2\theta+\frac{\pi}{3}\right)\cos\frac{\pi}{3}+\cos\left(2\theta+\frac{\pi}{3}\right)\right]=\frac{5}{4}\forall\theta$

$\therefore (fog)(x) = f[f(x)] = f\left(\frac{5}{4}\right) \qquad \left(\because f\left(\frac{5}{4}\right)=1\right)$
$= 1$

7. $2x + \{x + 1\} = 4[x + 1] - 6$
$\Rightarrow 2x + x + 1 - [x] - 1 = 4[x] - 2$
$(\because [x + n] = [x] + n; n \in I)$
$\Rightarrow 5[x] = 3x + 2$ ...(1)
$= 3([x] + \{x\}) + 2$
$\Rightarrow 3\{x\} = 2[x] - 2$ ...(2)
Now $0 \le \{x\} < 1 \Rightarrow 0 \le 3\{x\} < 3$
$\Rightarrow 0 \le 2[x] - 2 < 3 \Rightarrow 2 \le 2[x] < 5$
$\Rightarrow 1 \le [x] < 5/2 \Rightarrow [x] = 1, 2$
$\Rightarrow [x] = 1 \Rightarrow x = 1$ and $[x] = 2 \Rightarrow x=\frac{8}{3}$ ($\because$ from (1))

8. **(a) :** $D_{\cos^{-1}x}=[-1,1]$, $D_{[x]} = R$

$\therefore D_{\frac{\cos^{-1}x}{[x]}}=[-1,1]\cap R-\{x\,|\,[x]=0\} = [-1, 0) \cup \{1\}$

(b) $D_{\frac{1}{x}+2^{\sin^{-1}x}+\frac{1}{\sqrt{x-2}}} = R - \{0\}\cap[-1,1]\cap(2,\infty) = \phi$

$\therefore f(x)$ is not defined for any $x \in R$.

9. $\underset{x\to 0}{\text{Lt}}\frac{1-\cos x.\sqrt{\cos 2x}}{x^2}=\underset{x\to 0}{\text{Lt}}\frac{1-\cos^2 x.\cos 2x}{x^2\left(1+\cos x.\sqrt{\cos 2x}\right)}$

$=\underset{x\to 0}{\text{Lt}}\frac{1-\cos^2 x(2\cos^2 x-1)}{x^2\left(1+\cos x.\sqrt{\cos 2x}\right)}$

$=-\underset{x\to 0}{\text{Lt}}\frac{2\cos^4 x-\cos^2 x-1}{x^2\left(1+\cos x\sqrt{\cos 2x}\right)}$

$=\underset{x\to 0}{\text{Lt}}\left(\frac{2\cos^2 x+1}{1+\cos x\sqrt{\cos 2x}}\right)\left(\frac{\sin^2 x}{x^2}\right)=\frac{3}{2}$

10. Since f is continuous at $x = 0$

$\therefore \underset{x\to 0-}{\text{Lt}}f(x)=f(0)=\underset{x\to 0+}{\text{Lt}}f(x)$

$\Rightarrow e^p=q=e^{\frac{3}{5}} \Rightarrow p=\frac{3}{5},\ q=e^{\frac{3}{5}}$.

ANSWERS - CROSSWORD PUZZLE

ACROSS

2. Nested	4. Integer	7. Decade
8. Real	12. Ring	14. Radian
15. Bimodal	16. Degree	

DOWN

1. Circle	3. Ellipse	5. Zero
6. Common	9. Euclid	10. Concave
11. Surd	13. Couple	

A Discussion on Descartes' Rule of Signs

The statement: ***"We can determine also the number of true and false roots that any equation can have, as follows: An equation can have as many true roots as it contains changes of sign, from + to - or from - to +; and as many false roots as the number of times two + signs or two - signs are found in succession."*** is found in Descartes' revolutionary work, *La Geometrie*, but without hint of a proof. By the term "true roots" he means "positive roots" and by "false roots" he means "negative roots".

This statement is the basis of the attribution to Descartes of the proposition now known as **"Descartes' Rule of Signs"** which, in simpler words, can be said to be:

"The number of positive roots of a polynomial with real coefficients is equal to the number of "changes of sign" in the list of coefficients, or is less than this number by a multiple of 2." [Since the negative roots of the polynomial equation $f(x) = 0$ are positive roots of the equation $f(-x) = 0$, the rule can be readily applied to help count the negative roots as well.]

Before discussing on the rule, it may be helpful to examine some simple cases:

(1) If $f(x)$ is a linear polynomial (that is, of degree 1), say, $f(x) = mx + b$ then the only root of $f(x)$ is $x = -b/m$, which is positive only when m and b are of opposite sign.

(2) If $f(x)$ is a quadratic polynomial (that is, of degree 2), it is convenient to divide by the coefficient of the x^2 term, so as to obtain a new polynomial with leading coefficient equal to 1. This polynomial, say $f(x) = x^2 + bx + c$, shares the same roots, and the same pattern of variations in the signs of the coefficients, as the original polynomial. Suppose we have identified the two roots, say r and s. Then we may express $f(x)$ in factored form: $f(x) = (x - r)(x - s) = x^2 - (r + s)x + rs$. Compare coefficients $b = -(r + s)$ and $c = rs$. Thus c is positive only when r and s are of the same sign, that is, whenever $f(x)$ has two positive roots or two negative roots. But if c and x are both positive, the only way the expression $f(x) = x^2 + bx + c$ can equal zero is for b to be negative (in which case there are two variations in sign). Of course, according to the quadratic formula, the two roots are given by the formula $x = ½ [-b \pm \sqrt{(b^2 - 4c)}]$. And, if c is large enough, there will be no real roots at all. Thus, if $f(x)$ has two variations in sign, there will be either two positive real roots, or none at all. On the other hand, if c is negative, there will be one variation in sign (regardless of whether b is positive or negative), and there will be two real roots. Since $c = rs$, the roots will be of opposite sign – that is, there will be exactly one positive root. These arguments verify Descartes' Rule of Signs for linear and quadratic polynomials. Since there are no analogous formulas for the roots of polynomials of higher degree, a more systematic approach is needed in order to treat the general case. For further reference, let $f(x)$ is a polynomial, and let us denote the number of variations in sign of its coefficients by $V[f(x)]$, and denote the number of positive real roots of $f(x)$ by $P[f(x)]$. To simplify matters, we stipulate that in any polynomial $f(x)$, the leading coefficient (the coefficient of the highest power of the variable x in the polynomial) is never zero. Without loss of generality, we may assume our polynomial to have leading coefficient equal to 1, and that constant term a_0 in our polynomial is not zero; since other polynomials can be reduced to this kind by doing some simple mathematical steps (such as dividing by the leading coefficient, or by a suitable power of x) which will not change the pattern of variations in the signs of the coefficients.

Thus we may denote our polynomial in the form:

$f(x) = x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$

Now it is time to start counting the number of variations in the signs and number of roots. Begin with a simple observation:

I. Let $f(x) = x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$. If $a_0 < 0$, then $V[f(x)]$ is *odd*; if $a_0 > 0$, then $V[f(x)]$ is *even*.

In a two state system, if you change state an even number of times, you recover the original state. We can make a similar statement about the positive roots of $f(x)$:

II. Let $f(x) = x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$. If $a_0 < 0$, then $P[f(x)]$ is *odd*; if $a_0 > 0$, then $P[f(x)]$ is *even*.

We can give a simple proof *by induction* as follows:

We have already observed that this observation holds for polynomials of degree 1 and of degree 2.

Now suppose that **II** holds for polynomials of degree less than k. Consider a polynomial

$f(x) = x^k + a_{k-1} x^{k-1} + ... + a_1 x + a_0$, of degree k.

Suppose first that $a_0 < 0$. Then $f(0) = a_0 < 0$, while, for large enough values of x, the x^k term of the polynomial will dominate the others, so that for some positive value of x, $f(x) > 0$. Since every polynomial is a continuous function, we may conclude that $f(x)$ has a positive root, say at $x = p$. Then $(x - p)$ divides $f(x)$, and we may set $f(x) / (x - p) = g(x)$, where $g(x)$ is a monic polynomial of degree $k - 1$. Indeed, the constant term, say, b_0, of $g(x)$, must be positive (since $a_0 = -pb_0 < 0$). Applying the induction hypothesis, we know that $P[g(x)]$ is even. But $P[f(x)] = P[g(x) (x - p)] = P[g(x)] + 1$, so $P[f(x)]$ is odd. On the other hand, if $a_0 > 0$, we must consider two possibilities. If $P[f(x)] = 0$, then **II** holds, as 0 is an even number. If $P[f(x)] > 0$, then $f(x)$ has a positive root, say at $x = p$. Reasoning as in the previous case, the constant term of $g(x) = f(x) / (x - p)$ must be negative, and thus by the induction hypothesis, $P[g(x)]$ must be odd, and $P[f(x)]$ must be even. We may combine **I** and **II** in the form to say that if $f(x) = x^n + a_{n-1} x^{n-1} + ... + a_1 x + a_0$. Then $V[f(x)]$ and $P[f(x)]$ are both even or both odd – that is, $V[f(x)]$ and $P[f(x)]$ differ by an integer multiple of 2. ■■

by **Department of Mathematics**, MOMENTUM Jabalpur.

MATHS MUSING

SOLUTIONS (SET - 56)

1. (c) : If $CD = x$ and $DA = y$, then $x^2 + y^2 + xy = 19$
$xy = 6 \Rightarrow x + y = 5 \Rightarrow$ perimeter = 12.

2. (a) : The number are of the form p^9, pq^4, where p, q are primes
$3 \cdot 2^4 = 48$, $5 \cdot 2^4 = 80$, $7 \cdot 2^4 = 112$, $91 \cdot 2^4 = 176$
$2 \cdot 3^4 = 162$.

3. (c) : $x = e^t \Rightarrow I = \int_0^1 t^5 e^t dt$
$= e^t [t^5 - 5t^4 + 20t^3 - 60t^2 + 120t - 120]_0^1$
$= 120 - 44e \Rightarrow A = 120, B = -44 \quad \therefore \quad A + B = 76.$

4. (b) : The desired value is the distance of $(1, 3)$ from the point $(-1, 1)$ on the circle $|z - 2i| = \sqrt{2}$. Hence it is $2\sqrt{2}$.

5. (d) : $5 = f'\sin x - f\cos x\big|_0^{\pi} = 2 + f(0) \Rightarrow f(0) = 3$.

6. (d) : $x + \frac{1}{x} = t \Rightarrow t^2 - 3t - 4 = 0$
$t = 4, -1 \Rightarrow x = \omega, \omega^2, 2 \pm \sqrt{3}$.

7. (c) : The feet of normal at t_1, t_2, t_3 are given by $t^3 - t - 2 = 0 \Rightarrow \Sigma t_1 = 0$,
$\Sigma t_1 t_2 = -1$, $t_1 t_2 t_3 = 2$
centroid is $(-1/3, 0)$. Its distance from the focus is 4/3.

8. (d) : Orthocentre is $(-1, 2)$. Its distance from the focus is $2\sqrt{2}$.

9. (a) : Circumcentre is $(0, -1)$. Its distance from the focus is $\sqrt{2}$.

10. (a) - (q)
$-(1 + 2 + ... + 50) + 51^2 = 1326$
(b) - (s)
$1^3 + 2^3 + ... + 15^3 - 2(2^3 + 4^3 + ... + 14^3)$
$$= \left(\frac{15\times16}{2}\right)^2 - 16\left(\frac{7\times8}{2}\right)^2 = 1856$$
(c) - (p)
$1^2 + 2^2 + ... + 15^2 - (2^2 + 4^2 + ... + 14^2)$
$$= \frac{15\times16\times31}{6} - 4.\frac{7\times8\times15}{6} = 680$$
(d) - (r)
$1^3 + 2^3 + ... + 11^3 - (2^3 + 4^3 + ... + 10^3)$
$$= \left(\frac{11\times12}{2}\right)^2 - 8\left(\frac{5\times6}{2}\right)^2 = 2556$$

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Olympiad Enrichment Series-III

useful for IIT-JEE 2008-09

This series is selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solution.

1. Find the least positive integer m such that

$$\binom{2n}{n}^{\frac{1}{n}} < m$$

for all positive integers n.

2. Let a, b, c, d and e be positive integers such that $abcde = a + b + c + d + e$.

Find the maximum possible value of max $\{a, b, c, d, e\}$.

3. Evaluate

$$\frac{3}{1!+2!+3!}+\frac{4}{2!+3!+4!}+\ldots+\frac{2001}{1999!+2000!+2001!}$$

4. Let $x=\sqrt{a^2+a+1}-\sqrt{a^2-a+1},\ a\in\mathbb{R}$

Find all possible values of x.

5. Find all real numbers x for which $10^x + 11^x + 12^x = 13^x + 14^x$.

SOLUTIONS

1. Note that

$$\binom{2n}{n}<\binom{2n}{0}+\binom{2n}{1}+\ldots+\binom{2n}{2n}=(1+1)^{2n}=4^n$$

and for $n = 5$, $\binom{10}{5}=252>3^5$. Thus $m = 4$.

2. Suppose that $a \le b \le c \le d \le e$. We need to find the maximum value of e. Since $e < a + b + c + d + e \le 5e$, then $e < abcde \le 5e$, i.e. $1 < abcd \le 5$

Hence $(a, b, c, d) = (1, 1, 1, 2), (1, 1, 1, 3), (1, 1, 1, 4), (1, 1, 2, 2)$, or $(1, 1, 1, 5)$, which leads to max $\{e\} = 5$.

Alternative Solution : As before, suppose that $a \le b \le c \le d \le e$ Note that

$$1=\frac{1}{bcde}+\frac{1}{cdea}+\frac{1}{deab}+\frac{1}{eabc}+\frac{1}{abcd}$$

$$\le\frac{1}{de}+\frac{1}{de}+\frac{1}{de}+\frac{1}{e}+\frac{1}{d}=\frac{3+d+e}{de}$$

Therefore, $de \le 3 + d + e$ or $(d - 1)(e - 1) \le 4$

If $d = 1$, then $a = b = c = 1$ and $4 + e = e$, which is impossible.

Thus $d - 1 \ge 1$ and $e - 1 \le 4$ or $e \le 5$

It is easy to see that $(1, 1, 1, 2, 5)$ is a solution

Therefore max $\{e\} = 5$.

3. Note that

$$\frac{k+2}{k!+(k+1)!+(k+2)!}=\frac{k+2}{k![1+k+1+(k+1)(k+2)]}$$

$$=\frac{1}{k!(k+2)}=\frac{k+1}{(k+2)!}=\frac{(k+2)-1}{(k+2)!}$$

$$=\frac{1}{(k+1)!}-\frac{1}{(k+2)!}$$

By telescoping sum, the desired value is equal to

$$\frac{1}{2}-\frac{1}{2001!}$$

4. Since $\sqrt{a^2+|a|+1}>|a|$

and $x=\dfrac{2a}{\sqrt{a^2+a+1}+\sqrt{a^2-a+1}}$,

we have $|x| < |2a/a| = 2$

Squaring both sides of $x+\sqrt{a^2-a+1}=\sqrt{a^2+a+1}$

yields $2x\sqrt{a^2-a+1}=2a-x^2$

Squaring both sides of the above equation gives

$$4(x^2-1)a^2=x^2(x^2-4) \text{ or } a^2=\frac{x^2(x^2-4)}{4(x^2-1)}$$

Since $a^2 \ge 0$, we must have

$$x^2(x^2-4)(x^2-1)\ge 0,$$

Since $|x| < 2$, $x^2 - 4 < 0$ which forces $x^2 - 1 < 0$. Therefore, $-1 < x < 1$. Conversely, for every $x \in (-1, 1)$ there exists a real number a such that

$$x=\sqrt{a^2+a+1}-\sqrt{a^2-a+1}$$

Alternative Solution : Let $A = (-1/2, \sqrt{3}/2)$, $B = (1/2, \sqrt{3}/2)$, and $P = (a, 0)$. Then P is a point on the x-axis and we are looking for all possible value of $d = PA - PB$.

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CHINESE Olympiad Problems

SECTION - 1

1. Consider the following two statements about two positive integers p and q

P : The number $p^3 - q^3$ is even.

Q : The number $p + q$ is even.

Which of the following statements is true?

(a) P is necessary and sufficient for Q.
(b) P is necessary but not sufficient for Q.
(c) P is sufficient but not necessary for Q.
(d) P is neither necessary nor sufficient for Q.

2. In which of the following intervals does $\dfrac{1}{\log_{\frac{1}{2}} \frac{1}{3}} + \dfrac{1}{\log_{\frac{1}{5}} \frac{1}{3}}$ lie?

(a) $(-2, -1)$ (b) $(1, 2)$
(c) $(-3, -2)$ (d) $(2, 3)$

3. In triangle ABC, $AB = AC$. Both BC and the altitude AD from A to BC are integers. Which of the following statements is true?

(a) one of $\sin A$ and $\cos A$ is rational and the other irrational.
(b) both $\sin A$ and $\cos A$ are rational
(c) both $\sin A$ and $\cos A$ are irrational
(d) the rationality of $\sin A$ and $\cos A$ depends on the values of BC and AD.

4. Let $M = \{(x, y) \mid y \geq x^2\}$ and
$N = \{(x, y) \mid x^2 + (y - a)^2 \leq 1\}$.
What is a necessary and sufficient condition for $M \cap N = N$?

(a) $a \geq 5/4$ (b) $a = 5/4$
(c) $a \geq 1$ (d) $0 < a < 1$.

5. The function $f(x) = ax^2 - c$ satisfies $-4 \leq f(1) \leq -1$ and $-1 \leq f(2) \leq 5$. Which of the following statements is true?

(a) $7 \leq f(3) \leq 26$ (b) $-4 \leq f(3) \leq 15$
(c) $-1 \leq f(3) \leq 20$ (d) $-\dfrac{28}{3} \leq f(3) \leq \dfrac{35}{3}$

6. Let $P = \sqrt{ab} + \sqrt{cd}$ and $Q = \sqrt{ma + nc}\sqrt{\dfrac{b}{m} + \dfrac{d}{n}}$, where a, b, c, d, m and n are positive real numbers. Which of the following statements is true?

(a) $P \geq Q$ (b) $P \leq Q$
(c) $P < Q$
(d) the relative size of P and Q depend on the values of m and n.

7. P is a point on the plane of the square $ABCD$ such that each of PAB, PBC, PCD and PDA is an isosceles triangle. How many possible positions are there for such a point P?

(a) 9 (b) 17 (c) 1 (d) 5

8. Let ℓ, R and r be the perimeter, circumradius and inradius, respectively, of an arbitrary triangle. Which is the following statements is true?

(a) $\ell > R + r$ (b) $\ell \leq R + r$
(c) $\dfrac{\ell}{6} < R + r < 6\ell$
(d) none of (a), (b) and (c) is true

SECTION - 2

1. In triangle ABC, $\sin A = \dfrac{3}{5}$ and $\cos B = \dfrac{5}{13}$. What is the value of $\cos C$?

2. How many triangles are there such that each side has integral length and the longest side has length 11?

3. A function $f(x)$ defined on the interval $[-1, 1]$ satisfies $f(0) = f(1)$. If for any x_1 and x_2 in $[-1, 1]$, we have $|f(x_2) - f(x_1)| < |x_2 - x_1|$, prove that
$$|f(x_2) - f(x_1)| < 1/2$$

SOLUTIONS

SECTION - 1

1. Both statements are saying that p and q have the same parity. Hence they are equivalent to each other.

2. The given expression is equal to

$$\log_{\frac{1}{3}} \frac{1}{2} + \log_{\frac{1}{3}} \frac{1}{5} = \log_{\frac{1}{3}} \frac{1}{10} = \log_3 10$$

Now $2 = \log_3 9 < \log_3 10 < \log_3 27 = 3$.

3. Note that $\tan\frac{A}{2}, \frac{BD}{AD}, \frac{BC}{2AD}$ is rational. Since

$$\sin A = \frac{2\tan\frac{A}{2}}{1+\tan^2\frac{A}{2}} \text{ and } \cos A = \frac{1-\tan^2\frac{A}{2}}{1+\tan^2\frac{A}{2}},$$ both $\sin A$ and $\cos A$ are rational.

4. The desired condition is to have the circle $x^2 + (y-a)^2 = 1$ inside the convex region of the parabola $y = x^2$. In other words, $1 \le y + (y-a)^2$ or $y^2 + (1-2a)y + (a-1) \le 0$. This occurs when the discriminant is negative. From $(1-2a)^2 - 4(a-1) \le 0$, we have $a \ge \frac{5}{4}$.

5. We have $f(1) = a - c$ and $f(2) = 4a - c$. Hence $a = \frac{1}{3}(f(2) - f(1))$ and $c = \frac{1}{3}(f(2) - 4f(1))$. It follows that $f(3) = 9a - c = \frac{1}{3}(8f(2) - 5f(1))$ and

$$-1 = \frac{1}{3}(8(-1) - 5(-1)) \le f(3) \le \frac{1}{3}(8(5) - 5(-4)) = 20.$$

6. We have

$$Q^2 = ab + cd + \frac{nbc}{m} + \frac{mad}{n} \ge ab + cd + 2\sqrt{abcd} = P^2$$ by the Arithmetic Mean – Geometric Mean Inequality. We have $Q \ge P$ since both are positive.

7. The points P which makes at least one of those four triangles isosceles lie on two lines joining the midpoints of opposite sides of the squares, and four circles centered at the vertices and passing through adjacent vertices. Not counting the vertices themselves which would produce degenerate triangles, there are 9 positions of P which make all four triangles isosceles. One is the centre of the square. The other 8 are the points of triple intersection of two of the circles and one of the lines.

8. On a circle of radius R, choose, A, B and C sufficiently close to each others so that $R > 6\ell$. Then $R > 6\ell - r > \ell - r$. Alternatively, choose B and C sufficiently close together and A almost diametrically opposite to them. Then $\ell > AB + AC > 2R > R + r$. Hence none of (a), (b) and (c) is true.

SECTION - 2

1. We have $\cos A = \pm\sqrt{1-\sin^2 A} = \pm\frac{4}{5}$ and

$$\sin B = 4\sqrt{1-\cos^2 B} = \frac{12}{13}$$

Note that we have

$$\sin C = \sin(A+B) = \sin A\cos B + \cos A \sin B = \frac{3}{13} \pm \frac{48}{65}$$

Since $\sin C > 0$, we must have

$$\cos A = \frac{4}{5}$$

Hence

$$\cos C = -\cos(A+B) = \sin A \sin B - \cos A \cos B = \frac{16}{65}$$

2. Let the three sides be $a \le b \le c \le 11$. Then $6 \le b \le 11$ and $c - b < a \le b$. As b decreases by 1, the range of a decreases by 2. When $b = 11$, we have $1 \le a \le 11$. Hence the total number of triangles is $11 + 9 + 7 + 5 + 3 + 1 = 36$.

3. We may assume that $0 \le x_1 < x_2 \le 1$. Then $|f(x_2) - f(x_1)| < x_2 - x_1$.
Since $f(0) = f(1)$,
$|f(x_2) - f(x_1)| \le |f(x_2) - f(1)| + |f(0) - f(x_1)| < 1 - x_2 + x_1$.
Addition yields

$$|f(x_2) - f(x_1)| < \frac{1}{2}.$$

■■

Contd. from page no. 78

By the Triangle Inequality, $|PA - PB| < |AB| = 1$. And it is clear that all the values $-1 < d < 1$ are indeed obtainable. In fact, for such a, d, a half hyperbola of all point Q such that $QA - QB = d$ is well defined. (Point A and B are foci of the hyperbola).
Since line AB is parallel to the x-axis, this half hyperbola intersects the x-axis, i.e., P is well defined.

5. It is easy to check that $x = 2$ is a solution. We claim that it is the only one. In fact, dividing by 13^x on both sides gives

$$\left(\frac{10}{13}\right)^x + \left(\frac{11}{13}\right)^x + \left(\frac{12}{13}\right)^x = 1 + \left(\frac{14}{13}\right)^x$$

The left hand side is a decreasing function of x and the right hand side is an increasing function of x.
Therefore their graphs can have at most one point of intersection.

■■

Duplex (3) = 9, 9 × 2 = 18, 18 ÷ 5 gives quotient = 3 & remainder = 3

We write in the table $\bar{3}_3$, in the second row

Step 2. Duplex $(3, \bar{3}) = \bar{1}\bar{8}$, $\bar{1}\bar{8} \times 2 = \bar{3}\bar{6}$, $\bar{3}\bar{6} + 30 = \bar{6}$ *(for second row)*

$\bar{6} \div 5$ gives Quotient = 0 & remainder = $\bar{6}$.

We write in the table $0_{\bar{6}}$ in the second row.

Step 3. 3 × 3 = 9, 9 ÷ 2 gives quotient = 4 & remainder = 1 *(for third row)*

We write in the table 4_1 in the third row.

Step 4. Duplex $(3, \bar{3}, 4) = 33, 33 \times 2 = 66, 66 + \bar{6}\bar{0} = 6$ *(for second row)*

6 ÷ 5 gives Quotient = 0 & remainder = 6

We write in the table 0_6

Step 5. $\begin{pmatrix} 3 & 0 \\ 3 & \bar{3} \end{pmatrix} = \bar{9}, \bar{9} + 10 = 1$, 1 ÷ 2 gives Quotient = 1 & remainder = $\bar{1}$ *(for third row)*

We write in the table $1_{\bar{1}}$ in the third row.

Step 6. Duplex $(3, \bar{3}, 4, 1) = \bar{1}\bar{8}$, $\bar{1}\bar{8} \times 2 = \bar{3}\bar{6}$, $\bar{3}\bar{6} + 60 = 24$, *(for second row)*

24 ÷ 5 gives Quotient = 5 & remainder = $\bar{1}$.

We write in the table $\bar{5}_{\bar{1}}$, in the second row.

Step 7. $\begin{pmatrix} 3 & 0 & 0 \\ 3 & \bar{3} & 4 \end{pmatrix} = 12, 12 + \bar{1}\bar{0} = 2, 2 \div 2$ gives Quotient = 2 & remainder = $\bar{2}$ *(for third row)*

We write in the table $2_{\bar{2}}$ in the third row.

Thus, $x_1 = 0\,.\,3\bar{3}41\bar{3} = 0.27407$.

We can find the second root using transformed equation

$$3x^3 - 10x^2 - 4x + 2 = 0$$

This equation has roots which are reciprocals of the roots of the given equation.

Quadratic Equations : [General Method]

In finding square root of a number a we are actually solving quadratic equation $x^2 - a = 0$. In Vedic method of finding square root , we use $2x$ as divisor in the process of finding square root. Note that $2x$ is derivative of x^2 .

In solving quadratic equation $x^2 + bx + c = 0$ we use $(2x + b)$ as divisor.

Example 5. : Consider the equation $x^2 + x - 14 = 0$

$$x = \frac{-1 \pm \sqrt{57}}{2} = -0.5 \pm \frac{1}{2}\sqrt{57} = 3.274917218$$

or -4.274917218

We write $x^2 + x = 14$ and Note the following :

When $x = 1$, L. H. S. = 2

When $x = 2$, L. H. S. = 6

When $x = 3$, L. H. S. = 12

When $x = 4$, L. H. S. = 20

i.e. when $x = 3$, $x^2 + x$ is very close to 14. We consider 3 as our first digit in the answer and $\frac{d}{dx}(x^2 + x) = 2x + 1$ at $x = 3$ giving (2) (3) + 1 = 7 as the divisor. We have the solution as follows :

$$\begin{array}{ccccccccc} 14\,\cdot & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 7 & 2 & 6 & 0 & 3 & \bar{7} & 0 & \bar{3} & \bar{8} \\ \hline & 3\,\cdot & 2 & 8 & \bar{5} & \bar{1} & 2 & \bar{2} & \bar{8} \end{array}$$

Step 1. 20 ÷ 7 gives Quotient = 2 & remainder = 6.

Step 2. D(2) = 4, 60 – 4 = 56, 56 ÷ 7 gives Quotient = 8 & remainder = 0 .

Step 3. D(2, 8) = 32, $\bar{3}\bar{2} \div 7$ gives Quotient = $\bar{5}$ & remainder = 3

Step 4. D(2, 8, $\bar{5}$) = 44, 30 – 44 = $\bar{1}\bar{4}$, $\bar{1}\bar{4} \div 7$ gives Quotient = $\bar{1}$ & remainder = $\bar{7}$.

Step 5. D(2, 8, $\bar{5}$, $\bar{1}$) = $\bar{8}\bar{4}$, $\bar{7}\bar{0} - \bar{8}\bar{4} = 14$, 14 ÷ 7 gives Quotient = 2 & remainder = 0 .

Step 6. D(2, 8, $\bar{5}$, $\bar{1}$, 2) = 17, $\bar{1}\bar{7} \div 7$ gives Quotient = $\bar{2}$ & remainder = $\bar{3}$

Step 7. D(2, 8, $\bar{5}$, $\bar{1}$, 2, $\bar{2}$) = $\bar{3}\bar{4}$, $\bar{3}\bar{0} - \bar{3}\bar{4} = \bar{6}\bar{4}$, $\bar{6}\bar{4} \div 7$ gives Quotient = $\bar{8}$ & remainder = $\bar{8}$.

Required root is $3.28\bar{5}\bar{1}2\bar{2}\bar{8} = 3.2749172$

In the next article we shall solve different types of examples using the above procedure.

to be continued....

Mathematica Function

1. Domain of $f(x) = {}^{P(x)}C_{Q(x)}$, where $P(x) = 19x - 9 - 2x^2$ & $Q(x) = \sqrt{x-4}$ is
(a) $[4, 9]$ (b) $[0, 9]$
(c) $\{1, 2, 3, ..., 8\}$ (d) $\{4, 5, 8\}$

2. Range of the function $f(x) = \sin^4 x\,(1 + \sin^2 x) + \cos^4 x\,(1 + \cos^2 x)$ is
(a) $[0, 2]$ (b) $[0, 3/4]$ (c) $[3/4, 2]$ (d) R

3. If $f(x + 10) + f(x + 4) = 0$, then $f(x)$ is a periodic function with period
(a) 2 (b) 4 (c) 6 (d) 12

4. $f(x) = ax^3 + bx^2 + cx + d$ $(a \neq 0)$ is one-one onto function, if
(a) $b^2 - 4ac < 0$ (b) $b^2 - 3ac < 0$
(c) $b^2 - 4ac > 0$ (d) $b^2 - 3ac > 0$

5. Domain of $f(x) = \dfrac{1}{\sqrt{|[|x|-1]|-5}}$ (where $[\cdot]$ denotes the greatest integer function), is
(a) $(-\infty, \infty)$ (b) $(-4, 4)$
(c) $(-\infty, -7] \cup [7, \infty)$ (d) $(-\infty, -4] \cup [4, \infty)$

6. If $f: R \to R$ & $f(x) = ax + \sin x + a$, then
(a) $f(x)$ is one-one onto function If $a \in R$
(b) $f(x)$ is one-one onto function If $a \in R - [-1, 1]$
(c) $f(x)$ is one-one onto function If $a \in R - \{0\}$
(d) $f(x)$ is one-one onto function If $a \in R - \{-1\}$

7. If $f(x + y) = f(x) + f(y) - xy - 1 \;\forall\, x, y, \in R$ and $f(1) = 1$ then $\sum_{h=1}^{5} f(h)$ is equal to
(a) 15 (b) -15 (c) 17 (d) -17

8. If $f\left(x+\frac{1}{y}\right) + f\left(x-\frac{1}{y}\right) = 2\cdot f(x)\cdot f\left(\frac{1}{y}\right)$ $\forall\, x, y, \in R$ and $f(0) = \frac{1}{2}$ then $f(4)$ is
(a) 0 (b) 4 (c) -4 (d) 2

9. If $f(x + f(y)) = f(x) + y - 1 \;\forall\, x, y, \in R$ and $f(0) = 1$, then $f(1)$ is equal to
(a) 0 (b) -1 (c) 1 (d) 2

10. If $f(x) = x\,|x|$ then $f^{-1}(x)$ is equal to
(a) $\sqrt{|x|}$ (b) $(\text{Sgn}\, x)\sqrt{|x|}$
(c) $-\sqrt{|x|}$ (d) None of these.

11. For what values of a the equation $||x| - 1| = a$ has four solutions?
(a) $0 \le a \le 1$ (b) $0 < a < 1$
(c) $a > 1$ (d) $a \ge 1$

12. Let $f: R \to R$ & $g: R \to R$ be two one-one onto function such that they are mirror image of each other about the line $y = 0$, then $h(x) = f(x) + g(x)$ is
(a) one-one & onto (b) one-one but not onto
(c) not one-one but onto (d) None of these

13. If $[\sin^{-1} x] > [\cos^{-1} x]$, where $[\cdot]$ denotes the greatest integer function, the complete set of value of x is
(a) $[\cos 1, 1]$ (b) $[\sin 1, 1]$
(c) $[\cos 1, \sin 1]$ (d) $[0, 1]$

14. Period of $f(x) = x - [x + a] - b$, where $a, b \in R^+$ and $[\cdot]$ denotes the greatest integer function is
(a) a (b) b
(c) $|a - b|$ (d) None of these

15. Let $F: [-4, 4] - \{-\pi, 0, \pi\} \to R$, such that $f(x) = \cot(\sin x) + \left[\frac{x^2}{a}\right]$, where $[\cdot]$ denotes the greatest integer function, is an odd function. Complete set of values of 'a' is
(a) $(-\infty, -16) \cup [16, \infty)$ (b) $(-16, 16) - \{0\}$
(c) $(-\infty, -16) \cup (16, \infty)$ (d) $[-16, 16] - \{0\}$

16. If $af(x) + bf\left(\frac{1}{x}\right) = x + \frac{5}{x}$, $(a \neq b)$, then $f(x)$ is equal to

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(a) $\frac{1}{a^2-b^2}\left(x+\frac{1}{x}\right)$

(b) $\frac{1}{a^2-b^2}[x(5a-b)+\frac{1}{x}(5b-a)]$

(c) $\frac{1}{a^2-b^2}[x(a-5b)+\frac{1}{x}(5a-b)]$

(d) None of these

17. A real valued function $f(x)$ satisfies the functional equation $f(x-y)=f(x)f(y)-f(a-x)\cdot f(a+y)$ where 'a' is a given constant and $f(0) = 1$ then $f(2a-x)$ is equal to

(a) $f(-x)$ (b) $f(a)+f(a-x)$
(c) $f(x)$ (d) $-f(x)$

18. If $f(n) = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + ... + \frac{1}{n}$, then $\sum_{i=1}^{n} f(i)$ is equal to

(a) $(n-1)f(n)$ (b) $n(f(n)-1)+1$
(c) $n(f(n)-1)+f(n)$ (d) $n(f(n)-1)-1$

SOLUTIONS

1. **(d)** : $19x - 9 - 2x^2 > 0 \Rightarrow \frac{1}{2} < x < 9$
$\Rightarrow \{1, 2, 3, ..., 8\}$
$x - 4 \geq 0 \Rightarrow x \in \{4, 5, 6, ..., 8\}$
$19x - 9 - 2x^2 > \sqrt{x-4} \Rightarrow \{4, 5, 8\}$

2. **(c)** : $f(x) = \sin^4 x + \cos^4 x + \sin^6 x + \cos^6 x$
$\Rightarrow (\sin^2 x + \cos^2 x)^2 - 2\sin^2 x \cdot \cos^2 x + (\sin^2 x + \cos^2 x)^2 - 3\sin^2 x \cdot \cos^2 x (\sin^2 x + \cos^2 x)$

$\Rightarrow 1 - \frac{1}{2}\sin^2 2x + 1 - \frac{3}{4}\sin^2 2x$

$\Rightarrow 2 - \frac{5}{4}\sin^2 2x \Rightarrow 0 \leq \sin^2 2x \leq 1$

$\Rightarrow \frac{3}{2} \leq 2 - \frac{5}{4}\sin^2 2x \leq 2 \Rightarrow \left[\frac{3}{2}, 2\right]$.

3. **(d)** : Replace x by $x + 2$ we have
$f(x + 12) + f(x + 6) = 0$ ...(1)
again replace x by $x - 6$
$\Rightarrow f(x - 6 + 12) + f(x - 6 + 6) = 0$
$\Rightarrow f(x + 6) + f(x) = 0$ ....(2)
Solving (1) & (2) $\Rightarrow f(x + 12) = f(x)$
so period of $f(x) = 12$.

4. **(b)** : $f'(x) > 0$ or $f'(x) < 0$ $\forall$ x then $D < 0$
$\Rightarrow 4b^2 - 4 \cdot 3ac < 0 \Rightarrow b^2 - 3ac < 0$

5. **(c)** : $f(x) = \frac{1}{\sqrt{|[|x|-1]|-5}} \Rightarrow |[|x| - 1]| - 5 > 0$
$\Rightarrow$ $[|x| - 1] > 5$ and $[|x| - 1] < -5$
$\Rightarrow$ $|x| - 1 \geq 6$ and $|x| - 1 < -5$
$\Rightarrow$ $|x| \geq 7$ & $|x| < -4$ (Not possible)
$\Rightarrow$ $x \in (-\infty, -7] \cup [7, \infty)$.

6. **(b)** : If $a \neq 0$ Range of $f(x) = R$ Also $f'(x) = a + \cos x$. If $a > 1, f'(x) > 0$ the function is increasing, $a < -1, f'(x) < 0$
$\because$ the function is decreasing
so $f(x)$ to be one-one, onto it must be monotonic and is possible if $a < -1$ or $a > 1$.

7. **(b)** : $f(1 + 1) = f(1) + f(1) - 1 - 1 = 0$
$f(2 + 1) = f(2) + f(1) - 2 - 1 = -2$
$f(3 + 1) = -5$ and $f(4 + 1) = -9$
$\therefore$ Reqd sum $= 1 + 0 - 2 - 5 - 9 = -15$.

8. **(a)** : At $x = 0 \Rightarrow f\left(\frac{1}{y}\right) + f\left(\frac{-1}{y}\right) = 2f(0) \cdot f\left(\frac{1}{y}\right)$
$\Rightarrow f\left(\frac{-1}{y}\right) = 0$ since $f(0) = \frac{1}{2}$
at $y = -\frac{1}{4} \Rightarrow f(4) = 0$.

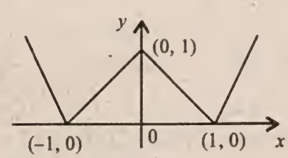
9. **(a)** : Put $x = y = 0 \Rightarrow f(0 + f(0)) = f(0) + 0 - 1$
$\because f(0) = 1$ given then $f(1) = f(0) + X - X \Rightarrow f(1) = 0$

10. **(b)** : $f(x) = y = x^2,\ x \geq 0; y \geq 0$
$= -x^2,\ x \leq 0; y \leq 0$
$\Rightarrow x = \sqrt{y},\ x \geq 0; y \geq 0$
$= -\sqrt{-y},\ x < 0; y < 0$

$$\Rightarrow f^{-1}(x) = \begin{cases} 1 \cdot \sqrt{x} & ; x > 0 \\ 0 & ; x = 0 \\ -1\sqrt{x} & ; x < 0 \end{cases}$$

$f^{-1}(\text{Sgn}(x)) = (\text{Sgn}\ x)\sqrt{|x|}$

11. **(b)** : Clearly a line parallel to x-axis
$\therefore$ $y = a$ cuts the curve at 4 points
If $0 < y < 1$;
$0 < a < 1$



$y = ||x| - 1|$

12. **(d)** : $f(x)$ & $g(x)$ are one-one onto and mirror image w.r.t. $y = 0$ then $f(x) = -g(x)$
$\therefore$ $h(x) = f(x) + g(x) = 0$ is a constant function, then $h(x)$ is many one into function.

13. **(b)** : $[\sin^{-1}x] = 0$ $\forall$ $x \in [0, \sin 1)$
$= 1$ $\forall$ $x \in [\sin 1, 1]$
and $\cos^{-1}x = 1$ $\forall$ $x \in [0, \cos 1]$
$= 0$ $\forall$ $x \in [\cos 1, 1]$

$\because [\sin^{-1}x] > [\cos^{-1}x] \Rightarrow x \in [\sin 1, 1]$.

14. (d) : $f(x) = x - [x+a] - b = x + a - [x+a] - a - b$
$\Rightarrow f(x) = \{x+a\} - (a+b)$ then period of $f(x)$ is 1.

15. (b) : For $f(x)$ to be odd, $\left[\frac{x^2}{|a|}\right]$ should not depend upon the value of x. since $x \in [-4, 4] \Rightarrow 0 \le x^2 \le 16$
$\Rightarrow \left[\frac{x^2}{|a|}\right] = 0$, If $a > 16 \Rightarrow a \in (-\infty, -16) \cup (16, \infty)$.

16. (c) : $af(x) + bf\left(\frac{1}{x}\right) = x + \frac{5}{x}$ ....(1)

$af\left(\frac{1}{x}\right) + bf(x) = \frac{1}{x} + 5x$ ....(2)

multiplying equ. (1) by a & equ. (2) by b then subtract.

$\Rightarrow (a^2 - b^2)f(x) = ax + \frac{5a}{x} - \frac{b}{x} - 5bx$

$\Rightarrow f(x) = \frac{1}{(a^2 - b^2)}\left[x(a - 5b) + \frac{1}{x}(5a - b)\right]$

17. (d) : Put $x = y = 0 \Rightarrow f(0) = (f(0))^2 - (f(a))^2$
$\because f(0) = 1 \Rightarrow \cancel{1} = \cancel{1} - (f(a))^2 \Rightarrow f(a) = 0$

$\Rightarrow \therefore f(2a - x) = f(a + a - x) = f(a - (x - a))$
$\Rightarrow f(a - (x - a)) = f(a)\cdot f(x - a) - f(\cancel{a} - \cancel{a})\cdot f(x)$
$\Rightarrow f(2a - x) = -f(x)$.

18. (c) : $f(1) + f(2) + f(3) + \ldots + f(n) = 1 + \left(1 + \frac{1}{2}\right) + \left(1 + \frac{1}{2} + \frac{1}{3}\right) + \ldots + \left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right)$

$f(1) + f(2) + f(3) + \ldots + f(n)$

$$= n + \left(\frac{n-1}{2}\right) + \left(\frac{n-3}{2}\right) + \ldots + \frac{1}{n}$$

$$= n\left(1 + \frac{1}{2} + \frac{1}{3} + \ldots + \frac{1}{n}\right) - \left(\frac{1}{2} + \frac{2}{3} + \frac{3}{4} + \ldots + \frac{n-1}{n}\right)$$

$$= nf(n) - \left\{\left(1 - \frac{1}{2}\right) + \left(1 - \frac{1}{3}\right) + \left(1 - \frac{1}{4}\right) + \ldots + \left(1 - \frac{1}{n}\right)\right\}$$

$$= nf(n) - \left\{(n-1) - \left(1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \ldots + \frac{1}{n}\right) - 1\right\}$$

$$= nf(n) - (n-1) - f(n) - 1$$

$$\sum_{i=1}^{n} f(i) \Rightarrow nf(n) - n + f(n).$$

EXERCISE

PART - I : Multiple choice – only one option correct

1. If $f(x + y, x - y) = xy$, then $f(x, y)$ is equal to

(a) xy (b) $x^2 - y^2$ (c) $\frac{x^2 - y^2}{4}$ (d) $x + y$

2. For a real number x, $[\cdot]$ denotes the greatest integer, the value of

$$\left[\frac{1}{2}\right] + \left[\frac{1}{2} + \frac{1}{100}\right] + \left[\frac{1}{2} + \frac{2}{100}\right] + \ldots + \left[\frac{1}{2} + \frac{99}{100}\right] \text{ is}$$

(a) 49 (b) 50 (c) 48 (d) 51

3. Let $f(x) = (1 + b^2)x^2 + 2bx + 1$ and let $m(b)$ be the minimum value of $f(x)$. As b varies, the range of $m(b)$ is

(a) $[0, 1]$ (b) $(0, 1/2]$ (c) $[1/2, 1]$ (d) $(0, 1]$

4. If f be an function satisfying $(f(x))^y + (f(y))^y = 2f(xy)$ $\forall\, x, y \in R$ and $f(1) = a \neq 1$ then $\sum_{r=1}^{n} f(r)$ is equal

(a) $\frac{a^n - 1}{a - 1}$ (b) $\frac{a^n}{a - 1}$ (c) $\frac{a^n + 1}{a + 1}$ (d) $\frac{a^n - 2}{a + 1}$

5. The range of

$$f(x) = \cos\left(\sin\left(\log\left(\frac{x^2 + e}{x^2 + 1}\right)\right)\right) + \sin\left(\cos\left(\log\left(\frac{x^2 + e}{x^2 + 1}\right)\right)\right)$$

is

(a) $[\cos(\sin 1), \sin(\cos 1)]$
(b) $[\sin(\cos 1) + 1, \cos(\sin 1)]$
(c) $[\cos(\sin 1) + \sin(\cos 1), 1 + \sin 1]$
(d) $(0, 1)$

6. If $f(x) = \int_{x^2}^{x^2+1} e^{-t^2}\,dt$ is one-one in the interval is

(a) $(0, \infty)$ (b) $(-\infty, 0)$
(c) $[-2, 2]$ (d) None of these

7. The function, $3\sin 3x + 4\sin 3x + 1$ is periodic with period

(a) $\frac{\pi}{3}$ (b) $\frac{2\pi}{3}$
(c) π (d) None of these

8. Let f be a real valued function with domain R. If $f(x + T) = 1[1 - 3f(x) + 3\{(f(x))^2 - (f(x))^3\}]^{1/3}$ $\forall\, x \in R$, where T is a fixed positive number then $f(x)$ is a periodic function with period

(a) T (b) $T/2$
(c) $2T$ (d) None of these

9. If $f: R \to R, g: R \to R$ be two given function, then $2 \min \{f(x) - g(x), 0\}$ is equal to
(a) $f(x) + g(x) - |g(x) - f(x)|$
(b) $|f(x) + g(x)| + |g(x) - f(x)|$
(c) $f(x) - g(x) + |g(x) - f(x)|$
(d) $f(x) - g(x) - |g(x) - f(x)|$

10. If $P(x) = x^4 + ax^3 + bx^2 + cx + d$ where $a, b, c, d \in R$. If $P(1) = 10$, $P(2) = 20$, $P(3) = 30$. Then the value of $\dfrac{P(12) - P(-8)}{10}$ is equal to
(a) 1984 (b) 1992 (c) 2004 (d) 2007

11. If for $x > 0, f(x) = (a - x^n)^{1/n}$, $g(x) = x^2 + px + q$, $p, q \in R$ and the equation $g(x) - x = 0$ has imaginary roots, then number of real roots of equation $g(g(x)) - f(f(x))$, is
(a) 0 (b) 2 (c) 4 (d) n

12. If $x^2 + y^2 = 1$, then the maximum & minimum value of $x + y$ are
(a) $-\sqrt{2}, \sqrt{2}$ (b) $-1, 1$
(c) $-\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}$ (d) $-\frac{1}{\sqrt{2}}, \sqrt{2}$

PART - II : Multiple choice – one or more than one options may be correct

1. Let $f(x) = \max \{1 + \sin x, 1 - \cos x\}$, $x \in [0, 2\pi]$ & $g(x) = \max \{1, |x - 1|\}$ $x \in R$, then
(a) $gf(0) = 1$ (b) $gf(1) = 1$
(c) $f(g(1)) = 1$ (d) $f(g(0)) = \sin 1$

2. Let $F: \left[-\frac{\pi}{3}, \frac{2\pi}{3}\right] \to [0, 4]$ be a function defined as $f(x) = \sqrt{3} \sin x - \cos x + 2$, then $f^{-1}(x)$ is given by
(a) $\sin^{-1}\left(\frac{x-1}{2}\right) - \frac{\pi}{6}$ (b) $\sin^{-1}\left(\frac{x-2}{2}\right) + \frac{\pi}{6}$
(c) $\frac{2\pi}{3} - \cos^{-1}\left(\frac{x-2}{x}\right)$ (d) None of these

3. Let $R = \{(x, y) : x, y \in R, x^2 + y^2 \le 25\}$ and $R' = \{(x, y) : x, y \in R, y \ge \frac{4}{9}x^2\}$ then
(a) dom $R \cap R' = [-3, 3]$
(b) Range $R \cap R' \supset [0, 4]$
(c) Range $R \cap R' = [0, 5]$
(d) $R \cap R'$ defined as function

4. Let n be a positive integer with $f(n) = 1! + 2! + 3! + ... + n!$ and $P(x)$ and $Q(x)$ be polynomials in x such that $f(n + 2) = P(n) \cdot f(n + 1) + Q(n) \cdot f(n)$ for all $n \ge 1$, then
(a) $P(x) = (x + 3)$ (b) $P(x) = -x - 2$
(c) $Q(x) = -x - 2$ (d) $Q(x) = x + 3$

ANSWERS

PART - I

1. (c) 2. (b) 3. (d) 4. (a) 5. (c) 6. (b)
7. (b) 8. (c) 9. (d) 10. (a) 11. (a) 12. (a)

PART - II

1. (a, b) 2. (b, c) 3. (a, b, c) 4. (a, c)

MBA MCA NDA

Quantitative Aptitude Test

NTSE GRE SAT

The aim of this test is to acclimatize you with real test-like experience and give you the required competitive edge.

1. The equation of the straight line through (–3, 5) and having equal positive intercepts is

(a) $2x + y + 1 = 0$ (b) $x + y - 2 = 0$
(c) $x + 2y - 7 = 0$ (d) none of these.

2. $$\frac{2+\dfrac{1}{3\frac{4}{5}}}{2+\dfrac{1}{3+\dfrac{1}{1+\frac{1}{4}}}} = ?$$

(a) 1 (b) 1/7
(c) 3/7 (d) 8/7.

3. Given triangle ABC where $a = 17$, $b = 25$, $c = 28$. The inradius of the circle is .... cms.

(a) 4.5 (b) 6
(c) 8 (d) none of these.

4. The point (2, 7) and (8, 5) are the 2 opposite vertices of a rectangle. The other two vertices lie on the line $y = 2x + c$. Therefore, the value of c is ....

(a) –4 (b) –1
(c) 0 (d) can't be determined.

5. The solution set (m, n) not satisfying the inequality, $\frac{1}{8} < \frac{m}{n} < \frac{1}{7}$ $(m, n \in I)$ is

(a) (7, 55) (b) (3, 22)
(c) (16, 116) (d) none of these.

6. A fibonacci sequence is formed by adding the previous 2 numbers e.g. 1, 2, 3, 5, 8 etc. or 2, 5, 7, 12, 19, 31 .... The eighth term of the sequence 1, 2, 3, 5, ... is

(a) 7 (b) 19
(c) 20 (d) 34.

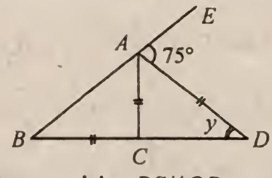
7. A man borrowed Rs. 3000/- at 8% p.a. interest. At the end of the year he repaid Rs. 1200 partly as interest and partly to reduce the debt. At the end of 2nd year he paid Rs. 1300. The sum he should pay to meet interest and clear off debt at the end of 3rd year is ....

(a) 1000 (b) 500
(c) 400 (d) none of these.

8. $PQRS$ is the quadrilateral formed by joining the midpoints of adjacent sides of quadrilateral $ABCD$. The ratio of areas $ABCD$ to $PQRS$ is ....

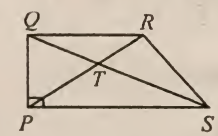
(a) 3 : 1 (b) 2 : 1
(c) 4 : 1 (d) can't be determined.

9. Given $CB = CA = AD$ and $\angle DAE = 75°$. The value of y is



(a) 45
(b) 50
(c) 60
(d) can't be determined.

10. $PQRS$ is a trapezium with $PS//QR$ and $\angle QPS = 90°$. QS and PS cut at T. The ratio of area of ΔTPQ to ΔTRS is



(a) 1
(b) 2
(c) 1/3
(d) can't be determined.

11. A cask contains 4 parts wine & 1 part water. How much of this mixture must be with drawn and substituted by water inorder that the resulting mixture be half wine and half water?

(a) 3/8 (b) 5/8
(c) 7/8 (d) none of these.

12. The area of the triangle PQR, where $P = (2, 6)$, $Q = (4, 7)$, $R = (6, 8)$ is

(a) 6 sq. units (b) 6.2 sq. units
(c) 8 sq. units (d) none of these.

13. If $f(x, y, z) = x^y y^2 + xyz - yz^y + 15y^z$, then $f(4, 3, 2) = ?$

(a) 641 (b) 711
(c) 982 (d) 563.

14. If $\log_{10} 3 = 0.477$ and $\log_{10} 7 = 0.8457$, the value of $\log_{10}\left(2\frac{1}{3}\right)^3$ is

Math Archives

10 Best Problems

Prof. Shyam Bhushan*

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE.

1. $\tan^2\frac{\pi}{16}+\tan^2\frac{2\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{6\pi}{16}+\tan^2\frac{7\pi}{16}$ is equal to

(a) 24 (b) 34

(c) 44 (d) none of these.

2. If $f(x) = e^{\sin(x-[x])\cos\pi x}$, then $f(x)$ is ($[x]$ denotes the greatest integer function)

(a) non-periodic

(b) periodic with no fundamental period

(c) periodic with period 2

(d) periodic with period π.

3. Which of the following homogeneous functions are of degree zero?

(a) $\frac{x}{y}\ln\frac{y}{x}+\frac{y}{x}\ln\frac{x}{y}$ (b) $\frac{x(x-y)}{y(x+y)}$

(c) $\frac{xy}{x^2+y^2}$ (d) all the above.

4. If θ is small and positive number then which of the following is/are correct?

(a) $\frac{\sin\theta}{\theta}=1$ (b) $\theta<\sin\theta<\tan\theta$

(c) $\frac{\tan\theta}{\theta}>\frac{\sin\theta}{\theta}$ (d) none of these.

5. Match the column. (Each entry of column I matches with exactly one entry of column II)

Column I	Column II
(A) $\lim\limits_{x\to0^+}\log_{\sin x}\sin 2x$	(P) -1
(B) $\lim\limits_{x\to0}\frac{\ln(\cos x)}{x}$	(Q) $-1/2$
(C) $\lim\limits_{x\to0}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right)$	(R) 1
(D) $\lim\limits_{x\to0}\frac{x-\sin x}{x-\tan x}$	(S) 0

(a) $A\to S, B\to P, C\to Q, D\to R$

(b) $A\to P, B\to Q, C\to R, D\to S$

(c) $A\to R, B\to S, C\to P, D\to Q$

(d) $A\to Q, B\to R, C\to S, D\to P$

6. If $y=\cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$ then prove that $\frac{dy}{dx}=\sqrt{\frac{3}{\cos x\cdot\cos 3x}}$.

7. Let $a\in R$, then prove that a function $f: R\to R$ is differentiable at a if a function $\phi: R\to R$ satisfies $f(x)-f(a)=\phi(x)(x-a)\ \forall\ x\in R$ and ϕ is continuous at a.

8. If $\beta, \gamma\in(0,\pi)$ such that
$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+\beta+\gamma)=0$ and
$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+\beta+\gamma)=0$.

Then evaluate $f'(\beta)$ and $\lim\limits_{x\to\gamma}g(x)$

where $f(x)=\sin 2x(1+\cos 2x)^{-1}$ and

$g(x)=\frac{1+\sin x-\cos x}{1+\sin x+\cos x}$

(Here $f'(x)$ denotes derivative of f with respect to x.)

9. Find area of the triangle formed with vertices (0, 0), $\left(\lim\limits_{x\to\pi/2}\left[\frac{x-\frac{\pi}{2}}{\cos x}\right], 0\right)$ and $\left(0, \lim\limits_{x\to0}\left(\frac{\tan x}{x}\right)^{1/x}\right)$ where $[\cdot]$ denotes the greatest integer function.

10. Prove that the straight lines whose direction cosines are given by the relations $al+bm+cn=0$ and $fmn+gnl+hlm=0$ are perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$.

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SOLUTIONS

1. **(b) :** $\theta = \frac{\pi}{6} \Rightarrow 8\theta = \frac{\pi}{2}$,

1st and last given $\tan^2\theta + \cot^2\theta$ simplifies to $\frac{8}{1-\cos 4\theta} - 2$ mutually other terms

$\Rightarrow$ Given expression = 34.

2. **(c) :** $f(x) = e^{\sin(x-[x])\cos \pi x}$

$\sin(x-[x]) = \sin\{x\}$ period is 1

$\cos\pi x$ period is 2, hence $f(x)$ period is 2.

3. **(d) :** $f(x, y)$ is homogeneous function of degree $n \in R$ in x, y if $f(kx, ky) = k^n f(x, y)$; where $k > 0$.

4. **(c)**

5. **(c) :**

(a) $\lim_{x\to 0} \frac{\ln \sin 2x}{\ln \sin x} = \frac{\ln 2 + \ln\cos x + \ln \sin x}{\ln \sin x}$

$= 0 + 0 + 1 \Rightarrow A \to R$

(b) $\lim_{x\to 0} \ln\left(1 - 2\sin^2\frac{x}{2}\right)^{\frac{1}{x}} = \ln\left[\left(1-2\sin^2\frac{x}{2}\right)^{\frac{-1}{2\sin^2\frac{x}{2}}}\right]^{\frac{-2\sin^2\frac{x}{2}}{x}}$

$= \lim_{x\to 0} \frac{-2\sin^2\frac{x}{2}}{x} = 0 \Rightarrow B \to S$

(c) $\lim_{x\to 0} \frac{1-x}{\ln x} = \lim_{x\to 0} \frac{h}{\ln(1-h)}$

$= \lim_{h\to 0} \frac{1}{\ln\left[(1-h)^{-1/h}\right]^{-1}} = -1 \Rightarrow C \to P$

(d) $\lim_{x\to 0} \frac{\frac{x-\sin x}{x^3}}{\frac{x-\tan x}{x^3}} = \frac{-1/6}{1/3} = -\frac{1}{2} \Rightarrow D \to Q.$

6. $\cos y = \sqrt{\frac{\cos 3x}{\cos^3 x}} = \sqrt{\frac{4\cos^3 x - 3\cos x}{\cos^3 x}}$ ....(1)

$= \sqrt{4 - 3\sec^2 x} \Rightarrow \cos^2 y = 4 - 3\sec^2 x$

$= 4 - 3(1+\tan^2 x) = 1 - 3\tan^2 x$

$\Rightarrow \sin^2 y = 3\tan^2 x \Rightarrow \sin y = \sqrt{3}\tan x$

$\Rightarrow \cos y \frac{dy}{dx} = \sqrt{3}\sec^2 x \quad \therefore \frac{dy}{dx} = \sqrt{\frac{3}{\cos x \cdot \cos 3x}}$

$\because$ from(1)

7. $\phi : R \to R$ is continuous at $x = a$ and satisfies

$f(x) - f(a) = \phi(x)(x-a) \ \forall \ x \in R$

$\Rightarrow \frac{f(x)-f(a)}{x-a} = \phi(x) \Rightarrow \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \phi(x)$

$\Rightarrow f'(a) = \phi(a) \ \because \lim_{x\to a}\phi(x) = \phi(a)$

$\Rightarrow f$ is differentiable at $x = a$.

8. Given

$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+\beta+\gamma) = 0$

$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+\beta+\gamma) = 0$

where $\beta, \gamma \in (0, \pi)$

$\Rightarrow [\cos\alpha + \cos(\alpha+\beta)]^2 + [\sin\alpha + \sin(\alpha+\beta)]^2 = 1$

$\Rightarrow 2 + 2 + [\cos(\beta)] = 1$

$\therefore \cos\beta = -\frac{1}{2}$ Similarly $\cos\gamma = -\frac{1}{2}$

$\therefore \beta = \gamma = \frac{2\pi}{3}$

But $f(x) = \frac{\sin 2x}{1+\cos 2x} = \tan x$ and $g(x) = \tan\frac{x}{2}$

$\therefore f'\left(\frac{2\pi}{3}\right) = \sec^2\frac{2\pi}{3}$ and $\lim_{x\to 2\pi/3} g(x) = \tan\frac{\pi}{3}$

$= 4 \qquad\qquad = \sqrt{3}$

9. Let $O = (0, 0)$

$A = \left(\lim_{x\to\frac{\pi}{2}}\left[\frac{x-(\pi/2)}{\cos x}\right], 0\right) = (-2, 0)$

$B = \left(0, \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{1/x}\right) = (0, 1)$

$\therefore$ area of $\Delta OAB = \frac{1}{2}|-2-0| = 1$ square units.

10. $al + bm + cn = 0$ ...(1)

$fmn + gnl + hlm = 0$ ...(2)

Eliminate n

$\Rightarrow ag\left(\frac{l}{m}\right)^2 + (af + bg - ch)\left(\frac{l}{m}\right) + bf = 0$ ...(3)

Now, if l_1, m_1, n_1 and l_2, m_2, n_2 are d.c.'s of two lines then roots of (3) and $\frac{l_1}{m_1}$ and $\frac{l_2}{m_2}$.

$\therefore$ Product of the roots $= \frac{l_1}{m_1} \cdot \frac{l_2}{m_2} = \frac{bf}{ag}$

$\therefore \frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b} \quad \therefore \frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c}$

$\because$ lines are perpendicular

$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = 0 \Rightarrow \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0.$

Very Similar Practice Paper

IIT-JEE 2008

By : Vidyalankar Institute*, Mumbai

(Two Dimensional and Three Dimensional Geometry)

Part I

Time : 1 hr.

Section I : Straight Objective Type

This section contains 9 multiple choice questions numbered 1 to 9. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. If $x + ky = 1$ and $x = a$ are the equations of the hypotenuse and a side of a right angled isosceles triangle, then

(a) $k = \pm 1$ (b) $k = \pm a$
(c) $k = \pm 1/a$ (d) $k = \pm 2$

2. Given two points $A \equiv (-2, 0)$ and $B \equiv (0, 4)$. The co-ordinates of a point M lying on the line $x = y$ so that the perimeter of the ΔAMB is least, is

(a) (1, 1) (b) (0, 0)
(c) (2, 2) (d) (3, 3)

3. A ray of light travels along the line $2x - 3y + 5 = 0$ and strikes a plane mirror lying along the line $x + y = 2$. The equation of the straight line containing the reflected ray is

(a) $2x - 3y + 3 = 0$ (b) $3x - 2y + 3 = 0$
(c) $21x - 7y + 1 = 0$ (d) $21x + 7y - 1 = 0$

4. If the inclination of the diameter PP' of the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, to the major axis is θ and PP'^2 is the A.M. of squares of major and minor axes, then $\tan\theta$ is equal to :

(a) b/a (b) a/b
(c) $\pi/4$ (d) $\pi/6$

5. If a circle of radius 'r' is concentric with ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, then the common tangent is inclined to major axis at an angle

(a) $\tan^{-1}\sqrt{\frac{r^2 - b^2}{a^2 - r^2}}$ (b) $\tan^{-1}\sqrt{\frac{r^2 - a^2}{b^2 - r^2}}$
(c) $\tan^{-1}\sqrt{\frac{r^2 - b^2}{r^2 - a^2}}$ (d) $\tan^{-1}\sqrt{\frac{r^2 - a^2}{b^2 - r^2}}$

6. The angle between the straight lines

$$\frac{x+1}{2} = \frac{y-2}{5} = \frac{z+3}{4} \text{ and } \frac{x-1}{1} = \frac{y+2}{2} = \frac{z-3}{-3} \text{ is}$$

(a) 45° (b) 30°
(c) 60° (d) 90°

7. The distance of the point where the line $\frac{x+1}{2} = \frac{y+1}{3} = \frac{z+1}{4}$ meets the plane $x + 2y + 3z = 14$, from the origin is

(a) $\sqrt{15}$ (b) $\sqrt{14}$
(c) 7 (d) $\sqrt{7}$

8. If the planes $x = cy + bz$, $y = az + cx$ and $z = bx + ay$ pass through one line, then $a^2 + b^2 + c^2 + 2abc =$

(a) ab (b) 1
(c) bc (d) 0

9. The length of perpendicular from (0, 0, 0) to the plane $ax + by + cz + d = 0$, where a, b, c, d are in AP, $(a, b, c, d > 0)$ is 1 unit then the value of $\frac{c-a}{b}$ is

(a) $2(\sqrt{2} - 1)$ (b) $2(\sqrt{2} + 1)$
(c) $\frac{1}{(\sqrt{2} - 1)}$ (d) 1

Section II : Assertion-Reason Type

This section contains 4 questions numbered 10 to 13. Each question contains STATEMENT-1 (Assertion) and STATEMENT-2 (Reason). Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct

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20 Challenging Problems

1. If $\lim_{x\to\infty} \cot^{-1}(1+n+n^2) = \lim_{x\to 0} \int_0^x \frac{t^2 dt}{(x-\sin t)\sqrt{a+t}}$

then value of a is

(a) $\frac{4}{\pi}$ (b) $\frac{16}{\pi^2}$

(c) $\frac{32}{\pi^2}$ (d) $\frac{32}{\pi}$

2. If $\begin{vmatrix} {}^{10}C_5 & {}^{10}C_4 & {}^{11}C_n \\ {}^{20}C_5 & {}^{20}C_4 & {}^{21}C_n \\ {}^{30}C_5 & {}^{30}C_4 & {}^{31}C_n \end{vmatrix} = 0$ then the value of $\frac{d^n f(x)}{dx^n}$

at $x = 1$ for $f(x) = (3x+1)^5$ for usual notations is equal to ${}^n p_3 \cdot \frac{\alpha}{\beta}$ then

(a) $\alpha = 21, \beta = 17$ (b) $\alpha = 81, \beta = 7$

(c) $\alpha = 81, \beta = 20$ (d) $\alpha = 20, \beta = 81$

3. Value of $\int_0^{\pi/2} \log(1 - x^2\cos^2\theta)\, d\theta$, $(x^2 \le 1)$ is

(a) $\log(1+\sqrt{1-x^2})$

(b) $\log(1+\sqrt{1-x^2}) - \pi\log 2$

(c) $\pi\log(1+\sqrt{1-x^2})$

(d) $\pi\log(1+\sqrt{1-x^2}) - \pi\log 2$

4. If $K = \int_0^\infty e^{-xy} \cdot \frac{\sin x}{x} dx$ ($y > 0$, is a parameter) then value of K is

(a) $\frac{\pi}{2} - \tan^{-1} y$ (b) $\frac{\pi}{2} + \tan^{-1} y$

(c) $\tan^{-1} y$ (d) $\pi - \tan^{-1} y$

5. Value of $\int_0^{\pi/2} \sin^3 x \cdot \cos^{11} x\, dx =$

(a) $\frac{1}{8}$ (b) $\frac{1}{56}$

(c) $\frac{1}{112}$ (d) none of these

6. $\int_0^\infty \frac{x}{1+e^x} dx =$

(a) $\frac{\pi^2}{6}$ (b) $\log 2$

(c) $\frac{\pi^2}{12}$ (d) $\frac{\pi^2}{8}$

7. If $\frac{d^2(f(x))}{dx^2} - 7\frac{d(f(x))}{dx} + 6f(x) = e^{2x}$ given that $y(0) = 0$ and $\frac{df(x)}{dx} = 1$ then $f(x) =$

(a) $e^{6x} - e^{2x}$ (b) $\frac{3}{4}e^{2x} - e^{6x}$

(c) $\frac{1}{4}(e^{6x} - e^{2x})$ (d) $\frac{1}{4}(e^{6x} + e^{2x})$

8. The curve whose differential equation is given as $\frac{d^2x}{dt^2} - 3\frac{dx}{dt} + 2x = 0$, given that $t = 0$, $x = 0$ and $\frac{dx}{dt} = 0$, is

(a) a circle (b) y-axis

(c) x-axis (d) a parabola

9. The area under the curve whose differential equation is $\frac{d^2y}{dx^2} + y = 0$, given $y = 2$, $x = 0$ and $x = \frac{\pi}{2}$, and the x-axis between $x = 0$ to $x = \frac{\pi}{2}$ (in square units) is

(a) 2 (b) 3

(c) 4 (d) $\frac{3}{4}$

10. Let $f(x)$ be a differentiable such that $f(x \cdot y) = f(x) + f(y)$ for all x and y then $f(e) + f\left(\frac{1}{e}\right) =$

(a) 0 (b) 1

(c) -1 (d) 2

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11. The function f defined on R is given by

$$f(x)=\begin{cases}1 & , \ x \text{ is rational}\\ -1 & , \ x \text{ is irrational}\end{cases}$$

(a) $f(x)$ is continuous and differentiable on R
(b) $f(x)$ is continuous but not differentiable on R
(c) $|f(x)|$ is continuous but not differentiable on R
(d) $|f(x)|$ is differentiable on R

12. Supreet and Kushween stand in a ring with ten other girls. If the arrangements of twelve girls is at random then the chance that there are exactly three girls between Supreet and Kushween, is

(a) $\frac{1}{10}$ (b) $\frac{2}{11}$
(c) $\frac{3}{11}$ (d) $\frac{3}{10}$

13. Three coins, identical in appearance, one of which is ideal and the two others are biased with probabilities $\frac{1}{3}$ and $\frac{2}{3}$ respectively for a head. One coin is taken at random and tossed twice, if a head appears both times then the probability that ideal coin is chosen, is

(a) $\frac{5}{19}$ (b) $\frac{3}{29}$
(c) $\frac{2}{29}$ (d) $\frac{11}{19}$

PARAGRAPH

Let $A=\begin{bmatrix}1&0&0\\2&1&0\\3&2&1\end{bmatrix}$ be a square matrix and C_1, C_2, C_3 be 3 column matrices satisfying $AC_1=\begin{bmatrix}1\\0\\0\end{bmatrix}$, $AC_2=\begin{bmatrix}2\\3\\0\end{bmatrix}$ and $AC_3=\begin{bmatrix}2\\3\\1\end{bmatrix}$ of a matrix B. If the matrix $C=\frac{1}{3}(A\cdot B)$ then

14. Value of sum of elements of B^{-1} is
(a) -1 (b) 0
(c) 4 (d) 2

15. The ratio of the trace of the matrix A to the matrix B is
(a) 1 : 3 (b) 2 : 3
(c) 1 : 1 (d) 3 : 1

16. Value of $\sin^{-1}|A| + \cos^{-1}|C|$ is
(a) $\frac{\pi}{2}$ (b) $\frac{\pi}{3}$
(c) $\frac{\pi}{4}$ (d) 1

17. If $n! = n(n-1)(n-2) \ldots 3.2.1$ and 5! contains 3 digits then the number of digits in the number after expansion 100! is
(a) 98 (b) 49
(c) 97 (d) 47

18. If $a_1, a_2, \ldots, a_{19}$ represent first 19 natural numbers then sum of square roots of all products taken two at a time of these natural numbers is
(a) less than 1710 (b) more than 1710
(c) less than 1510 (d) more than 1525

19. Match the following Column I with Column II

Column I	Column II
(i) If $f(x)=\frac{1}{x+1}+\frac{2}{x^2+1}+\frac{4}{x^4+1}+\ldots+\frac{2^n}{x^{2^n}+1}$ then domain of $f(x)$	(p) $\frac{4}{\pi}$
(ii) Let $S=\{(x,y): \lvert x-3\rvert<1, \lvert y-3\rvert<1\}$ $T=\{(x,y): 4x^2+4y^2-32x-54y+109\le 0\}$	(q) $(-\infty,\infty)-\{1\}$
(iii) For $\lambda \in R$, the function $f(x)$ defined on $\left[0,\frac{\pi}{2}\right]$ as $f(x)=\lambda\int_0^{\pi/2}\sin x\cdot\sin y\, f(y)dy$	(r) $S\cap T=\phi$ or $S\cap T=S$
(iv) $\int_0^\infty \frac{\tan^{-1}(ax)dx}{x(1+x^2)}$, where a is a parameter	(s) $\frac{\pi}{2}\log(1+a)$
	(t) $\pi\log a$
	(u) $\frac{\pi}{4}$

20. If $x_1^2+x_2^2+x_3^2+\ldots+x_{20}^2=400$ and $y_1^2+y_2^2+y_3^2+\ldots+y_{20}^2=900$ then value of $\frac{x_1}{y_1}+\frac{x_2}{y_2}+\frac{x_3}{y_3}+\ldots+\frac{x_{20}}{y_{20}}$ is
(a) $\frac{2}{3}$ (b) $\frac{3}{4}$
(c) $\frac{4}{9}$ (d) $\frac{3}{2}$

Answers will be published in the next issue.

10 Subjective PROBLEMS

Classical Geometry Revisited

by Er. Tapas Kumar Yogi

1. Each side of a ΔABC has length 2 units. A circle with centre at A and radius 1 unit cuts AB at M. A tangent to the circle from B and lying outside the triangle meets the circle at P. What is the area of the region bounded by BP, BM and minor arc MP.

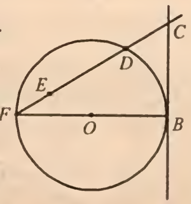
2. Let P denote the perimeter of a right triangle and Q the sum of the squares of its three sides. Find the ordered triple of coefficients (x, y, z) such that the area of the triangle equals $xP^2 + yP\sqrt{Q} + zQ^2$?

3. A semicircle is inscribed in a quadrilateral $ABCD$ in such a way that the midpoint of BC coincides with the centre of the semicircle, and the diameter of the semicircle lies along a portion of BC. If $AB = 4$ units, $CD = 5$ units. Then what is BC?

4. In a cartesian co-ordinate system, the two tangent lines from $P(39, 52)$ meet the circle defined by $x^2 + y^2 = 625$ at points Q and R. Find the length QR.

5. Points A, B, C, D are on a cartesian plane with $A(0, 0)$, $B(2, 0)$, $C(2, 1)$ and $D(0, 1)$. Compute the minimum possible value of $PA + PB + PC + PD$ for all position of P.

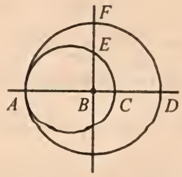
6. In the figure, AB is a diameter of the circle with centre O and radius r. A chord AD is drawn and extended to meet the tangent at B, in point C. The point E is chosen on AC such that $AE = DC$. Let x be the minimum distance from E to the tangent through A and y be the minimum distance from E to the diameter AB. Find a relation between x, y and r.



7. A rhombus of side length x has the property that there is a point on its longer diagonal such that the distances from that point to the vertices are 1, 1, 1 and x. what is the value of x?

8. Persons A and B stand at point P on line l. Point Q lies at a distance of 10 units form point P in the direction perpendicular to l. Both persons initially face towards Q. Person A walks forward and to the left at an angle of 15° with l, when he is again at a distance of 10 units from point Q, he stops, turns 90° to the right and continues walking. And at the same time, person B walks forward and to the right at an angle of 65° with the line l; when he is again at a distance of 10 units from point Q, he stops, turns 90° to the left and continues walking. Their paths cross at point R. Find the distance PR.

9. In the figure the circles are tangent at A, the centre of the larger circle is at B, $CD = 42$ units and $EF = 24$ units. What are the radii of the circles?



10. Consider a hexagon inscribed in a circle of radius r. If the hexagon has two sides of length 2, two sides of length 7 and two sides of length 11, what is r?

ANSWERS

1. Required area = area of right ΔAPB – area of minor sector AMP

Now ΔAPB has $AP = 1$, $AB = 2 \Rightarrow BP = \sqrt{3}$ and $\angle PAB = 60° = \pi/3$

so, Area $(\Delta APB) = \frac{1}{2} \times 1 \times \sqrt{3} = \frac{\sqrt{3}}{2}$

and area of sector $AMP = \frac{1}{2} \times 1^2 \times \frac{\pi}{3} = \frac{\pi}{6}$

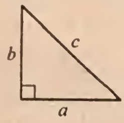
so required area $= \frac{\sqrt{3}}{2} - \frac{\pi}{6}$.

2. $C = \sqrt{a^2 + b^2}$, $P = a + b + \sqrt{a^2 + b^2}$

$Q = 2(a^2 + b^2)$, Δ = area = $(1/2)\, ab$

so, $P = a + b + \sqrt{\frac{Q}{2}}$,

$\frac{Q}{2} = a^2 + b^2$ and $ab = 2\Delta$



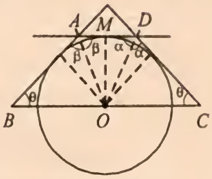
or, $a+b = P-\dfrac{\sqrt{Q}}{\sqrt{2}}$, $ab = 2\Delta$ and $a^2 + b^2 = Q/2$

Now $(a + b)^2 = a^2 + b^2 + 2ab$

$$\Rightarrow \left(P-\frac{\sqrt{Q}}{\sqrt{2}}\right)^2 = \frac{Q}{2} + 2\times 2\Delta$$

simplifying, $\Delta = \dfrac{1}{4}P^2 - \dfrac{1}{2\sqrt{2}}P\sqrt{Q}$

so, comparing, $x = \dfrac{1}{4}$, $y = \dfrac{-1}{2\sqrt{2}}$, $z = 0$.



3. Using similar triangles,
$\angle B = \angle C = \theta$ (say)
Let $\angle ODC = \alpha = \angle ODM$
$\angle OAM = \beta = \angle OAB$
Now for quadrilateral $ABCD$:
$2\theta + 2\alpha + 2\beta = 360° \Rightarrow \theta + \alpha + \beta = 180°$
so, in ΔODC : $\angle COD = \beta$ and similarly $\angle AOB = \alpha$

i.e. $\Delta AOB \sim \Delta ODC \Rightarrow \dfrac{AB}{OB} = \dfrac{OC}{DC}$

or, $\dfrac{4}{BC/2} = \dfrac{BC/2}{5} \Rightarrow BC = 4\sqrt{5}$ units.

4. For right ΔOQP; $OP = \sqrt{39^2+52^2} = 65$

and radius $= OQ = 25 \Rightarrow PQ = \sqrt{OP^2-OQ^2} = 60$

Let QR meet OP in S then $QS = SR$

Now area of $\Delta OPQ = \dfrac{1}{2}\times OQ\times PQ = \dfrac{1}{2}\times QS\times OP$

$\Rightarrow \dfrac{1}{2}\cdot 25\cdot 60 = \dfrac{1}{2}\cdot QS\cdot 65 \Rightarrow QS = \dfrac{300}{13}$

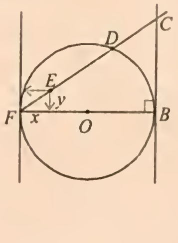
so, $QR = 2QS = 600/13$ units.

5. By the triangle inequality, $PA + PC$ is minimized when P lies on the line segment AC. Similarly $PB + PD$ is minimum when P lies on line segment BD. $\Rightarrow$ the given sum is minimum when P lies on both AC and BD i.e., it is the point of intersection of the diagonals AC and BD. So, sum (min.) $= PA + PB + PC + PD$

$= AC + BD = \sqrt{5}+\sqrt{5} = 2\sqrt{5}$ units.

6. From similar triangles, $\dfrac{y}{x} = \dfrac{BC}{AB} = \dfrac{BC}{2r}$

$\Rightarrow BC = 2ry/x$



Now, $BC^2 = CD \times CA$

$\Rightarrow BC^2 = AE \times CA$

or, $\left(\dfrac{2ry}{x}\right)^2 = \sqrt{x^2+y^2}\times CA$

or, $CA = \dfrac{4r^2y^2}{x^2\sqrt{x^2+y^2}}$

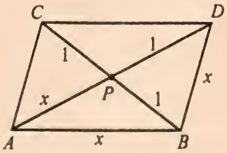
Now, in right ΔABC, $AB^2 + BC^2 = AC^2$

$$\Rightarrow (2r)^2 + \left(\frac{2ry}{x}\right)^2 = \left(\frac{4r^2y^2}{x^2\sqrt{x^2+y^2}}\right)^2$$

simplifying, $x(x^2 + y^2) = 2ry^2$.

7. Triangles BPC and ABC are similar.

Thus $\dfrac{BC}{CP} = \dfrac{AC}{BC} \Rightarrow \dfrac{x}{1} = \dfrac{x+1}{x}$



or, $x^2 - x - 1 = 0$

Neglecting –ve value of x :

$x = \dfrac{\sqrt{5}+1}{2}$ (Golden ratio)

Note : This ratio is most commonly used in tile designs.

8. Let A', B' be the points at which persons A and B turn. Let R be the point where A and B meet.
Note that P, A', B', R lie on a circle.
(Turning by 90°) and PR is the diameter of the circle.
so, $PR = 20$ units.

9. Let O be the centre of the smaller circle.
Let r = radius of smaller circle.
R = radius of larger circle
$x = BC$, $y = BE$
Now, $BF = BE + EF \Rightarrow R = y + 24 \Rightarrow y = R - 24$
And, $BD = BC + CD \Rightarrow R = x + 42$ or $x = R - 42$
and $AB = OA + OB \Rightarrow R = r + (r - x)$
$\Rightarrow r = R - 21$
Now in right ΔOBE, $(r - x)^2 + y^2 = r^2$
Now substituting, r, x and y in terms of R
and simplifying, we have $R = 96$ units and $r = 75$ units.

10. Let $ABCDEF$ be the given hexagon. Since, the order of the sides will not affect r, we take
$AB = AF = 2$, $BC = FE = 11$ and $CD = ED = 7$
So, by symmetry AD must be the diameter of the circle,

$\Rightarrow AD = 2r$, and $BD = \sqrt{AD^2-AB^2} = \sqrt{4r^2-4}$

and $AC = \sqrt{AD^2-CD^2} = \sqrt{4r^2-49}$

Now, since $ABCD$ is cyclic,
$AB \cdot CD + AD \cdot BC = AC \cdot BD$ (Ptolemy's theorem)

i.e., $2 \times 7 + 2r \times 11 = \sqrt{4r^2-49} \times \sqrt{4r^2-4}$

simplifying, $2r^3 - 87r - 77 = 0$
$(r - 7)(2r^2 + 14r + 11) = 0$
since r is +ve, $2r^2 + 14r + 11$ can never be zero.
Hence, $r = 7$ units.

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- All students preparing for PET examinations can participate in mtG **Math-Genius Contest**.
- Answers marked only on the entry form of the magazine / photocopy of form will be accepted.
- More than one response to a question will be disqualified.
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  - 3rd Prize - MTG Books (worth Rs. 500/–)
- The entries with maximum number of correct answers for **three consecutive months** (August '07 to October '07) will be awarded 1st prize. 2nd and 3rd prize will be given to the next maximum scorers. In case of a tie, the winners will be decided through a lucky draw.
- The decision of the editor will be final and binding in all cases and will not be a matter for consideration of any court and no correspondence will be entertained.
- Name and photograph of the prize winners of this contest along with the answers will be published in the November issue.
- MTG is not responsible for any postal delays, transit losses or mutilation of entries.
- **Last Date**
  The entries should reach **on/before 31st October '07** to – mtG **Math-Genius Contest**, 406, Taj Apartment, Ring Road, Near Safdarjung Hospital, New Delhi-29.
  [***Note :*** *Enclosures include a passport size photograph and a photocopy of age proof.*]

1. If $f(x) = p|\sin x| + qe^{|x|} + r|x|^3$ and if $f(x)$ is differentiable at $x = 0$, then
(a) $p = q = r = 0$
(b) $p = 0$; $q = 0$; r is any real number
(c) $q = 0$; $r = 0$; p is any real number
(d) $r = 0$; $p = 0$; q is any real number.

2. If $f(x) = x^2 + \dfrac{x^2}{(1+x^2)} + \dfrac{x^2}{(1+x^2)^2} + \ldots + \dfrac{x^2}{(1+x^2)^n} + \ldots$
then at $x = 0$
(a) $f(x)$ has no limit
(b) $f(x)$ is discontinuous
(c) $f(x)$ is continuous but not differentiable
(d) $f(x)$ is differentiable.

3. The solution set of $f'(x) > g'(x)$ where $f(x) = (1/2)\, 5^{2x+1}$ and $g(x) = 5^x + 4x \log 5$ is
(a) $(1, \infty)$ (b) $(0, 1)$
(c) $[0, \infty)$ (d) $(0, \infty)$

4. Let $f(x) = \prod_{k=1}^{n} (\cos(2k-1)x + i\sin(2k-1)x)$ then $(\mathrm{Re}\, f(x))'' + i\,(\mathrm{Im}\, f(x))''$ is equal to

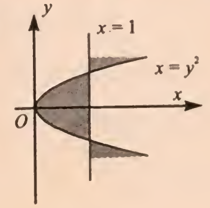
CHINESE Olympiad Problems

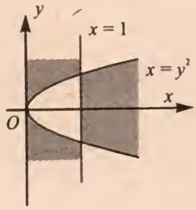
SECTION - 1

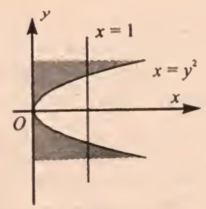
1. Let α be a constant. What curve in the complex plane represents the set $\{\overline{z^2} \mid \arg z = \alpha\}$?

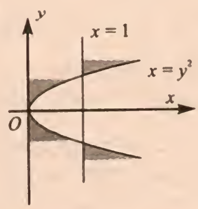
(a) the ray $\arg z = 2\alpha$.
(b) the ray $\arg z = -2\alpha$.
(c) the ray $\arg z = -\alpha$.
(d) None of (a), (b) and (c) is correct.

2. Let x be a positive real number not equal to 1. In each of the following diagrams, the shaded region does not include its boundary. Which of these regions represents the set $\{(x, y) \mid \log_x (\log_x y^2) > 0\}$?

(a)


(b)


(c)


(d)


3. How many different hyperbolas are represented by the polar equation

$$\frac{1}{r} = 1 - \binom{m}{n}\cos\theta,$$

where m and n are integers satisfying $1 \le n \le m \le 5$?

(a) 15 (b) 10
(c) 7 (d) 6

4. How many real roots does $\sin x = \log_{10} x$ have?

(a) 0 (b) 1
(c) 2 (d) more than 2

5. Let $G(x) = \left(\frac{1}{a^x - 1} + \frac{1}{2}\right) F(x)$,

where a is a positive real number not equal to 1 and $F(x)$ is an odd function. Which of the following statements is true?

(a) $G(x)$ is an odd function
(b) $G(x)$ is an even function
(c) $G(x)$ is neither an odd function nor an even function.
(d) Whether $G(x)$ is an odd or even function depends on the value of a.

6. Let $F(x)$ be such that $F\left(\frac{1-x}{1+x}\right) = x$ for all $x \ne -1$. Which of the following statements is true?

(a) $F(-2 - x) = -2 - F(x)$ (b) $F(-x) = F\left(\frac{1+x}{1-x}\right), x \ne 1$

(c) $F\left(\frac{1}{x}\right) = F(x), x \ne 0$ (d) $F(F(x)) = -x$

7. If the point (x, y) is moving counterclockwise along the unit circle at constant angular speed ω, how is the point $(-2xy, y^2 - x^2)$ moving?

(a) clockwise along the unit circle at angular speed ω.
(b) counterclockwise along the unit circle at angular speed ω.
(c) clockwise along the unit circle at angular speed 2ω.
(d) counterclockwise along the unit circle at angular speed 2ω.

8. In a tetrahedron, all sides are of length 1 except possibly one whose length is denoted by x. It volume is denoted by $V(x)$. Which of the following statements is true?

(a) $V(x)$ is an increasing function which has no maximum.
(b) $V(x)$ is an increasing function which has a maximum.
(c) $V(x)$ is not an increasing function and has no maximum.
(d) $V(x)$ is not an increasing function but has a maximum.

SECTION - 2

1. Let the coordinates of A, B and D be $(1, 0)$, $(-1, 0)$ and $(x, 0)$ respectively. A line through D perpendicular

to AB cuts the unit circle at a point C. In which interval does x lie if the segments AD, BD and CD can form an acute triangle?

2. What is the general solution of $\cos \frac{x}{4} = \cos x$ and how many distinct solutions are there in the interval $(0, 24\pi)$?

3. Prove that

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_3} + \ldots + \frac{x_{n-1}^2}{x_n} + \frac{x_n^2}{x_1} \geq x_1 + x_2 + \ldots + x_n$$

where $x_1, x_1, \ldots, x_n$ are positive real numbers.

SOLUTIONS

SECTION - 1

1. Note that $0 \leq \arg z < 2\pi$ for all z. Hence (b) and (c) are both incorrect. Let $\alpha = \arg z = \frac{\pi}{6}$. Then $\arg z^2 = \frac{\pi}{3}$ and $\arg \overline{z^2} = 2\pi - \frac{\pi}{3} = \frac{5\pi}{3} \neq 2\left(\frac{\pi}{6}\right)$. Hence (a) is also incorrect.

2. When $0 < x < 1$, $\log_x (\log_x y^2) > 0$ is equivalent to $0 < \log_x y^2 < 1$ or $x < y^2 < 1$. When $x > 1$, we have $\log_x y^2 > 1$ or $y^2 > x$. Hence the shaded region in diagram (d) is correct.

3. The polar equation represents a conic section with eccentricity $\binom{m}{n}$, which must be greater than 1 if we are to have a hyperbola. Since $\binom{m}{n} = \binom{m}{m-n}$, there are 6 distinct values of $\binom{m}{n}$ for $1 \leq n \leq m \leq 5$, namely, $\binom{2}{1}, \binom{3}{1}, \binom{4}{1}, \binom{5}{1}, \binom{4}{2}$ and $\binom{5}{2}$.

4. We have $\sin 1 > 0 = \log_{10} 1$ and

$\sin \frac{3\pi}{2} = -1 < 0 < \log_{10} \frac{3\pi}{2}$.

Since $\pi < 4$, $\frac{5\pi}{2} < 10$ and $\log_{10}\frac{5\pi}{2} < 1 = \sin\frac{5\pi}{2}$.

Also $\log_{10}\frac{7\pi}{2} > 0 > -1 = \sin\frac{7\pi}{2}$

Since both functions are continuous, we have at least one real roots in each of the intervals $\left(1, \frac{3\pi}{2}\right)$, $\left(\frac{3\pi}{2}, \frac{5\pi}{2}\right)$ and $\left(\frac{5\pi}{2}, \frac{7\pi}{2}\right)$.

5. Let $f(x) = \frac{1}{a^x - 1} + \frac{1}{2}$

Then $f(-x) = \frac{a^x}{1-a^x} + \frac{1}{2} = -f(x)$

Hence $f(x)$ is an odd function. Since so is $F(x)$ and $G(x) = f(x)\,F(x)$, $G(x)$ is an even function.

6. Let $y = \frac{1-x}{1+x}$

Then $F(y) = x = \frac{1-y}{1+y}$

Hence $F(-2-x) = -\frac{3+x}{1+x} = -2 - F(x)$

Putting $x = 0, 1, 2$ and -2, we have $F(1) = 0$, $F(0) = 1$, $F\left(-\frac{1}{3}\right) = 2$ and $F(-3) = -2$. When $x = 0$, $F(-x) = F\left(\frac{1+x}{1-x}\right)$ becomes $1 = 0$. When $x = -3$, $F\left(\frac{1}{x}\right) = F(x)$ becomes $2 = -2$. When $x = 1$, $F(F(x)) = -x$ becomes $1 = -1$. All of these are false.

7. Let $x = \cos \omega t$ and $y = \sin \omega t$. Then

$-2xy = -\sin 2\omega t = \cos\left(-2\omega t + \frac{3\pi}{2}\right)$

and $y^2 - x^2 = -\cos 2\omega t = \sin\left(2\omega t + \frac{3\pi}{2}\right)$

Hence the point $(-2xy, y^2 - x^2)$ is moving clockwise along the unit circle at angular speed 2ω.

8. Let $AB = x$ and consider the face BCD as the base. As x increases from 0, the altitude increases, and hence so does the volume. The volume reaches a maximum when the face ACD is perpendicular to the base. As x continues to increase towards 2, the altitude decreases, and hence so does the volume.

SECTION - 2

1. We have $AD = 1 + x$, $BD = 1 - x$ and $CD = \sqrt{1-x^2}$. If $x \geq 0$, then AD is the longest of the three. In order that these segments form an acute triangle, we must have $BD^2 + CD^2 > AD^2$ or $x^2 + 4x - 1 > 0$. It follows that $0 \leq x < \sqrt{5} - 2$. By symmetry, if $x \leq 0$, we have $2 - \sqrt{5} < x \leq 0$. Hence the desired interval is $(2 - \sqrt{5}, \sqrt{5} - 2)$.

2. The given equation may be rewritten as

$$0 = \cos x - \cos\frac{x}{4} = -2\sin\frac{5x}{8}\sin\frac{3x}{8}$$

From $\sin\frac{5x}{8} = 0$, we have $x = \frac{8m\pi}{5}$ for any integer m. From $\sin\frac{3x}{8} = 0$, we have $x = \frac{8n\pi}{3}$ for any integer n. These are the general solutions of the original equation. The roots in $[0, 8\pi]$ are $1 \le m \le 14$ and $1 \le n \le 8$, but the same root is given by $(m, n) = (5, 3)$ and by $(m, n) = (10, 6)$. Hence the total number of roots in $[0, 8\pi]$ is $14 + 8 - 2 = 20$.

3. **First Solution :**

We use induction on n. For $n = 2$, we have

$$\frac{x_1^2}{x_2} + \frac{x_2^2}{x_1} = (x_1 + x_2)\left(\frac{x_1^2 + x_2^2}{x_1x_2} - 1\right) \ge x_1 + x_2$$

since $x_1^2 + x_2^2 \ge 2x_1x_2$ by the Arithmetic Mean – Geometric Mean Inequality. Suppose the result holds for some $n \ge 2$. Consider the next case with $n + 1$ numbers. Because of the cyclic symmetry, we may assume that x_{n+1} is the largest among them.

Then $\dfrac{x_{n+1}^2 - x_n^2}{x_1} \ge \dfrac{x_{n+1}^2 - x_n^2}{x_{n+1}}$

or $\dfrac{x_n^2}{x_{n+1}} + \dfrac{x_{n+1}^2}{x_1} - \dfrac{x_n^2}{x_1} \ge x_{n+1}$

By the induction hypothesis,

$$\left(\sum_{i=1}^{n-1}\frac{x_i^2}{x_{i+1}}\right) + \frac{x_n^2}{x_1} \ge \sum_{i=1}^{n} x_i$$

Addition yields the result which completes the inductive argument.

Second Solution :

Take $x_{n+1} = x_1$. It follows from the Arithmetic Mean – Geometric Mean Inequality that for $1 \le i \le n$, $\dfrac{x_i^2}{x_{i+1}} + x_{i+1} \ge 2x_i$. Summation yields the desired result.

Third Solution :

Take $x_{n+1} = x_1$. By Cauchy's Inequality,

$$\left(\sum_{i=1}^{n} x_i\right)^2 = \left(\sum_{i=1}^{n}\sqrt{x_{i+1}}\,\frac{x_i}{\sqrt{x_{i+1}}}\right)^2 \le \left(\sum_{i=1}^{n} x_i\right)\left(\sum_{i=1}^{n-1}\frac{x_i^2}{x_{i+1}}\right)$$

from which the desired result shows.

Fourth Solution :

Take $x_{n+1} = x_1$. By the Rearrangement Inequality,

$$\sum_{i=1}^{n}\frac{x_i^2}{x_{i+1}} \ge \sum_{i=1}^{n}\frac{x_i^2}{x_i} = \sum_{i=1}^{n} x_i$$

■■

Olympiad Enrichment Series-IV

useful for IIT-JEE 2008-09

This series is selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solution.

1. Let $f : \mathbb{N} \times \mathbb{N} \to \mathbb{N}$ be a function such that
$f(1, 1) = 2$
$f(m + 1, n) = f(m, n) + m$ and
$f(m, n + 1) = f(m, n) - n$
for all $m, n, \in \mathbb{N}$.
Find all pairs (p, q) such that $f(p, q) = 2001$.

2. Let f be a function defined on $[0, 1]$ such that $f(0) = f(1) = 1$ and $|f(a) - f(b)| < |a - b|$, for all $a \neq b$ in the interval $[0, 1]$.
Prove that $|f(a) - f(b)| < \frac{1}{2}$.

3. Find all pairs of integers (x, y) such that $x^3 + y^3 = (x + y)^2$.

4. Let $f(x) = \dfrac{2}{4^x + 2}$ for real numbers x.
Evaluate $f\left(\frac{1}{2001}\right) + f\left(\frac{2}{2001}\right) + ... + f\left(\frac{2000}{2001}\right)$.

5. Prove that for $n \geq 6$ the equation
$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + ... + \frac{1}{x_n^2} = 1$$
has integer solutions.

SOLUTIONS

1. We have
$$\begin{aligned} f(p, q) &= f(p-1, q) + p - 1 \\ &= f(p-2, q) + (p-2) + (p-1) \\ &= ... \\ &= f(1, q) + \frac{p(p-1)}{2} \\ &= f(1, q-1) - (q-1) + \frac{p(p-1)}{2} \\ &= ... \\ &= f(1, 1) - \frac{q(q-1)}{2} + \frac{p(p-1)}{2} \\ &= 2001. \end{aligned}$$
$\therefore \dfrac{p(p-1)}{2} - \dfrac{q(q-1)}{2} = 1999.$
i.e. $(p - q)(p + q - 1) = 2 \cdot 1999$.
Note that 1999 is a prime number and that $p - q < p + q - 1$ for $p, q \in \mathbb{N}$.
We have the following two cases:
1. $p - q = 1$ and $p + q - 1 = 3998$.
Hence $p = 2000$ and $q = 1999$.

2. $p - q = 2$ and $p + q - 1 = 1999$.
Hence $p = 1001$ and $q = 999$.
Therefore $(p, q) = (2000, 1999)$ or $(1001, 999)$.

2. We consider the following cases.
1. $|a - b| \le 1/2$.
Then $|f(a) - f(b)| < |a - b| \le 1/2$, as desired.
2. $|a - b| > 1/2$.
By symmetry, we may assume that $a > b$. Then

$$\begin{aligned}|f(a)-f(b)| &= |f(a)-f(1)+f(0)-f(b)| \\ &\le |f(a)-f(1)| + |f(0)-f(b)| \\ &< |a-1| + |0-b| \\ &= 1-a+b-0 \\ &= 1-(a-b) < 1/2 \text{ as desired.}\end{aligned}$$

3. Since $x^3 + y^3 = (x + y)(x^2 - xy + y^2)$,
all pairs of integers $(n, -n)$, $n \in \mathbb{Z}$, are solutions.
Suppose that $x + y \neq 0$. Then the equation becomes
$x^2 - xy + y^2 = x + y$
i.e. $x^2 - (y + 1)x + y^2 - y = 0$.
Treated as a quadratic equation in x, we calculate the discriminant
$\Delta = y^2 + 2y + 1 - 4y^2 + 4y = -3y^2 + 6y + 1$.
Solving for $\Delta \ge 0$ yields

$$\frac{3-2\sqrt{3}}{3} \le y \le \frac{3+2\sqrt{3}}{3}.$$

Thus the possible values for y are 0, 1 and 2, which lead to the solutions (1, 0), (0, 1), (1, 2), (2, 1) and (2, 2).
Therefore, the integer solutions of the equation are $(x, y) = (1, 0), (0, 1), (1, 2), (2, 1), (2, 2)$, and $(n, -n)$, for all $n \in \mathbb{Z}$.

4. Note that f has a half-turn symmetry about point (1/2, 1/2). Indeed,

$$f(1-x) = \frac{2}{4^{1-x}+2} = \frac{2\cdot 4^x}{4+2\cdot 4^x} = \frac{4^x}{4^x+2},$$

from which it follows that $f(x) + f(1 - x) = 1$.
Thus the desired sum is equal to 1000.

5. Note that $\frac{1}{a^2} = \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2} + \frac{1}{(2a)^2}$,
from which it follows that if
$(x_1, x_2, ..., x_n) = (a_1, a_2, ..., a_n)$
is an integer solution to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots + \frac{1}{x_n^2} = 1$$

then $(x_1, x_2, ..., x_{n-1}, x_n, x_{n+1}, x_{n+2}, x_{n+3})$
$= (a_1, a_2, ..., a_{n-1}, 2a_n, 2a_n, 2a_n, 2a_n)$
is an integer solution to

$$\frac{1}{x_1^2} + \frac{1}{x_2^2} + \ldots + \frac{1}{x_{n+3}^2} = 1$$

Therefore we can construct the solutions inductively if there are solutions for $n = 6$, 7 and 8.
Since $x_1 = 1$ is a solution for $n = 1$, (2, 2, 2, 2) is a solution for $n = 4$, and (2, 2, 2, 4, 4, 4, 4) is a solution for $n = 7$.
It is easy to check that (2, 2, 2, 3, 3, 6) and (2, 2, 2, 3, 4, 4, 12, 12) are solutions for $n = 6$ and $n = 8$, respectively. This completes the proof.

IIT's a dream

These days getting a seat in any of the Indian Institutes of Technology (IIT) has become a status symbol for students, more for the parents. The two years after the Class X becomes a rigorous journey not just for the aspirants but also their parents and sometimes even grandparents. We find most parents over-indulging in their child's IIT ambitions, right from selecting the coaching class, the reference material, mapping a proper time schedule to counselling them for tackling stress.

Being an IIT alumnus, I like other 'IIT frenzied' mothers decided to chart a path that could hone my status symbol. I withdrew my daughter from Welham, only for 'grooming' her for the Joint Entrance Engineering exam. I would, sometimes, dream of my daughter going for her counselling to one of the IITs and me bragging about it to friends and relatives. For two years, the IIT fever soared high in my house. My husband and my father, also IITians, were roped in. I would sometimes take one of her books and try to understand her notes, so that I could help her. I would keep awake till late night to 'slyly' watch her academic movements. But despite the genetic leanings, my daughter was never serious about IIT. She was confused, see-sawing from school to coaching to school. I could see the mistake I had made by thrusting my own desires on her. She wrote the exam last year but could not get a berth in the intellectual mansion. Surprisingly, I was not disappointed. Last month, as I was bidding her adieu while she was on her way back to the Australian National University, Canberra, where she is studying finance, I felt my eyes well up. I felt sorry for having inflicted so much pressure on her.

Back home from the airport, I started yelling at my son who is in Class IX : "If this is the way you study you will never be selected in IIT." Some dreams just don't go away.

Courtesy : TOI

SOLVED PAPER – Paper I (April)

NDA 2007

1. For any two vectors $\vec{a}$ and $\vec{b}$, consider the following statements :

   1. $|\vec{a}+\vec{b}| = |\vec{a}-\vec{b}| \Leftrightarrow \vec{a},\vec{b}$ are orthogonal.
   2. $|\vec{a}+\vec{b}| = |\vec{a}|+|\vec{b}| \Leftrightarrow \vec{a},\vec{b}$ are orthogonal.
   3. $|\vec{a}+\vec{b}|^2 = |\vec{a}|^2+|\vec{b}|^2 \Leftrightarrow \vec{a},\vec{b}$ are orthogonal.

   Which of the above statements are correct?
   (a) 2 and 3 only (b) 1, 2 and 3
   (c) 1 and 2 only (d) 1 and 3 only

2. Two vectors $2\hat{i}+m\hat{j}-3n\hat{k}$ and $5\hat{i}+3m\hat{j}-n\hat{k}$ are such that their magnitudes are respectively $\sqrt{14}$ and $\sqrt{35}$, where m, n are integers. Which one of the following is correct?
   (a) m takes 2 values, n takes 1 value
   (b) m takes 2 values, n takes 2 values
   (c) m takes 1 values, n takes 1 value
   (d) m takes 1 values, n takes 2 values

3. If $\int_{\ln 2}^{x} (e^x-1)^{-1}dx = \ln\frac{3}{2}$, then what is the value of x?
   (a) ln 4 (b) 1
   (c) e^2 (d) $\frac{1}{e}$

4. If $\int_{-3}^{2} f(x)dx = \frac{7}{3}$ and $\int_{-3}^{9} f(x)dx = -\frac{5}{6}$, then what is the value of $\int_{2}^{9} f(x)dx$?
   (a) $\frac{3}{2}$ (b) $-\frac{3}{2}$
   (c) $-\frac{19}{6}$ (d) $\frac{19}{6}$

5. What is the value of $\int \frac{dx}{(x^2+a^2)(x^2+b^2)}$
   (a) $\frac{[\{\tan^{-1}(x/a)\}/a-\{\tan^{-1}(x/b)\}/b]}{(b^2-a^2)}+c$
   (b) $\frac{[\{\tan^{-1}(x/a)\}/a+\{\tan^{-1}(x/b)\}/b]}{(b^2-a^2)}+c$
   (c) $\frac{[\{\tan^{-1}(x/a)\}/a-\{\tan^{-1}(x/b)\}/b]}{(a^2+b^2)}+c$
   (d) $\frac{[\{\tan^{-1}(x/a)\}/a+\{\tan^{-1}(x/b)\}/b]}{(a^2+b^2)}+c$

6. What is the equation of the curve passing through the origin and satisfying the differential equation $dy = (y \tan x + \sec x)\, dx$?
   (a) $xy = \cos x$ (b) $y \sin x = x$
   (c) $y = x \cos x$ (d) $y \cos x = x$

7. What is the solution of the differential equation $\frac{dy}{dx} = \sec(x+y)$?
   (a) $y+\tan\left\{\frac{(x+y)}{2}\right\}=c$
   (b) $y+\tan\left\{\frac{(x-y)}{2}\right\}=c$
   (c) $x + \tan(x + y) = c$
   (d) $y-\tan\left\{\frac{(x+y)}{2}\right\}=c$

8. For what value of k, does the differential equation $\frac{dy}{dx} = ky$ represent the law of natural decay?
   (a) 0.01 (b) $(10)^{-1}$
   (c) – 5 (d) 0

9. What is/are the critical point(s) of the function $f(x) = x^{2/3}(5-2x)$ on the interval [–1, 2]?
   (a) $\frac{3}{2}$ only (b) 0, $\frac{3}{2}$
   (c) 1 only (d) 0, 1

Math Archives

10 Best Problems

Prof. Shyam Bhushan*

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x}{2}\right)^{\frac{2}{x}}$ $a,b\in R^+$ is equal to

(a) ab (b) e^{ab} (c) $(1/2)ab$ (d) $e^{\ln ab}$

2. $\lim\limits_{x\to\infty}\dfrac{e^{\frac{1}{x^2}}-1}{2\tan^{-1}x^2-\pi}$ has the value equal to

(a) $-1/2$ (b) $1/2$ (c) 1 (d) zero

3. A and B are two fixed points whose co-ordinates are (3, 2) and (5, 4) respectively. The co-ordinates of a point P if ABP is an equilateral triangle, is/are

(a) $(4-\sqrt{3}, 3+\sqrt{3})$ (b) $(4+\sqrt{3}, 3-\sqrt{3})$

(c) $(3-\sqrt{3}, 4+\sqrt{3})$ (d) $(3+\sqrt{3}, 4-\sqrt{3})$

4. If the sum of the distances of a point from two perpendicular lines in a plane is 1, then its locus is

(a) square (b) circle

(c) a straight line (d) two intersecting lines

5. If the point $\left(\dfrac{a^3}{a-1}, \dfrac{a^2-3}{a-1}\right)$; $\left(\dfrac{b^3}{b-1}, \dfrac{b^2-3}{b-1}\right)$ & $\left(\dfrac{c^3}{c-1}, \dfrac{c^2-3}{c-1}\right)$ are collinear for three distinct values of a, b & c then $(ab + bc + ca) + 3(a + b + c)$

(a) abc (b) $2\,abc$ (c) $3\,abc$ (d) $4\,abc$

6. Let $f(x)=\begin{cases} x+a & ;\ \text{if } x<0 \\ |x-1| & ;\ \text{if } x\ge 0\end{cases}$ and

$g(x)=\begin{cases} x+1 & ;\ \text{if } x<0 \\ (x-1)^2+b & ;\ \text{if } x\ge 0\end{cases}$ where a and b are non $-$ve real numbers. Determine (gof). If $(gof)(x)$ is continuous for all x determine the value of a and b

7. If $f\left(\dfrac{x+y}{3}\right)=\dfrac{2+f(x)+f(y)}{3}\ \forall\ x, y\in R$ and $f'(2) = 2$ determine $f(x)$ where $f'(x)$ denotes derivation of f with respect to x.

8. If $x^2+y^2=t-\dfrac{1}{t}$ and $x^4+y^4=t^2+\dfrac{1}{t^2}$ then prove that $\dfrac{dy}{dx}=\dfrac{1}{x^3y}$

9. Evaluate the following

(a) $\lim\limits_{x\to 0}\left(\dfrac{1^x+2^x+\ldots+n^x}{n}\right)^{\frac{a}{x}}$

(b) $\lim\limits_{x\to\infty} n^{-n^2}\left\{(n+1)\left(n+\dfrac{1}{2}\right)\left(n+\dfrac{1}{2^2}\right)\ldots\left(n+\dfrac{1}{2^{n-1}}\right)\right\}^2$

10. Find the domain of the function

$f(x)=\dfrac{1}{[|x-2|]+[|6-x|]-8}$ where $[\cdot]$ denote- the greatest integer function.

SOLUTIONS

1. **(a) :** We have $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x}{2}\right)^{\frac{1}{x}}=\sqrt{ab}$

2. **(a):** $\lim\limits_{x\to\infty}\dfrac{e^{\frac{1}{x^2}}-1}{2\tan^{-1}x^2-\pi} \Rightarrow \lim\limits_{x\to\infty}-\dfrac{1}{2}\left(\dfrac{\dfrac{e^{\frac{1}{x^2}}-1}{\frac{1}{x^2}}}{\dfrac{x^2}{\tan^{-1}x^2}}\right)=\dfrac{-1}{2}$

3. **(a) :** Third vertex

$$\left[\left(\frac{x_1+x_2\pm\sqrt{3}(y_1-y_2)}{2}\right),\left(\frac{y_1+y_2\mp\sqrt{3}(x_1-x_2)}{2}\right)\right]$$

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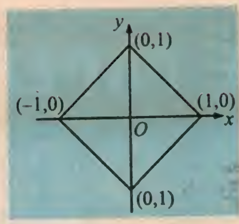
4. **(a)** : Locus is $|x| + |y| = 1$ represents square



5. **(c)** : Let the given points lie on the line

$$lx+my+n=0 \Rightarrow l\frac{t^3}{t-1}+m\frac{t^3-3}{t-1}+n=0$$

When $t = a, b, c$ this simplifies to

$$lt^3 + mt^2 + nt - (3m + n) = 0 \Rightarrow a+b+c=-\frac{m}{l};$$

$$ab+bc+ca=\frac{n}{l};\quad abc=\frac{3m+n}{l} \Rightarrow \text{result]}$$

6. $$f(x)=\begin{cases} x+a & ; \quad x<0 \\ 1-x & ; \quad 0\le x\le 1 \\ x-1 & ; \quad x\ge 1 \end{cases}$$

and $$g(x)=\begin{cases} x+1 & x<0 \\ (x-1)^2+b & x\ge 0 \end{cases}$$

$$\therefore\ (gof)(x)=\begin{cases} x+a+1 & ; \quad x<-a \\ (x+a-1)^2+b & ; \quad -a\le x<0 \\ x^2+b & ; \quad 0\le x<1 \\ (x-2)^2+b & ; \quad x\ge 1 \end{cases}$$

$\because$ $(gof)(x)$ is continuous $\forall x$

$\therefore$ it is continuous at $x = -a$, 0 and 1 also

i). **Continuity at $x = -a$:–**

$$\underset{x\to -a-}{Lt}(gof)(x)=\underset{x\to -a+}{Lt}(gof)(x)=(gof)(-a)$$

$\Rightarrow -a + a + 1 = (-a + a - 1)^2 + b \Rightarrow b = 0$

ii). **Continuity at $x = 0$:–**

$$\underset{x\to 0-}{Lt}(gof)(x)=\underset{x\to 0+}{Lt}(gof)(x)=(gof)(0)$$

$\Rightarrow (a - 1)^2 + b = b \Rightarrow a = 1$

$\therefore$ $a = 1$ and $b = 0$ are the required values

7. $$f\left(\frac{x+y}{3}\right)=\frac{2+f(x)+f(y)}{3}\ \forall x, y \in R$$

Putting $y = 0$, $\Rightarrow f\left(\frac{x}{3}\right)=\frac{2+f(x)+f(0)}{3}$

$$\Rightarrow 3\cdot f\left(\frac{x}{3}\right)=2+f(x)+f(0)$$

Replacing x with $3x$, $\Rightarrow 3f(x)=2+f(3x)+f(0)$ ... (i)

$$\therefore f'(x)=\underset{h\to 0}{lt}\frac{f(x+h)-f(x)}{h}=\underset{h\to 0}{lt}\frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$=\underset{h\to 0}{lt}\frac{\frac{2+f(3x)+f(3h)}{3}-f(x)}{h}$$

$$=\underset{h\to 0}{lt}\frac{f(3h)-f(0)}{3h}\ \because \text{from (i)}$$

$= f'(0)$

$\therefore f'(x) = f'(0)\ \forall x \Rightarrow f'(2) = f'(0) = 2\ \forall x$

$\Rightarrow f'(x) = 2\ \forall x \Rightarrow f(x) = 2x + c\ \forall x$

Put $x = 0$ in (i) $\Rightarrow f(0) = 2$

Now, $f(x) = 2x + c \Rightarrow f(0) = c$

$\Rightarrow c = 2 \quad \therefore f(x) = 2x + 2$

8. $$(x^2+y^2)^2=\left(t-\frac{1}{t}\right)^2=t^2+\frac{1}{t^2}-2=x^4+y^4-2$$

$\Rightarrow 2x^2y^2 = -2 \quad \therefore y^2 = -x^{-2}$

$$\therefore 2y\frac{dy}{dx}=2x^{-3} \qquad \therefore \frac{dy}{dx}=\frac{1}{x^3y}$$

9. **(a)** $$\lim_{x\to 0}\left(\frac{1^x+2^x+\dots n^x}{n}\right)^{\frac{a}{x}}$$

$$=e^{\lim\limits_{x\to 0}\left(\frac{1^x+2^x+\dots+n^x-n}{n}\right)\left(\frac{a}{x}\right)}\ \Big|\because \text{ it is in the form } 1^\infty$$

$$=e^{\frac{a}{n}\lim\limits_{x\to 0}\left\{\left(\frac{1^x-1}{x}\right)+\left(\frac{2^x-1}{x}\right)\dots+\left(\frac{n^x-1}{x}\right)\right\}}$$

$$=e^{\frac{a}{n}\{\log 1+\log 2+\log 3+\dots\log n\}}=e^{\frac{a}{n}\log(n!)}=(n!)^{\frac{a}{n}}$$

(b) $$\lim_{n\to\infty}\left\{\frac{(n+1)\left(n+\frac{1}{2}\right)\left(n+\frac{1}{2^2}\right)\dots\left(n+\frac{1}{2^{n-1}}\right)}{n^n}\right\}^n$$

$$\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\left(1+\frac{1}{2n}\right)^n\left(1+\frac{1}{2^2n}\right)^n\dots\left(1+\frac{1}{2^{n-1}n}\right)^n$$

$$=e\cdot e^{\frac{1}{2}}\cdot e^{\frac{1}{2^2}}\dots e^{\frac{1}{2^{n-1}}}\dots=e^{1+\frac{1}{2}+\frac{1}{2^2}+\dots+\frac{1}{2^{n-1}}+\dots}=e^2$$

10. f is not defined if $[|x-2|] + [|6-x|] = 8$

Now consider the following cases

Case I : If $x \le 2$

$|x - 2| = 2 - x$ and $|6 - x| = 6 - x$

$\therefore [2 - x] + [6 - x] = 8$

$\Rightarrow 8 + 2[-x] = 8 \Rightarrow [-x] = 0 \Rightarrow 0 \le -x < 1$

$\Rightarrow -1 < x \le 0 \quad \therefore x \in (-1, 0]$

Case II : If $2 < x \le 6$

$|x - 2| = x - 2$ and $|6 - x| = 6 - x$

$\therefore [x - 2] + [6 - x] = 8$

$\Rightarrow [x] + [-x] = 4 \ \therefore x \in \phi$

Case III : If $x > 6$

$[x - 2] + [x - 6] = 8 \Rightarrow 2[x] = 16$

$\Rightarrow [x] = 8 \Rightarrow x \in [8, 9)$

$\therefore D_f = R - (-1, 0] \cup [8, 9)$

PRACTICE PAPER IIT-JEE 2008

Exam on 13th April 2008

PHYSICS

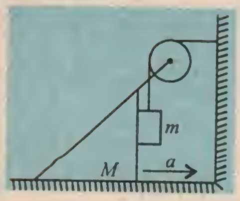
SECTION - I
(Straight Objective Type)

This section contains 9 multiple choice questiones numbered 1 to 9. Each question has 4 choices (a), (b), (c), (d) out of which only one is correct.

1. In a clock what is the time period of meeting of the minute hand and the second hand

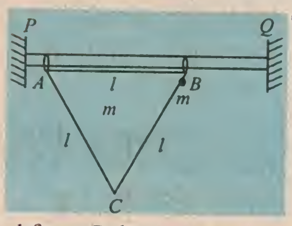
(a) 59 sec (b) 60/59 minutes
(c) 59/60 minutes (d) none of these

2. If wedge is moving with acceleration a as shown in the figure and friction coefficient between two blocks is μ then value of net force on m is



(a) ma
(b) $\sqrt{2}\,ma$
(c) $mg - \mu\, ma$ (d) zero

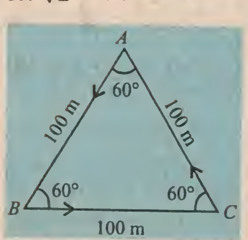
3. A rigid uniform triangular frame ABC of mass m is hanging from a rigid long horizontal rod PQ. The frame is constrained to move along horizontal without friction.



A bead of mass m is released from B that moves along BC. Displacement of frame when bead reaches C is

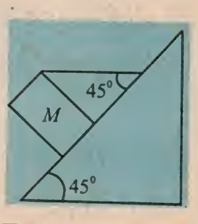
(a) $l/2$ (b) $l/4$ (c) $3l/\sqrt{2}$ (d) none

4. Three boys are running on a equitriangular track with same speed 5 m/s. At start, they were at the three corners with velocity along indicated directions. The velocity of approach of any one of them towards another, at $t = 10$ s equals



(a) 7.5 m/s (b) 10 m/s
(c) 5 m/s (d) 0 m/s

5. A block of mass 15 kg is resting on a rough inclined plane as shown in the figure. The block is tied up by a horizontal string which has a tension of 50 N. The friction force between the surfaces of contact is ($g = 10$ m/s^2)



(a) $50\sqrt{2}$ N (b) $100\sqrt{2}$ N
(c) 50 N (d) none of these

6. The distance between an object and screen is d. A convex lens of focal length f is placed between the object and the screen. If m is the transverse magnification of image then

(a) $f = \dfrac{md}{(1+m)^2}$ (b) $f = \dfrac{d}{(1-m)}$

(c) $f = \dfrac{md}{1+m}$ (d) $f = \dfrac{(1+m)^2}{m}d$

7. A frog sits on the end of a long board of length L. The board rests on a frictionless horizontal table. The frog wants to jump to the opposite end of the board. What is the minimum take-off speed (relative to ground) v that allows the frog to do the trick ? The board and the frog have equal masses.

(a) $\sqrt{\dfrac{gL}{2}}$ (b) $\sqrt{\dfrac{gL}{4}}$ (c) $\sqrt{\dfrac{2gL}{3}}$ (d) $\sqrt{\dfrac{4gL}{3}}$

8. To a man running upwards on the hill, the rain appears to fall vertically downwards with 4 m/s. The velocity vector of the man w.r.t. earth is $(2\hat{i} + 3\hat{j})$ m/s. If the man starts running down the hill with the same speed, then determine the relative speed of the rain w.r.t. man.

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CHINESE Olympiad Problems

SECTION - 1

1. Consider the following two statements about two real numbers a and b :

- P : The number a is positive.
- Q : Both $a > b$ and $\frac{1}{a} > \frac{1}{b}$ are true.

Which of the following statements is true?

(a) P is necessary and sufficient for Q.
(b) P is necessary but not sufficient for Q.
(c) P is sufficient but not necessary for Q.
(d) P is neither necessary nor sufficient for Q.

2. PQ is a focal chord of the parabola $y^2 = 2px$. MN is the projection of PQ on the directrix l. S_1 is the surface area generated by revolving PQ once around l, while S_2 is the surface area of the sphere with diameter MN. Which of the following statements is true?

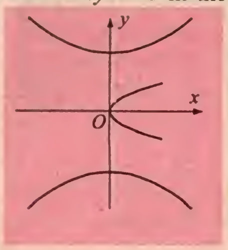
(a) $S_1 > S_2$ (b) $S_1 < S_2$
(c) $S_1 \geq S_2$ and $S_1 = S_2$ for at least one position of PQ
(d) The relative sizes of S_1 and S_2 depend on the position of PQ

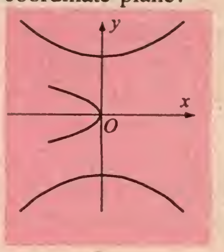
3. What is the value of x if

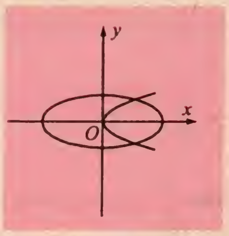
$$\arcsin x = \arccos \frac{4}{5} - \arccos\left(-\frac{4}{5}\right)?$$

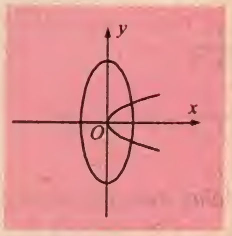
(a) $\frac{24}{25}$ (b) $-\frac{24}{25}$
(c) 0 (d) nonexistent

4. Let m and n be non zero real numbers. Which of the following diagrams represents the curves $mx + ny^2 = 0$ and $mx^2 + ny^2 = 1$ in the same coordinate plane?

(a)


(b)


(c)


(d)


5. Consider the following four statements about the equation $\bar{z} - \lambda z = w$, where z is a complex variable while w and λ are complex numbers with $|\lambda| \neq 1$:

- P : A solution is given by $z = \frac{\bar{\lambda} w + \bar{w}}{1 - |\lambda|^2}$
- Q : There is only one solution.
- R : There are two solution.
- S : There are infinitely many solution.

Which of the following statements is true?

(a) Only P and Q are true
(b) Only P and R are true
(c) Only P and S are true
(d) None of (a), (b) and (c) is true

6. Let a be a real number such that $0 < a < 1$. The sequence $\{x_n\}$ is defined by $x_1 = a$ and $x_n = a^{x_{n-1}}$ for $n > 1$. Which of the following statements is true?

(a) The sequence is increasing.
(b) The sequence is decreasing.
(c) The odd terms of the sequence are increasing, but the even terms are decreasing.
(d) The even terms of the sequence are increasing, but the odd terms are decreasing.

SECTION - 2

1. In triangle ABC, $AB = c$, $BC = a$, $CA = b$, $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$. If $b^2 - a^2 = ac$ and $\beta^2 = \alpha\gamma$, what is the value of β?

2. Consider the equation $2x_1 + x_2 + x_3 + x_4 + x_5 + x_6 + x_7 + x_8 + x_9 + x_{10} = 3$. How many non-negative integral solutions does it have?

3. For any two real numbers x and y, define $x \star y = ax + by + cxy$, where a, b and c are constants. It is known that $1 \star 2 = 3$, $2 \star 3 = 4$, and there is a non-zero real number d such that $x \star d = x$ for any real number x. What is the value of d?

4. Each of sixteen cities entered an A team and a B team in a soccer tournament. Any two teams were to play each other once, except for teams from the same city which would not play each other. At some point during the tournament, the A team of a certain city noticed that every other team had played a different number of games. How many games had been played by the B team of this city?

SOLUTIONS

SECTION - 1

1. If $a > b > 0$ or $0 > a > b$, then $\frac{1}{a} < \frac{1}{b}$. In order for Q to hold, we must have $a > 0 > b$. Hence P is necessary for Q but not sufficient.

2. We have $S_1 = \pi PQ(PM + QN) = \pi PQ^2 \geq \pi MN^2 = S_2$, with equality if and only if $PQ = MN$. This occurs when PQ is parallel to MN.

3. We have

$$\arcsin x = 2 \arccos \frac{4}{5} - \pi < 2 \arccos \frac{1}{\sqrt{2}} - \pi = -\frac{\pi}{2}$$

which is impossible.

4. If m and n are both negative, then $mx^2 + ny^2 = 1$ cannot hold. If one of them is positive and the other negative, the two curves are a hyperbola and a parabola which opens to the right. If both m and n are positive, the two curves are an ellipse and a parabola which opens to the left.

5. Taking conjugates, we have $z - \overline{\lambda z} = \overline{w}$. Multiplying both sides by λ and adding to the original equation, we have $\overline{z}(1 - |\lambda|^2) = \lambda\overline{w} + w$. Taking conjugates again yields $z(1 - |\lambda|^2) = \overline{\lambda}w + \overline{w}$. Since $|\lambda| \neq 1$, the unique solution is

$$z = \frac{\overline{\lambda}w + \overline{w}}{1 - |\lambda|^2}$$

6. Define $x_0 = 1$. We claim that

$$x_{2n} > x_{2n+2} > x_{2n+3} > x_{2n+1}$$

for all $n \geq 0$. Since $0 < a < 1$, we have $a^0 > a^a > a^1$ or $x_0 > x_2 > x_1$. Also $a^{x_0} < a^{x_2} < a^{x_1}$ or $x_1 < x_3 < x_2$. Thus the basis $n = 0$ is established. Suppose the claim holds for some $n \geq 0$. Then $a^{x_{2n+2}} < a^{x_{2n+3}} < a^{x_{2n+1}}$ or $x_{2n+3} < x_{2n+4} < x_{2n+2}$. Also, $a^{x_{2n+3}} > a^{x_{2n+4}} > a^{x_{2n+2}}$ or $x_{2n+4} > x_{2n+5} > x_{2n+3}$. This completes the inductive argument. It follows that the odd terms are increasing and the even terms are decreasing.

SECTION - 2

1. Note that $C > B > A$. Extend CB to D so that $BD = c$. Since $b^2 = a(c + a)$, triangles CAB and CDA are similar, so that $\angle CDA = \alpha$. It follows that $\angle BAD = \alpha$ also and $\beta = 2\alpha$. From $\gamma = 2\beta$ and $\alpha + \beta + \gamma = \pi$, we have $\beta = \frac{2\pi}{7}$.

2. Clearly, $x_1 = 0$ or 1. If $x_1 = 0$, then

$$x_2 + x_3 + \ldots + x_{10} = 3$$

There are $\binom{11}{3}$ sequences of eight 1's and three 0's. For each such sequence, interpret the 1's as dividers which partition the 0's into nine ordered groups. These corresponds to a solution $(x_2, x_3, \ldots, x_{10})$.
Similarly, if $x_1 = 1$, then

$$x_2 + x_3 + \ldots + x_{10} = 1$$

and we consider sequences of eight 1's and one 0. Hence the total number of solutions is

$$\binom{11}{3} + \binom{9}{1} = 174$$

3. We have $0 = 0 \star d = bd$. Since $d \neq 0$, we must have $b = 0$. Now $3 = 1 \star 2 = a + 2c$ and $4 = 2 \star 3 = 2a + 6c$. Hence $a = 5$ and $c = -1$. Now $1 = 1 \star d = 5 - d$ or $d = 4$.

4. Let us solve the general problem with n cities. We claim that the B team of the host city has played $n - 1$ games. The case $n = 1$ is trivial. Suppose the claim holds for some $n \geq 1$. Consider a tournament with $n + 1$ cities. These are $2n + 1$ teams other than the A team of the host city. Each has played from 0 to $2n$ games. Since the number of games each has played is distinct, these numbers must be $0, 1, \ldots, 2n$ respectively.

Now the team which has played $2n$ games has completed its schedule. The only team which it has not played must be the team which has played 0 games, so that these two teams are from the same city. Eliminating them, we have a tournament with n cities, where each remaining team has played exactly one game against the teams eliminated. Hence the teams other than the A team of the host city still have played different numbers of games. By the induction hypothesis, the B team of the host city has played $n - 1$ games in the reduced tournament, and hence n games in the original one. This completes the inductive argument. When $n = 16$, the B team of the host city has played 15 games.

Olympiad Enrichment Series-V

useful for IIT-JEE 2008-09

This series is selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solution.

1. Find all pairs of integers (a, b) such that the polynomial $ax^{17} + bx^{16} + 1$ is divisible by $x^2 - x - 1$.

2. Given a positive integer n, let $p(n)$ be the product of the non-zero digits of n. (If n has only one digit, then $p(n)$ is equal to that digit.) Let

$$S = p(1) + p(2) + ... + p(999)$$

What is the largest prime factor of S?

3. Let x_n be a sequence of nonzero real numbers such that

$$x_n = \frac{x_{n-2}x_{n-1}}{2x_{n-2} - x_{n-1}} \text{ for } n = 3, 4, ...$$

Establish necessary and sufficient conditions on x_1 and x_2 for x_n to be an integer for infinitely many values of n.

4. Solve the equation $x^3 - 3x = \sqrt{x+2}$

5. For any sequence of real number $A = \{a_1, a_2, a_3, ...\}$, define ΔA to be the sequence $\{a_2 - a_1, a_3 - a_2, a_4 - a_3, ...\}$. Suppose that all of the terms of the sequence $\Delta(\Delta A)$ are 1, and that $a_{19} = a_{92} = 0$. Find a_1.

SOLUTIONS

1. Let p and q be the roots of $x^2 - x - 1 = 0$. By **Vieta's theorem**, $p + q = 1$ and $pq = -1$. Note that p and q must also be the roots of $ax^{17} + bx^{16} + 1 = 0$. Thus

$$ap^{17} + bp^{16} = -1 \text{ and } aq^{17} + bq^{16} = -1$$

Multiplying the first of these equations by q^{16}, the second one by p^{16}, and using the fact that $pq = -1$, we find

$$ap + b = -q^{16} \text{ and } aq + b = -p^{16} \quad ... (1)$$

Thus

$$a = \frac{p^{16} - q^{16}}{p - q} = (p^8 + q^8)(p^4 + q^4)(p^2 + q^2)(p + q)$$

Since

$$p + q = 1$$
$$p^2 + q^2 = (p + q)^2 - 2pq = 1 + 2 = 3,$$
$$p^4 + q^4 = (p^2 + q^2)^2 - 2p^2q^2 = 9 - 2 = 7,$$
$$p^8 + q^8 = (p^4 + q^4)^2 - 2p^4q^4 = 49 - 2 = 47,$$

it follows tha $a = 1 \cdot 3 \cdot 7 \cdot 47 = 987$

Likewise, eliminating a in (1) gives

$$-b = \frac{p^{17} - q^{17}}{p - q} = p^{16} + p^{15}q + p^{14}q^2 + ... + q^{16}$$

$$= (p^{16} + q^{16}) + pq(p^{14} + q^{14}) + p^2q^2(p^{12} + q^{12}) + ... + p^7q^7(p^2 + q^2) + p^8q^8$$

$$= (p^{16} + q^{16}) - (p^{14} + q^{14}) + ... - (p^2 + q^2) + 1$$

For $n \geq 1$, let $k_{2n} = p^{2n} + q^{2n}$

Then $k_2 = 3$ and $k_4 = 7$, and

$$k_{2n+4} = p^{2n+4} + q^{2n+4}$$
$$= (p^{2n+2} + q^{2n+2})(p^2 + q^2) - p^2q^2(p^{2n} + q^{2n})$$
$$= 3k_{2n+2} - k_{2n}$$

for $n \geq 3$, Then $k_6 = 18$, $k_8 = 47$, $k_{10} = 123$, $k_{12} = 322$, $k_{14} = 843$, $k_{16} = 2207$

Hence

$$-b = 2207 - 843 + 322 - 123 + 47 - 18 + 7 - 3 + 1 = 1597$$

or $(a, b) = (987, -1597)$

Alternative Solution

The other factor is of degree 15 and we write

$$(c_{15}x^{15} - c_{14}x^{14} + ... + c_1x - c_0)(x^2 - x - 1) = ax^{17} + bx^{16} + 1$$

Comparing coefficients :

x^0 : $c_0 = 1$,

x^1 : $c_0 - c_1 = 0$, $c_1 = 1$

x^2 : $-c_0 - c_1 + c_2 = 0$, $c_2 = 2$,

and for $3 \leq k \leq 15$, $\quad x^k : -c_{k-2} - c_{k-1} + c_k = 0$

It follows that for $k \leq 15$, $c_k = F_{k+1}$

(The Fibonacci number)

Thus $a = c_{15} = F_{16} = 987$

and $b = -c_{14} - c_{15} = -F_{17} = -1597$

or $(a, b) = (987, -1597)$

2. Consider each positive integer less than 1000 to be a three-digit number by prefixing 0s to numbers with fewer than three digits. The sum of the products of the digits of all such positive numbers is

$$(0 \cdot 0 \cdot 0 + 0 \cdot 0 \cdot 1 + ... + 9 \cdot 9 \cdot 9) - 0 \cdot 0 \cdot 0$$
$$= (0 + 1 + ... + 9)^3 - 0$$

However, $p(n)$ is the product of non-zero digits of n. The sum of these products can be found by replacing 0 by 1 in the above expression, since ignoring 0's is equivalent to thinking of them as 1's in the products. (Note that the final 0 in the above expression becomes a 1 and compensates for the contribution of 000 after it is changed to 111.) Hence

$S = 46^3 - 1 = (46 - 1)(46^2 + 46 + 1) = 3^3 \cdot 5 \cdot 7 \cdot 103$ and the largest prime factors is 103

3. We have $\dfrac{1}{x_n} = \dfrac{2x_{n-2} - x_{n-1}}{x_{n-2}x_{n-1}} = \dfrac{2}{x_{n-1}} - \dfrac{1}{x_{n-2}}$

Let $y_n = 1/x_n$. Then $y_n - y_{n-1} = y_{n-1} - y_{n-2}$, i.e., y_n is an arithmetic sequence. If x_n is a nonzero integer when n is in an infinite set S, the y_n's for $n \in S$ satisfy $-1 \le y_n \le 1$. Since an arithmetic sequence is unbounded unless the common difference is 0, $y_n - y_{n-1} = 0$ for all n, which in turn implies that $x_1 = x_2 = m$, a nonzero integer. Clearly, this condition is also sufficient.

Alternative Solution

An easy induction shows that

$$x_n = \frac{x_1x_2}{(n-1)x_1 - (n-2)x_2} = \frac{x_1x_2}{(x_1 - x_2)n + (2x_2 - x_1)}$$

for $n = 3, 4, \ldots..$

In this form we see that x_n will be an integer for infinitely many values of n if and only if $x_1 = x_2 = m$ for some nonzero integer m.

4. It is clear that $x \ge -2$.

We consider the following cases.

1. $-2 \le x \le 2$. Setting $x = 2\cos a$, $0 \le a \le \pi$, the equation becomes

$$8\cos^3 a - 6\cos a = \sqrt{2(\cos a + 1)}$$

or $2\cos 3a = \sqrt{4\cos^2 \frac{a}{2}}$

from which it follows that $\cos 3a = \cos\frac{a}{2}$

Then $3a - \frac{a}{2} = 2m\pi$, $m \in \mathbb{Z}$, or $3a + \frac{a}{2} = 2n\pi$, $n \in \mathbb{Z}$

Since $0 \le a \le \pi$, the solution in this case is $x = 2\cos 0 = 2$, $x = 2\cos\frac{4\pi}{5}$, and $x = 2\cos\frac{4\pi}{7}$

2. $x > 2$. Then $x^3 - 4x = x(x^2 - 4) > 0$ and $x^2 - x - 2 = (x - 2)(x + 1) > 0$

or $x > \sqrt{x+2}$

It follows that $x^3 - 3x > x > \sqrt{x+2}$

Hence there are no solutions in this case

Therefore, $x = 2$, $x = 2\cos 4\pi/5$, and $x = 2\cos 4\pi/7$

Alternative Solution

For $x > 2$, there is a real number $t > 1$ such that

$$x = t^2 + \frac{1}{t^2}$$

The equation becomes

$$\left(t^2 + \frac{1}{t^2}\right)^3 - 3\left(t^2 + \frac{1}{t^2}\right) = \sqrt{t^2 + \frac{1}{t^2} + 2}$$

i.e. $t^6 + \dfrac{1}{t^6} = t + \dfrac{1}{t}$

i.e. $(t^7 - 1)(t^5 - 1) = 0$

which has no solutions for $t > 1$

Hence there are no solution for $x > 2$

For $-2 \le x \le 2$, refer to the first solution

5. Suppose that the first term of the sequence ΔA is d

Then $\Delta A = \{d, d + 1, d + 2, \ldots\}$

with the n^{th} term given by $d + (n - 1)$

Hence $A = (a_1, a_1 + d, a_1 + d + (d + 1), a_1 + d + (d + 1) + (d + 2) \ldots\}$

with the n^{th} term given by

$$a_n = a_1 + (n - 1)d + \frac{1}{2}(n - 1)(n - 2)$$

This shows that a_n is a quadratic polynomial in n with leading coefficient 1/2

Since $a_{19} = a_{92} = 0$, we must have

$$a_n = \frac{1}{2}(n - 19)(n - 92)$$

so $a_1 = (1 - 19)(1 - 92)/2 = 819$.

Required integral $= t \tan t - \log (\sec t) + c$

$= x \tan^{-1} x - \frac{1}{2} \log (1 + x^2) + c$

Example 7. : Evaluate $\int \sin^{-1} x \, dx$

Vedic method : Substituting $\sin^{-1} x = t$, $x = \sin t$

we get given Integral $I = \int t \cos t \, dt$

$$\begin{array}{ccccc} t & & 1 & & \\ & \searrow^{+} & & \searrow^{-} & \\ \cos t & & \sin t & & -\cos t \end{array}$$

Required integral $= t \sin t + \cos t + c$

$= x \sin^{-1} x - \sqrt{1 - x^2} + c$

D. Example of only one Logarithmic function as integrand : Here first substitution is made and then integration by parts using Vedic procedure is used

Example 8. : Evaluate $\int (\log x)^2 dx$

Traditional method : $\int (\log x)^2 dx = \int 1 \cdot (\log x)^2 dx$

$$= x \cdot (\log x)^2 - \int x \cdot 2(\log x) \frac{1}{x} dx$$

$$= x \cdot (\log x)^2 - 2\int 1 \cdot (\log x)\, dx$$

$$= x \cdot (\log x)^2 - 2\left\{x \cdot (\log x) - 2\int x \cdot \left(\frac{1}{x}\right) dx\right\}$$

$$= x \cdot (\log x)^2 - 2\{x \cdot (\log x) - x\} + C$$

$$= x \cdot (\log x)^2 - 2x \log x + 2x + C$$

Vedic method : Substitute $\log x = t$, $x = e^t$ gives the given integral as $\int t^2 e^t dt$

$$\begin{array}{ccccccc} t^2 & & 2t & & 2 & & \xleftarrow{\text{(Derivative)}} \\ & \searrow^{+} & & \searrow^{-} & & \searrow^{+} & \\ e^t & & e^t & & e^t & e^t & \xleftarrow{\text{(Integral)}} \end{array}$$

Integral $= t^2 e^t - 2t\, e^t + 2e^t = x (\log x)^2 - 2x(\log x) + 2x + c$

III. Differentiation of Parametric functions (only Polynomial type) : Here vedic method works well for quotient of polynomials where procedure deals only with coefficients of powers of the variable.

Example 9. : If $x = \dfrac{2at}{1+t^3}$, $y = \dfrac{2at^2}{(1+t^3)^2}$ then show that $\dfrac{dy}{dx} = \dfrac{x}{a}$

Vedic method : $x = \dfrac{2at}{1+t^3} = \dfrac{0 + 2a \cdot t + 0 \cdot t^2 + 0 \cdot t^3}{1 + 0 \cdot t + 0 \cdot t^2 + 1 \cdot t^3}$

We use **'Vertically Crosswise'** for coefficient of powers of t in numerator and denominator which is multiplied by the difference in powers of t to get coefficients of powers of t in answer. Only non-zero cross products are written below. Zero cross products can be identified easily and need not be expressed. For finding derivative of x with respect to t, we have

$[(1)(2a) - (0)(0)]\ [1 - 0]\ t^0$ gives $2a$ as coefficient of t^0

$[(0)(0) - (1)(2a)]\ [3 - 1]\ t^3$ gives $-4a$ as coefficient of t^3

gives $\dfrac{dx}{dt} = \dfrac{2a - 4at^3}{(1+t^3)^2} = \dfrac{2a(1 - 2t^3)}{(1+t^3)^2}$

next,

$$y = \frac{2at^2}{(1+t^3)^2} = \frac{0 + 0 \cdot t + 2a \cdot t^2 + 0 \cdot t^3 + 0 \cdot t^4 + 0 \cdot t^5 + 0 \cdot t^6}{1 + 0 \cdot t + 0 \cdot t^2 + 2 \cdot t^3 + 0 \cdot t^4 + 0 \cdot t^5 + 1 \cdot t^6}$$

Now for finding derivative of y with respect to t, we have

$[(1)(2a) - (0)(0)]\ [2 - 0]\ t^1$ gives $4a$ as coefficient of t^1

$[(0)(0) - (2)(2a)]\ [3 - 2]\ t^4$ gives $-4a$ as coefficient of t^4

$[(0)(0) - (1)(2a)]\ [6 - 2]\ t^7$ gives $-8a$ as coefficient of t^7

gives $\dfrac{dy}{dt} = \dfrac{4at(1 - t^3 - 2t^6)}{(1+t^3)^4} = \dfrac{4at(1 - t^3)(1 - 2t^3)}{(1+t^3)^4}$

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{2at}{a(1+t^3)} = \frac{x}{a}$$

In traditional method we use quotient formula and then take the quotient of two derivatives. Same procedure is true in case of the following example. These methods are quite elementary but lengthy and need not be given here. Vedic procedure is fast and easy to handle. Also possibility of making error by the student is quite less.

Example 10. : If $x = \dfrac{a(1 - t^2)}{1 + t^2}$, $y = \dfrac{2bt}{1 + t^2}$ then show that $\dfrac{dy}{dx} = -\dfrac{b^2 x}{a^2 y}$

We write, $x = \dfrac{a - at^2}{1 + t^2} = \dfrac{a + 0 \cdot t - a \cdot t^2}{1 + 0 \cdot t + 0 \cdot t^2}$

We use 'Vertically Crosswise' for coefficient of powers of t in numerator and denominator which is multiplied by the difference in powers of t to get coefficients of powers of t in answer. Only non-zero cross products are written below. Zero cross products can be identified easily and need not be expressed. For finding derivative of x with respect to t, we have

$[(1)(-a) - (1)(a)]\ [2 - 0]\ t^1$ gives $-4a$ as coefficient of t^1

gives $\dfrac{dx}{dt} = \dfrac{-4at}{(1+t^2)^2}$

Next, $y = \dfrac{2bt}{1 + t^2} = \dfrac{0 + 2b \cdot t + 0 \cdot t^2}{1 + 0 \cdot t + t^2}$

For finding derivative of y with respect to t, we have

$[(1)(2b) - (0)(0)]\ [1 - 0]\ t^0$ gives $2b$ as coefficient of t^0

$[(0)(0) - (1)(2b)]\ [2 - 1]\ t^2$ gives $-2b$ as coefficient of t^2

gives $\dfrac{dy}{dt} = \dfrac{2b - 2bt^2}{(1+t^2)^2}$

$$\frac{dy}{dx} = \frac{(dy/dt)}{(dx/dt)} = \frac{2b(1 - t^2)}{(1+t^2)^2} \div \frac{-4at}{(1+t^2)^2} = -\frac{b^2 x}{a^2 y}$$

Challenging Problems

Definite Integral

1. Match Column I with Column II

Col. I	Col. II
(a) $\int_{-1}^{1}\frac{dx}{1+x^2}$	(p) $\frac{1}{2}\log\frac{2}{3}$
(b) $\int_{0}^{1}\frac{dx}{\sqrt{1-x^2}}$	(q) $2\log\frac{2}{3}$
(c) $\int_{2}^{3}\frac{dx}{1-x^2}$	(r) $\frac{\pi}{3}$
(d) $\int_{1}^{2}\frac{dx}{x\sqrt{x^2-1}}$	(s) $\frac{\pi}{2}$

2. Fill in the blanks

(a) $\int_{-\pi}^{\pi}\frac{\cos^2 x}{1+a^x}dx =$ ....

(b) $\left|\int_{0}^{1}(1-t^2)dt\right| + \left|\int_{1}^{0}(t^2-1)dt\right| =$ ....

(c) $\int_{0}^{\pi/2}(\sin x)^{\cos x}[\cos x\cot x - \log(\sin x)^{\sin x}] =$ ....

3. Value of integral $\int_{0}^{\pi/2}\sqrt{\sin x}\cdot\cos^5 x\,dx$ is

(a) 64/231 (b) 64/131 (c) 64/331 (d) 6/231

4. Value of $\int_{0}^{\pi/2}\sin^6 x\,dx$ is

(a) $\pi/32$ (b) $3\pi/32$ (c) $5\pi/32$ (d) none

5. The value of $\int_{0}^{\pi/2}\log\left(\frac{4+3\sin x}{4+3\cos x}\right)dx$ is

(a) 2 (b) 3/4 (c) 0 (d) – 2

6. Value of $\int_{0}^{\pi}\frac{x}{1+\sin x}dx$ is

(a) $\pi/2$ (b) $\pi^2/4$ (c) $\pi/4$ (d) π

7. $\int_{-5}^{5}|x+2|\,dx =$

(a) 27 (b) 29 (c) 9 (d) 19

8. Value of $\int_{0}^{1}\tan^{-1}\left(\frac{2x-1}{1-\sin x-x^2}\right)dx =$

(a) 1 (b) 0 (c) – 1 (d) $\pi/4$

9. $\int_{-1/2}^{1/2}\left\{[x]+\log\left(\frac{1+x}{1-x}\right)\right\}dx =$, $[x]$ = greatest integer

(a) –1/2 (b) – 1 (c) 1 (d) 0

10. With usual notations, $\int_{1}^{2}\{[x^2]-[x]^2\}dx =$

(a) $4+\sqrt{2}-\sqrt{3}$ (b) $4-2+\sqrt{3}$
(c) $4-\sqrt{3}$ (d) $4-\sqrt{2}-\sqrt{3}$

11. Value of $\int_{0}^{1}\sqrt{\frac{1-x}{1+x}}\cdot dx =$

(a) $(\pi/2) + 1$ (b) $(\pi/2) - 2$
(c) $(\pi/2) - 1$ (d) $\pi - 1$

12. Value of $5050\cdot\frac{\int_{0}^{1}(1-x^{50})^{100}dx}{\int_{0}^{1}(1-x^{50})^{101}dx} =$

(a) 5050 (b) 5151 (c) 5150 (d) 5051

13. $\int_{0}^{\pi/2}\sin^3 x\cos^{7/2}x\,dx =$

(a) 1/99 (b) 11/9 (c) 2/99 (d) none

14. Value of $\int_{\pi/6}^{\pi/3}\frac{dx}{1+\sqrt{\tan x}} =$

(a) $\pi/4$ (b) $\pi/8$ (c) $\pi/12$ (d) $2\pi/3$

15. $\int_{0}^{\infty}\frac{\tan^{-1}(ax)}{x(1+x^2)}dx$ (where 'a' is a parameter) =

(a) $\frac{\pi}{2}\log(1+a)$ (b) $\frac{\pi}{2}\log a$
(c) $\pi\log a$ (d) none

16. The integral $\int_{0}^{1}\frac{x^c-1}{\log x}dx\,(c>-1) =$

(a) $c\log c$ (b) $(1/c)\log c$
(c) $\log(1+c)$ (d) none

ANSWERS

1. (a)-(s), (b)-(s), (c)-(p), (d)-(r) 2. (a) $\pi/2$, (b) –4/3, (c) 1
3. (a) 4. (c) 5. (c) 6. (d) 7. (b) 8. (b)
9. (a) 10. (d) 11. (c) 12. (d) 13. (b) 14. (c)
15. (a) 16. (c)

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Solutions of 20 Challenging Problems

Questions published in October '07 issue on page no. 19

1. (c) : $\lim_{n\to\infty} \cot^{-1}(1+n+n^2) = \lim_{n\to\infty} \tan^{-1}\frac{(n+1)-n}{1-n(n+1)}$

$$= \lim_{n\to\infty} \tan^{-1}\left(\frac{n}{n+1}\right) = \frac{\pi}{4} \quad \text{... (i)}$$

$$\lim_{x\to 0}\int_0^x \frac{t^2 dt}{(x-\sin x)\sqrt{a+t}} = \lim_{x\to 0}\frac{\int_0^x \frac{t^2}{\sqrt{a+t}}}{(x-\sin x)}$$

$$= \lim_{x\to 0}\frac{x^2}{\sqrt{a+x}(1-\cos x)} = \frac{2}{\sqrt{a}} \quad \text{... (ii)}$$

From (i) and (ii), $\frac{2}{\sqrt{a}} = \frac{\pi}{4} \Rightarrow a = \frac{32}{\pi^2}$

2. (c) : $\Delta = 0 \Rightarrow n = 5$ so $\frac{d^5 f(x)}{dx} = 5!\cdot 3^5 = {}^5P_3 \cdot \frac{\alpha}{\beta}$

so $\alpha = 81$, $\beta = 1/6$

3. (d) : Let $f(x) = \int_0^{\pi/2} \log(1-x^2\cos^2\theta)d\theta$

$$\Rightarrow f'(x) = \int_0^{\pi/2} \frac{-2x\cos^2\theta}{1-x^2\cos^2\theta}d\theta$$

$$f'(x) = \frac{1}{x}\int_0^{\pi/2} \frac{2(1-x^2\cos^2\theta)-2}{1-x^2\cos^2\theta}d\theta$$

$$= \frac{1}{x}\int_0^{\pi/2}\left[2-\frac{2}{(1+x\cos\theta)(1-x\cos\theta)}\right]d\theta$$

$$= \frac{1}{x}\left[2\theta - \frac{2}{\sqrt{1-x^2}}\tan^{-1}\sqrt{\frac{1-x}{1+x}}\cdot\tan\left(\frac{\theta}{2}\right)\right.$$

$$\left. - \frac{2}{\sqrt{1-x^2}}\tan^{-1}\sqrt{\frac{1+x}{1-x}}\cdot\tan\left(\frac{\theta}{2}\right)\right]_0^{\pi/2}$$

$$= \frac{1}{x}\left[\pi - \frac{2}{\sqrt{1-x^2}}\cdot\frac{\pi}{2}\right]$$

$$f'(x) = \pi\left[\frac{1}{x} - \frac{1}{x\sqrt{1-x^2}}\right]$$

$$f(x) = \pi\left[\log x - \log\left(\frac{1-\sqrt{1-x^2}}{x}\right)\right] + C$$

$$= \pi\log\left(\frac{x^2}{1-\sqrt{1-x^2}}\right) + C = \pi\log(1+\sqrt{1-x^2}) + C$$

But $f(0) = 0$, $\therefore C = -\pi\log 2$

Thus $f(x) = \pi\log(1+\sqrt{1-x^2}) - \pi\log 2$

4. (a) : Let $f(y) = \int_0^\infty e^{-xy}.\frac{\sin x}{x}dx$

$$f'(y) = \int_0^\infty -x\cdot e^{-xy}.\frac{\sin x}{x}dx = -\int_0^\infty e^{-xy}\sin x\,dx$$

$$= -\left[\frac{-y\sin x - \cos x}{1+y^2}\cdot e^{-xy}\right]_0^\infty = \frac{-1}{1+y^2}$$

$\Rightarrow f(y) = -\tan^{-1}y + C;\ y\to\infty,\ f(y)\to\frac{\pi}{2}$

$\therefore f(y) = \frac{\pi}{2} - \tan^{-1}y$

5. (d) : $I = \int_0^{\pi/2}\sin^3 x\cdot\cos^{11}x\,dx$

$$= \frac{1}{2}\cdot\frac{\lceil 2\lceil 6}{\lceil 8} = \frac{1}{2}\cdot\frac{1.5!}{7!} = \frac{1}{84}$$

6. (c) : $I = \int_0^\infty \frac{x}{1+e^x}dx = \int_0^\infty \frac{x\cdot e^{-x}}{e^{-x}+1}dx$

$= \int_0^1 \frac{\log t}{t+1}dt$, put $e^{-x} = t$ and integrate by parts

$$= \int_0^1 \frac{\log(1+t)}{t}dt$$

$$= \frac{\pi^2}{12} \qquad \left[\because 1 - \frac{1}{2^2} + \frac{1}{3^2} - \frac{1}{4^2} + \ldots = \frac{\pi^2}{12}\right]$$

7. (c) : Auxiliary equation $D^2 - 7D + 6 = 0$ give c.f. $= c_1e^x + c_2e^{6x}$

Now P.I. $= \frac{1}{D^2-7D+6}\cdot e^{2x} = -\frac{1}{4}e^{2x}$

$\therefore f(x) = \text{C.F.} + \text{P.I.} = c_1e^x + c_2e^{6x} - \frac{1}{4}e^{2x}$

when $x = 0$, $f(x) = 0$, $f'(x) = 1$

$\Rightarrow c_1 + c_2 - \frac{1}{4} = 0 \Rightarrow c_1 + c_2 = \frac{1}{4}$

and $c_1 + 6c_2 = \frac{3}{2} \Rightarrow c_1 = 0$ and $c_2 = \frac{1}{4}$

$\therefore f(x) = \frac{1}{4}(e^{6x} - e^{2x})$

8. **(b)** : Auxiliary equation is $D^2 - 3D + 2 = 0$

$\Rightarrow \quad D = 1, 2$

$\therefore$ Complete solution is $x = c_1e^t + c_2e^{2t}$

when $x = 0,\ t = 0,\ \dfrac{dx}{dt} = 0$

$\Rightarrow \quad c_1 + c_2 = 0$ and $c_1 + 2c_2 = 0$

$\Rightarrow c_1 = 0,\ c_2 = 0$

$\therefore$ Solution is $x = 0$ which is y axis.

9. **(c)** : Auxiliary equation is $D^2 + 1 = 0 \Rightarrow D = \pm i$

$\therefore$ Equation of curve is $y = c_1\cos x + c_2\sin x$

when $x = 0,\ y = 2 \Rightarrow c_1 = 2$

$x = \pi/2,\ y = 2 \Rightarrow c_2 = 2$

$\therefore$ Area under the curve $y = 2\cos x + 2\sin x$ is

$$2\int_0^{\pi/2} (\cos x + \sin x)dx$$

$= 2[\sin x - \cos x]_0^{\pi/2} = 4$ sq. units

10. **(a)** : $f(x) = k\log x$ satisfying given condition

$\therefore \quad f(e) + f(1/e) = 0$

11. **(d)** : On R, $|f(x)|$ is clearly a constant single valued function so it is differentiable

12. **(b)** : Three girls out of 10 can be selected in $^{10}C_3$ ways. Fixing the position of Supreet and Kushween, 3 girls can be arranged in $^3P_3 = 6$ ways. Similarly 7 other girls can be arranged in 7! ways and Supreet and Kushween can be arranged in 2! ways. Total number of ways of arranging 12 girls in a ring = 11!

$\therefore$ Required probability $= \dfrac{^{10}C_3 \cdot 3! \cdot 7! \cdot 2!}{11!} = \dfrac{2}{11}$

13. **(c)** : Let E_1, be the event of choosing the ideal coin with prob. 1/2 for a head turning up. E_2, the event of choosing coin with prob. 1/3 and E_3 with prob. 2/3. Let E be event of head appearing when the selected coin is tossed twice.

$\therefore P(E_1) = P(E_2) = P(E_3) = 1/3$

Required prob. (Using Baye's theorem) =

$$\frac{P(E_1)\cdot P(E/E_1)}{P(E_1)\cdot P(E/E_1) + P(E_2)\cdot P(E/E_2) + P(E_3)\cdot P(E/E_3)}$$

$$= \frac{\frac{1}{3}\cdot\frac{1}{4}}{\frac{1}{3}\cdot\frac{1}{4} + \frac{1}{3}\cdot\frac{1}{9} + \frac{1}{3}\cdot\frac{4}{9}} = \frac{9}{29}$$

14. **(b)** :

$$\text{Matrix } B = \begin{bmatrix} 1 & 2 & 2 \\ -2 & -1 & -1 \\ 1 & -4 & -3 \end{bmatrix} \Rightarrow B^{-1} = \frac{1}{3}\begin{bmatrix} -1 & -2 & 0 \\ -7 & -5 & -3 \\ 9 & 6 & 3 \end{bmatrix}$$

15. **(c)** : Trace of matrix $A = 3$, Trace of matrix $B - 3$

$\therefore$ Ratio $= 3 : -3 \Rightarrow 1 : -1$

16. **(a)** : $\sin^{-1}1 + \cos^{-1}1 = \pi/2$

17. **(c)** : No. of zeros in $n! = (n - 3)$ if $n \geq 6$

18. **(a)** : Sum of square roots of all products taken n positive numbers

$$a_1, a_2, \ldots, a_n < \left(\frac{n-1}{2}\right)(a_1 + a_2 + a_3 + \ldots + a_n)$$

$$\therefore\ 9\left(\frac{20\times 19}{2}\right) = 90\times 19 = 1710$$

19. **(i) → (q), (ii) → (r), (iii) → (p), (iv) → (s)**

(i) $\displaystyle f(x) = \frac{1}{x+1} + \frac{2}{x^2+1} + \frac{4}{x^4+1} + \ldots + \frac{2^n}{x^{2^n}+1}$

$\displaystyle = \frac{1}{x-1} - \frac{2^{n+1}}{x^{2^{n+1}}-1}$

$\therefore\ D(f) = (-\infty, \infty) - \{1\}$

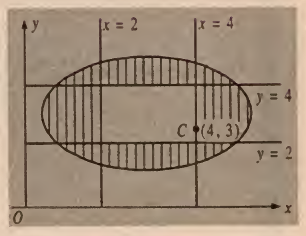
(ii) $E \equiv 4x^2 + 9y^2 - 32x - 54y + 109 = 0$

$\Rightarrow 4(x-4)^2 + 9(y-3)^2 = 36$

$\displaystyle \Rightarrow \frac{(x-4)^2}{9} + \frac{(y-3)^2}{4} = 1$

which is an ellipse having centre at $C(4, 3)$

$\therefore\ S \subset E$



(iii) $f(x) = \lambda \sin x \cdot C$

where $\displaystyle C = \int_0^{\pi/2} \sin y f(y)dy$

$\displaystyle \therefore\ C = \lambda C\int_0^{\pi/2} \sin^2 y\, dy = \lambda C\cdot\frac{1}{2}\cdot\frac{\pi}{2} \Rightarrow \lambda = \frac{4}{\pi}$

(iv) Let $\displaystyle f(a) = \int_0^\infty \frac{\tan^{-1}(ax)}{x(1+x^2)}dx$

$$f'(a) = \int_0^\infty \frac{x}{1+a^2x^2}\cdot\frac{1}{x(1+x^2)}dx$$

$$= \frac{1}{1-a^2}\int_0^\infty\left(\frac{1}{1+x^2} - \frac{a^2}{1+a^2x^2}\right)dx$$

$$= \frac{1}{1-a^2}[\tan^{-1}x - a\tan^{-1}(ax)]_0^\infty$$

$$= \frac{(1-a)\pi/2}{1-a^2} = \frac{\pi}{2}\cdot\frac{1}{1+a}$$

$\displaystyle \Rightarrow f(a) = \frac{\pi}{2}\log(1+a) + K$

when $a = 0$, so $K = 0$

$\displaystyle \therefore\ f(a) = \frac{\pi}{2}\log(1+a)$

20. **(a)** : $\displaystyle \frac{x_1}{y_1} + \frac{x_2}{y_2} + \ldots + \frac{x_{20}}{y_{20}} = \frac{20}{30} = \frac{2}{3}$

CHINESE Olympiad Problems

SECTION 1

1. If $-1 < \alpha < 0$ and $\theta = \arcsin \alpha$, what is the solution set of the inequality $\sin x < \alpha$?
(a) $\{x \mid 2n\pi + \theta < x < (2n+1)\pi - \theta, n \in \mathbb{Z}\}$
(b) $\{x \mid 2n\pi - \theta < x < (2n+1)\pi + \theta, n \in \mathbb{Z}\}$
(c) $\{x \mid (2n-1)\pi + \theta < x < 2n\pi - \theta, n \in \mathbb{Z}\}$
(d) $\{x \mid (2n-1)\pi - \theta < x < 2n\pi + \theta, n \in \mathbb{Z}\}$

2. Let $M = \{z \mid (z-1)^2 = |z-1|^2,\ z \in \mathbb{C}\}$. Which of the following statements is true?
(a) $M = \mathbb{C} - \mathbb{R}$ (b) $M = \mathbb{R}$
(c) $\mathbb{R} \subset M \subset \mathbb{C}$ (d) $M = \mathbb{C}$

3. Let a, b and c be real numbers such that
$a^2 - bc - 8a + 7 = 0$ and $b^2 + c^2 + bc - 6a + 6 = 0$.
What are the possible values of a?
(a) $(-\infty, \infty)$ (b) $(-\infty, 1] \cup [9, \infty)$
(c) $(0, 7)$ (d) $[1, 9]$

4. If none of the faces of a tetrahedron is an isosceles triangle, what is the minimum number of edges no two of which have the same length?
(a) 3 (b) 4 (c) 5 (d) 6

5. On the plane is a point set M. For $1 \le i \le 7$, a circle C_i passes through exactly i points in M. What is the minimum number of points in M?
(a) 11 (b) 12 (c) 21 (d) 28

6. The side lengths of a triangle are a, b and c, its area is $\frac{1}{4}$ and its circumradius is 1. Let $s = \sqrt{a} + \sqrt{b} + \sqrt{c}$ and $t = \frac{1}{a} + \frac{1}{b} + \frac{1}{c}$. Which of the following statements is true?
(a) $s > t$ (b) $s = t$ (c) $s < t$
(d) The relative sizes of s and t depends on a, b and c.

SECTION 2

1. The distance between the centres of two spheres of radii 6 inside a cylinder of radius 6 is equal to 13. A plane tangent to both spheres intersects the cylinder in an ellipse. What is the sum of the major and minor axes of this ellipse?

2. Let $f(x) = \dfrac{4^x}{4^x + 2}$. What is the value of $\displaystyle\sum_{i=1}^{1000} f\left(\frac{i}{1001}\right)$?

3. An equilateral triangle ABC of side n is divided into n^2 equilateral triangles of side 1 by lines parallel to the sides of ABC. Each point which is a vertex of at least one of these unit triangles is labeled with a real number. The points A, B and C are labeled with a, b and c respectively. In each rhombus composed of two unit triangles with a common side, the sums of the labels on the two sets of opposite vertices are equal.
(a) Determine the shortest distance between a point with the largest label and a point with the smallest label.
(b) Determine the sum of all labels.

4. In a tournament, every two players have exactly one game between them. There are no ties. A player A is awarded a prize if for every other player B, either A defeats B or A defeats some player C who defeats B. Prove that if only one player is awarded a prize, then this player defeats every other player.

Solutions on page no.73

Olympiad Enrichment Series VI

useful for IIT-JEE 2008-09

This series is selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solution.

1. Find all real numbers x satisfying the equation
$$2^x + 3^x - 4^x + 6^x - 9^x = 1$$

2. Prove that $16 < \sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 17$

3. Determine the number of ordered pairs of integers (m, n) for which $mn \geq 0$ and
$$m^3 + n^3 + 99mn = 33^3$$

4. Let a, b and c be positive real numbers such that $a + b + c \leq 4$ and $ab + bc + ca \geq 4$.
Prove that at least two of the inequalities
$|a - b| \leq 2$, $|b - c| \leq 2$, $|c - a| \leq 2$ are true.

5. Evaluate $\sum_{k=0}^{n} \frac{1}{(n-k)!(n+k)!}$

SOLUTION

1. Setting $2^x = a$ and $3^x = b$, the equation becomes
$1 + a^2 + b^2 - a - b - ab = 0$
Multiplying both sides of the last equation by 2 and completing the squares gives
$(1 - a)^2 + (a - b)^2 + (b - 1)^2 = 0$
Therefore $1 = 2^x = 3^x$, and $x = 0$ is the only solution.

2. Note that $2(\sqrt{k+1} - \sqrt{k}) = \frac{2}{\sqrt{k+1} + \sqrt{k}} < \frac{1}{\sqrt{k}}$

Therefore $\sum_{k=1}^{80} \frac{1}{\sqrt{k}} > 2\sum_{k=1}^{80} (\sqrt{k+1} - \sqrt{k}) = 16$

which proves the lower bound.
On the other hand,
$$2(\sqrt{k} - \sqrt{k-1}) = \frac{2}{\sqrt{k} + \sqrt{k-1}} > \frac{1}{\sqrt{k}}$$
Therefore
$$\sum_{k=1}^{80} \frac{1}{\sqrt{k}} < 1 + 2\sum_{k=2}^{80} (\sqrt{k} - \sqrt{k-1}) = 2\sqrt{80} - 1 < 17$$
which proves the upper bound. Our proof is complete.

3. Note that $(m + n)^3 = m^3 + n^3 + 3mn(m + n)$.
If $m + n = 33$,
then $33^3 = (m + n)^3 = m^3 + n^3 + 3mn(m + n)$
$= m^3 + n^3 + 99mn$
Hence $m + n - 33$ is a factor of $m^3 + n^3 + 99mn - 33^3$
We have $m^3 + n^3 + 99mn - 33^3$
$= (m + n - 33)(m^2 + n^2 - mn + 33m + 33n + 33^2)$
$= \frac{1}{2}(m + n - 33)[(m - n)^2 + (m + 33)^2 + (n + 33)^2]$
Hence there are 35 solutions altogether :
$(0, 33), (1, 32), ..., (33, 0)$ and $(-33, -33)$.

4. We have $(a + b + c)^2 \leq 16$
i.e. $a^2 + b^2 + c^2 + 2(ab + bc + ca) \leq 16$,
i.e. $a^2 + b^2 + c^2 \leq 8$,
i.e. $a^2 + b^2 + c^2 - (ab + bc + ca) \leq 4$,
i.e. $(a - b)^2 + (b - c)^2 + (c - a)^2 \leq 8$,
and the desired result follows.

5. Let S_n denote the desired sum. Then
$$S_n = \frac{1}{(2n)!}\sum_{k=0}^{n} \frac{(2n)!}{(n-k)!(n+k)!}$$
$$= \frac{1}{(2n)!}\sum_{k=0}^{n} \binom{2n}{n-k}$$
$$= \frac{1}{(2n)!}\sum_{k=0}^{n} \binom{2n}{k}$$
$$= \frac{1}{(2n)!} \cdot \frac{1}{2}\left[\sum_{k=0}^{2n} \binom{2n}{k} + \binom{2n}{n}\right]$$
$$= \frac{1}{(2n)!} \cdot \frac{1}{2}\left[2^{2n} + \binom{2n}{n}\right]$$
$$= \frac{2^{2n-1}}{(2n)!} + \frac{1}{2(n!)^2}$$

■■

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. If $f(x)=\begin{cases}|x| & \text{when } x\le 2\\ [x] & \text{when } x>2\end{cases}$ then

(a) $\lim\limits_{x\to 2^-} f(x)=-2$ (b) $\lim\limits_{x\to 2^+} f(x)=-2$
(c) $\lim\limits_{x\to 2^+} f(x)=f(2)$ (d) $\lim\limits_{x\to 2} f(x)$ does not exist

2. If $n\in N$ then $\lim\limits_{x\to\infty}\frac{x^n}{e^x}=0$

(a) when n is even only (b) for no value of n
(c) for all values of n (d) when n is odd only

3. If $y=\sin^{-1}\left[\frac{1-x^2}{1+x^2}\right]$, then $\frac{dy}{dx}=$

(a) $\frac{-2}{1+x^2}$ (b) $\frac{2}{1+x^2}$ (c) $\frac{1}{2+x^2}$ (d) $\frac{2}{2-x^2}$

4. The derivative of $\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x = 0$ is

(a) $\frac{1}{8}$ (b) $\frac{1}{4}$ (c) $\frac{1}{2}$ (d) 1

5. If $y=\tan^{-1}\left(\frac{\log(e/x^2)}{\log(ex^2)}\right)+\tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$ then $\frac{d^2y}{dx^2}$ is

(a) 2 (b) 1 (c) 0 (d) -1

6. Let $f(x)=\begin{cases} xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & ; x\neq 0\\ 0 & ; x=0\end{cases}$, Check whether $f(x)$ is differentiable at $x = 0$. Is it continuous at $x = 0$? Justify

7. A triangle ABC, right angle at C, with $CA = b$ and $CB = a$, moves such that the angular points A and B slide along x-axis and y-axis respectively. Find locus of C

8. Prove that $\sin\theta\cdot\sec 3\theta=\frac{1}{2}(\tan 3\theta-\tan\theta)$ and hence find the sum to 'n' terms of the series
$\sin\theta\cdot\sec 3\theta+\sin 3\theta\cdot\sec 3^2\theta+\sin 3^2\theta\cdot\sec 3^3\theta+...$

9. Consider a real valued function $f(x)$ satisfying $2f(xy)=(f(x))^y+(f(y))^x\ \forall\ x,y\in R$ and $f(1)=p$ where $p\neq 1$ then find $(p-1)\sum\limits_{r=1}^{n} f(r)$

10. A line makes angles $\alpha, \beta, \gamma, \delta$ with four diagonals of a cube. Prove that $\sum\limits_{r\in\{\alpha,\beta,\gamma,\delta\}}\cos^2(r)=\frac{4}{3}$

SOLUTIONS

1. **(c)** : $\lim\limits_{x\to 2^+} f(x)=\lim\limits_{x\to 0^+}[2+h]=2=f(2)$

$\lim\limits_{x\to 2^-} f(x)=\lim\limits_{x\to 0^+}|2-h|=2$

Hence limit exists

2. **(c)** : Use L' Hospital rule

$\lim\limits_{x\to\infty}\frac{x^n}{e^x}=\frac{n!}{e^\infty}=0$

3. **(a)** : Put $x = \tan\theta \Rightarrow y=\frac{\pi}{2}-2\theta$

4. **(b)** : $P=\tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ Put $x = \tan\theta$

$P=\frac{\theta}{2}$

Hence $\frac{dP}{dx}=\frac{1}{1+x^2}$ ... (1)

For $Q=\tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ Put $x = \sin\phi$

$\frac{dQ}{dx}=\frac{2}{\sqrt{1-x^2}}$ ... (2)

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from (1) & (2)

$$\frac{dP}{dQ}=\frac{\frac{1}{2(1+x^2)}}{\frac{2}{\sqrt{1-x^2}}} \qquad \text{Put } x=0$$

$$\left.\frac{dP}{dQ}\right|_{x=0}=\frac{1}{4}$$

5. **(c) :** In given function put $\log x^2=\tan\theta$

Therefore $y=\frac{\pi}{4}+\tan^{-1}3$

Hence $\frac{d^2y}{dx^2}=0$

6. $$f(x)=\begin{cases} xe^{\frac{-2}{x}} & ; \; x>0 \\ 0 & ; \; x=0 \\ x & ; \; x<0 \end{cases}$$

$\therefore\ f(0)=0$

L.H.D.

$$f'(0-)=\lim_{h\to 0^+}\frac{f(0-h)-f(0)}{-h}=\lim_{h\to 0^+}\frac{-h}{-h}=1$$

R.H.D.

$$f'(0+)=\lim_{h\to 0^+}\frac{f(0+h)-f(0)}{h}=\lim_{h\to 0^+}\frac{he^{-2/h}}{h}=e^{-\infty}=0$$

Since **L.H.D.** $\neq$ **R.H.D.**

$\therefore$ f is not differentiable at $x=0$

Since **L.H.D.** and **R.H.D.** exist

$\therefore$ f is continuous at $x=0$

7. Let $A=(p,0)$, $\quad B=(0,q)$

Let $C(h,k)$ be the any point on the locus

$$CB=a=\sqrt{h^2+(k-q)^2} \qquad \text{... (i)}$$

$$CA=b=\sqrt{(h-p)^2+k^2} \qquad \text{... (ii)}$$

$$AB=\sqrt{p^2+q^2} \qquad \text{... (iii)}$$

Since $\angle C=90°$

$\therefore\ AB^2=AC^2+BC^2$

$\Rightarrow\ p^2+q^2=a^2+b^2 \qquad$ ... (iv)

from (i) and (ii) $q=k\pm\sqrt{a^2-h^2}$

$p=h\pm\sqrt{b^2-k^2}$

$\therefore$ from (iv) Locus is $bx\pm ay=0$

8. $$\frac{\sin\theta}{\cos 3\theta}=\frac{\cos\theta\cdot\sin\theta}{\cos\theta\cdot\cos 3\theta}=\frac{1}{2}\frac{\sin(3\theta-\theta)}{\cos\theta\cdot\cos 3\theta}$$

$$=\frac{1}{2}(\tan 3\theta-\tan\theta)$$

$$\therefore \sum_{r=1}^{n}\sin 3^{r-1}\theta\cdot\sec 3^r\theta$$

$$=\sum_{r=1}^{n}\frac{\sin 3^{r-1}\theta}{\cos 3^r\theta}=\frac{1}{2}\sum_{r=1}^{n}\frac{\sin(3^r\theta-3^{r-1}\theta)}{\cos 3^r\theta\cdot\cos 3^{r-1}\theta}$$

$$=\frac{1}{2}\sum_{r=1}^{n}(\tan 3^r\theta-\tan 3^{r-1}\theta)=\frac{1}{2}[\tan 3^n\theta-\tan\theta]$$

9. $2f(xy)=(f(x))^y+(f(y))^x\ \forall\ x,y\in R$

Put $y=1$

$\Rightarrow\ 2f(x)=f(x)+(f(1))^x \Rightarrow\ f(x)=p^x\Rightarrow f(1)=p$

$$\therefore \sum_{r=1}^{n}f(r)=\sum_{r=1}^{n}p^r=\frac{p^{n+1}-p}{p-1}$$

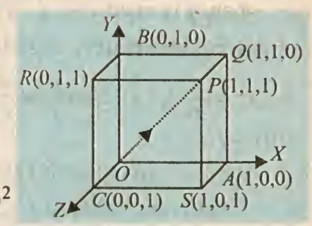
$$(p-1)\sum_{r=1}^{n}f(r)=(p^{n+1}-p)$$

10. D.C's of $\overline{OP}=\left(\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right)$

D.C's of $\overline{AR}=\left(-\frac{1}{\sqrt3},\frac{1}{\sqrt3},\frac{1}{\sqrt3}\right)$

D.C's of $\overline{BS}=\left(\frac{1}{\sqrt3},\frac{-1}{\sqrt3},\frac{1}{\sqrt3}\right)$

D.C's of $\overline{CQ}=\left(\frac{1}{\sqrt3},\frac{1}{\sqrt3},-\frac{1}{\sqrt3}\right)$



Let l, m, n be the D.C's of required line

$\therefore\ \cos^2\alpha+\cos^2\beta+\cos^2\gamma+\cos^2\delta$

$$=\frac{1}{3}((l+m+n)^2+(l-m+n)^2+(l+m-n)^2$$

$$=\frac{4}{3}$$

CHALLENGING
Subjective Problems
CALCULUS

by Er. Tapas Kr. Yogi

1. Evaluate $\lim_{n\to\infty}\int_0^2\left(1+\frac{t}{n+1}\right)^n dt$

2. Evaluate $\sum_{i=0}^{\infty} i(i+1)p^{i-2},\ p\in(0,1)$

3. Let us define a new function $\delta(x)$ with the properties $\delta(x)=0$ for all $x\neq 0$ and $\int_{-\infty}^{\infty}\delta(x)f(x)dx=f(0)$ for every function $f(x)$. Evaluate $\int_{-\infty}^{\infty}\delta'(x)\sin x\,dx$ (Assume $\delta(x)$ is differentiable).

4. Evaluate $\int_0^{\pi/2}\sin x\,dx+\int_0^1\sin^{-1}x\,dx$

5. Given $f(x)=x^5+2x^3+2x$. Find $[f^{-1}(-5)]$.

6. Let $I(n)=\int_0^{\pi}\sin(nx)\,dx$. Find $\sum_{n=0}^{\infty} I(5^n)$.

(Instructions for Qns. 7 – 9) **:** Let $\theta_k(x)$ be 0 for $x<k$ and 1 for $x\geq k$. The Dirac delta function (more appropriately distribution) is defined to be $\delta_k(x)=\frac{d}{dx}\theta_k(x)$. Suppose $\frac{d^2f(x)}{dx^2}=\delta_1(x)+\delta_2(x)$ and $f(0)=f'(0)=0$.

7. What is the value of $f(5)$?

8. What is the value of $f'(5)$?

9. Find the number of points where $f'(x)$ is not differentiable?

SOLUTIONS

1. $\lim_{n\to\infty}\int_0^2\left(1+\frac{t}{n+1}\right)^n dt=\lim_{n\to\infty}\left(1+\frac{t}{n+1}\right)^{n+1}\Bigg|_0^2$

$=\lim_{n\to\infty}\left(1+\frac{2}{n+1}\right)^{n+1}-\lim_{n\to\infty}\left(1+\frac{0}{n+1}\right)^{n+1}$

$=e^2-1$ [Since, $\lim_{n\to\infty}\left(1+\frac{x}{n+1}\right)^{n+1}=e^x$]

2. $S=\sum_{i=0}^{\infty} i(i+1)p^{i-2}=\sum_{i=0}^{\infty} i(i-1)p^{i-2}+2\sum_{i=0}^{\infty} ip^{i-2}$

Now, $\sum_{i=0}^{\infty} i(i-1)p^{i-2}=\frac{d^2}{dp^2}\sum_{i=0}^{\infty}p^i=\frac{d^2}{dp^2}\left(\frac{1}{1-p}\right)$

$=\frac{d}{dp}\left(\frac{1}{(1-p)^2}\right)=\frac{2}{(1-p)^3}$ and $\sum_{i=0}^{\infty} ip^{i-2}=\frac{1}{p}\sum_{i=0}^{\infty} ip^{i-1}$

$=\frac{1}{p}\times\frac{d}{dp}\sum_{i=0}^{\infty}p^i=\frac{1}{p}\times\frac{d}{dp}\left(\frac{1}{1-p}\right)=\frac{1}{p(1-p)^2}$

Hence, $S=\frac{2}{(1-p)^3}+\frac{1}{p(1-p)^2}=\frac{2}{p(1-p)^3}$

3. $I=\int_{-\infty}^{\infty}\delta'(x)\sin x\,dx=\delta(x)\sin x\Big|_{-\infty}^{\infty}-\int_{-\infty}^{\infty}f(x)\cos x\,dx$

[Integration by parts]

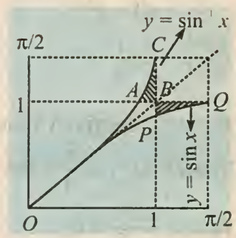
$=0-\cos(0)=-1$ [Since $f(x)=0$ at $\pm\infty$]

4. Note that area ABC is same as area PBQ.

So, the net required area

= Area of rectangle $(O1Q\pi/2)$

$=\frac{\pi}{2}\times 1=\frac{\pi}{2}$



5. Now, $f(f^{-1}(x))=x$

Differentiating this, $f'(f^{-1}(x))\times(f^{-1}(x))'=1$

$\Rightarrow (f^{-1}(x))'=\frac{1}{f'(f^{-1}(x))}$

Now, $f^{-1}(5)$ is the solution to $-5=x^5+2x^3+2x$

Which has the unique solution of $x=-1$

So, $(f^{-1}(-5))'=\frac{1}{f'(-1)}=\frac{1}{5+6+2}=\frac{1}{13}$

6. For odd n, $I(n)=\int_0^{\pi}\sin nx\,dx=-\frac{\cos nx}{n}\Big|_0^{\pi}=\frac{2}{n}$

So, $\sum_{n=0}^{\infty} I(5^n)=\sum_{n=0}^{\infty}\frac{2}{5^n}=\frac{5}{2}$ [using sum of infinite G.P.]

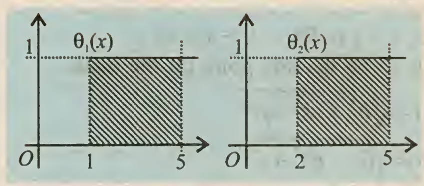
7. When integrated once $f'(x)=\int(\delta_1(x)+\delta_2(x))dx$

$=\theta_1(x)+\theta_2(x)+c$ and $f'(0)=c=0$

So, $f'(x)=\theta_1(x)+\theta_2(x)$

Now to find $f(x)$, we integrate the above eqn. or take the area covered by $\theta_1(x)$ and $\theta_2(x)$.

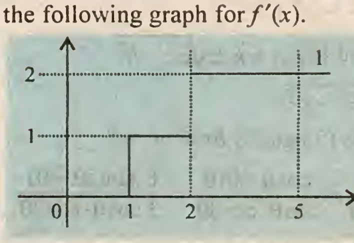
Now the graphs of $\theta_1(x)$ and $\theta_2(x)$ are



Now, $f(5)$ = Area of shaded region in both the graphs

$=4\times 1+3\times 1=7$

8. Combining the graphs of both $\theta_1(x)$ and $\theta_2(x)$ we have the following graph for $f'(x)$.



So, $f'(5)=2$

9. From the graph it is clear that $f'(x)$ is not differentiable at two points $x=1$ and $x=2$.

CHINESE Olympiad Problems

SOLUTIONS

Section 1.

1. We have $-\pi/2 < \theta < 0$. On the interval $(-\pi, \pi)$, the solutions of $\sin x = \alpha$ are $x = \theta$ and $x = -\pi - \theta$. At $x = -\pi/2$, $\sin x = -1 < \alpha$. Hence the solution set of $\sin x < \alpha$ on this interval is $\{x| -pi - \theta < x < \theta\}$. Translated into the whole line, it becomes

$$\{x|(2n-1)\pi - \theta < x < 2n\pi + \theta\}.$$

2. From $(z-1)^2 = |z-1|^2 = (z-1)(\overline{z}-1)$,
we have $(z-1)(z-\overline{z}) = 0$. Hence either $z = 1$ or $z = \overline{z}$, so that z is a real number. Moreover, it can be any real number.

3. The equations may be rewritten as $bc = a^2 - 8a + 7$ and $b^2 + c^2 + bc = 6a - 6$. Subtracting three times the first from the second, we have $(b-c)^2 = -3(a-1)(a-9)$. Since $(b-c)^2 \geq 0$, we must have $1 \leq a \leq 9$.

4. Clearly, we need three distinct edge lengths to make even one face non-isosceles. We can achieve the desired result with three by making two edges equal in length if and only if they are opposite to each other.

5. There are 7 points on C_7. At most 2 of the points on C_6 are on C_7. At most 4 of the points on C_5 are on C_6 or C_7. Hence M has at least 12 points. Draw four circles, no three concurrent and every two intersect at two points except for two which do not intersect at all. Call these two C_4 and C_5. Put all the points of intersections of these four circles into M, plus an additional point on each of C_5 and C_7. Then M has 12 points and C_i passes through exactly i points for $4 \leq i \leq 7$. It is easy to draw additional circles C_i for $1 \leq i \leq 3$, so that C_i passes through exactly i points already in M.

6. The area of the triangle is given by abc divided by four times the circumradius. Hence $abc = 1$. By the Arithmetic Mean – Geometric Mean Inequality,

$$t = \frac{1}{2}\left(\frac{1}{a}+\frac{1}{b}\right)+\frac{1}{2}\left(\frac{1}{b}+\frac{1}{c}\right)+\frac{1}{2}\left(\frac{1}{c}+\frac{1}{a}\right)$$

$$\geq \frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{bc}}+\frac{1}{\sqrt{ca}} = \frac{\sqrt{a}+\sqrt{b}+\sqrt{c}}{\sqrt{abc}} = s.$$

Equality cannot hold as otherwise $a = b = c = 1$, but then the circumradius should be $\frac{1}{\sqrt{3}}$.

Section 2.

1. Consider a cross-section containing the axis of the cylinder and the major axis of the ellipse. A vertex of the ellipse is at a distance 6 from the axis of the cylinder, while the centre of a sphere is also at a distance 6 from the major axis of the ellipse. Hence the length of the major axis is equal to the distance between the centres of the spheres. Since the length of the minor axis is equal to the diameter of the cylinder, the desired sum is $13 + 12 = 25$.

2. Note that

$$f(x)+f(1-x) = \frac{4^x}{4^x+2}+\frac{4^{1-x}}{4^{1-x}+2} = 1$$

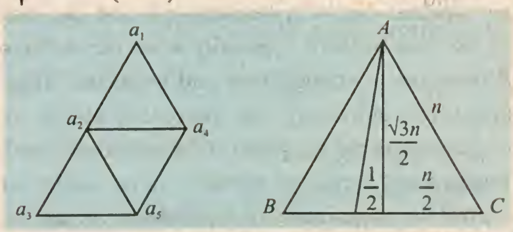
for all x. Hence

$$\sum_{i=1}^{1000} f\left(\frac{i}{1000}\right) = \sum_{i=1}^{500}\left(f\left(\frac{i}{1000}\right)+f\left(1-\frac{i}{1000}\right)\right) = 500.$$

3. Consider any three small triangles in a row, and let the five points be labeled a_1, a_2, a_3, a_4 and a_5 as shown in Figure 1. From $a_1 + a_5 = a_2 + a_4$ and $a_2 + a_5 = a_3 + a_4$, we have $a_1 - a_2 = a_2 - a_3$. It follows that the labels along any line parallel to side of ABC are in arithmetic progression, so that the extreme values of the labels lie on the perimeter of ABC.

(a) Denote the desired distance by r. If $a = b = c$, then all labels are the same, and $r = 0$. If $a \neq b \neq c \neq a$, then the extreme values lie on the vertices of ABC only and $r = n$. Suppose $b = c \neq a$. Then the extreme values lie on A and BC. Figure 2 shows that $r = \frac{\sqrt{3}n}{2}$ if n is even and

$$r = \sqrt{\left(\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}n}{2}\right)^2} = \frac{\sqrt{3n^2+1}}{2}$$ if n is odd.



(b) Obtain two other labeled triangles by rotating ABC 120° and 240° respectively. Superimpose them and label each point by the sum of its labels in the three triangles. Since the rhombic rules holds for each triangle, it also hold for the composite triangle. Hence all labels are $a + b + c$. Since there are

$$1+2+\ldots+(n+1) = \frac{(n+1)(n+2)}{2}$$

points, the sum of all labels in ABC is

$$\frac{(n+1)(n+2)(a+b+c)}{6}.$$

4. We claim that a player A who has the most wins will be awarded a prize. For any player B who beats A, at least one of the players whom A beats must beat B, as otherwise B will have more wins than A. This justifies the claim. Suppose A is the only player awarded a prize. Denote by S and T the sets of players who beat A and those who lose to A, respectively. Suppose S is non-empty. In the mini-tournament among the players in S, there will be a player B with the most wins, and hence a prize-winner. Since B beats A and A beats every player in T, B is also a prize-winner in the overall tournament, contradicting the uniqueness hypothesis. Hence S is empty, and A beats other player. ■■

How to Overcome JEE STRESS

Are you in the age group of sixteen to eighteen? A period that should be full of fun, frolic and laughter, when you have just begun to understand life better and develop your point of view. But wait, the picture is not as rosy as it looks because this is the time when you have to think about your career, which in itself is very tough as there are umpteen number of options before you. Amongst these options one of the most popular is getting through IIT-JEE. Popular it is, but remember that it is also the toughest exam in the country. It demands heavy preparation, at times, for almost a year or two combined with single minded devotion to achieve the goal and an enormous amount of concentration. These 3 factors, namely, preparation, devotion and concentration generate a lot of stress which can be nerve wrecking. There are several cases of candidates dropping the proposition of facing the JEE midway, unable to combat the stress thereby leading to a wastage of precious time and money.

No doubt taking competitive exams with hoards of other aspirants can be a daunting task and can unnerve even the most brilliant, especially when the students are still teenagers – young, fresh and protected. Thus the process of countering the generated stress needs understanding of the age group of the individuals, the level of maturity and the factors prevalent in the society which can affect the commitment of the students. As said earlier the students are young and fresh with the maturity level not being very high. The innumerable opportunities available in the form of outings, movies – picnics and TV act as the hindering factors. So what do you see? I am sure many of you do not see a very positive picture. Surely this exam sounds like a trip to slavery of books and of course stress. Dear young friends, do not worry because here are some tips (Aw! Not again!) that will help you cope with the stress and come out with flying colours.

1. **Time management :** Time, as they say, can be your best friend or your worst enemy – its just the way you treat it. In our analysis we have found out that the major stress generation is due to the lack of time. Better time management can always reduce the stress. We don't advise you to give up entertainment and lock yourself in a room. In fact, if one starts the preparation early, say a year or two in advance one can always find time for better things in life. You have to manage time in such a way that the breaks that you take for relaxation can relax you enough to take up the coming hours of hard work with fresh vigour and enthusiasm.

2. **Interaction :** Another tendency that has been noted in the candidates appearing for JEE is that they tend to go into a shell as they start their preparation. They refuse to speak (this is their stress speaking for them) to anyone including their parents and siblings. This is a very dangerous tendency. If the amount of stress generated is x units while preparing it becomes x^2 if one goes into a shell. Often a pep talk by parents, friends or guides can act as a tonic and bring down stress. The topic need not be and if possible should not be related to JEE. Verbal interaction with respect to something interesting happening or the latest movie or about the popular soap on the tube could do the trick more often than not. So go ahead, unwind and talk away examination blues.

3. **Fixing a schedule :** Scheduling of the subjects to be covered is another way of reducing stress. Here everyone has his own way of scheduling. Some people prefer to do a single subject, say, mathematics at a stretch for ten to fifteen days and then go to, say physics. The problem in this case is of money. Some others prefer to do a single subject in a day. Say two hours for maths, two hours for physics and two hours for chemistry. It is advised in this connection to stick to the schedule with which you are most comfortable. It is often observed that just because the previous year's topper has stuck to a certain schedule others tend to blindly follow it. Avoid doing this lest it be a stress increment factor.

4. **Be firm and determined :** Last but not the least is "do not allow your mind to wavier". Students come from various strata of society. Each strata has its own problems, those of a plenty and those of a few. But the secret to success lies in putting aside all those things and striving towards the goal with untiring efforts. There can never be stumbling block for those with a sense of purpose.

So friends, "Awake, arise and rest not till the goal is achieved".

EXAMINER's MIND

by R.K. Tyagi, HOD Maths, Samarth Shiksha Samiti, New Delhi.

LIMITS, CONTINUITY AND DIFFERENTIABILITY

The questions given in this column has been prepared strictly on the basis of NCERT for XII class. Over the last few years IIT-JEE / AIEEE has drawn its paper heavily from NCERT books.

STRAIGHT OBJECTIVE TYPE I

In each of the questions given below, four choices are given of which only one is correct. You have to select the correct choice which is most appropriate.

1. The value of $\lim\limits_{x\to 1}\dfrac{x^{1/4}-x^{1/5}}{x^3-1}$ is

(a) $\frac{1}{20}$ (b) $\frac{1}{40}$ (c) $\frac{1}{60}$ (d) $\frac{3}{20}$

2. Let the function $f(x)=\dfrac{3-(243-5x)^{1/5}}{(5x+27)^{1/3}-3}$, $(x\neq 0)$ is continuous every where, then the value of $f(0)$ equals,

(a) $\frac{1}{15}$ (b) $\frac{2}{15}$ (c) $\frac{4}{15}$ (d) $\frac{1}{45}$

3. The value of the $\lim\limits_{n\to\infty}[(2^n+1)(7^n+10^n)]^{1/n}$ is equal to

(a) $\frac{10!}{6!}$ (b) ${}^{10}C_5$ (c) ${}^{10}C_7$ (d) ${}^{6}C_3$

4. Let $f(x)=\left[\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)\right]$, $0<x\le 2\pi$ (where $[x]$ denotes the greatest integer $\le x$). The number of points of discontinuity of $f(x)$ are

(a) 5 (b) 6 (c) 4 (d) 3

5. If $f(n+1)=\frac{1}{5}\left\{f(n)+\dfrac{1}{f(n+2)}\right\}, \forall\, n\in N, f(n)>0$ then $\lim\limits_{n\to\infty} f(n+1)$ is equal to

(a) -3 (b) 3 (c) $\frac{1}{5}$ (d) 36

6. Let $f(x)=\dfrac{1}{\sqrt{36-x^2}}$ then $\lim\limits_{x\to 3}\dfrac{f(x)-f(3)}{x-3}$ equals

(a) $27\sqrt{3}$ (b) $9\sqrt{3}$ (c) $\frac{1}{27\sqrt{3}}$ (d) $\frac{1}{9\sqrt{3}}$

7. $\lim\limits_{t\to 0}\left[\min\,(x^2+4x+6)\dfrac{\sin t}{t}\right]$ is equal to (where [ ] denotes the greatest integer function)

(a) 1 (b) -1
(c) 0 (d) does not exist

8. The value of $\lim\limits_{x\to\beta}(1+ax^2+bx+c)^{\frac{1}{x-\beta}}$ assuming α, β are roots of the equation $ax^2+bx+c=0$, is

(a) $a(\beta-\alpha)$ (b) $e^{a(\beta-\alpha)}$ (c) $a(\alpha+\beta)$ (d) $e^{a(\alpha+\beta)}$

9. The value of the limit $\lim\limits_{x\to\frac{\pi}{2}}[1^{1/\cos^2 x}+2^{1/\cos^2 x}+3^{1/\cos^2 x}+...+10^{1/\cos^2 x}]^{\cos^2 x}$ is

(a) 10! (b) ${}^{10}C_{10}$ (c) ${}^{10}C_1$ (d) $(10)^{10}$

10. The value of λ if $\lim\limits_{x\to 1}\dfrac{x^4-1}{x-1}=\lim\limits_{x\to\lambda}\dfrac{x^3-\lambda^3}{x^2-\lambda^2}$ $(\lambda\neq 0)$

(a) $\frac{3}{8}$ (b) $\frac{8}{3}$ (c) $\frac{4}{3}$ (d) $\frac{3}{4}$

11. Let $f(x)=\begin{cases}\dfrac{3x}{|x|+2x^2}, & x\neq 0\\ 0, & x=0\end{cases}$ then $\lim\limits_{x\to 0} f(x)$ is

(a) 3 (b) 1/3
(c) ± 3 (d) does not exist

STRAIGHT OBJECTIVE TYPE II

(One or more than one correct choice(s))

In each of the questions given below, four choices are given of which one or more than one are correct. You have to select the correct choice(s) accordingly.

12. The $\lim\limits_{x\to 0}\left(\dfrac{a^x+b^x+c^x+d^x}{4}\right)^{\lambda/x}$, $(a, b, c, d, \lambda>0)$ is equal to

(a) $abcd$ if $\lambda=4$ (b) 1 if $\lambda=1$
(c) $abcd$ if $\lambda=2$ (d) $(abcd)^{3/4}$ if $\lambda=3$

13. The point at which $P(x)=\dfrac{\log x}{\sqrt{1-4x^2}}$ is continuous

(a) $\frac{1}{2}$ (b) $\frac{1}{3}$ (c) $\frac{1}{4}$ (d) $\frac{1}{10}$

14. Let $f(x)=x^2+xg'(1)+g''(2)$ and $g(x)=f(1)x^2+xf'(x)+f''(x)$ then $f'(1)+f'(2)$

(a) – 4 (b) 0 (c) 2 (d) 4

15. Let $h(x)=\min\{x, x^2\}\ \forall\, x\in R$ then

(a) h is continuous every where

(b) h is differentiable everywhere

(c) h is nondifferentiable at two values of x

(d) $h'(x)$ is a constant function $\forall\, x>1$

16. Let $f(x)=\begin{cases} |x|+3 & , \forall \quad x\le -3 \\ -2x & , \forall \quad -3<x<3 \\ 6x+2 & , \forall \quad x\ge 3 \end{cases}$, then

(a) f is continuous at $x = -3$

(b) f is differentiable at $x = 3$

(c) f is continuous for $-3 < x < 3$

(d) f is continuous for $x > 3$

LINKED COMPREHENSION TYPE

In these questions, a passage has been given followed by questions based on each of the passages. You have to answer the questions based on the passage given.

Passage for question 17.

Let the functions $f(x)$ is differentiable at the point $x = c$ where $f(x)=\begin{cases} x^2 & ,\forall \quad x\le c \\ ax+b & \forall \quad x>c \end{cases}$

on the bases of above information, answer the following questions :

i. The value of a is

(a) 0 (b) c^2 (c) $2c^2$ (d) $2c$

ii. The value of b is

(a) $-c^2$ (b) $\frac{-c^2}{2}$ (c) $-2c$ (d) c^2

iii. The value of $\lim\limits_{x\to c^-} f(x)$ is

(a) $-c^2$ (b) c^2 (c) $\frac{a}{c}$ (d) $\frac{c}{a}$

iv. For same values of a & b the value of $\lim\limits_{x\to a}(ax+b)$ equals

(a) c^2 (b) $3b$ (c) $-3b$ (d) $-c^2$

Passage for question 18.

Let the $f(x)=\dfrac{a\cos x+bx\sin x-6}{x^5}$ and $\lim\limits_{x\to 0} f(x)$ is finite value. ($a, b, \in R$)

on the basis of the above information, answer the following questions :

i. The value of a is

(a) 4 (b) 6 (c) 8 (d) 3

ii. The value of b is

(a) 3 (b) 6 (c) 4 (d) 2

iii. Value of $\lim\limits_{x\to 0} f(x)$ for same values of a & b is

(a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $-\frac{1}{4}$ (d) $\frac{1}{4}$

iv. If $f(0) = c$ then c equals,

(a) $\frac{a+b}{4}$ (b) $\frac{a-b}{4}$ (c) $\frac{b-a}{4}$ (d) $\frac{b-a}{12}$

Passage for question 19.

If $y=f(x)$ be a differentiable function of x such that its second, third ... n^{th} order derivative exist and whose n^{th} order derivative is denoted by given below.

$$y_n, \frac{d^n y}{dx^n}, f^n(x), y^n, D^n y, d^n y \text{ which means}$$

$$\lim_{h\to 0}\frac{f^{n-1}(x+h)-f^{n-1}(x)}{h}=f^n(x)$$

on the basis of above information, answer the following questions :

i. If $y = \log_a x$ and $x \in I^+$, then $y_n(e)$ equals.

(a) $(-1)^n\, n!\, e^{-n}$ (b) $\frac{(-1)^{n-1}\, n!}{ne^n}\log_a e$

(c) $\left(\frac{-1}{e}\right)^n n!$ (d) $(n-1)!\, e^{-n}$

ii. If $y=\sin 2x\ \forall\ x\in R$ then $y_n\left(\frac{\pi}{4}\right)$ equals,

(a) $2^{n-1}\cos n\pi$ (b) $2^{n-1}\cos\frac{n\pi}{2}$

(c) $2^n\sin\frac{n\pi}{2}$ (d) $2^n\cos\frac{n\pi}{2}$

iii. If $y=\frac{1}{(a+bx)}$ then $y_n(1)$ when $a = 1, b = -1$ is

(a) $\frac{(-1)^n n!}{(1+x)^n}$ (b) $\frac{(-1)^n(n-1)!}{(1-x)^n}$

(c) $\frac{n!}{(1-x)^{n+1}}$ (d) $\frac{(-1)^{n-1}n!}{(1+x)^n}$

iv. If $x^y = e^{x-y}$ then $y_1(e)$ equals

(a) $\frac{1}{4}$ (b) $\frac{1}{2}$ (c) log 2 (d) $-\frac{1}{4}$

v. If $n = 4t - 3$, $t \in I$ and $y = \cot^{-1} x$ then $y_n(0)$ is

(a) 0 (b) $n!$

(c) $(4n - 1)!$ (d) $-(n - 1)!$

vi. If $y = e^{mx}$ then $y_n(0)$

(a) mn (b) m/n (c) m^n (d) n^m

MATRIX-MATCH TYPE

Problems 20 to 24.

Match the columns. Left & right column have mathematical statements and expression. Read the expression or statement of left column & find the corresponding expression or statement in the right column for their relationship. An item of column I can be matched with more than one items of column II. All items of column II may be matched.

20. **Column I** — **Column II**

(A) If the value of $\lim\limits_{x\to -1}\dfrac{x+x^2+x^3+...+x^n-n}{x-1}=5050$ then $n=$ (p) 1035

(B) $\lim\limits_{x\to 1}256(\pi-\pi x)\tan\left(\dfrac{\pi x}{2}\right)=$ (q) 729

(C) If $\lim\limits_{x\to e}\dfrac{\log x^{729e}-729e}{x-e}=\lambda$ then $\lambda=$ (r) 512

(D) If (s) 100

$$f(x)=\begin{cases}\dfrac{27^x-9^x-3^x+1^x}{x^2}, & x\neq 0\\ e^{2x}\sin x+\pi x^2+\mu\log_e 3, & x=0\end{cases}$$

is continuous at $x = 0$ then value of $115e^{\mu}=$

(a) A → s, B → r, C → q, D → p

(b) A → p, B → q, C → r, D → s

(c) A → q, B → p, C → s, D → r

(d) None of these

21. **Column I** — **Column II**

(A) If $y = \sin x + \cos x$ and $f(x)=\dfrac{d^{22}y}{dx^{22}}$, then the $1024y\left(\dfrac{\pi}{4}\right)\cdot f\left(\dfrac{5\pi}{4}\right)=$ (p) 2564

(B) If $f(x) = x^n$ & the value of $f(1)+\dfrac{f'(1)}{1!}+\dfrac{f''(1)}{2!}+\dfrac{f'''(1)}{3!}+...+\dfrac{f^n(1)}{n!}$ is λ^n the value of $\lambda^{12}=$ (q) 6561

(C) If $x = at^2$, $y = 2at$ and $\dfrac{d^2y}{dx^2}\div\left(\dfrac{dy}{dx}\right)^{n-1}$ is a constant then (r) 4096

(D) If $x = a(\theta - \sin\theta)$ & $y = a(1 + \cos\theta)$ then the value of $2564a\left(\dfrac{d^2y}{dx^2}\right)_{\frac{\pi}{2}}=$ (s) 2048

(a) A → s, B → r, C → q, D → p

(b) A → p, B → q, C → r, D → s

(c) A → q, B → p, C → s, D → r

(d) None of these

22. **Column I** — **Column II**

(A) If $f(x)=\begin{cases}2x+3 & \text{if } x\le 2\\ 2x-3 & \text{if } x>2\end{cases}$, then number of discontinuity = (p) 2880

(B) If $f(x)=\begin{cases}x^2+3x+a & ,\text{for } x\le 1\\ bx+2 & ,\text{for } x>1\end{cases}$ is differentiable and $a + b = 2^{\lambda}$ then $\lambda =$ (q) 243

(C) If $f(3) = 4$, $f'(3) = 1$ and $\lim\limits_{x\to 2}\dfrac{xf(3)-3f(x)}{x-3}=\lambda$ then $2880\lambda=$ (r) 6

(a) A → r, B → q, C → p

(b) A → q, B → p, C → r

(c) A → r, B → p, C → q

(d) None of these

23. **Column I** — **Column II**

(A) If $y\sqrt{1-x^2}+x\sqrt{1-y^2}=1$ then $\dfrac{dy}{dx}=$ (p) $\sqrt{\dfrac{1-y^2}{1-x^2}}$

(B) If $\sqrt{1-x^2}+\sqrt{1-y^2}=a(x-y)$ then $\dfrac{dy}{dx}=$ (q) $\dfrac{y-x^2}{y^2-x}$

(C) If $x = y\log(xy)$ then $\dfrac{dy}{dx}=$ (r) $\dfrac{2xy-y^2}{2xy-x^2}$

(D) If $xy^2 - x^2y$ = constant (K say) then $\dfrac{dy}{dx}=$ (s) $\dfrac{y^2}{x(y-x)}$

(E) If $x^3 + y^3 = 3xy$ then $\dfrac{dy}{dx}=$ (t) $-\sqrt{\dfrac{1-y^2}{1-x^2}}$

(F) If $y = x\log y$ then $\dfrac{dy}{dx}=$ (u) $\dfrac{y(x-y)}{x(x+y)}$

(a) A → t, B → p, C → q, D → r, E → s, F → u

(b) A → t, B → p, C → u, D → r, E → q, F → s

(c) A → t, B → p, C → u, D → s, E → q, F → r

(d) None of these

24. **Column I** — **Column II**

(A) If $y = \sin^{-1}(3x - 4x^3)$ and $-\dfrac{1}{2}\le x\le 1$ then $\dfrac{dy}{dx}=$

(B) If $y = \cos^{-1}(4x^3 - 3x)$ and $-\dfrac{1}{2}\le x\le 1$

(p) $\dfrac{3}{1+x^2}\ \forall\, |x|<\dfrac{1}{\sqrt{3}}$

(q) $\dfrac{3}{1+x^2}\ \forall\, x<-\dfrac{1}{\sqrt{3}}$

(r) $-\dfrac{3}{\sqrt{1-x^2}}\ \forall x\in \{|x|<1\}-\left\{|x|\le\dfrac{1}{2}\right\}$

(s) $\dfrac{3}{\sqrt{1-x^2}}\ \forall\, |x|\le\dfrac{1}{2}$

then $\frac{dy}{dx}$ = (t) $\frac{3}{1+x^2}\ \forall x>\frac{1}{\sqrt{3}}$

(C) If $y=\tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)$ and $x\in R$ then $\frac{dy}{dx}$ = (u) $\frac{3}{1+x^2}\ \forall x\in R-\left\{-\frac{1}{\sqrt{3}},\frac{1}{\sqrt{3}}\right\}$

(a) A → r, s B → r, s, q C → p, q, t, u
(b) A → p, r, s B → r, s, u C → q, r, u
(c) A → r, s B → p, r, s C → p, q, t, u
(d) A → r, s B → r, s C → p, q, t, u

ASSERTION & REASON

Problems 25 to 29

Two statements are given, one statement is Assertion (A) & the other statement is Reason (R). Select the correct choice for these statements as per following codes.

(A) Both Assertion (A) & Reason (R) are individually true & Reason (R) is feasible explanation of Assertion (A).
(B) Both Assertion (A) & Reason (R) are individually true but Reason (R) is not proper explanation of Assertion (A).
(C) Assertion (A) is true but Reason (R) is false.
(D) Assertion (A) is false but Reason (R) is true.

25. **Assertion (A) :** $\lim\limits_{x\to 1}\frac{x^{15}-1}{x^{10}-1}=\frac{3}{2}$

Reason (R) : $\lim\limits_{x\to a}\frac{x^n-a^n}{x-a}=n\cdot a^{n-1}$

(a) A (b) B (c) C (d) D

26. **Assertion (A) :** $\lim\limits_{x\to 0}\frac{\sin ax}{\sin bx}=a/b,\ a,b\neq 0$

Reason (R) : $\lim\limits_{x\to 0}\frac{\sin x}{x}=1$

(a) A (b) B (c) C (d) D

27. **Assertion (A) :** Let $f(x)=\cos^2 x+\cos^2(x+\pi/3)+\cos^2(x-\pi/3)$ then $f'(x)\neq 0$.

Reason (R) : Derivative of a constant function is always zero.

(a) A (b) B (c) C (d) D

28. **Assertion (A) :** If $x=a(1-\cos\theta)$, $y=a(\theta+\sin\theta)$ then $\left(\frac{dy}{dx}\right)_{(\text{at }\theta=\pi/2)}=1$

Reason (R) : If $x=f(t)$ & $y=g(t)$ then

$$\frac{dy}{dx}=\frac{dy}{dt}\Big/\frac{dx}{dt},\ \&\ \frac{dx}{dt}\neq 0$$

(a) A (b) B (c) C (d) D

29. **Assertion (A) :** The function $f(x)=|x|$ is discontinuous at origin.

Reason (R) : The function $f(x)=|x|$ is non differentiable at origin.

(a) A (b) B (c) C (d) D

SOLUTIONS

1. **(c) :** $\lim\limits_{x\to 1}\frac{x^{1/4}-x^{1/5}}{x^3-1}$

$$=\lim_{x\to 1}\frac{(x^{1/4}-1)-(x^{1/5}-1)}{x^3-1}$$

$$=\lim_{x\to 1}\left(\frac{\frac{x^{1/4}-1^{1/4}}{x-1}}{\frac{x^3-1}{x-1}}-\frac{\frac{x^{1/5}-1^{1/5}}{x-1}}{\frac{x^3-1}{x-1}}\right)$$

$$=\frac{1}{12}-\frac{1}{15}\qquad\text{using }\lim_{x\to a}\frac{x^n-a^n}{x-a}=na^{n-1}$$

$$=\frac{1}{60}$$

2. **(a) :** As $f(x)$ is continuous $\forall x\in R$

$\therefore f(0)=\lim\limits_{x\to 0^+}f(x)=\lim\limits_{x\to 0^-}f(x)$

$$\therefore f(0)=\lim_{x\to 0}f(x)=\lim_{h\to 0}f(0+h)=\lim_{h\to 0}\frac{3-(243-5h)^{1/5}}{(5h+27)^{1/3}-3}$$

$$=\lim_{h\to 0}\frac{(243-5h)^{1/5}-(243)^{1/5}}{(5h+27)^{1/3}-(27)^{1/3}}$$

$$=-\frac{\lim\limits_{h\to 0}\frac{(243-5h)^{1/5}-(243)^{1/5}}{(243-5h)-243}\times(-5h)}{\lim\limits_{h\to 0}\frac{(5h+27)^{1/3}-(27)^{1/3}}{(5h+27)-27}\times(5h)}$$

$$=\frac{\frac{1}{5}(243)^{-4/5}}{\frac{1}{3}\times(27)^{-2/3}}=\frac{1}{15}$$

3. **(d) :** $\lim\limits_{n\to\infty}[(2^n+1)(7^n+10^n)]^{1/n}$

$$=\lim_{n\to\infty}\left[(20)^n\left(1+\frac{1}{2^n}\right)\left(1+\left(\frac{7}{10}\right)^n\right)\right]^{1/n}$$

$$=20\left[\lim_{n\to\infty}\left(1+\frac{1}{2^n}\right)\cdot\lim_{n\to\infty}\left(1+\left(\frac{7}{10}\right)^n\right)\right]^{1/n}$$

$$=20\times 1\times 1={}^6C_3$$

4. **(b) :** Fact $[x]$ is discontinues at all integer numbers

$f(x)=\sqrt{2}\cos\left(x+\frac{\pi}{4}\right)$ is an integer for

$$x+\frac{\pi}{4}=\frac{\pi}{2},\frac{3\pi}{4},\frac{5\pi}{4},\frac{3\pi}{2},\frac{7\pi}{4},\frac{9\pi}{4}$$

$$\Rightarrow x=\frac{\pi}{4},\frac{\pi}{2},\pi,\frac{5\pi}{4},\frac{3\pi}{2},2\pi\qquad(\text{as }0<x\le 2\pi)$$

⇒ Number of points of discontinuity of $f(x)$ are 6.

5. **(b) :** As $n\to\infty$,

Let $\lim\limits_{n\to\infty}f(n)=\lim\limits_{n\to\infty}f(n+1)=\lim\limits_{n\to\infty}f(n+2)=\lambda$

Now $f(n+1)=\frac{1}{5}\left[f(n)+\frac{36}{f(n+2)}\right]$

$$5\lambda = \lambda + \frac{36}{\lambda} \Rightarrow 4\lambda^2 = 36 \Rightarrow \lambda = \pm 3.$$

$\therefore \lambda = 3$, as $f(n) > 0 \ \therefore \lim_{n\to\infty} f(n+1) = 3.$

6. **(c) :** $\lim_{x\to a} \frac{f(x)-f(a)}{x-a} = f'(a)$

Now $\lim_{x\to 3} \frac{f(x)-f(3)}{x-3} = f'(3)$

Now $f(x) = \frac{1}{\sqrt{36-x^2}} \quad \therefore f'(x) = \frac{x}{(36-x^2)^{3/2}}$

$$f'(3) = \frac{3}{(27)^{3/2}} = \frac{1}{27\sqrt{3}}$$

7. **(a) :** Minimum of $(x^2 + 4x + 6)$
$= \text{Minimum of } (x+2)^2 + 2 \geq 2$

$$\therefore \lim_{t\to 0} \text{minimum}\left[(x^2+4x+6)\frac{\sin t}{t}\right]$$

$$= \lim_{t\to 0}\left[2\frac{\sin t}{t}\right] = f(t)(\text{say})$$

Now $\lim_{t\to 0^+} f(t) = 1$

$\therefore$ for $t > 0$

$$\sin t < t \Rightarrow \frac{\sin t}{t} < 1 \Rightarrow 2\frac{\sin t}{t} < 2, \quad \therefore \left[2\frac{\sin t}{t}\right] = 1$$

$\therefore$ R.H.L. $= \lim_{t\to 0^+} f(t) = 1.$

Again for $t < 0$, $\sin t > t$

$$\frac{\sin t}{t} < 1 \Rightarrow 2\frac{\sin t}{t} < 2 \Rightarrow \left[2\frac{\sin t}{t}\right] = 1 = \lim_{t\to 0^-} f(t)$$

$$\therefore \lim_{t\to 0}\left[\min(x^2+4x+6)\frac{\sin t}{t}\right] = 1.$$

8. **(b) :** The value of $\lim_{x\to\beta}(1 + ax^2 + bx + c)^{\frac{1}{x-\beta}}$

Defination (i)

as α, β are roots of $ax^2 + bx + c = 0$

$\therefore ax^2 + bx + c = a(x-\alpha)(x-\beta)$

Defination (ii)

If $\lim_{x\to a} f(x) = 1$ and $\lim_{x\to a} g(x) = \infty$

then $\lim_{x\to a}[f(x)]^{g(x)} = e^{\boxed{\lim_{x\to a}(f(x)-1)\times g(x)}}$

Now $\lim_{x\to\beta}(1 + ax^2 + bx + c)^{\frac{1}{x-\beta}}$

$$= \lim_{x\to\beta}[1 + a(x-\alpha)(x-\beta)]^{\frac{1}{x-\beta}} \qquad (1^\infty \text{ form})$$

$$= e^{\boxed{\lim_{x\to\beta}\frac{a(x-\alpha)(x-\beta)}{(x-\beta)}}} = e^{a(\beta-\alpha)}$$

9. **(c) :** Let $\frac{1}{\cos^2 x} = \lambda \geq 1$ (as $0 \leq \cos^2 x \leq 1$)

$$\therefore \lim_{x\to\frac{\pi}{2}}(1^{1/\cos^2 x} + 2^{1/\cos^2 x} + 3^{1/\cos^2 x} + \dots + 10^{1/\cos^2 x})^{\cos^2 x}$$

$$= \lim_{\lambda\to\infty}(1^\lambda + 2^\lambda + \dots + 10^\lambda)^{1/\lambda}$$

$$= (10^\lambda)^{1/\lambda}\lim_{\lambda\to\infty}\left[\left(\frac{1}{10}\right)^\lambda + \left(\frac{2}{10}\right)^\lambda + \dots + 1\right]^{\frac{1}{\lambda}}$$

$$= 10[0 + 0 + \dots + 1]^0 = 10$$

10. **(b) :** $\lim_{x\to 1}\frac{x^4-1}{x-1} = \lim_{x\to\lambda}\frac{x^3-\lambda^3}{x^2-\lambda^2}$

$$\Rightarrow 4 = \lim_{x\to\lambda}\frac{x^3-\lambda^3}{x-\lambda}\cdot\frac{x-\lambda}{x^2-\lambda^2} \Rightarrow 4 = \frac{3\lambda^2}{2\lambda} \Rightarrow \lambda = \frac{8}{3}$$

11. **(d) :** $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^-}\frac{x}{-x+2x^2} = -3$

and $\lim_{x\to 0^+} f(x) = \lim_{x\to 0^+}\frac{3x}{x+2x^2} = 3$

$\lim_{x\to 0^-} f(x) \neq \lim_{x\to 0^+} f(x)$

$\Rightarrow \lim_{x\to 0} f(x)$ does not exist

12. **(a, d) :** Using $\lim_{x\to a}(f(x))^{g(x)} = 1^\infty$ form

$= e^{\lim_{x\to a}(f(x)-1)\times g(x)}$ as $\lim_{x\to a} f(x) = 1$, $\lim_{x\to a} g(x) = \infty$

$$\therefore \lim_{x\to 0}\left(\frac{a^x+b^x+c^x+d^x}{4}\right)^{\lambda/x}, \qquad (1^\infty \text{ form})$$

$$= e^{\lim_{x\to 0}\left(\frac{a^x+b^x+c^x+d^x}{4}-1\right)\frac{\lambda}{x}} = e^{\lim_{x\to 0}\left(\frac{a^x-1}{x}+\frac{b^x-1}{x}+\frac{c^x-1}{x}+\frac{d^x-1}{x}\right)\frac{\lambda}{4}}$$

$$= e^{\frac{\lambda}{4}\log(abcd)} = e^{\log(abcd)^{\lambda/4}} = (abcd)^{\lambda/4}$$

$$= \begin{cases} abcd & \text{if } \lambda = 4 \ \therefore \text{choice (a) is correct} \\ (dabc)^{3/4} & \text{if } \lambda = 3 \ \therefore \text{choice (d) is correct} \end{cases}$$

13. **(b, c, d) :** Let $f(x) = \log x$ which is continuous $\forall x > 0$ ie $(0, \infty)$

$g(x) = \sqrt{1-4x^2}$ will be continuous $\forall$ x satisfying the condition $1 - 4x^2 \geq 0$ ie $|x| \leq \frac{1}{2}$, which means

$g(x)$ is continuous $\forall x \in \left[-\frac{1}{2}, \frac{1}{2}\right]$

Now $P(x) = \frac{f(x)}{g(x)}$ will be continuous for

{Domain of continuity of $\log x \cap$ Domain of continuity $g(x)$} – {set of values for which $g(x)$ is zero.}

$$= (0,\infty) \cap \left[-\frac{1}{2}, \frac{1}{2}\right] - \left\{-\frac{1}{2}, \frac{1}{2}\right\} = \left(0, \frac{1}{2}\right] - \left\{\frac{1}{2}\right\} = \left(0, \frac{1}{2}\right)$$

Now $\left\{\frac{1}{10}, \frac{1}{4}, \frac{1}{3}\right\} \in \left(0, \frac{1}{2}\right)$

$\therefore$ $P(x)$ is continuous at $x = \frac{1}{10}, \frac{1}{4}, \frac{1}{3}$

14. **(a) :** $f(x) = x^2 + ax + b$ where $g'(1) = a$, $g''(2) = b$

$\therefore$ $g(x) = f(1)x^2 + xf'(x) + f''(x)$

$\Rightarrow$ $g(x) = (1 + a + b)x^2 + x(2x + a) + 2$

$\Rightarrow$ $g(x) = (3 + a + b)x^2 + ax + 2$

$g'(x) = 2(3 + a + b)x + a$ and

$g''(x) = 2(3 + a + b) = g'(2)$

$\therefore$ $g'(1) = 2(3 + a + b) + a$

$\Rightarrow$ $3 + a + b = 0$ as $g'(1) = a$ ... (*)

and $g'(2) = b = 2(3 + a + b) \Rightarrow 6 + 2a + b = 0$ ... (**)

solving (*) & (**) $\therefore b = 0, \quad a = -3$

$\therefore \quad f(x) = x^2 - 3x \ \& \ g(x) = -3x + 2$

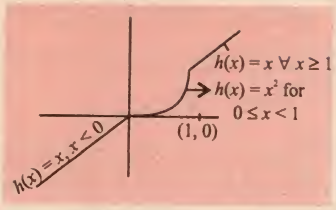
$\therefore \quad f'(x) = 2x - 3, g'(x) = -3$

$\Rightarrow \quad f'(1) = 2 - 3 = -1, g'(1) = -3$

$\therefore \quad f'(1) + g'(1) = -1 - 3 = -4$

15. (a, c, d) : $h(x) = \min\{x, x^2\} = \begin{cases} x, & x < 0 \\ x^2, & 0 \le x < 1 \\ x, & x \ge 1 \end{cases}$

observe the graph, we note that $h(x)$ is continuous every where



Again the graph has two corner points $x = 0$ & $x = 1$ so $h(x)$ is not differentiable at these points. At these corner point we note that left hand derivative limit $Lh'(0) \neq Rh'(0)$ and $Lh'(1) \neq Rh'(1)$ which means $h(x)$ is non differential at $x = 0, x = 1$

Further, for $x \ge 1$, $h(x) = x \Rightarrow h'(x) = 1$ which is a constant

16. (a, c, d) : At $x = -3, f(-3) = 6$

LHL $= \lim\limits_{x \to -3^-} |x| + 3 = 6$

RHL $= \lim\limits_{x \to -3^+} |x| + 3 = 6$

LHL = RHL $= f(-3) = 6$

$\therefore f$ is continuous at $x = -3$

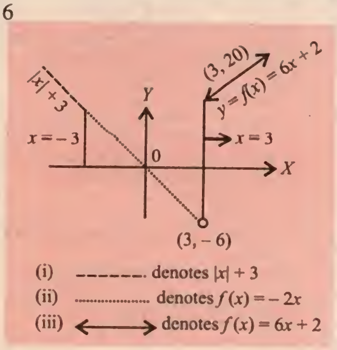
$\therefore$ (a) is correct

Again at $x = 3$

LHL $= \lim\limits_{x \to 3^-} (-2x) = -6$

RHL $= \lim\limits_{x \to 3^+} f(x)$

$= \lim\limits_{x \to 3^+} (6x + 2) = 20$



$\therefore$ LHL at $x = 3 \neq$ RHL at $x = 3$

$\Rightarrow f$ is not continuous at $x = 3$

$\Rightarrow f$ can not be differentiable at $x = 3$ $\therefore$ (b) false

Further At $x = x_0$ where $-3 < x_0 < 3$ $\lim\limits_{x \to x_0} f(x) = f(x_0)$

$\therefore$ (c) is correct

and At $x = x_0 > 3$ $\lim\limits_{x \to x_0} 6x + 2 = 6x_0 + 2 = f(x_0)$

$\therefore$ (d) is correct

Alternative Method :

$$f(x) = \begin{cases} -2x & , \forall \ -3 < x < 3 \\ 6x + 2 & , \forall \quad x \ge 3 \end{cases}$$

$$f'(x) = \begin{cases} -2 & \forall \quad -3 < x < 3 \\ 6 & \forall \quad x \ge 3 \end{cases}$$

$\Rightarrow$ L.H. Derivative at $x = 3 \neq$ R.H. Derivative at $x = 3$

$\Rightarrow f(x)$ can be differentiable at $x = 3$

Further, with the help of graph we note that the function f is continuous at $x = -3$, continuous for $-3 < x < 3$ & $x > 3$

17. i. (d) : As function is differentiable at $x = c$, it means it must be continuous at $x = c$

$\therefore \lim\limits_{x \to c^-} f(x) = f(c) = \lim\limits_{x \to c^+} f(x)$

$\Rightarrow c^2 = ac + b$ ... (i)

Further f is differentiable at $x = c$

$\therefore Lf'(c) = Rf'(c)$

$$\Rightarrow \lim_{h \to 0^-} \frac{f(c+h) - f(c)}{h} = \lim_{h \to 0^+} \frac{f(c+h) - f(c)}{h}$$

$$\Rightarrow \lim_{h \to 0^-} (h + 2c) = \lim_{h \to 0^+} \frac{ah}{h} \Rightarrow 2c = a$$

ii. (a) : Now from equation (i) on putting $a = 2c$ we get $b = -c^2$

iii. (b)

iv. (c) : Now $\lim\limits_{x \to a} ax + b$

$= a^2 + b = a^2 - c^2 = 4c^2 - c^2 = 3c^2 = -3b$

18. i. (b) : $\lim\limits_{x \to 0} f(x)$ is finite

$$\Rightarrow \lim_{x \to 0} \frac{a\cos x + bx\sin x - 6}{x^4} = \text{finite}$$

$\Rightarrow a - 6 = 0 \Rightarrow a = 6$

ii. (a) : $\therefore \lim\limits_{x \to 0} \dfrac{a\cos x + bx\sin x - 6}{x^4}$

$$= \lim_{x \to 0} \frac{6\cos x + bx\sin x - 6}{x^4} \quad (0/0 \text{ form})$$

$$= \lim_{x \to 0} \frac{-6\sin x + bx\cos x + b\sin x}{x^4} \quad (0/0)$$

$$= \lim_{x \to 0} \frac{-6\cos x + b[x(-\sin x) + \cos x + \cos x]}{x^4}$$

$$= \frac{2b - 6}{0} = \infty \text{ if } 2b - 6 \neq 0$$

But limit is finite $\therefore b = 3$

iii. (c) : $\therefore \lim\limits_{x \to 0} \dfrac{a\cos x + bx\sin x - 6}{x^4}$

$$= \lim_{x \to 0} \frac{6\cos x + 3x\sin x - 6}{x^4} = \lim_{x \to 0} \frac{6x\sin\frac{x}{2}\cos\frac{x}{2} - 6 \times 2\sin^2\frac{x}{2}}{x^4}$$

$$\left(\text{as} \quad \begin{array}{l} \sin x = 2\sin\frac{x}{2}\cos\frac{x}{2} \\ 6\cos x - 6 = 6\left(-2\sin^2\frac{x}{2}\right) \end{array} \right)$$

$$= \lim_{x \to 0} \frac{6\left(x\cos\frac{x}{2} - 2\sin\frac{x}{2}\right)}{2x^3}$$

$= -\dfrac{1}{4}$ (by using L Hospital Rule)

iv. (d) : Again f (0) = c

$$\Rightarrow c = -\frac{1}{4} = \frac{(3-6)}{12} = \frac{b-a}{12}$$

19. i. (b) : $y = \log_a x = \log_e x \cdot \log_a e$

$\therefore y_1 = \log_a e\left(\dfrac{1}{x}\right)$

$y_2 = \log_a e \dfrac{(-1)}{x^2}$

$y_3 = \log_a e(-1)(-2)x^{-3} = \log_a e(-1)^2 \cdot 2! x^{-3}$

$\vdots \quad \vdots$

$\therefore y_n = \log_a e \dfrac{(-1)^{n-1} n!}{n} x^{-n}$

$\therefore\ y_n(e) = \log_a e\dfrac{(-1)^{n-1}n!}{n}\left(\dfrac{1}{e}\right)^n$

ii. (d) : $y = \sin 2x$

$y_1 = 2\cos 2x = 2\sin\left(\dfrac{\pi}{2}+2x\right)$

$y_2 = -2^2\sin 2x = 2^2\sin\left(\dfrac{2\pi}{2}+2x\right)$

$y_3 = -2^3\cos 2x = 2^3\sin\left(\dfrac{3\pi}{2}+2x\right)$

$\vdots \quad \vdots$

$y_n = 2^n\sin\left(\dfrac{n\pi}{2}+2x\right)$

$y_n\left(\dfrac{\pi}{4}\right) = 2^n\sin\left(\dfrac{n\pi}{2}+\dfrac{\pi}{2}\right) = 2^n\cos\dfrac{n\pi}{2}$

iii. (c) : $y = \dfrac{1}{(a+bx)}$

$y_1 = \dfrac{(-1)b}{(a+bx)^2}, \quad y_2 = \dfrac{b^2(-1)(-2)}{(a+bx)^3}$

$y_3 = \dfrac{(-1)^3 3!b^3}{(a+bx)^4}, \quad y_n = \dfrac{(-1)^n n!b^n}{(a+bx)^{n+1}}$

$\therefore\ y_n(1) = \dfrac{(-1)^n n!b^n}{(a+bx)^{n+1}} = \dfrac{n!}{(1-x)^{n+1}}$

iv. (a) : $x^y = e^{x-y}$

$\Rightarrow\ y\log_e x = x - y$ (taking logarithm both side at base e)

$\Rightarrow\ y = \dfrac{x}{1+\log_e x} \quad \therefore\ y_1 = \dfrac{\log_e x}{(1+\log_e x)^2}$

$\Rightarrow\ y_1(e) = \dfrac{1}{4}$

v. (d) : $y = \cot^{-1}x$

$y_1 = -\dfrac{1}{1+x^2} = \dfrac{1}{2i}\left[\dfrac{1}{x+i} - \dfrac{1}{x-i}\right]$

$y_2 = \dfrac{1}{2i}\left[\dfrac{(-1)^1}{(x+i)^2} - \dfrac{(-1)^1}{(x-i)^2}\right]$

$y_3 = \dfrac{1}{2i}\left[\dfrac{(-1)^2 2!}{(x+i)^3} - \dfrac{(-1)^2(2!)}{(x-i)^3}\right]$

$\vdots \quad \vdots$

$y_n(x) = \dfrac{1}{2i}\left[\dfrac{(-1)^{n-1}}{(x+i)^n} - \dfrac{(-1)^{n-1}}{(x-i)^n}\right](n-1)!$

$= \dfrac{(-1)^{n-1}(n-1)!}{2i}\left[\dfrac{1}{(x+i)^n} - \dfrac{1}{(x-i)^n}\right]$

$\therefore\ y_n(0) = \dfrac{(-1)^{n-1}(n-1)!}{2i}\left[\dfrac{1}{i^n} - \dfrac{1}{(-i)^n}\right]$

Putting $n = 4t-3$

$\therefore\ y_n(0) = \dfrac{(-1)^{4(t-1)}\cdot(4t-4)!}{2i}\left[\dfrac{1}{(i)^{4t-3}} - \dfrac{1}{(-i)^{4t-3}}\right]$

$= \dfrac{(n-1)!}{2i}\left[\dfrac{1}{i}+\dfrac{1}{i}\right] = -(n-1)!$

vi. (c) : $y = e^{mx} \quad \therefore\ y_1 = m\,e^{mx},\ y_2 = m^2e^{mx}$

$\therefore\ y_n = m^n e^{mx}$

$\therefore\ y_n(0) = m^n e^0 = m^n$

20. (a) A → s : $\lim\limits_{x\to1}\dfrac{x+x^2+x^3+\ldots+x^n-n}{x-1} = 5050$

$\Rightarrow \lim\limits_{x\to1}\dfrac{(x-1)+(x^2-1)+(x^3-1)+\ldots+x^n-1}{x-1} = 5050$

$\Rightarrow \lim\limits_{x\to1}\left\{\dfrac{x-1}{x-1}+\dfrac{x^2-1}{x-1}+\dfrac{x^3-1}{x-1}+\ldots+\dfrac{x^n-1}{x-1}\right\} = 5050$

$\Rightarrow\ 1+2+3+\ldots+n = 5050$

$\Rightarrow\ \dfrac{n(n+1)}{2} = \dfrac{100\times101}{2}$

$\Rightarrow\ n = 100$

B → r : $\lim\limits_{x\to1}256\pi(1-x)\tan\dfrac{\pi x}{2}$

$= 256\pi\lim\limits_{h\to0}\left(-h\tan\left(\dfrac{\pi}{2}+\dfrac{\pi h}{2}\right)\right) = 256\pi\lim\limits_{h\to0}h\cot\dfrac{\pi h}{2}$

$= 256\pi\lim\limits_{h\to0}\dfrac{h}{\tan\dfrac{\pi h}{2}} = \dfrac{256\pi\times2}{\pi} = 512$

C → q : The $\lim\limits_{x\to e}\dfrac{\log x^{729e}-729e}{x-e} = \lambda$

$= 729e\lim\limits_{x\to e}\dfrac{\log x-1}{x-e} = \lambda$

$\Rightarrow 729e\lim\limits_{h\to0}\dfrac{\log(e+h)-\log e}{h} = \lambda$

$\Rightarrow 729e\lim\limits_{h\to0}\dfrac{\log\left(\dfrac{e+h}{e}\right)}{h} = \lambda \Rightarrow 729e\lim\limits_{h\to0}\dfrac{\log\left(1+\dfrac{h}{e}\right)}{e\cdot\dfrac{h}{e}} = \lambda$

$\Rightarrow 729e\times\dfrac{1}{e} = \lambda \quad \Rightarrow\ \lambda = 729$

D → p : Function is continuous at $x = 0$

$\therefore\ f(0) = \lim\limits_{x\to0^+}f(x) = \lim\limits_{x\to0^-}f(x) = \lim\limits_{x\to0}f(x)$

$\mu\log_e 3 = \lim\limits_{x\to0}\dfrac{27^x-9^x-3^x+1^x}{x^2}$

$\mu\log_e 3 = \lim\limits_{x\to0}\dfrac{(3^{2x}-1)}{x}\cdot\dfrac{(3^x-1)}{x}$

$\mu\log_e 3 = 2\log_e 3\cdot\log_e 3$

$\mu = \log_e 9 \quad \therefore\ e^\mu = 9 \quad \therefore\ 115e^\mu = 9\times115 = 1035$

21. (a) A → s : The 4th order derivative of $y = \sin x + \cos x$ is equal to y.

$\therefore\ \dfrac{d^{20}y}{dx^{20}} = y = \sin x+\cos x$

$\Rightarrow f(x) = \dfrac{d^{22}y}{dx^{22}} = \dfrac{d^2y}{dx^2} = \dfrac{d}{dx}(\cos x-\sin x) = -(\sin x+\cos x)$

$\Rightarrow f\left(\dfrac{5\pi}{4}\right) = \sqrt{2}$ and $y\left(\dfrac{\pi}{4}\right) = \sqrt{2}$

$\Rightarrow 1024\,y\left(\dfrac{\pi}{4}\right)f\left(\dfrac{5\pi}{4}\right) = 1024\sqrt{2}\,\sqrt{2} = 2048$

B → r : $f(x) = x^n \Rightarrow f'(x) = nx^{n-1}, f''(x) = n(n-1)x^{n-2}$

$\Rightarrow$ r^{th} order derivative is $f^r(x) = \dfrac{n!}{(n-r)!}x^{n-r}\ (r = 1, 2, \ldots n)$

$\therefore\ \lambda^n = f(1)+\dfrac{f'(1)}{1!}+\dfrac{f''(1)}{2!}+\dfrac{f'''(1)}{3!}+\ldots+\dfrac{f^n(1)}{n!}$

$$= 1 + \frac{n}{1!} + \frac{n(n-1)}{2!} + \frac{n(n-1)(n-2)}{3!} + \ldots + \frac{n!}{n!}$$
$$= {}^nC_0 + {}^nC_1 + {}^nC_2 + {}^nC_3 + \ldots + {}^nC_n$$

$\Rightarrow \lambda^n = 2^n$ (using binomial theorem)

$\Rightarrow \lambda = 2$

$\Rightarrow \lambda^{12} = 2^{12} = 4096$

C → q : $y = 2at$ & $x = at^2$

$$\Rightarrow \frac{dy}{dt} = 2a \;\&\; \frac{dx}{dt} = 2at \Rightarrow \frac{dy}{dx} = \frac{2a}{2at} = \frac{1}{t}$$

$$\Rightarrow \frac{d^2y}{dx^2} = -\frac{1}{t^2}\frac{dt}{dx} = -\frac{1}{t^2}\frac{1}{2at} = -\frac{1}{2at^3}$$

Again $\frac{dy}{dx} = \frac{1}{t}$

$$\Rightarrow \left(\frac{dy}{dx}\right)^{n-1} = \frac{1}{t^{n-1}}$$

Now $\frac{d^2y}{dx^2} \div \left(\frac{dy}{dx}\right)^{n-1} = \frac{1}{2at^3} \times t^{n-1} = -\frac{1}{2a}t^{n-4}$

But $\frac{d^2y}{dx^2} \Big/ \left(\frac{dy}{dx}\right)^{n-1}$ is a constant

$\Rightarrow n - 4 = 0$ ie $n = 4$

$\therefore 3^{2n} = 9^4 = 6561$

D → p : $x = a(\theta - \sin\theta)$, $y = a(1 + \cos\theta)$

$$\Rightarrow \frac{dx}{d\theta} = a(1-\cos\theta), \quad \frac{dy}{d\theta} = -a\sin\theta$$

$$\therefore \frac{dy}{dx} = \frac{-a\sin\theta}{a(1-\cos\theta)} = -\cot\theta/2$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{2}\text{cosec}^2\theta/2 \cdot \frac{d\theta}{dx} = \frac{1}{2}\text{cosec}^2\theta/2\left(\frac{1}{a(1-\cos\theta)}\right)$$

$$\Rightarrow \frac{d^2y}{dx^2} = \frac{1}{4a}\text{cosec}^4\theta/2$$

$$\therefore 2564a\frac{d^2y}{dx^2}\bigg|_{\theta=\frac{\pi}{2}} = \frac{2564a}{a} = 2564.$$

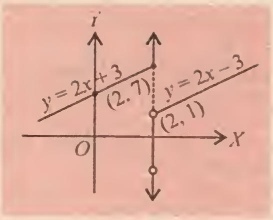
22. (a) A → r : LHL $= \lim_{x\to 2^-} f(x)$

$= \lim_{x\to 2^-} 2x + 3$

$= \lim_{h\to 0} 2(2-h) + 3 = 7$

RHL $= \lim_{x\to 2^+} f(x) =$

$\lim_{h\to 0} 2(2+h) - 3 = 1$



$\therefore$ Number of discontinuity

$= |\text{LHL} - \text{RHL}|$ at $x = 2$

$= |7 - 1|$

$= 6$ (The graph portion shows number of jumps)

B → q : As $f(x)$ is differentiable every where so $f(x)$ is continuous at $x = 1$

$\therefore \lim_{x\to 1^-} f(x) = f(1) = \lim_{x\to 1^+} f(x)$

$\Rightarrow a + 2 - b = 0$

$\Rightarrow a - b = -2$ ... (i)

Again $f(x)$ is differentiable at $x = 1$

$\therefore$ LHD at $x = 1$ = RHD at $x = 1$

$$\Rightarrow \lim_{x\to 1^-}\frac{f(x)-f(1)}{x-1} = \lim_{x\to 1^+}\frac{f(x)-f(1)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{x^2+3x+a-(4+a)}{x-1} = \lim_{x\to 1}\frac{(bx+2)-(4+a)}{x-1}$$

$$\Rightarrow \lim_{x\to 1}\frac{x^2+3x-4}{x-1} = \lim_{x\to 1}\frac{bx-b}{x-1} \quad \text{using (i)}$$

$\Rightarrow 5 = b$ $\therefore$ on using (i) we get $a = 3$

Now $a + b = 2^\lambda$

$\Rightarrow 8 = 2^\lambda$

$\Rightarrow \lambda = 3$

$\therefore \lambda^5 = 3^5 = 243$

C → p : $\lambda = \lim_{x\to 3}\frac{xf(3) - 3f(x)}{x-3}$

$$\lambda = \lim_{x\to 3}\frac{xf(3) - 3f(3) + 3f(3) - 3f(x)}{x-3}$$

[By adding & subtracting $3f(3)$]

$$\lambda = \lim_{x\to 3}\frac{(x-3)f(3)}{x-3} - 3\lim_{x\to 3}\frac{f(x)-f(3)}{x-3}$$

$\lambda = f(3) - 3f'(3)$

$\lambda = 4 - 3(1) = 1$

$\therefore 2880\lambda = 2880$

23. (b) A → t : Putting $x = \sin\phi$, $y = \sin\theta$

$\therefore y\sqrt{1-x^2} + x\sqrt{1-y^2} = 1$

$\Rightarrow \sin\theta\cos\phi + \cos\theta\sin\phi = 1$

$\Rightarrow \sin(\theta + \phi) = 1$

$\Rightarrow \theta + \phi = \sin^{-1}1$

$\Rightarrow \sin^{-1}x + \sin^{-1}y =$ constant

$$\Rightarrow \frac{dy}{dx} = -\sqrt{\frac{1-y^2}{1-x^2}}$$

B → p : Putting $x = \sin\phi$, $y = \sin\theta$

then $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

$\Rightarrow \cos\phi + \cos\theta = a(\sin\phi - \sin\theta)$

$\Rightarrow \phi - \theta = K$ (constant)

$\Rightarrow \sin^{-1}x - \sin^{-1}y = K$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

C → u : $x = y\log(xy)$

$\Rightarrow x = y(\log x + \log y)$ or $\frac{x}{y} = \log x + \log y$ ... (i)

on differentiating w.r. to x we get

$$\Rightarrow \frac{x-y}{x} = \frac{y+x}{y}dy$$

$$\Rightarrow \frac{dy}{dx} = \frac{y(x-y)}{x(x+y)}$$

D → r : $xy^2 - x^2y = K$

Differentiating w.r. to x both sides, we get

$$\left(2xy\frac{dy}{dx} + y^2\right) - \left(x^2\frac{dy}{dx} + 2xy\right) = 0$$

$$\Rightarrow \frac{dy}{dx}(2xy - x^2) = 2xy - y^2$$

$$\Rightarrow \frac{dy}{dx} = \frac{2xy - y^2}{2xy - x^2}$$

E → q : $x^3 + y^3 = 3xy$

on differentiating w.r. to x both sides we get

$$(y^2 - x)\frac{dy}{dx} = y - x^2$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y - x^2}{y^2 - x}$$

F → s : $y = x \log y$ ... (i)

on differentiating both sides w.r. to x, we get

$$\frac{dy}{dx} = \frac{x}{y}\frac{dy}{dx} + \log y$$

$$\frac{dy}{dx}\left(1 - \frac{x}{y}\right) = \frac{y}{x} \qquad \text{using (i)}$$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{y^2}{x(y - x)}$$

24. (d) A → r, s :

$$y = \sin^{-1}(3x - 4x^3) = \begin{cases} 3\sin^{-1} x & , \forall \quad |x| \le \frac{1}{2} \\ \pi - 3\sin^{-1} x & , \forall \quad x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$\therefore \frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1 - x^2}} & , \forall \quad |x| \le \frac{1}{2} \\ -\frac{3}{\sqrt{1 - x^2}} & , \forall \quad x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

B → r, s :

$$y = \cos^{-1}(4x^3 - 3x) = \begin{cases} 2\pi - 3\cos^{-1} x & , \forall \quad |x| \le \frac{1}{2} \\ 3\cos^{-1} x & , \forall \quad x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

$$\therefore \frac{dy}{dx} = \begin{cases} \frac{3}{\sqrt{1 - x^2}} & , \forall \quad |x| \le \frac{1}{2} \\ -\frac{3}{\sqrt{1 - x^2}} & , \forall \quad x \in \left[\frac{1}{2}, 1\right] \end{cases}$$

C → p, q, t, u :

$$y = \tan^{-1}\left(\frac{3x - x^3}{1 - 3x^2}\right) = \begin{cases} \pi + 3\tan^{-1} x & , \forall \quad x < -\frac{1}{\sqrt{3}} \\ 3\tan^{-1} x & , \forall \quad |x| < \frac{1}{\sqrt{3}} \\ -\pi + 3\tan^{-1} x & , \forall \quad x > \frac{1}{\sqrt{3}} \end{cases}$$

$$\therefore \frac{dy}{dx} = \left\{\frac{3}{1 + x^2} \qquad \forall x \in R - \left\{-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right\}\right.$$

25. (a) : $\lim_{x\to 1}\frac{x^{15} - 1}{x^{10} - 1} = \lim_{x\to 1}\frac{x^{15} - 1}{x - 1} \Big/ \lim_{x\to 1}\frac{x^{10} - 1}{x - 1}$

$$= \frac{15(1)^{14}}{10(1)^{10}} \quad \text{using } \lim_{x\to a}\frac{x^n - a^n}{x - a} = n \cdot a^{n-1}$$

$$= \frac{3}{2}$$

⇒ Assertion (A) & Reason (R) both are true & Reason (R) is proper explanation of Assertion (A)

26. (a) : $\lim_{x\to 0}\frac{\sin ax}{\sin bx}$

$$= \lim_{x\to 0}\frac{\frac{\sin ax}{ax}}{\frac{\sin bx}{bx}}\left(\frac{ax}{bx}\right)$$

$= a/b$ ∴ Assertion (A) is true.

and Reason (R) $\lim_{x\to 0}\frac{\sin x}{x} = 1$ is true

as it is used to solve the problem.

27. (d) : $f(x) = \cos^2 x + \cos^2(x + \pi/3) + \cos^2(x - \pi/3)$

$= \cos^2 x + \cos^2(x + \pi/3) + 1 - \sin^2(x - \pi/3)$

$= 1 + \cos^2 x + \cos^2(x + \pi/3) - \sin^2(x - \pi/3)$

$= 1 + \cos^2 x + \cos 120° \cos 2x$

$= 1 + \cos^2 x - \frac{1}{2}(2\cos^2 x - 1)$

$f(x) = \frac{3}{2}$ (constant function)

$$\Rightarrow \quad \frac{d}{dx}(f(x)) = \frac{d}{dx}\left(\frac{3}{2}\right)$$

⇒ $f'(x) = 0$ ∴ Assertion (A) is false.

⇒ Reason (R) is true but Assertion (A) is false.

28. (a) : Given $x = a(1 - \cos\theta)$, $y = a(\theta + \sin\theta)$

$$\Rightarrow \quad \frac{dy}{dx} = \frac{dy}{d\theta}\Big/\frac{dx}{d\theta}, \left(\frac{dx}{d\theta} \ne 0\right)$$

∴ Reason (R) is true & proper explanation for Assertion (A).

$$= \frac{1 + \cos\theta}{\sin\theta}$$

$$\frac{dy}{dx} = \cot\theta/2$$

$$\left(\frac{dy}{dx}\right)_{\theta = \frac{\pi}{2}} = \cot\pi/4 = 1$$

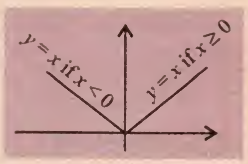
Both Assertion (A) & Reason (R) are true & Reason (R) is proper explaration for Assertion (A)

29. (d) : $f(x) = |x| = \begin{cases} x, x & \ge 0 \\ -x, & x < 0 \end{cases}$

and $\lim_{x\to 0^-} f(x) = \lim_{x\to 0^+} f(x) = f(0) = 0$

⇒ $f(x)$ continuous at origin

⇒ Assertion (A) is false



Again $f(x) = \begin{cases} x & , \forall \quad x \ge 0 \\ -x & , \forall \quad x < 0 \end{cases}$

$$\Rightarrow f'(x) = \begin{cases} 1 & \forall \quad x \ge 0 \\ -1 & \forall \quad x < 0 \end{cases}$$

⇒ $LHD \ne RHD$ at $x = 0$

⇒ $f(x)$ is non differentiable at origin.

⇒ Reason (R) is true.

MTG Math GENIUS Contest-II

- All students preparing for PET examinations can participate in MTG Math-Genius Contest-II.
- Answers marked only on the entry form of the magazine / photocopy of form will be accepted.
- More than one response to a question will be disqualified.
- **Prizes**
  - 1st Prize - LG Mobile phone
  - 2nd Prize - Adidas Bag
  - 3rd Prize - MTG Books (worth Rs. 500/–)
- The entries with maximum number of correct answers for **three consecutive months** (November '07 to January '08) will be awarded 1st prize. 2nd and 3rd prize will be given to the next maximum scorers. In case of a tie, the winners will be decided through a lucky draw.
- The decision of the editor will be final and binding in all cases and will not be a matter for consideration of any court and no correspondence will be entertained.
- Name and photograph of the prize winners of this contest along with the answers will be published in the March issue.
- MTG is not responsible for any postal delays, transit losses or mutilation of entries.
- **Last Date**

  The entries should reach **on/before 30th January '08** to – **MTG Math-Genius Contest-II**, 406, Taj Apartment, Ring Road, Near Safdarjung Hospital, New Delhi-29.
  [***Note :*** *Enclosures include a passport size photograph and a photocopy of age proof.*]

1. If the direction cosines of a ray are $\frac{1}{2}, \frac{1}{2}, n$ and $n < 0$ then $n =$

   (a) $\frac{1}{2}$ (b) $-\frac{1}{2}$ (c) $\frac{1}{\sqrt{2}}$ (d) $-\frac{1}{\sqrt{2}}$

2. The direction cosines of $\overrightarrow{OX}$ are given by the triad

   (a) (1, 0, 0) (b) (–1, 0, 0)
   (c) (0, 1, 0) (d) (0, 0, 1)

3. In a cube, an angle between any two diagonals is

   (a) $\cos^{-1}\left(\frac{1}{\sqrt{3}}\right)$ (b) $\cos^{-1}\left(\frac{1}{3}\right)$
   (c) $\frac{\pi}{3}$ (d) $\frac{\pi}{2}$

4. If $P(-1, 2, 4)$ $Q(1, 0, 5)$, $R(3, 4, 5)$ and $S(4, 6, 3)$ are four points in space, the projection of $\overline{PQ}$ along the ray $\overrightarrow{RS}$ is

   (a) $\frac{4}{3}$ (b) $-\frac{4}{3}$ (c) $\frac{8}{3}$ (d) -2

5. If a ray makes angles $\alpha, \beta, \gamma, \delta$ with the four diagonals of a cube then $\cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta =$

   (a) 1 (b) 2 (c) $\frac{3}{4}$ (d) 0

6. The angle between the line L_1 with direction numbers x, 3, 5 and the line L_2 with direction numbers 2, – 1, 2 is 45°. Then the value of x are

   (a) – 4, – 52 (b) 3, 42 (c) 4, 52 (d) – 3, 32

$<20> = 0,\ 0 + \bar{2}\,\bar{0} = \bar{2}\,\bar{0},\ \bar{2}\,\bar{0} \div 6 = \bar{3}$

($\bar{2}$ is the remainder)

$<\bar{2}\,0\,3> = \bar{1}\,\bar{2},\ \bar{1}\,\bar{2} + \bar{2}\,\bar{0} = \bar{3}\,\bar{2},\ \bar{3}\,\bar{2} \div 6 = \bar{5}$

($\bar{2}$ is the remainder)

$<\bar{2}\,0\,3\,\bar{4}> = 16,\ 16 + \bar{2}\,\bar{0} = \bar{4},\ \bar{4} \div 6 = 0$

($\bar{4}$ is the remainder)

$<\bar{2}\,0\,3\,\bar{4}\,\bar{1}> = 13,\ 13 + \bar{4}\,\bar{0} = \bar{2}\,\bar{7},\ \bar{2}\,\bar{7} \div 6 = \bar{4}$

($\bar{3}$ is the remainder)

$$\frac{x^4}{4!} = \left(-\frac{x^2}{2!}\right)^2 \times \frac{1}{6} = 1\,\bar{3}\,\bar{5}\,0\,\bar{4}$$

Using series expansion terms,

$$\begin{array}{ll} 1-\dfrac{x^2}{2!} & 1\,.\,0\,\bar{2}\,0\,3\,\bar{4}\,\bar{1}\,2\,4\,\bar{5} \\ \dfrac{x^4}{4!} = \left(-\dfrac{x^2}{2!}\right)^2 \times \dfrac{1}{6} & \,.\,\phantom{0\,\bar{2}\,0\,3}\,1\,\bar{3}\,\bar{5}\,0\,\bar{4} \\ \left(-\dfrac{x^2}{2!}\right)\times\left(\dfrac{x^4}{4!}\right)\times\dfrac{1}{15} & \,.\,\phantom{0\,\bar{2}\,0\,3\,\bar{4}\,\bar{1}}\,\bar{1} \\ \hline & 1\,.\,0\,\bar{2}\,0\,4\,\bar{7}\,\bar{6}\,1\,\bar{8}\,\bar{5} \\ \text{(using Vinculum)} & 0\,.\,9\,8\,0\,3\,2\,4\,0\,1\,8 \end{array}$$

Example 5. To find the value of cos (0.23^c)

Using calculator we get cos $(0.23^c) = 0.973666395$

Now let us calculate the above value using series expansion terms. We have,

$$(0.23)^2 = 0.0529,\ \frac{(0.23)^2}{2} = 0.02645 \text{ and}$$

$$-\frac{(0.23)^2}{2} = 0.0\bar{2}\bar{6}\bar{4}\bar{5}$$

$$\frac{x^4}{4!} = \left(-\frac{x^2}{2!}\right)^2 \times \frac{1}{6} = 0\,.\,0\,0\,0\,1\,1\,6\,6\,0\,0\,4\,1$$

Working of $\frac{x^4}{4!} = \left(-\frac{x^2}{2!}\right)^2 \times \frac{1}{6}$ making simultaneous operations, squaring and division, is as follows : (< > stands for duplex of a number)

$<2> = 4,\ 4 \div 6 = 1$ ($\bar{2}$ is the remainder)

$<2\,6> = 24,\ 24 + \bar{2}\,\bar{0} = 4,\ 4 \div 6 = 1$

($\bar{2}$ is the remainder)

$<2\,6\,4> = 52,\ 52 + \bar{2}\,\bar{0} = 32,\ 32 \div 6 = 6$

($\bar{4}$ is the remainder)

$<2\,6\,4\,5> = 68,\ 68 + \bar{4}\,\bar{0} = 28,\ 28 \div 6 = 6$

($\bar{8}$ is the remainder)

$<6\,4\,5> = 76,\ 76 + \bar{8}\,\bar{0} = \bar{4},\ \bar{4} \div 6 = 0$

($\bar{4}$ is the remainder)

$<4\,5> = 40,\ 40 + \bar{4}\,\bar{0} = 0,\ 0 \div 6 = 0$

$<5> = 25,\ 25 \div 6 = 4$ (1 is the remainder)

If we use the fourth term of the expansion i.e. $\left(-\frac{x^2}{2!}\right)\times\left(\frac{x^4}{4!}\right)\times\frac{1}{15}$ then the calculations are as follows (referring to above steps for the product)

$P\begin{pmatrix}\bar{2}\\1\end{pmatrix} = \bar{2},\ \bar{2} \div 15 = 0$ ($\bar{2}$ is the remainder)

$P\begin{pmatrix}\bar{2} & \bar{6}\\1 & 1\end{pmatrix} = \bar{8},\ \bar{8} + \bar{2}\bar{0} = \bar{2}\bar{8},\ \bar{2}\bar{8} \div 15 = \bar{2}$

(2 is the remainder)

$P\begin{pmatrix}\bar{2} & \bar{6} & \bar{4}\\1 & 1 & 6\end{pmatrix} = \bar{2}\bar{2},\ \bar{2}\bar{2} + 20 = \bar{2},\ \bar{2} \div 15 = 0$

($\bar{2}$ is the remainder)

$P\begin{pmatrix}\bar{2} & \bar{6} & \bar{4} & \bar{5}\\1 & 1 & 6 & 6\end{pmatrix} = \bar{5}\bar{7},\ \bar{5}\bar{7} + \bar{2}\bar{0} = \bar{7}\bar{7},\ \bar{7}\bar{7} \div 15 = \bar{5}$

($\bar{2}$ is the remainder)

$P\begin{pmatrix}\bar{2} & \bar{6} & \bar{4} & \bar{5} & \bar{0}\\1 & 1 & 6 & 6 & 0\end{pmatrix} = \bar{6}\bar{5},\ \bar{6}\bar{5} + \bar{2}\bar{0} = \bar{8}\bar{5},\ \bar{8}\bar{5} \div 15 = \bar{6}$

(5 is the remainder)

With these calculations we have, Using series expansion terms,

$$\begin{array}{ll} 1-\dfrac{x^2}{2'} & 1\,.\,0\,\bar{2}\,\bar{6}\,\bar{4}\,\bar{5} \\ \dfrac{x^4}{4!} = \left(-\dfrac{x^2}{2!}\right)^2 \times \dfrac{1}{6} & \,.\,0\,0\,0\,1\,1\,6\,6\,0\,0\,4\,1 \\ \left(-\dfrac{x^2}{2!}\right)\times\left(\dfrac{x^4}{4!}\right)\times\dfrac{1}{15} & \,.\,0\,0\,0\,0\,0\,0\,\bar{2}\,0\,\bar{5}\,\bar{6} \\ \hline & 1\,.\,0\,\bar{2}\,\bar{6}\,\bar{3}\,\bar{4}\,6\,4\,0\,\bar{5}\,\bar{2}\,1 \\ \text{using vinculum} & 0\,.\,9\,7\,3\,6\,6\,6\,3\,9\,4\,8\,1 \end{array}$$

In the next article we shall take up the problems of sine function, tangent function using the series expansion. We shall use this method in solving transcendental equations after completing the methods for hyperbolic functions and exponential functions. *to be continued ...*

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. If $x\sqrt{1+y}+y\sqrt{1+x}=0$ then $\frac{dy}{dx}=$

(a) $\frac{1}{1+x^2}$ (b) $\frac{-1}{1+x^2}$

(c) $\frac{1}{1+x}$ (d) none of these

2. If $\sin y = x\sin(a+y)$, then $\frac{dy}{dx}=$

(a) $\frac{\sin^2(a+y)}{\sin a}$ (b) $\sin(a+y)$

(c) $\sin^2(a+y)$ (d) $\frac{\sin(a+y)}{\sin a}$

3. $\lim\limits_{x\to\infty}\sqrt{\left(\frac{x+\sin x}{x-\cos x}\right)}=$

(a) 0 (b) 1

(c) -1 (d) none of these

4. If $f(x)=\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$, then $\lim\limits_{x\to\infty} f(x)$ is

(a) e^4 (b) e^3 (c) e^2 (d) 2^4

5. Let α and β be the roots of $ax^2+bx+c=0$ then $\lim\limits_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to

(a) 0 (b) $\frac{1}{2}(\alpha-\beta)^2$

(c) $\frac{a^2}{2}(\alpha-\beta)^2$ (d) $-\frac{a^2}{2}(\alpha-\beta)^2$

6. Find the domain of the function

$$f(x)=\frac{2}{\left[\frac{x}{3}\right]}-5^{\sin^{-1}x^2}+\frac{3x+1!}{\sqrt{x+1}}$$

where [.] denotes the greatest integer function.

7. If $\alpha_1, \alpha_2, \alpha_3 \ldots \alpha_n$ are the roots of equation $x^n-nax-b=0$ and $(\alpha_1-\alpha_2)\cdot(\alpha_1-\alpha_3)\ldots(\alpha_1-\alpha_n)=A$, then find the value of $A-n\alpha_1^{\,n-1}$.

8. If left hand derivative and right hand derivative of a function f at 'a' are finite, then show that f is continuous at 'a'.

9. Find the value(s) of 'a' for which

$$\lim_{x\to 0}\frac{\sin 3x+a\sin 2x}{x^3}$$

exists finitely. Find the value of the limit also.

10. If α, β are distinct real roots of the quadratic equation $ax^2+bx+c=0$, then show that

$$\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}=\frac{b^2-4ac}{2}.$$

SOLUTIONS

1. (d) : Squaring the given equation we will get $y=\frac{-x}{1+x}$. Hence $\frac{dy}{dx}=\frac{1}{(1+x)^2}$.

2. (a)

3. (b) : $\lim\limits_{x\to\infty}\sqrt{\frac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}}}$ and $\lim\limits_{x\to\infty}\frac{\sin x}{x}=\lim\limits_{x\to\infty}\frac{\cos x}{x}=0$

4. (a) : Use the fact $\lim\limits_{x\to\infty}(f(x))^{g(x)}$ $e^{\lim\limits_{x\to\infty}(f(x)-1)g(x)}$

Where $f(x)\to 1$ as $x\to\infty$, $g(x)\to\infty$

5. (c) : Use the fact $\lim\limits_{x\to 0}\frac{1-\cos ax}{x^2}=\frac{a^2}{2}$

6. As $\left[\frac{x}{3}\right]=0 \Rightarrow 0\le\frac{x}{3}<1 \Rightarrow 0\le x<3$

i.e., $\left[\frac{x}{3}\right]$ is defined for $x\in R-[0, 3)$.

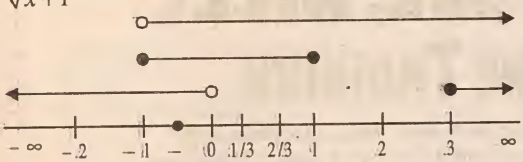
$\sin^{-1}x^2$ is defined for $0\le x^2\le 1 \Rightarrow x\in[-1, 1]$

By : **Prof. Shyam Bhushan, Director, Narayana Institute, Jamshedpur.** Mobile: 09334870021

$3x + 1!$ is defined for $3x + 1 \geq 0$ and

$(3x + 1) \in \{0, 1, 2, 3, ...\} \Rightarrow x \in \left\{\frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}, ...\right\}$

$\frac{1}{\sqrt{x+1}}$ is defined for $x \in (-1, \infty)$



$\therefore f(x)$ is defined for $x \in \left\{\frac{-1}{3}\right\}$

7. Given $x^n - nax - b = 0$

Root $\rightarrow \alpha_1, \alpha_2, \alpha_3, ... \alpha_n$

$\therefore x^n - nax - b = (x - \alpha_1)(x - \alpha_2)(x - \alpha_3) ... (x - \alpha_n)$

or $\frac{x^n - nax - b}{(x - \alpha_1)} = (x - \alpha_2)(x - \alpha_3) ... (x - \alpha_n)$

or $\lim_{x \to \alpha_1} \frac{x^n - nax - b}{x - \alpha_1} = \lim_{x \to \alpha_1} (x - \alpha_2)(x - \alpha_3) ... (x - \alpha_n)$

or $\lim_{x \to \alpha_1} \frac{nx^{n-1} - na}{1} = \lim_{x \to \alpha_1} (x - \alpha_2)(x - \alpha_3) ... (x - \alpha_n)$

or $n\alpha_1^{n-1} - na = (\alpha_1 - \alpha_2)(\alpha_1 - \alpha_3)(\alpha_1 - \alpha_4) ... (\alpha_1 - \alpha_n)$

$\Rightarrow n\alpha_1^{n-1} - na = A$

$\therefore A - n\alpha_1^{n-1} = -na$

8. $\lim_{h \to 0^+} (f(a) - f(a-h)) = \lim_{h \to 0^+} \frac{f(a) - f(a-h)}{h} \cdot h$

$= \lim_{h \to 0^+} f'(a-) \cdot h$ (as $f'(a-)$ is finite) $= 0$

Similarly

$\lim_{h \to 0^+} (f(a+h) - f(a)) = \lim_{h \to 0^+} f'(a+) \cdot h = 0$

Hence $\lim_{h \to 0^+} f(a-h) = \lim_{h \to 0^+} f(a+h) = f(a)$

Hence f is continuous at a.

9. Let $l = \lim_{x \to 0} \frac{\sin 3x + a \sin 2x}{x^3} \left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{3\cos 3x + 2a \cos 2x}{3x^2}$

we should have $g(0) = 0$,

where $g(x) = 3\cos 3x + 2a \cos 2x$,

as the given limit exists finitely $\Rightarrow a = -3/2$.

Further $l = \lim_{x \to 0} \frac{3\cos 3x - 3\cos 2x}{3x^2}$

$= \lim_{x \to 0} \frac{\cos 3x - \cos 2x}{x^2} \left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{2\sin 2x - 3\sin 3x}{2x} \left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x \to 0} \frac{\cos 2x - 9\cos 3x}{2} = -\frac{5}{2}$

Thus $a = -3/2$ and limit value $= -5/2$.

10. $\lim_{x \to \alpha} \frac{1 - \cos(ax^2 + bx + c)}{(x - \alpha)^2} = \lim_{x \to \alpha} \frac{2\sin^2 \frac{ax^2 + bx + c}{2}}{(x - \alpha)^2}$

$= \lim_{x \to \alpha} \frac{2\sin^2 \frac{a(x-\alpha)(x-\beta)}{2}}{\left(\frac{a(x-\alpha)(x-\beta)}{2}\right)^2} \cdot \frac{a^2}{4}(x-\beta)^2$

$= \frac{a^2(\alpha - \beta)^2}{2} = \frac{a^2}{2}[(\alpha + \beta)^2 - 4\alpha\beta]$

$= \frac{a^2}{2}\left[\frac{b^2}{a^2} - \frac{4c}{a}\right] = \frac{b^2 - 4ac}{2}$

EXAMPLE DATES 2008

CBSE PMT(Prelims)	6th April
IIT-JEE	13th April
WB JEE	20th April
MGIMS	20th April
CBSE AIEEE	27th April
CBSE PMT (Mains)	11th May
DCE	1st June
BHU Screening	7th May
BHU Mains	15th June

CHINESE Olympiad Problems

SECTION 1

1. Let n be any positive integer. A positive integer a is such that $n^6 + 3a$ is the cube of a positive integer. Which of the following statements is true
(a) There are always infinitely many such a.
(b) There are always at least one but finitely many such a.
(c) There are never such a.
(d) The existence of such a depends on n.

2. In a rhombus of side length 5, the length of one of the diagonals is at least 6, and the length of the other is at most 6. What is the maximum value of the sum of the lengths of the diagonals?
(a) $10\sqrt{2}$ (b) 14 (c) $5\sqrt{6}$ (d) 12.

3. Let a be an irrational number. How many of the lines through the point $(a, 0)$ contain at least two points both coordinates of which are rational?
(a) infinitely many
(b) at least two but finitely many
(c) only one (d) none.

4. $A_1A_2A_3$ is a triangle with $A_1A_2 = A_2A_3 = 1$ and $\angle A_1A_2A_3 = \alpha$, $0 < \alpha < \frac{\pi}{3}$. Initially, A_2 is at the origin while A_1 and A_3 are both on the unit circle. In the n-th operation, $1 \le n \le 100$, the triangle is rotated counterclockwise about A_n until A_{n+1} is on the unit circle, where A_n is taken to be A_{n-3} if $n > 3$. At the end of the 100-th operation, what is the total distance traveled by the point A_1?
(a) $22\pi(1 + \sin\alpha) - 66\alpha$ (b) $\frac{67\pi}{3}$
(c) $22\pi + \frac{68\pi\sin\alpha}{3} - 66\alpha$ (d) $33\pi - 66\alpha$.

SECTION 2

1. The sets $\{x, xy, \log(xy)\}$ and $\{0, |x|, y\}$ are identical. What is the value of $\sum_{i=1}^{2001} (x^i + y^{-i})$?

2. The intersection of the sets $\{(x, y) \mid |x| + |y| = a, a > 0\}$ and $\{(x, y) \mid |xy| + 1 = |x| + |y|\}$ consists of the vertices of a regular octagon. What are the possible values of a?

3. Let k be some integer greater than 1, and a be a root of $x^2 - kx + 1 = 0$. For any integer $n > 10$, the units digit of $a^{2^n} + a^{-2^n}$ is always 7. What are the possible values of the units digit of k?

4. We have two copies of a triangle with side lengths 3, 4 and 5, four copies of a triangle with side lengths 4, 5 and $\sqrt{41}$, and six copies of a triangle with side lengths $\frac{5\sqrt{2}}{6}$, 4 and 5. Using these triangles as faces, at most how many tetrahedra can we construct simultaneously?

5. Five brother-sister pairs took part in k activities. No brother-sister pair participated in the same activity. Any two people who were not a brother-sister pair were together in exactly one activity. One of the people took part in only two activities. What is the minimum value of k?

Solutions on page no.72

Contd. from page no. 26

$\therefore a = \frac{1}{2}, \quad r = \frac{1}{3} \quad \therefore S_n = a\left(\frac{1-r^n}{1-r}\right)$ (as $r < 1$)

$$\therefore S_n = \frac{1}{2}\left(\frac{1-\left(\frac{1}{3}\right)^n}{\frac{2}{3}}\right) = \frac{3}{4}(1-3^{-n})$$

v. **(b) :** $\therefore S_\infty = \lim_{n\to\infty}\frac{3}{4}\left(1-\frac{1}{3^n}\right) = \frac{3}{4}$

16. i. (a) : Using AM ≥ GM

$\therefore \frac{1+a_1}{2} \ge (a_1)^{1/2}$ ... (*)

$\left(\frac{1+a_2}{2}\right) \ge (a_2)^{1/2}$ ... (**)

$\vdots \quad \vdots$

$\frac{1+a_n}{2} \ge (a_n)^{1/2}$ ... (*** ... *)

now multiplying the above relations we have

$$\left(\frac{1+a_1}{2}\right)\left(\frac{1+a_2}{2}\right)\ldots\left(\frac{1+a_n}{2}\right) \ge (a_1\, a_2 \ldots a_n)^{1/2}$$

$\Rightarrow \quad (1+a_1)(1+a_2)\ldots(1+a_n) \ge 2^n(1)$ (as $a_1\, a_2 \ldots a_n = 1$)

ii. **(c) :** $\frac{a^{1/3}+b^{1/3}+c^{1/3}}{3} < \left(\frac{a+b+c}{3}\right)^{1/3}$ as $0 < m < 1$

$\Rightarrow \quad a^{1/3}+b^{1/3}+c^{1/3} < 3\cdot 3^{-1/3}$

$\Rightarrow \quad P < \frac{1}{9^3}$

iii. **(a) :** $(a^{n/2}-c^{n/2})^2 > 0$

$\Rightarrow \quad a^n + c^n > 2(ac)^{n/2}$

$\Rightarrow \quad \frac{a^n+c^n}{2} > (ac)^{n/2}$ ... (i)

But GM > HM $\Rightarrow (ac)^{1/2} > b \Rightarrow (ac)^{n/2} > b^n$ ... (ii)

$\therefore$ from (i) & (ii) we get

$\Rightarrow \frac{a^n+c^n}{2} > b^n$

$\Rightarrow \frac{a^5+c^5}{2} > b^5$

$\Rightarrow \frac{a^5+c^5}{b^5} > 2$ By putting $n = 5$

$\Rightarrow \quad \lambda > 2$

iv. **(b) :** Using the concept of mean of m^{th} powers for $0 < m < 1$

$$\frac{(a^3)^{1/3}+(b^3)^{1/3}+(c^3)^{1/3}}{3} \le \left(\frac{a^3+b^3+c^3}{3}\right)^{1/3}$$

$\Rightarrow \quad \frac{a+b+c}{3} \le \left(\frac{27}{3}\right)^{1/3} = 9^{1/3}$

$\Rightarrow \quad a+b+c \le 3\cdot 9^{1/3}$

$\Rightarrow \quad a+b+c \le 3^{5/3}$

$\therefore \quad a+b+c \in (0, 3^{5/3}]$ ie $0 < a+b+c \le 3^{5/3}$

v. **(c) :** Using AM ≥ GM

$\therefore \frac{\tan a + \tan b + \tan c}{3} \ge (\tan a \cdot \tan b \cdot \tan c)^{1/3}$

$\Rightarrow \quad \frac{\tan a \cdot \tan b \cdot \tan c}{3} \ge (\tan a \cdot \tan b \cdot \tan c)^{1/3}$

(using given condition)

$\Rightarrow \quad (\tan a \cdot \tan b \cdot \tan c)^{1-1/3} \ge 3$

$\Rightarrow \quad (\tan a \cdot \tan b \cdot \tan c)^{2/3} \ge 3$

$\Rightarrow \quad \tan a \cdot \tan b \cdot \tan c \ge (3)^{3/2}$

$\Rightarrow \quad \cot a \cdot \cot b \cdot \cot c \le \frac{1}{3\sqrt{3}}$

$\cot a \cdot \cot b \cdot \cot c \in \left(0, \frac{1}{3\sqrt{3}}\right]$ (as $0° < a, b, c < 90°$)

17. A → p, t : $S_n = an^2 + bn$

$t_n = S_n - S_{n-1}$
$= an^2 + bn - a(n-1)^2 - b(n-1)$
$= a(2n-1) + b$
$= 2na + b - a$

Again common difference : If sum of n terms of an A.P. is of the form $An^2 + Bn$ then common difference = 2 × coefficient of n^2

$\therefore \quad$ Common Difference = $2 \times a = 2a$

Note : Common difference is generally calculated as $D = t_n - t_{n-1}$

B → p, t, y : $S_n = an^2 + bn + c$

$\therefore \quad$ As the sequence is in AP $\therefore c = 0$

$\therefore \quad t_n = S_n - S_{n-1} = a(n^2-(n-1)^2) + b(n-n+1)$
$= a(2n-1) + b$
$= a(2n-1) + b + c$

$\therefore t_n = a(2n-1) + b + c = a(2n-1) + b$ as $c = 0$

and common difference $D = t_n - t_{n-1} = \ldots = t_2 - t_1 = 2a$

$\therefore D = 2a$

C → u, x : Given $S_n = na + \frac{n(n-1)}{2}b$

$\therefore \quad t_n = S_n - S_{n-1}$
$= na + \frac{n(n-1)}{2}b - (n-1)a - \frac{(n-1)(n-2)}{2}b$
$= a + (n-1)b$

$\therefore \quad$ Common difference $D = t_n - t_{n-1} = b = t_2 - t_1$

D → q, s : Given $t_a = b$ & $t_b = a$

Let $t_1 = A$ and common difference is

$\therefore \quad t_a = A + (a-1)D = b$
$t_b = A + (b-1)D = a$

$\Rightarrow \quad A = a + b - 1$ & $D = -1$

$\therefore \quad t_n = a + b - n$

E → r, v, w, z : $S_n = (a-2)n^3 + (b-1)n^2 + (c-3)n + d$

Fact sum of n term of AP with first term A & common difference D is given by $\frac{n}{2}[2A + (n-1)D]$ which is pure quadratic with no constant number

$\therefore \quad a - 2 = 0$ & $d = 0$

$\therefore \quad S_n = (b-1)n^2 + (c-3)n$

$\therefore \quad t_n = S_n - S_{n-1} = (b-1)[2n-1] + (c-3)$
$= b(2n-1) - 2n + 1 + c - 3$
$= b(2n-1) - 2n + c - a$
$= b(2n-1) - (2n+4) + c + 2$
$= b(2n-1) - 2(n+2) + c + 2$
$= b(2n-1) - a(n+2) + c + a \quad (\because a = 2)$
$= b(2n-1) - a(n+1) + c$
$= b(2n-1) - a(n+1) + c + d$

and common difference = 2 × coefficient of $n^2 = 2(b-1)$

18. (a) : Given $a, b, c, \in$ AP $\Rightarrow a + c = 2b$

Now $\frac{a+b+c}{3} \geq (abc)^{1/3}$ $\because$ AM $\geq$ GM

$\Rightarrow \frac{3b}{3} \geq (64)^{1/3}$

$b \geq 4$

$\Rightarrow$ Minimum value of $b = 4$

$\Rightarrow$ Assertion (A) & Reason (R) both are individually correct & Reason (R) is correct explanation of Assertion (A).

19. (d) : $\because$ AM > GM

$\Rightarrow \frac{a^2+b^2}{2} > \sqrt{a^2b^2}$

$\Rightarrow a^2 + b^2 > 2ab$ ... (i)

Similarly $b^2 + c^2 > 2bc$ ... (ii)

and $c^2 + a^2 > 2ca$ ... (iii)

on adding (i), (ii) & (iii)

$2(a^2 + b^2 + c^2) > 2(ab + bc + ca)$

$\Rightarrow ab + bc + ca < 1$

$\Rightarrow$ Assertion (A) false but Reason (R) is true

20. (d) : Sum of n term of AP is given by $S_n = \frac{n}{2}[2A + (n-1)D]$ where A is first term & D is common difference. Hence, sum of n terms A.P. is always of the form. $pn^2 + qn$. So Reason (R) is true.

Now $2n^2 + 3n + 1$ is not an A.P.

$\Rightarrow$ Assertion is false.

21. (b) : Given $a^{2/3} + b^{2/3} + c^{2/3} + d^{2/3} = 2$

$m = \frac{2}{3}, n = 4$ (Number of variables are 4)

Using Arithmetic mean of $\frac{2}{3}$ power < $\frac{2}{3}$ power of arithmetic means

$\therefore \frac{a^{2/3}+b^{2/3}+c^{2/3}+d^{2/3}}{4} < \left(\frac{a+b+c+d}{4}\right)^{2/3}$ as $0<m<1$

$\Rightarrow \frac{1}{2} \times 4^{2/3} < (a+b+c+d)^{2/3}$

$\Rightarrow 4 \times \left(\frac{1}{2}\right)^{3/2} < a+b+c+d$

$2^{2-3/2} < a+b+c+d \Rightarrow a+b+c+d > \sqrt{2}$

Assertion (A) & Reason (R) both are true but Reason (R) is not the proper explanation of Assertion (A)

22. (a) : Given $3x + 4y = 5$ ie $ax + by = K$

the expression $x^m y^n$ ($m, n, \geq 1$) will be maximum when $\left(\frac{ax}{m}\right)^m \cdot \left(\frac{by}{n}\right)^n$ is maximum and this is maximum

$\therefore$ at $\frac{ax}{m} = \frac{by}{n} = \frac{ax+by}{m+n} = \frac{K}{m+n}$

(Here $a = 3, b = 4, m = 2, n = 3, K = 5$)

$\therefore$ x^2y^3 be maximum at

$\Rightarrow \frac{3x}{2} = \frac{4y}{3} = \frac{5}{5}$

$\Rightarrow x = \frac{2}{3}, y = \frac{3}{4} \Rightarrow \frac{x}{y} = \frac{8}{9}$ or $9x = 8y$

$\therefore$ Maximum value of $x^2y^3 = \left(\frac{2}{3}\right)^2\left(\frac{3}{4}\right)^3 = \frac{3}{16}$

$\therefore$ both Assertion (A) & Reason (R) are true and Reason (R) is correct explanation of Assertion (A).

23. (a) : It is well known fact if $S_n = An^2 + Bn$ then common difference is twice the coefficient of n^2

ie $S_n = An^2 + Bn$

$\therefore t_n = S_n - S_{n-1} = A(n^2 - (n-1)^2) + B(n - n + 1)$

$= A(2n-1) + B$

$\therefore t_1 = A + B, t_2 = 3A + B$

$\therefore$ Common difference $= t_2 - t_1 = 2A = 2 \times$ coefficient of n^2

(b) : If $ax + by + cz = K$ then $a^m b^n c^p$ will be maximum at $\frac{ax}{m} = \frac{by}{n} = \frac{cz}{p} = \frac{K}{m+n+p}$ ... (*)

In our case $x = y = z = 1, K = 18, m = 2, n = 3, p = 4$

$\therefore$ By (*) we have

$\frac{a}{2} = \frac{b}{3} = \frac{c}{4} = \frac{18}{2+3+4}$

$\Rightarrow a = 4, b = 6, c = 8$

$\therefore$ Maximum (greatest) value of $a^2b^3c^4 = 4^2 \cdot 6^3 \cdot 8^4 = 2^{19} \cdot 3^3$

(c) : Sum of 20 terms of an AP $a_1\, a_2 \ldots a_{20} = \frac{20}{2}[a_1 + a_{20}]$

Now $a_3 + a_6 + \ldots + a_{18} = 120$

$\Rightarrow 3(a_3 + a_{18}) = 120$

$\Rightarrow (a_3 + a_{18}) = 40$

$\Rightarrow a_1 + a_{20} = 40$

$\therefore S_{20} = \frac{20}{2}[a_1 + a_{20}] = 10(40) = 400$

(d) : $S_n = 5 + 55 + 555 + \ldots$ up to n terms

$= \frac{5}{9}[9 + 99 + \ldots \text{ up to } n \text{ terms}]$

$= \frac{5}{9}[(10-1) + (10^2 - 1) + (10^3 - 1) + \ldots + (10^n - 1)]$

$= \frac{5}{9}\left[10\left(\frac{10^n - 1}{9}\right) - n\right]$

$= K\left[\frac{10}{9}(10^n - 1) - n\right]$

$\therefore K = \frac{5}{9}$.

(e) : $\lambda = \sum_{x=1}^{\infty} \frac{1}{x^4} = \frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} \ldots \infty$

$\therefore \sum_{x=1}^{\infty} \frac{1}{(2x-1)^4} = \frac{1}{1^4} + \frac{1}{3^4} + \frac{1}{5^4} \ldots \infty$

$= \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \frac{1}{4^4} + \frac{1}{5^4} + \ldots \infty\right) - \left(\frac{1}{2^4} + \frac{1}{4^4} + \frac{1}{6^4} + \ldots \infty\right)$

$= \left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots \infty\right) - \frac{1}{16}\left(\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots \infty\right)$

$= \left(1 - \frac{1}{16}\right)\left[\frac{1}{1^4} + \frac{1}{2^4} + \frac{1}{3^4} + \ldots \infty\right] = \frac{15}{16}\lambda.$

Olympiad Enrichment Series VII

useful for IIT-JEE 2008-09

This series is selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solution.

1. Let $0 < a < 1$. Solve $x^{a^x} = a^{x^a}$ for positive numbers x.

2. What is the coefficient of x^2 when
$$(1 + x)(1 + 2x)(1 + 4x) \dots (1 + 2^n x)$$
is expanded?

3. Let m and n be distinct positive integers. Find the maximum value of $|x^m - x^n|$, where x is a real number in the interval $(0, 1)$.

4. Prove that the polynomial
$$(x - a_1)(x - a_2) \dots (x - a_n) - 1,$$
where $a_1, a_2, \dots, a_n$ are distinct integers, cannot be written as the product of two non-constant polynomials with integer coefficients, i.e., it is irreducible.

5. Find all ordered pairs of real numbers (x, y) for which : $(1 + x)(1 + x^2)(1 + x^4) = 1 + y^7$
and $(1 + y)(1 + y^2)(1 + y^4) = 1 + x^7$.

SOLUTIONS

1. Taking $\log_a$ yields, $a^x \log_a x = x^a$

Consider functions from $\mathbb{R}^+ \to \mathbb{R}$,
$$f(x) = a^x, \quad g(x) = \log_a x, \quad h(x) = x^a.$$
Then both f and g are decreasing and h is increasing. It follows that $f(x)\, g(x) = h(x)$ has unique solution $x = a$.

2. Let $f_n(x) = a_{n,0} + a_{n,1} x + \dots + a_{n,n} x^n$
$$= (1 + x)(1 + 2x) \dots (1 + 2^n x).$$
It is easy to see that $a_{n,0} = 1$ and
$$a_{n,1} = 1 + 2 + \dots + 2^n = 2^{n+1} - 1$$
Since
$$\begin{aligned} f_n(x) &= f_{n-1}(x)(1 + 2^n x) \\ &= (1 + (2^n - 1)x + a_{n-1,2}\, x^2 + \dots)(1 + 2^n x) \\ &= 1 + (2^{n+1} - 1)x + (a_{n-1,2} + 2^{2n} - 2^n)x^2 + \dots, \end{aligned}$$
We have
$$\begin{aligned} a_{n,2} &= a_{n-1,2} + 2^{2n} - 2^n \\ &= a_{n-2,2} + 2^{2n-2} - 2^{n-1} + 2^{2n} - 2^n \\ &= \dots \\ &= a_{1,2} + (2^4 + 2^6 + \dots + 2^{2n}) - (2^2 + 2^3 + \dots + 2^n) \\ &= 2 + \frac{2^4(2^{2n-2} - 1)}{3} - 4(2^{n-1} - 1) \\ &= \frac{2^{2n+2} - 3 \cdot 2^{n+1} + 2}{3} = \frac{(2^{n+1} - 1)(2^{n+1} - 2)}{3}. \end{aligned}$$

3. By symmetry, we can assume that $m > n$.
Let $y = x^{m-n}$
Since $0 < x < 1$, $x^m < x^n$ and $0 < y < 1$. Thus
$$|x^m - x^n| = x^n - x^m = x^n(1 - x^{m-n}) = (y^n(1 - y)^{m-n})^{\frac{1}{m-n}}$$
Applying the AM-GM inequality yields
$$y^n(1 - y)^{m-n} = \left(\frac{n}{m-n}\right)^n \left(\frac{(m-n)y}{n}\right)^n (1 - y)^{m-n}$$
$$\le \left(\frac{n}{m-n}\right)^n \left(\frac{n \cdot \frac{(m-n)y}{n} + (m-n)(1-y)}{n + m - n}\right)^{n+m-n}$$
$$= \frac{n^n (m-n)^{m-n}}{m^m}$$
Therefore
$$|x^m - x^n| \le \left(\frac{n^n (m-n)^{m-n}}{m^m}\right)^{\frac{1}{m-n}} = (m-n)\left(\frac{n^n}{m^m}\right)^{\frac{1}{m-n}}$$
Equality holds if and only if
$$\frac{(m-n)y}{n} = 1 - y \quad \text{or} \quad x = \left(\frac{n}{m}\right)^{\frac{1}{m-n}}$$

4. For the sake of contradiction, suppose that
$$f(x) = (x - a_1)(x - a_2) \dots (x - a_n) - 1$$
is not irreducible. Let $f(x) = p(x)q(x)$ such that $p(x)$ and $q(x)$ are two polynomials with integral coefficients having degree less than n. Then $g(x) = p(x) + q(x)$ is a polynomial with integral coefficients having degree less than n.

Since $\quad p(a_i)q(a_i) = f(a_i) = -1$
and both $p(a_i)$ and $q(a_i)$ are integers,
$$|p(a_i)| = |q(a_i)| = 1 \text{ and } p(a_i) + q(a_i) = 0$$
Thus $g(x)$ has at least n roots. But $\deg g < n$, so $g(x) = 0$.
Then $p(x) = -q(x)$ and $f(x) = -p(x)^2$,

which implies that the leading coefficient of $f(x)$ must be a negative integer, which is impossible, since the leading coefficient of $f(x)$ is 1.

5. We consider the following cases.

1. $xy = 0$. Then it is clear that $x = y = 0$ and $(x, y) = (0, 0)$ is a solution.
2. $xy < 0$. By the symmetry, we can assume that $x > 0 > y$. Then $(1 + x)(1 + x^2)(1 + x^4) > 1$ and $1 + y^7 < 1$. There are no solutions in this case.
3. $x, y > 0$ and $x \neq y$.
   By the symmetry, we can assume that $x > y > 0$. Then $(1 + x)(1 + x^2)(1 + x^4) > 1 + x^7 > 1 + y^7$. showing that there are no solutions in this case.
4. $x, y < 0$ and $x \neq y$.
   By the symmetry, we can assume that $x < y < 0$. Multiplying by $1 - x$ and $1 - y$ the first and the second equation, respectively, the system now reads
   $$1 - x^8 = (1 + y^7)(1 - x) = 1 - x + y^7 - xy^7$$
   $$1 - y^8 = (1 + x^7)(1 - y) = 1 - y + x^7 - x^7y$$
   Subtracting the first equation from the second yields
   $$x^8 - y^8 = (x - y) + (x^7 - y^7) - xy(x^6 - y^6) \quad (1)$$
   Since $x < y < 0$, $x^8 - y^8 > 0$, $x - y < 0$, $x^7 - y^7 < 0$, $-xy < 0$ and $x^6 - y^6 > 0$. Therefore, the left-hand side of (1) is positive while the right-hand side of (1) is negative
   Thus there are no solutions in this case.
5. $x = y$. Then solving
   $$1 - x^8 = 1 - x + y^7 - xy^7 = 1 - x + x^7 - x^8$$
   leads to $x = 0, 1, -1$, which implies that
   $$(x, y) = (0, 0) \text{ or } (-1, -1).$$
   Therefore $(x, y) = (0, 0)$ and $(-1, -1)$ are the only solutions to the system.

CHINESE Olympiad Problems

SOLUTIONS

SECTION 1

1. For any positive integer k, take $a = 3n^4k + 9n^2k^2 + 9k^3$. Then $n^6 + 3a = (n^2 + 3k)^3$.

2. We may assume that $BD \le 6 \le AC$. Let $\angle ABD = \theta$. Then $\arccos\frac{3}{5} \le \theta < \frac{\pi}{2}$.

Now $AC + BD = 10(\sin\theta + \cos\theta) = 10\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$.

Note that $\frac{\pi}{2} < \frac{\pi}{4} + \arccos\frac{3}{5} \le \frac{\pi}{4} + \theta < \frac{3\pi}{4}$.

In the second quadrant, $\sin x$ is a decreasing function. Hence the maximum value of $AC + BD$ is attained when $\theta = \arccos\frac{3}{5}$. Then $AC + BD = 8 + 6 = 14$.

3. Of the lines passing through $(a, 0)$, $y = 0$ has the desired property but $x = a$ does not. Suppose another line passes through two points (x_1, y_1) and (x_2, y_2) with all coordinates rational. Then $x_1 \ne x_2$, and $y_1 \ne 0$. The slope of this line is given by $\frac{y_2 - y_1}{x_2 - x_1} = \frac{y_1}{x_1 - a}$. This is impossible since the left side is rational but the right side is irrational.

4. A_1 does not move during the first rotation. In the second rotation, the angle between the initial position of A_2A_1 and the final position of A_2A_3 is $\frac{2\pi}{3}$. Hence A_1 describes an arc with radius 1 and central angle $\frac{2\pi}{3} - 2\alpha$. During the third rotation, the angle between the initial and final positions of A_3A_1 is the same as the angle between the initial and final positions of A_3A_2, which is $\frac{\pi}{3}$. Hence A_1 describes an arc with radius $A_1A_3 = 2\sin\alpha$ and central angle $\frac{\pi}{3}$. It follows that in one cycle of three rotations, the distance traveled by A_1 is $\frac{2\pi}{3}(1+\sin\alpha) - 2\alpha$. Since it does not move in the 100-th rotation, the total distance traveled by A_1 after 33 cycles is

$$22\pi(1 + \sin\alpha) - 66\alpha.$$

SECTION 2

1. Exactly one of x, xy and $\log(xy)$ is 0. Since $\log(xy)$ is undefined if $xy = 0$, we must have $\log(xy) = 0$ or $y = \frac{1}{x}$. Now exactly one of $|x|$ and $\frac{1}{x}$ is 1. Hence $x = -1$ and $y = -1$. It follows that

$$\sum_{i=1}^{2001}(x^i + y^{-i}) = 2\sum_{i=1}^{2001}(-1)^i = -2.$$

2. The second set consists of the lines $x = \pm 1$ and $y = \pm 1$, defining a square of side length 2. Suppose the regular octagon contains this square. Then its side length is also 2. One of the vertices has coordinates $(1, 1+\sqrt{2})$ and we have $a = 2+\sqrt{2}$. Suppose the regular octagon is contained in the square. Let its side length be b. Then $b+\sqrt{2}b = 2$ or $b = 2(\sqrt{2}-1)$. One of the vertices has coordinates $\left(1, 1-\frac{b}{\sqrt{2}}\right)$ and we have $a = 2-\frac{b}{\sqrt{2}} = \sqrt{2}$. These two are the only possible valus of a.

3. The other root of $x^2 - kx + 1 = 0$ is $\frac{1}{a}$, and we have

$$a+\frac{1}{a} = k. \text{ Let } x_n = a^{2^n} + a^{-2^n}.$$

Then $x_0 = k$ and $x_n = x_{n-1}^2 - 2$ for all $n \ge 1$. Let y_n be the units digit of x_n, so that y_0 is the units digit of k.
If y_0 is even, all subsequent y's are even.
If $y_0 = 1$ or 9, all subsequent y's are 9's.
If $y_0 = 3$ or 7, all subsequent y's are 7's.
If $y_0 = 5$, then $y_1 = 3$ and all subsequent y's are 7's.
The desired condition is satisfied if and only if

$$y_0 = 3, 5 \text{ or } 7.$$

4. Since each triangle has two sides of lengths 4 and 5, any constructible tetrahedron has a pair of opposite sides of length 4 and another pair of length 5. The remaining two side lengths are chosen from $\frac{5\sqrt{2}}{6}$, 3 and $\sqrt{41}$. By Ptolemy's Inequality, the product of these two side lengths is less than $5^2 + 4^2 = 41$ and greater than $5^2 - 4^2 = 9$. The only possible combination consists of 3 and $\sqrt{41}$. Since there are only two copies of the triangle with sides of lengths 3, exactly one tetrahedron may be constructed at a time.

5. Designate the brother-sister pair as (A_1, A_2), (B_1, B_2), (C_1, C_2), (D_1, D_2) and (E_1, E_2). We may assume that A_1 takes part in only two activities, and that A_1's companions in one activity are B_1, C_1, D_1 and E_1, while those in the other are B_2, C_2, D_2 and E_2. Not counting A_2, no other activity may involve three or more of the others, as otherwise two of them will have the same sub-script and have been together in an activity with A_1. Hence there must be at least 12 other activities, involving separately the pairs (B_1, C_2), (B_2, C_1), (B_1, D_2), (B_2, D_1), (B_1, E_2), (B_2, E_1), (C_1, D_2), (C_2, D_1), (C_1, E_2), (C_2, E_1), (D_1, E_2) and (D_2, E_1). Hence $k \ge 14$. By adding A_2 to the first two and the last two of these 12 activities, all conditions are fulfilled. It follows that the minimum value of k is 14.

Olympiad Enrichment Series VIII

useful for IIT-JEE 2008-09

This series is selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solution.

1. Solve the equation

$2(2^x - 1)x^2 + (2^{x^2} - 2)x = 2^{x+1} - 2$ for real numbers x.

2. Let a be an irrational number and let n be an integer greater than 1. Prove that

$$(a + \sqrt{a^2 - 1})^{1/n} + (a - \sqrt{a^2 - 1})^{1/n}$$

is an irrational number.

3. Solve the system of equations

$$(x_1 - x_2 + x_3)^2 = x_2(x_4 + x_5 - x_2)$$
$$(x_2 - x_3 + x_4)^2 = x_3(x_5 + x_1 - x_3)$$
$$(x_3 - x_4 + x_5)^2 = x_4(x_1 + x_2 - x_4)$$
$$(x_4 - x_5 + x_1)^2 = x_5(x_2 + x_3 - x_5)$$
$$(x_5 - x_1 + x_2)^2 = x_1(x_3 + x_4 - x_1)$$

for real numbers x_1, x_2, x_3, x_4, x_5.

4. let x, y and z be complex numbers such that $x + y + z = 2$, $x^2 + y^2 + z^2 = 3$ and $xyz = 4$.

Evaluate $\dfrac{1}{xy + z - 1} + \dfrac{1}{yz + x - 1} + \dfrac{1}{zx + y - 1}$.

5. Given that the real numbers a, b, c, d and e satisfy simultaneously the relations $a + b + c + d + e = 8$ and $a^2 + b^2 + c^2 + d^2 + e^2 = 16$, determine the maximum and minimum value of a.

SOLUTION

1. Rearranging terms by powers of 2 yields

$$2^{x^2} x + 2^{x+1}(x^2 - 1) - 2(x^2 + x - 1) = 0 \quad ...(1)$$

Setting $y = x^2 - 1$ and dividing by 2 on both side, (1) becomes $2^y x + 2^x y - (x + y) = 0$

or $x(2^y - 1) + y(2^x - 1) = 0$ ...(2) Since $f(x) = 2^x - 1$ and x always have the same sign, $x(2^y - 1) \cdot y(2^x - 1) \geq 0$

Hence if the terms on the left-hand side of (2) are nonzero, they must have the same sign, which in turn implies that their sum is not equal to 0.

Therefore (2) is ture if and only if $x = 0$ or $y = 0$, which leads to solutions $x = -1$, 0 and 1.

2. Let $N = (a + \sqrt{a^2 - 1})^{1/n} + (a - \sqrt{a^2 - 1})^{1/n}$

and let $b = (a + \sqrt{a^2 - 1})^{1/n}$

Then $N = b + 1/b$. For the sake of contradiction, assume that N is rational. Then by using the identity

$$b^{m+1} + \frac{1}{b^{m+1}} = \left(b + \frac{1}{b}\right)\left(b^m + \frac{1}{b^m}\right) - \left(b^{m-1} + \frac{1}{b^{m-1}}\right)$$

repeatedly for $m = 1, 2, ..,$ we obtain that $b^m + (1/b^m)$ is rational for all $m \in N$. In particular,

$$b^n + \frac{1}{b^n} = a + \sqrt{a^2 - 1} + a - \sqrt{a^2 - 1} = 2a$$

is rational, in contradiction with the hypothesis. Therefore our assumption is wrong and N is irrational.

3. Let $x_{k+5} = x_k$. Adding the five equation gives

$$\sum_{k=1}^{5} (3x_k^2 - 4x_k x_{k+1} + 2x_k x_{k+2}) = \sum_{k=1}^{5} (-x_k^2 + 2x_k x_{k+2})$$

It follows that $\sum_{k=1}^{5} (x_k^2 - x_k x_{k+1}) = 0$.

Multiplying both sides by 2 and completing the squares yields $\sum_{k=1}^{5} (x_k - x_{k+1})^2 = 0$, from which $x_1 = x_2 = x_3 = x_4 = x_5$. Therefore the solutions to the system are $(x_1, x_2, x_3, x_4, x_5) = (a, a, a, a, a)$ for $a \in \mathbb{R}$.

4. Let S be the desired value. Note that

$xy + z - 1 = xy + 1 - x - y = (x - 1)(y - 1)$

Likewise, $yz + x - 1 = (y - 1)(z - 1)$

and $zx + y - 1 = (z - 1)(x - 1)$

Hence

$$S = \frac{1}{(x-1)(y-1)} + \frac{1}{(y-1)(z-1)} + \frac{1}{(z-1)(x-1)}$$
$$= \frac{x + y + z - 3}{(x-1)(y-1)(z-1)} = \frac{-1}{(x-1)(y-1)(z-1)}$$
$$= \frac{-1}{xyz - (xy + yz + zx) + x + y + z - 1}$$
$$= \frac{-1}{5 - (xy + yz + zx)}$$

But $2(xy + yz + zx) = (x + y + z)^2 - (x^2 + y^2 + z^2) = 1$

Therefore $S = -2/9$.

5. Since the total of b, c, d and e is $8 - a$, their averag is $x = (8 - a)/4$.

Let $b = x + b_1$, $c = x + c_1$, $d = x + d_1$, $e = x + e_1$

Then $b_1 + c_1 + d_1 + e_1 = 0$ and

$$16 = a^2 + 4x^2 + b_1^2 + c_1^2 + d_1^2 + e_1^2 \geq$$
$$a^2 + 4x^2 = a^2 + (8 - a)^2/4 \quad ...(1)$$

or $0 \geq 5a^2 - 16a = a(5a - 16)$. Therefore $0 \leq a \leq 16/$ where $a = 0$ if and only if $b = c = d = e = 2$ and $a = 16$ if and only if $b = c = d = e = 6/5$.

70. The value of $P(S \text{ or } T)$

(a) 0.6900 (b) 1.19 (c) 0.8450 (d) 0

$P_{71\text{-}73}$: Paragraph for Question Nos. 71 to 73

Three players A, B and C alternately throw a die in that order. The first player to throw a 6 will be the winner. A's die is fair whereas B and C throw a biased die with probability of throwing a 6 being p_1 and p_2 respectively.

71. Probability of winning of A is

(a) $\dfrac{1}{1-5q_1q_2}$ (b) $\dfrac{1}{6-5q_1q_2}$

(c) $\dfrac{5p_1}{6-5q_1q_2}$ (d) none of these

72. Ratio of Probability of winning of B to Probability of winning of C is

(a) $\dfrac{q_1}{p_1p_2}$ (b) $\dfrac{p_1}{q_1}$ (c) $\dfrac{p_1}{q_1q_2}$ (d) $\dfrac{q_1}{p_1}$

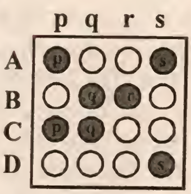
73. Ratio of p_1 and p_2 so that the game becomes equally likely for all the three players.

(a) 5 : 4 (b) 1 : 2 (c) 4 : 5 (d) 2 : 1

SECTION-IV

Matrix-Match Type

This section contains 2 questions. Each question contains statements given in two columns which have to be matched. Statements (A, B, C, D) in Column I have to be matched with statements (p, q, r, s) in Column II. The answers to these questions have to be appropriately bubbled as illustrated in the following example.



If the correct matches are A-p, A-s, B-q, B-r, C-p, C-q and D-s, then the correctly bubbled 4 × 4 matrix should be as shown.

74. COLUMN - I **COLUMN - II**

(A) Let A be the set of 4-digit number abcd, where $a > b > c > d$ then $n(A)$ equal (p) 210

(B) On a railway there are 20 station. The number of different tickets required in order that it may be possible to travel every station to every station is ? (q) 300

(C) How many numbers consisting of 5 digits can be formed in which the digits 3,4 and 7 are used only once and the digit 5 is used twice ? (r) 60

(D) How many different nine-digit numbers can be formed from the number 223355888 by rearranging its digits, So that the old digits occupy odd positions ? (s) 180

75. COLUMN - I **COLUMN - II**

(A) If $\tan\alpha = \dfrac{1}{3}$ and $\tan\dfrac{\beta}{2} = \dfrac{1}{2}$, then $(\alpha + \beta) =$ (p) 30°

(B) In a scalene triangle ABC, if $a \cos A = b \cos B$ then $\angle C$ equals (q) 45°

(C) In a triangle ABC, $BC = 1$ and $AC = 2$. The maximum possible value which the $\angle A$ can have is (r) 60°

(D) In a ΔABC $\angle B = 75°$ and $BC = 2AD$ where AD is the altitude from A, then $\angle C$ equals (s) 90°

ANSWERS

1. a	2. b	3. b	4. c
5. b	6. b	7. c	8. a
9. a	10. a,c	11. a,b,c,d	12. a,b,c
13. a,b,c	14. a,c	15. b	16. a
17. b	18. c	19. d	20. a
21. d	22. b	23. b	

24. (A) - q; (B) - r; (C) - p; (D) - s

25. (A) - p; (B) - q, s; (C) - q; (D) - q, r

26. d	27. b	28. d	29. c
30. b	31. d	32. a	33. b
34. a	35. a,b,d	36. b,c	37. a,b
38. c,d	39. c	40. b	41. b
42. d	43. c	44. d	45. b
46. c	47. a	48. b	

49. (A) - q, r; (B) - q, s; (C) - p, s; (D) - p, s

50. (A) - q, s; (B) - q, s; (C) - p, s; (D) - p, r

51. c	52. a	53. a	54. a
55. b	56. d	57. a	58. c
59. d	60. a,b,c	61. c,d	62. a,b,d
63. b,c,d	64. a,d	65. b	66. b
67. c	68. b	69. a	70. c
71. b	72. c	73. c	

74. (A) - p; (B) - q; (C) - r; (D) - q

75. (A) - q; (B) - s; (C) - p; (D) - p

For Paper - I, refer Physics For You

CHINESE Olympiad Problems

SECTION 1

1. What is the function whose graph is the reflection about the line $x + y = 0$ of the inverse function $f^{-1}(x)$ of a function $f(x)$?
(a) $-f(x)$ (b) $-f(-x)$ (c) $-f^{-1}(x)$ (d) $-f^{-1}(-x)$.

2. The origin is inside the ellipse $k^2x^2 + y^2 - 4kx + 2ky + k^2 - 1 = 0$. Which of the following statements is true?
(a) $|k| > 1$ (b) $|k| \neq 1$ (c) $-1 < k < 1$ (d) $0<|k|< 1$.

3. Consider the sets $M = \{(x, y) \mid |x| + |y| < 1\}$,

$$N = \left\{(x, y) \mid \sqrt{\left(x-\frac{1}{2}\right)^2 + \left(y+\frac{1}{2}\right)^2} + \sqrt{\left(x+\frac{1}{2}\right)^2 + \left(y-\frac{1}{2}\right)^2} < 2\sqrt{2}\right\}$$

and $P = \{(x, y) \mid |x + y| < 1, |x| < 1, |y| < 1\}$
Which of the following statements is true?
(a) $M \subset P \subset N$ (b) $M \subset N \subset P$
(c) $P \subset N \subset M$
(d) none of (a), (b) and (c) is true.

4. The dihedral angle between any two of the three planes α, β and γ is θ. Let $\alpha \cap \beta = a$, $\beta \cap \gamma = b$ and $\gamma \cap \alpha = c$. Consider the following two statements.
$P : \theta > \pi/3$
$Q : a, b, c$ concurrent.
Which of the following statements is true?
(a) P is necessary and sufficient for Q
(b) P is necessary but not sufficient for Q
(c) P is sufficient but not necessary for Q
(d) P is neither necessary nor sufficient for Q.

5. In the plane, let I be the set of all lines, M the set of lines each passing through exactly one lattice point, N the set of lines not passing through any lattice point, and P the set of lines each passing through infinitely many lattice points. Consider the following statements.
$P : M \cup N \cup P = I$
$Q : M \neq \phi$
$R : N \neq \phi, S : P \neq \phi$
How many of them are true?
(a) 1 (b) 2 (c) 3 (d) 4.

SECTION 2

1. In the arithmetic progressions x, a_1, a_2, a_3, y and b_1, x, b_2, b_3, y, b_4 we have $x \neq y$. What is the value of $\dfrac{b_4 - b_3}{a_2 - a_1}$?

2. In the expansion of $(\sqrt{x} + 2)^{2n+1}$, what is the sum of the coefficients of all the integral powers of x?

3. CD and BE are altitudes of triangle ABC. If $\angle BAC = \alpha$, what is the value of $\dfrac{DE}{BC}$?

4. Each of two teams has seven players numbered 1 to 7. In the first game, the two players numbered 1 play each other. The loser of each game is eliminated and replaced by the next player of the same team. Until all players from one team have been eliminated. What is the number of possible sequences of games?

Solutions will be published in March issue.

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The reader's comments and suggestions regarding the problems and solutions offered are always welcome.

1. Discuss the continuity of the function $f(x)=\lim_{n\to\infty}\left\{\dfrac{\cos\pi x - x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}\right\}$ at $x = 1$.

2. Show that $\lim_{n\to\infty}\sum_{K=0}^{n}\dfrac{{}^nC_K}{n^K(K+3)} = e-2.$

3. Suppose $\phi(.)$ is a differentiable function. If $\phi(x+y)=\phi(x).\phi(y)$ and $\phi(5)=2$, $\phi'(0)=3$, then find the value of $\phi'(1)$.

4. Find the domain and range of the function f, defined by $f(x)=\dfrac{x^2+1}{\ln(x^2+1)}$.

5. Show that $\sin[x]$, where [.] denotes the greatest integer function, is non-periodic. Also show that there is no x for which $\sin[x] = \cos[x]$ but there are infinitely many x for which $\sin[x] = \tan[x]$.

6. Let $f(x+y)=f(x)-f(y)+2xy-1$, $\forall\, x, y\in R$. If f is differentiable and $f'(0)=b$, then find the set of values of b, if $f(x)>0$, $\forall\, x$.

7. A function f is defined by

$$f(x)=\begin{cases}\dfrac{(b^2-a^2)}{2}, & 0\le x\le a\\[2mm] \dfrac{b^2}{2}-\dfrac{x^2}{6}-\dfrac{a^3}{3x}, & a<x\le b\\[2mm] \left(\dfrac{b^3-a^3}{3x}\right), & x>b\end{cases}$$

Discuss the continuity of f, f' and f'' in $[0,\infty)$.

8. A function $f(x)$ is defined for $x\in[0,1]$ and $f(x)+f(y)=f(xy-\sqrt{1-x^2}.\sqrt{1-y^2})$ and $f(0)=\dfrac{\pi}{2}$, $f\left(\dfrac{1}{\sqrt{2}}\right)=\dfrac{\pi}{4}$. Find the function $f(x)$.

9. Discuss the continuity and differentiability of the function f defined by $f(x)=[x^2[1/x^2]]$, $x\ne 0$, where $[\cdot]$ denotes the G.I.F.

10. Find the set of values of 'a' for which the function $f:[-3,3]\to R$ defined by $f(x)=\left[\dfrac{x^2}{a}\right]\tan ax+\sec ax$ is an (i) even function (ii) odd function.
([·] denotes the G.I.F.)

SOLUTIONS

1. For, $0<x<1$

$$f(x)=\lim_{n\to\infty}\left\{\frac{\cos\pi x-x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}\right\}=\cos\pi x$$

For $x = 1$

$$f(x)=\lim_{n\to\infty}\left\{\frac{\cos\pi x-x^{2n}\cdot\sin(x-1)}{1+x^{2n+1}-x^{2n}}\right\}=\cos\pi=-1$$

For $x > 1$

$$f(x)=\lim_{n\to\infty}\left\{\frac{\cos\pi x-x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{\dfrac{\cos\pi x}{x^{2n+1}}-\dfrac{1}{x}\sin(x-1)}{\dfrac{1}{x^{2n+1}}+1-\dfrac{1}{x}}\right\}=\frac{\sin(x-1)}{(1-x)}$$

Thus, we have

$$f(x)=\begin{cases}\cos\pi x, & 0<x<1\\ -1, & x=1\\ \dfrac{\sin(x-1)}{(1-x)}, & x>1\end{cases}$$

$$\text{L.H.L.}=\lim_{h\to0}f(1-h)=\lim_{h\to0}\cos\pi(1-h)$$
$$=\lim_{h\to0}\cos(\pi-\pi h)=-1$$

R.H.L.

$$=\lim_{h\to0}f(1+h)=\lim_{h\to0}\frac{\sin[(1+h)-1]}{[1-(1+h)]}=\lim_{h\to0}\frac{\sin h}{-h}=-1$$

and $f(1)=-1$

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Thus $\lim_{h\to 0} f(1-h) = \lim_{h\to 0} f(1+h) = f(1)$

Hence $f(x)$ is continuous at $x = 1$.

2. $\lim_{n\to\infty} \sum_{K=0}^{n} \frac{{}^nC_K}{n^K(K+3)} = \lim_{n\to\infty} \sum_{K=0}^{n} \frac{1}{(K+3)}\, {}^nC_K \frac{1}{n^K}$

$$= \lim_{n\to\infty} \sum_{K=0}^{n} {}^nC_k \frac{1}{n^K} \cdot \int_0^1 x^{K+2} dx$$

$$= \int_0^1 \left(x^2 \lim_{n\to\infty} \sum_{K=0}^{n} {}^nC_K \left(\frac{x}{n}\right)^K \right) dx$$

$$= \int_0^1 x^2 \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^n dx$$

$$= \int_0^1 x^2 \lim_{n\to\infty} \left(1+\frac{x}{n}\right)^{\frac{n}{x}.x} dx = \int_0^1 x^2 e^x dx$$

$$= \left[x^2 \int e^x dx - \int \frac{d}{dx}(x^2)(\int e^x dx)\ \ dx \right]_0^1$$

$$= \left[x^2 e^x - 2\int xe^x dx \right]_0^1 = e - 2\int_0^1 xe^x dx$$

$$= e - 2\left[x\int e^x dx - \int \frac{dx}{dx}(\int e^x dx)\ \ dx \right]_0^1$$

$$= e - 2\left[x \cdot e^x - e^x \right]_0^1 = e - 2 = \text{R.H.S. Proved}$$

3. Given $\phi(x+y) = \phi(x) \,.\, \phi(y)$ ....(1)

or $\phi(x) = \phi(x) \,.\, \phi(0)$

or $\phi(0) = 1$

since $\lim_{h\to 0} \frac{\phi(x+h)-\phi(x)}{h} = \phi'(x)$

$\therefore \lim_{h\to 0} \frac{\phi(x)\cdot\phi(h)-\phi(x)}{h} = \phi'(x)$

or $\lim_{h\to 0} \frac{\phi(x)\cdot[\phi(h)-1]}{h} = \phi'(x)$

or $\lim_{h\to 0} \frac{\phi(x)\cdot[\phi(h)-\phi(0)]}{h} = \phi'(x)$

or $\phi(x)\cdot\phi'(0) = \phi'(x)$

or $3\phi(x) = \phi'(x)$ ...(2)

or $3\phi(x) = \frac{d[\phi(x)]}{dx}$ or $3dx = \frac{d[\phi(x)]}{\phi(x)}$

Integrating both sides

$\therefore\ 3x = \ln\phi(x) + C$ ...(3)

Put $x = 5$

$15 = \ln\phi(5) + C$

or $15 = \ln 2 + C$

$\therefore\ C = 15 - \ln 2$

Putting the value of 'C' in (3)

$\therefore\ 3x = \ln\phi(x) + 15 - \ln 2$ or $\ln\phi(x) = 3x + (\ln 2) - 15$

$\therefore\ \phi(x) = e^{3x+(\ln 2)-15} = e^{3x-15} \cdot e^{\ln 2}$

$\therefore\ \phi(x) = 2e^{3x-15}$ then $\phi(1) = 2e^{-12}$

From (2)

$\Rightarrow\ 3\phi(x) = \phi'(x)\ \therefore\ \phi'(1) = 3\phi(1) = 6e^{-12}$

4. $f(x)$ is defined if $\ln|\,x^2 + 1| \neq 0$

$\Rightarrow\ x^2 + 1 \neq 1 \Rightarrow x \neq 0.$

Thus domain of $f = R - \{0\}$

Now let $t = x^2 + 1$, then $t > 1$.

Let $g(t) = \frac{t}{\ln t} \Rightarrow g'(t) = \frac{\ln t - t.\frac{1}{t}}{\ln^2 t} = \frac{\ln t - 1}{\ln^2 t}$

Thus $g(t)$ decreases for $t \in (1, e]$ and increases for $t \in [e, \infty)$

$$g(e) = \frac{e}{\ln e} = e$$

We observe that $\lim_{t\to 1^+} g(t) = \infty$ and

$$\lim_{t\to\infty} g(t) = \lim_{t\to\infty} \frac{t}{\ln t} = \lim_{t\to\infty} \frac{t}{1/t} = \infty$$

Thus range of g is $[e, \infty)$. Hence range of $f\ [e, \infty)$

5. Let $f(x) = \sin[x]$. If f is periodic and $T > 0$ is one of the periods, then $f(x+T) = f(x),\ \forall\ x$

$\Rightarrow\ \sin[x+T] = \sin[x],\ \forall\ x$

If $x = 0$, then $\sin[T] = 0 \Rightarrow [T] = n\pi$, for some integer n.

Since $[T]$ is an integer and $n\pi$ is an integer only when $n = 0$, $[T] = 0$

If $x = T$, then $\sin[2T] = \sin[T] = 0 \Rightarrow [2T] = 0$ proceeding this way.

$[T] = [2T] = [3T] = \ldots\ldots. = 0$, which is not possible for any $T > 0$

Hence $\sin[x]$ is not a periodic function.

If $\sin[x] = \cos[x]$, then $[x] = n\pi + \pi/4,\ n \in I$, which is not possible for any x as L.H.S. is an integer and R.H.S. is never an integer.

However if $\sin[x] = \tan[x]$, then $[x] = n\pi,\ n \in I$, which is possible only where

$[x] = 0 \Rightarrow 0 \le x < 1$

6. Putting $x = 0 = y$ in the given functional equation, we get $f(0) = -1$.

Now $f'(x) = \lim_{h\to 0} \frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0} \frac{f(x)-f(h)+2xh-1-f(x)}{h}$$

$$= 2x - \lim_{x\to 0} \frac{f(h)-f(0)}{h} = 2x - f'(0) = 2x - b$$

$\Rightarrow\ f(x) = x^2 - bx + c$

Putting $x = 0$ we have $c = -1$.

Hence $f(x) = x^2 - bx - 1$

The discriminant $D = b^2 + 4$.

For $f(x) > 0,\ \forall\ x,\ D < 0$, which is not possible for any b.

Hence there is no such b.

7. $f(0)=f(0^+)=\dfrac{b^2-a^2}{2}$

$$f(a)=f(a^-)=\frac{b^2-a^2}{2},\ f(a^+)=\frac{b^2}{2}-\frac{a^2}{6}-\frac{a^3}{3a}=\frac{b^2-a^2}{2}$$

$$f(b)=f(b^-)=\frac{b^2}{2}-\frac{b^2}{6}-\frac{a^3}{3b}=\frac{b^2}{3}-\frac{a^3}{3b},$$

$$f(b^+)=\frac{b^3-a^3}{3b}=\frac{b^3-a^3}{3b}=\frac{b^3}{3}-\frac{a^3}{3b}.$$

Hence f is continuous every where in the domain i.e. $[0, \infty)$

$$\text{Now } f'(x)=\begin{cases} 0, & 0<x<a \\ -\dfrac{x}{3}+\dfrac{a^3}{3x^2}, & a<x<b \\ \dfrac{(a^3-b^3)}{3x^2}, & x>b \end{cases}$$

$$f'(a^-)=0,\ f'(a^+)=0,\ f'(b^-)=-\frac{b}{3}+\frac{a^3}{3b^2},$$

$$f'(b^+)=\frac{a^3-b^3}{3b^2}$$

Thus f is differentiable at 'a' and 'b'. Hence

$$f'(x)=\begin{cases} 0, & 0<x\le a \\ -\dfrac{x}{3}+\dfrac{a^3}{3x^2}, & a<x\le b \\ \dfrac{a^3-b^3}{3x^2}, & x>b \end{cases}$$

Thus f' is continuous every where in $(0, \infty)$.

$$\text{Again } f''(x)=\begin{cases} 0, & 0<x<a \\ -\dfrac{1}{3}-\dfrac{2a^3}{3x^3}, & a<x<b \\ \dfrac{2(b^3-a^3)}{3x^3}, & x>b \end{cases}$$

$$f''(a^-)=0,\qquad f''(a^+)=\frac{-1}{3}-\frac{2a^3}{3a^3}\ne f''(a^-)$$

$$f''(b^-)=-\frac{1}{3}-\frac{2a^3}{3b^3},\ f''(b^+)=\frac{2(b^3-a^3)}{3b^3}$$

$$=\frac{2}{3}-\frac{2a^3}{3b^3}\ne f''(b^-)$$

Hence $f''(a)$ and $f''(b)$ does not exist

Hence $f''(x)$ is continuous every where in $[0, \infty)$ expect at 'a' and 'b'.

8. $f'(x)=f'(xy-\sqrt{1-x^2}\sqrt{1-y^2})\left[y-\dfrac{\sqrt{1-y^2}(-2x)}{2\sqrt{1-x^2}}\right]$

Put $x=0$

$$f'(0)=f'(-\sqrt{1-y^2})\,(y)$$

$$\Rightarrow\ f'(-\sqrt{1-y^2})=\frac{f'(0)}{y}$$

Put $-\sqrt{1-y^2}=t\Rightarrow y=\sqrt{1-t^2}\ \Rightarrow f'(t)=\dfrac{f'(0)}{\sqrt{1-t^2}}$

$$\Rightarrow\ f(t)=-f'(0)\cos^{-1}t+c$$

Put $t=0$

$$\frac{\pi}{2}=-f'(0)\frac{\pi}{2}+c \qquad \ldots(1)$$

Put $t=\dfrac{1}{\sqrt{2}},\ \dfrac{\pi}{4}=-f'(0)\dfrac{\pi}{4}+c \qquad \ldots(2)$

from (1) & (2), $f(x) = \cos^{-1}x$.

9. $f(x) = [x^2\,[1/x^2]]$. Obviously f is an even function. Hence it suffices to discuss the continuity and differentiability of f for $x > 0$ (f is not defined for $x = 0$). If $x^2 > 1$, then $[1/x^2] = 0$ and hence $f(x) = 0$, which is every where continuous and differentiable.

Further for $0<x\le 1,\ \dfrac{1}{x^2}\ge 1$

If $n<\dfrac{1}{x^2}<n+1$, for some $n \in N$, then $[1/x^2] = n$ and hence for such x, $f(x) = [x^2\cdot n]$

But $\dfrac{1}{n+1}<x^2<\dfrac{1}{n}\Rightarrow\dfrac{n}{n+1}<x^2n<1\Rightarrow[x^2n]=0$

$\Rightarrow f(x) = 0$, which is every where continuous and differentiable.

If $\dfrac{1}{x^2}=n$, for some $n \in N$, then $f(x)=\left[\dfrac{1}{n}n\right]=1$

Hence for $x > 0$ and $n \in N$, $f(x)=\begin{cases}1, & \text{if } x^2=\dfrac{1}{n}\\ 0, & \text{otherwise}\end{cases}$

Thus f is not continuous at $x=\pm\dfrac{1}{\sqrt{n}},\ n \in N$. Hence f is not differentiable at $x=\pm\dfrac{1}{\sqrt{n}}$. Further for $x \ne 0$, $x\ne\pm\dfrac{1}{\sqrt{n}}$, f is a constant function, therefore f is continuous and differentiable.

10. (i) If $f(x)$ is even, then $f(-x)=f(x)$

$$\Rightarrow\ 2\left[\frac{x^2}{a}\right]\tan ax=0\Rightarrow\left[\frac{x^2}{a}\right]=0,\ \forall\ x\in[-3,3]$$

$$\Rightarrow 0\le\frac{x^2}{a}<1,\ \forall\ x\in[-3,3]\Rightarrow\ 0\le x^2<a \text{ (as } a>0)$$

$$\forall\ x\in[-3,3]\Rightarrow\ a>3^2$$

(ii) If f is an odd function, then $f(-x)=-f(x)\Rightarrow 2\sec ax=0,\ \forall\ x\in[-3,3]$, which is not possible for any a.

CHINESE Olympiad Problems

SECTION 1

1. ABC is an acute triangle. In which quadrant of the complex plane does the complex number

$$(\cos B - \cos A) + i\,(\sin B - \cos A)\text{ lie?}$$

(a) first (b) second (c) third (d) fourth.

2. What is the range of the function

$f(x) = \text{arc}\tan x + \dfrac{1}{2}\text{arc}\sin x$?

(a) $(-\pi, \pi)$ (b) $\left[-\dfrac{3\pi}{4}, \dfrac{3\pi}{4}\right]$

(c) $\left(-\dfrac{3\pi}{4}, \dfrac{3\pi}{4}\right)$ (d) $\left[-\dfrac{\pi}{2}, \dfrac{\pi}{2}\right]$

3. Let $f(x)$ be any function. About which line are the graphs of $y = f(x-1)$ and $y = f(-x+1)$ symmetric?

(a) $y = 0$ (b) $x = 1$

(c) $x = -1$ (d) $x = 0$.

4. How many of the triangles whose vertices are chosen from the vertices of a rectangular block are acute?

(a) 0 (b) 6

(c) 8 (d) 24.

5. How many elements are in the intersection of the sets

$$\left\{z \mid z = \frac{t}{t+1} + i\frac{1+t}{t},\ t \in \mathbf{R},\ t \neq 0,\ -1\right\} \text{ and}$$

$$\{z \mid z = \sqrt{2}(\cos(\text{arc}\sin t) + i\cos(arc\cos t)),\ t \in \mathbf{R},\ |t| \le 1\}?$$

(a) 0 (b) 1

(c) 2 (d) 4.

6. Let $M = \{u \mid u = 12m + 8n + 4l,\ m, n, l \in \mathbf{Z}\}$ and $N = \{u \mid u = 20p + 16q + 12r,\ p, q, r \in \mathbf{Z}\}$

Which of the following statement is true?

(a) $M = N$ (b) $N \subset M$

(c) $M \subset N$ (d) $M \not\subseteq N,\ N \not\subseteq M$

SECTION 2

1. What are the possible values of a if $\log_a \sqrt{2} < 1$?

2. The equation of line l is $2x + y = 10$. A line l' passing through the point $(-10, 0)$ is perpendicular to l. What are the coordinates of the point of intersection of l and l'?

3. Let $f_0(x) = |x|$, $f_1(x) = |f_0(x) - 1|$ and $f_2(x) = |f_1(x) - 2|$. What is the area of the finite region enclosed between the x-axis and the graph of $y = f_2(x)$?

4. Which positive real number x has the property that x, $\lfloor x \rfloor$ and $x - \lfloor x \rfloor$ form a geometric progression ?

Solutions on page no.81

CHINESE Olympiad Problems

SOLUTION

1. We have $A+B > \frac{\pi}{2}$ so that $0 < \frac{\pi}{2} - A < B < \frac{\pi}{2}$.

Hence $\cos B < \cos\left(\frac{\pi}{2} - A\right) = \sin A$

and $\sin B > \sin\left(\frac{\pi}{2} - A\right) = \cos A$

Since $\cos B - \sin A < 0$ while $\sin B - \cos A > 0$, the point is in the second quadrant.

2. The domain of $f(x)$ is $[-1, 1]$. On this interval,
$-\frac{\pi}{4} \le \text{arc}\tan x \le \frac{\pi}{4}$ and $-\frac{\pi}{4} \le \frac{1}{2}\text{arc}\sin x \le \frac{\pi}{4}$.
Moreover, both arc tan$x$ and arc sin$x$ are continuous and increasing, so that the range of $f(x)$ is $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$.

3. Since $-x+1 = -(x-1)$, the two graphs are symmetric about the line $x - 1 = 0$ or $x = 1$.

4. The eight vertices of the block determine $\binom{8}{3} = 56$ triangles, each of which is either an acute triangle or a right triangle. Each vertex of the block serves as the vertex of the right angle in three triangles whose hypotenuses are face diagonals of the block, and three triangles whose hypotenuses are space diagonals of the block. Hence there are $8(3 + 3) = 48$ right triangles and $56 - 48 = 8$ acute triangles.

5. For the first set, let $x = \frac{t}{t+1}$ and $y = \frac{t+1}{t}$.
Then $xy = 1$ with $x \ne 0$ or 1. In the second set, let

$x = \sqrt{2}\cos(arc\sin t) = \sqrt{2}\sqrt{1-t^2}$

and $u = \sqrt{2}t$

Then $x^2 + y^2 = 2$ with $0 \le x \le 2$. The hyperbola and the circle intersect only at the points (1, 1) and (–1, –1), both of which are excluded. Hence the two sets have empty intersection.

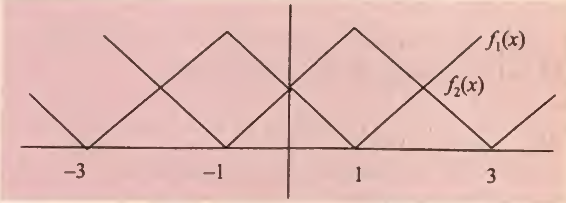
6. Since $20p + 16q + 12r = 12r + 8(2q) + 4(5p)$, every element of N is an element of M. Since $12m + 8n + 4l = 20n + 16l + 12(m - n - l)$, every element of M is an element of N. Hence $M = N$.

Section 2.

1. Let $y = \log_a \sqrt{2}$. Then $a^y = \sqrt{2}$.
Note that $a > 0$ and $a \ne 1$. If $0 < a < 1$, then $y < 1$ is equivalent to $a^y > a^1$ or $\sqrt{2} > a$, which certainly holds.
If $a > 1$, then $y < 1$ is equivalent to $a^1 > a^y$ or $a > \sqrt{2}$.
Hence the desired values are $0 < a < 1$ and $a > \sqrt{2}$.

2. The slope of l' is $-1/2$ and its equation is $x - 2y = -10$. Combined with $2x + y = 10$, we have $(x, y) = (2, 6)$.

3. The graphs of $f_1(x)$ and $f_2(x)$ are plotted in the diagram below.



4. We have $x(x-[x]) = [x]^2$

or $\left(\frac{x}{[x]}\right)^2 - \frac{x}{[x]} - 1 = 0$

Hence $\frac{x}{[x]} = \frac{1+\sqrt{5}}{2}$

Since the negative root must be rejected. Now

$0 < x - [x] = \frac{\sqrt{5}-1}{2}[x] = \frac{2}{1+\sqrt{5}}[x] < 1$

so that $0 < [x] < \frac{1+\sqrt{5}}{2} < 2$.

It follows that $[x] = 1$ and $x = \frac{1+\sqrt{5}}{2}$.

Long night studies do not help you to remember. After you have studied, think about it intensely, and see whether you can get all the logical steps of the derivation. Every step should be understood.

MATHS MUSING

SOLUTIONS (SET - 61)

1. **(d)** : Only III is false.

2. **(c)** : $F_1 = \dfrac{F \cdot (\vec{b} \times (\vec{a} \times \vec{b}))}{[\vec{a}\ \vec{b}\ \vec{c}]} = \dfrac{4}{3}\bar{F} \cdot \bar{a} - \dfrac{2}{3}\bar{F} \cdot \bar{b}.$

3. **(b)** : The conjugate of $iz^2 = (\bar{z})^2 + z$ ...(1)

is $-i(\bar{z})^2 = z^2 + \bar{z}$ ....(2)

(1), (2) $\Rightarrow z = -i\bar{z} \Rightarrow |z| = 1,$

$z = \pm\dfrac{(1-i)}{\sqrt{2}} \Rightarrow z^2 = -i.$

4. **(b)** : $x^2 + y^2 - a^2 = \lambda y \Rightarrow (x^2 - a^2 - y^2)y_1 = 2xy.$
The desired curves are the solution of

$2yy_1 - \dfrac{y^2}{x} = \dfrac{a^2}{x} - x$ namely, $x^2 + y^2 = cx - a^2.$

5. **(c)** : The number of 5-digit numbers is
$5 \cdot 5 \cdot 4 \cdot 3 \cdot 2 = 600$
The number of them divisible by 6 is 108.
Probability = 9/50.

6. **(b)** : Area of triangle $ABC = abc/4$
The height of the triangle = $ab/2$

$= \dfrac{1}{2}|z_2 - z_3||z_1 - z_3|.$

7. **(c)** : Area = $4\left[\int_0^{2/\sqrt{3}} \sqrt{1 - \dfrac{x^2}{4}}dx + \int_{2/\sqrt{3}}^{\sqrt{2}} \sqrt{2 - x^2}dx\right]$

$= 4\tan^{-1}2\sqrt{2}.$

8. **(c)** : The points $\left(2\sqrt{\dfrac{2}{3}}, \dfrac{1}{\sqrt{3}}\right)$ on the ellipse and $\left(\sqrt{\dfrac{2}{3}}, \dfrac{2}{\sqrt{3}}\right)$ on the circle are the ends of a common tangent.

9. **(a)**

10. **(a) - (q), (b) - (r), (c) - (s), (d) - (p)**

(a) Range is $\left[-\dfrac{1}{4}, \dfrac{3}{4}\right]$

(b) Range is $\left[\dfrac{1}{8}, \dfrac{5}{4}\right]$

(c) Range is $\left[-\dfrac{1}{8}, \dfrac{9}{8}\right]$

(d) Range is $\left[\dfrac{1}{32}, \dfrac{7}{8}\right]$

SOLUTIONS (SET - 62)

1. **(d)** : $\prod_{r=2}^{n}\left(\dfrac{r^3+1}{r^3-1}\right) = \dfrac{3}{2}\left(\dfrac{n(n+1)}{n^2+n+1}\right) \rightarrow \dfrac{3}{2}$ as $n \rightarrow \infty$

2. **(c)** : The mean value theorem for $f(x) = x^{1/5}$ in [3125, 3126] yields $N = 3126 = 2.3\ 521$. The sum of the divisors of N is 6264

3. **(b)** : Let $p = a\alpha$, $q = b\beta$
$a^2 p + b^2 q = a^3 + h^3$, $p^3 + q^3 = a^3 + h^3$
$\Rightarrow \alpha - \beta = \beta^2 - \alpha^2 \Rightarrow \alpha + \beta + 1 = 0 \Rightarrow hp + aq + ab = 0$

4. **(c)** : $(z_1 \pm i z_2)^3 = 2 \pm 11i$

$z_1^2 + z_2^2 = (z_1 + i z_2)(z_1 - i z_2)$
$= (2 + 11i)^{1/3}(2 - 11i)^{1/3} = 125^{1/3} = 5$

5. **(d)** : $[\vec{a}\ \vec{h}\ \vec{c}]^2 = 1 + 2(\vec{a}\cdot\vec{h})(\vec{h}\cdot\vec{c})(\vec{c}\cdot\vec{a})$

$-(\vec{a} - \vec{h})^2 - (\vec{h}\cdot\vec{c})^2 = \dfrac{\sqrt{3}-\sqrt{2}}{2\sqrt{2}}$

$\Rightarrow [\vec{a}\ \vec{h}\ \vec{c}] = \dfrac{\sqrt{\sqrt{6}-2}}{2}$

6. **(d)** : The number of ways of getting the sum atmost 12 is the coefficient of x^8 in

$(1 - x^6)^4(1 - x)^{-5} = \dbinom{12}{8} - 4\dbinom{6}{2} = 435$

The desired number is $6^4 - 435 = 861$.

7. **(a)** : The d.r's of the lines are – 3, 5, 7 and 1, 1, 0.

The desired angle is $\cos^{-1}\sqrt{\dfrac{2}{83}}$

8. **(b)** : The lines meet at the point (– 1, 4, 4) whose distance from the origin is $\sqrt{33}$

9. **(c)** : The plane through the lines is $7x - 7y + 8z + 3 = 0$. The distance of the origin from it is $\dfrac{1}{3\sqrt{2}}$

10. **(a) - (S), (b) - (P), (c) - (Q), (d) - (R)**

(a) $\dfrac{y}{y'} = a \Rightarrow y = A\,e^{x/a}$ (b) $yy' = a \Rightarrow y^2 = 2ax + c$

(c) $\dfrac{y}{y'} = 2x \Rightarrow y^2 = cx$ (d) $yy' = x \Rightarrow y^2 - x^2 = c$

$\vec{b} = \vec{a} - 2(d - \vec{a} \cdot \hat{n})\hat{n} \Rightarrow \dfrac{\vec{b} + \vec{a}}{2} = \vec{a} - (d - \vec{a} \cdot \hat{n})\hat{n}$

For this $\vec{r}$, $\vec{r} \cdot \hat{n} = d$

$\Rightarrow \quad \vec{r} \cdot \hat{n} = \vec{a} \cdot \hat{n} + (d - \vec{a} \cdot \hat{n})(\hat{n} \cdot \hat{n}) = d$

$\Rightarrow$ the point $\dfrac{\vec{b} + \vec{a}}{2}$ lies on the plane $\vec{r} \cdot \hat{n} = d$ also

$\vec{b} - \vec{a} = 2(d - \vec{a} \cdot \hat{n})(\hat{n}) \Rightarrow (\vec{b} - \vec{a}) \times \hat{n} = 0$

Thus the point in choice (c), both condition for any image are satisfied $\Rightarrow$ (c) is correct.

21. (a) : If we take line through $\vec{a}$ & along normal to the plane then the line $\vec{r} = \vec{a} + t\vec{c} \Rightarrow \vec{b} = \vec{c}, \vec{c} = \hat{n}$ since the correct answer should be $d - \vec{a} \cdot \hat{n}$ (perpendicular distance of $\vec{a}$ from the plane) we conclude the choice (a) is correct.

22. (a) : Let $y = mx$ be any chord through (0, 0) this will meet conic at point whose x-coordinate are given by

$x^2 + m^2x^2 + mx^2 = 0 \Rightarrow x^2(1 + m + m^2) - 1 = 0$

$\Rightarrow \quad \dfrac{x_1 + x_2}{2} = 0 \Rightarrow y_1 = m \cdot x_1$ and $y_2 = m \cdot x_2$

$\Rightarrow \quad y_1 + y_2 = m(x_1 + x_2) \Rightarrow \dfrac{y_1 + y_2}{2} = 0.$

23. (a) : Let E_1 be the event of both getting the correct answer and E_2 the event of both getting wrong answer. Let E be the event of both obtaining the same answer.

$$P(E_1) = \frac{1}{8} \cdot \frac{1}{12} = \frac{1}{96},$$

$$P(E_2) = \left(1 - \frac{1}{8}\right)\left(1 - \frac{1}{2}\right) = \frac{77}{68}$$

$$P\left(\frac{E}{E_1}\right) = 1, P\left(\frac{E}{E_2}\right) = \frac{1}{1001}$$

$$P\left(\frac{E_1}{E}\right) = \frac{P\left(\frac{E}{E_1}\right) \cdot P(E_1)}{P\left(\frac{E}{E_1}\right)P(E_1) + P\left(\frac{E}{E_2}\right)P(E_2)}$$

$$= \frac{1 \cdot \frac{1}{96}}{1 \cdot \frac{1}{96} + \frac{1}{1001} \cdot \frac{77}{96}} = \frac{13}{14}$$

24. (a) : The equation of circle contains three independent constants if it passes through three non-collinear points, therefore A is true & follows from R.

25. (d) : The Assertion A is false since the lines $x = 1$, $x = 2$, $x = 3$ satisfies.

$$\begin{vmatrix} 1 & 0 & -1 \\ 1 & 0 & -2 \\ 1 & 0 & -3 \end{vmatrix} = 0$$

but they are not concurrent.
The Reason R is true
(Note we must note the determinant)

$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$$

(does not represent the area of any triangle related with given line)

CBSE head blames 3Cs

Cricket, Cinema, Computers

In a society where students are expected to at the least secure A-grades, Cs are obviously looked down on. Hence Central Board of Secondary Education (CBSE) chairman Ashok Ganguly should logically frown at 3 Cs. However, when Mr Ganguly makes the point at an annual day function of a Delhi school that the three Cs he wants parents to control their children's indulgence in are cricket, cinema and computers, there is scope for discussion. He cites the instance of a student ringing him up to request postponement of the 2007 CBSE exams so that cricket fans could watch the 50-over World Cup which comes once in four years. The student, he adds, rang him up after the results were out to thank him for his advice that the CBSE exam came once in life. However, what Mr Ganguly does not mention is that the over-indulgence in what he terms as the three Cs could itself be a reaction to the intense pressure on a student in the final years of school. Even while simultaneously trying to top the CBSE exams, many students are also preparing for the joint entrance exams to the IITs.

The problem is simply that the number of seats in prestigious institutes like the IITs are very few when compared with the overwhelming response from applicants (less than 1 : 100). Which is why the cinema offers some kind of escape from the unremitting routine of preparation followed by more preparation. In fact, the biggest beneficiary of the existing system is not so much the student as the promoter of cramming classes who claims that his coaching can improve the applicant's chances of not only getting into an IIT but also securing the subject of choice. At the graduate level, there are the cramming shops which claim to improve the applicant's chances of getting into the IIMs (admission-ratio is 1 : 140). Given the intensity of preparation, it is not surprising that many Indian students rely on cricket, cinema and the computer to try and get away from the rat race!

Courtesy : Economic Times

Olympiad Enrichment Series-VIII

useful for IIT-JEE 2008-09

This series is selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solution.

1. Find the real zeros of the polynomial
$$P_a(x) = (x^2+1)(x-1)^2 - ax^2,$$
where a is a given real number.

2. Prove that $\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} < \frac{1}{\sqrt{3n}}$
for all positive integers n.

3. Let $P(x) = a_0x^n + a_1x^{n-1} + \dots + a_n$
be a non zero polynomial with integral coefficients such that $P(r) = P(s) = 0$ for some integers r and s, with $0 < r < s$. Prove that $a_k \le -s$ for some k.

4. Let m be a given real number.
Find all complex numbers x such that
$$\left(\frac{x}{x+1}\right)^2 + \left(\frac{x}{x-1}\right)^2 = m^2 + m.$$

5. The sequence given by $x_0 = a$, $x_1 = b$, and
$$x_{n+1} = \frac{1}{2}\left(x_{n-1} + \frac{1}{x_n}\right)$$
is periodic. Prove that $ab = 1$.

SOLUTIONS

1. We have $(x^2+1)(x^2-2x+1) - ax^2 = 0$
Dividing by x^2 yields
$$\left(x+\frac{1}{x}\right)\left(x-2+\frac{1}{x}\right) - a = 0$$
By setting $y = x + 1/x$, the last equation becomes
$$y^2 - 2y - a = 0$$
It follows that $x + \frac{1}{x} = 1 \pm \sqrt{1+a}$,
which in turn implies that, if $a \ge 0$, then the polynomial $P_a(x)$ has the real zeros
$$x_{1,2} = \frac{1+\sqrt{1+a} \pm \sqrt{a+2\sqrt{1+a}-2}}{2}$$
In addition, if $a \ge 8$, then $P_a(x)$ also has the real zeros
$$x_{3,4} = \frac{1-\sqrt{1+a} \pm \sqrt{a-2\sqrt{1+a}-2}}{2}.$$

2. We prove a stronger statement :
$$\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2n-1}{2n} \le \frac{1}{\sqrt{3n+1}}$$
We use induction. For $n = 1$, the result is evident.
Suppose the statement is true for some positive integer k,
i.e. $\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2k-1}{2k} < \frac{1}{\sqrt{3k+1}}$

Then $\frac{1}{2}\cdot\frac{3}{4}\cdots\frac{2k-1}{2k}\cdot\frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2}$
In order for the induction step to pass it suffices to prove that $\frac{1}{\sqrt{3k+1}}\cdot\frac{2k+1}{2k+2} < \frac{1}{\sqrt{3k+4}}$

This reduces to $\left(\frac{2k+1}{2k+2}\right)^2 < \frac{3k+1}{3k+4}$
i.e. $(4k^2+4k+1)(3k+4) < (4k^2+8k+4)(3k+1)$
i.e. $0 < k$, which is evident. Our proof is complete.
Comment : By using Stirling numbers, the upper bound can be improved to $1/\sqrt{\pi n}$ for sufficiently large n.

3. Write $P(x) = (x-s)\,x^cQ(x)$ and
$$Q(x) = b_0x^m + b_1x^{m-1} + \dots + b_m,$$
where $b_m \ne 0$. Since Q has a positive root, by Descarte's rule of signs, either there must exist some k for which $b_k > 0 \ge b_{k+1}$ or $b_m > 0$.
If there exists a k for which $b_k > 0 \ge b_{k+1}$, then
$$a_{k+1} = -sb_k + b_{k+1} \le -s.$$
If $b_m > 0$, then $a_m = -sb_m \le -s$
In either case, there is a k such that $a_k \le -s$, as desired.

4. Completing the square gives
$$\left(\frac{x}{x+1}+\frac{x}{x-1}\right)^2 = \frac{2x^2}{x^2-1} + m^2 + m,$$
i.e. $\left(\frac{2x^2}{x^2-1}\right)^2 = \frac{2x^2}{x^2-1} + m^2 + m$
Setting $y = 2x^2/(x^2-1)$, the above equation becomes
$$y^2 - y - (m^2+m) = 0,$$
i.e. $(y-m-1)(y+m) = 0$.
Thus $\frac{2x^2}{x^2-1} = -m$ or $\frac{2x^2}{x^2-1} = m+1$,
which leads to solutions
$$x = \pm\sqrt{\frac{m}{m+2}} \text{ if } m \ne -2 \text{ and } x = \pm\sqrt{\frac{m+1}{m-1}} \text{ if } m \ne 1.$$

5. Multiplying by $2x_n$ on both sides of the given recursive relation yields
$2x_n x_{n+1} = x_{n-1}x_n + 1$ or $2(x_n x_{n+1} - 1) = x_{n-1}x_n - 1$.
Let $y_n = x_{n-1}x_n - 1$ for $n \in \mathbb{N}$. Since $y_{n+1} = y_n/2$, $\{y_n\}$ is a geometric sequence. If x_n is periodic, then so is y_n, which implies that $y_n = 0$ for all $n \in \mathbb{N}$. Therefore
$$ab = x_0x_1 = y_1 + 1 = 1.$$

CHINESE Olympiad Problems

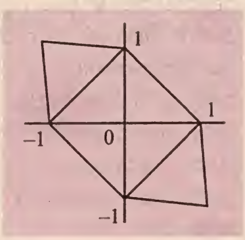
Questions Published in Feb. Issue on Page-80

SOLUTIONS

1. The graphs of $f(x)$ and $f^{-1}(x)$ are symmetric about the line $x - y = 0$ while the graphs of $f^{-1}(x)$ and the desired function are symmetric about the line $x + y = 0$. Hence the graphs of $f(x)$ and the desired function are symmetric about the origin. It follows that this function must be $-f(-x)$.

2. There are points (x, y) outside the ellipse that are sufficiently far from the origin, so that the left side of the equation is positive. It follows that for points inside the ellipse such as the origin, the left side of the equation is negative. Hence $k^2 - 1 < 0$. However, if $k = 0$, the curve is a pair of straight lines rather than an ellipse. It follows that we must ihave $0 < |k| < 1$.

3. In the diagram below, the region M is the square symmetric about the origin while P is the hexagon. It is easy to verify that the four vertices of the square are inside the ellipse N while the other two vertices of the hexagon are on its boundary. It follows that $M \subset P \subset N$.



4. If a, b and c are not concurrent, then they must be parallel, and we have $\theta = \pi/3$. Hence P is the sufficient for Q. Assume now that they are concurrent. From any point inside the trihedral angle, drop perpendiculars to α, β and γ. Then the angle between any two of these lines is $\pi - \theta$. Since the lines are not copalnar, the sum of these three angles is less than 2π. Hence $\theta > \pi/3$ and P is also necessary for Q.

5. The lines $y = \sqrt{2}x$, $y = \frac{1}{2}$ and $y = 0$ shows that Q, R and S are all true. Suppose a line l passes through two lattice points (x_1, y_1) and (x_2, y_2). If $x_1 = x_2$, it obviously passes through infinitely many other lattice points. If $x_1 \neq x_2$, then the slope of l is

$$\frac{y_2 - y_1}{x_2 - x_1}.$$

Consider the lattice point $(x_2 + k(x_2 - x_1), y_2 + k(y_2 - y_1))$ where k is an arbitrary integer. Its x-coordinate is different from x_1 as otherwise we will have $x_1 = x_2$. Together with (x_1, y_1), it determines a line of slope.

$$\frac{y_2 + k(y_2 - y_1) - y_1}{x_2 + k(x_2 - x_1) - x_1} = \frac{y_2 - y_1}{x_2 - x_1}.$$

Hence the point also lies on l, which passes through infinitely many lattice points. It follows that P is also true.

6. We have

$$b_4 - b_3 = 2(b_2 - b_1) = \frac{2}{3}(y - x) = \frac{8}{3}(a_2 - a_1),$$

Hence $\dfrac{b_4 - b_3}{a_2 - a_1} = \dfrac{8}{3}$.

7. Let $(2 + \sqrt{x})^{2n+1} = f(x) + \sqrt{x}g(x)$

where $f(x)$ and $g(x)$ are polynomials in x. Then we have

$$(2 - \sqrt{x})^{2n+1} = f(x) - \sqrt{x}g(x).$$

The desired sum is given by $f(1) = \frac{1}{2}(3^{2n+1} + 1)$.

8. Since $BDEC$ is cyclic, $\angle ADE = \angle ACB$ so that triangles ADE and ACB are similar. Hence

$$\frac{DE^2}{BC^2} = \frac{[ADE]}{[ACB]} = \frac{\frac{1}{2}AD \cdot AE\sin\alpha}{\frac{1}{2}AC \cdot AB\sin\alpha} = \cos^2\alpha$$

It follows that $\dfrac{DE}{BC} = |\cos\alpha|$.

9. There are at most 13 games. Of these, exactly 7 are won by the winning team. Since either team can win, the total number of possible sequences is $2\binom{13}{7} = 3432$. ■

35. Use S-44

36. Using S-46, the equation $x^8 - x^5 + x^2 - x + 1 = 0$ has no negative real root as $f(-x) = x^8 + x^5 + x^2 + x + 1$ has no change in sign. Therefore the equation may have only positive real roots.
Now, for all $x \geq 1$, $x^8 > x^5$, $x^2 > x$,
therefore $x^8 - x^5 + x^2 - x + 1 > 0$ for all $x \geq 1$
i.e. the equation has no root greater than 1.
Also, for all $0 < x < 1$, $x^5 < x^2$, $x < 1$, $x^8 > 0$, therefore $x^8 - x^5 + x^2 - x + 1 > 0$ for all $0 < x < 1$ i.e. the equation has no root lying between 0 and 1. Thus the equation has no real root.

37. Use S-46

38. The given equation $\left(\sqrt{2}\right)^x + \left(\sqrt{3}\right)^x = \left(\sqrt{13}\right)^x$ may be written as

$$\left(\sqrt{\frac{2}{13}}\right)^x + \left(\sqrt{\frac{3}{13}}\right)^x = 1$$

$$\Rightarrow \left(\frac{2}{13}\right)^{x/2} + \left(\frac{3}{13}\right)^{x/2} = 1,$$

which is of the form $\sin^2\theta + \cos^2\theta = 1$. Thus, only one solution i.e. $x/2 = 2$ is possible.

41. Use S-50 42. Use S-48 43. Use S-49

44. Use Table II 45. Use Table II 46. Use Table II

48. Put $\dfrac{6 - x + 8 - x}{2} = y$ i.e. $7 - x = y$ and solve it.

49. Use S-45.

50. Let $\dfrac{p}{a} = x, \dfrac{q}{b} = y, \dfrac{r}{c} = z$ and
use $(x + y + z)^2 = x^2 + y^2 + z^2 + 2xy + 2yz + 2zx$

52. The given equation may be written as
$x^4 + 4x^3 - 6x^2 + 7x - 9 = (x - \alpha)(x - \beta)(x - \gamma)(x - \delta)$
Also, $(1 + \alpha^2)(1 + \beta^2)(1 + \gamma^2)(1 + \delta^2) = |(\alpha + i)(\beta + i)(\gamma + i)(\delta + i)|^2$.
Therefore, put $x = i$ in above equation and then take its modulus.

53. The given equation is of the form $a^4 + b^4 = (a + b)^4$, which on simplifying implies that $2ab(2a^2 + 3ab + 2b^2) = 0$, where $a = 3 - x$ and $b = 2 - x$. Clearly it has 2 real and 2 imaginary roots.

54. $\alpha + \beta = -p$, $\alpha\beta = 1$,
$\gamma + \delta = -q$, $\gamma\delta = 1$.
Now,
$$(\alpha - \gamma)(\beta - \gamma)(\alpha + \delta)(\beta + \delta) = \{\alpha\beta - (\alpha + \beta)\gamma + \gamma^2\} \cdot \{\alpha\beta + (\alpha + \beta)\delta + \delta^2\}$$

55. Observing the given equations, we find that q and r satisfies the equation $a(p + x)^2 + 2bpx + c = 0$. Rewrite it as $ax^2 + 2(bp + ap)x + ap^2 + c = 0$. Now apply S-1 to obtain the result.

Tense about IIT, meet the Tensors

The most dreaded entrance exams of all time, IIT JEE, is just around the corner, thousands of students with IIT dreams are on pins and needles now. And who would understand their plight better than IITians themselves.

Keeping up the annual seven-year-old tradition, eight Hyderabadi students of IIT Madras in December to host Tensors-2007, a mock JEE session.

This test, which helps students clear a lot of doubts about the exam and their future, is conducted by second year students, Ashok Varma, Narsimha Rao, Sharad Giri, Susheel Kumar, Sai Praveen, Naveen Reddy, Aditya and Surya Sudheer.

"Tensors was started by a Hyderabadi alumnus of IIT Madras in 2000.

An event that IIT aspirants look forward to throughout the year, Tensors is considered one of the most unbiased ways of knowing where a students stands at this juncture. Explaining this Sharad Giri says, "Students in our city begin IIT preparations right from Class 8. Everyone from parents and peers to teachers allow them to see just one goal before them – cracking JEE. By taking this test they could get a clearer picture of where they stand." Though thousands of students sacrifice the best part of their high school lives to achieve the elusive dream of getting into IIT, the irony is that only about "0.95 per cent" of them make it that far.

"Through Tensors we hope to tell these young people what reality is. IIT is good, but it's not the end of the world. There are other options if you fail to crack JEE. For this, you need to fare well in your Board exams too," says Surya Sudhir.

The Tensors question papers which are set by the brightest minds in IIT Madras, are similar in pattern to the JEE papers. "Since we all went through the same situation, we can identify with all their apprehensions, hopes and dreams. There a small things we wished we knew after we wrote JEE. We hope to tell these students all those things, in a fun, casual manner, so they can make the best of the time that's left in a wise manner," adds Aditya.

Lessons they learnt the hard way and mistakes they ended up making can be avoided, believe the Tensors team of all 19-year-olds.

"Students ignore their Board exams entirely, some barely manage to get two hours of sleep a day, other slip into depression if they don't crack JEE simply because they see no other option before them. All this can be avoided if there's someone to guide them, and we believe we can do that," says Susheel Kumar.

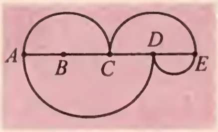
HIGH SCHOOL

CONTEST PROBLEMS

PART A

1. Each edge of a cube is coloured either red or black. If every face of the cube has at least one black edge, the smallest possible number of black edge is :
(a) 6 (b) 5 (c) 4 (d) 3.

2. Line AE is divided into four equal parts by the points B, C and D. Semicircles are drawn on segments AC, CE, AD and DE creating semicircular regions as shown. The ratio of the area enclosed above the line AE to the area enclosed below the line is



(a) 4 : 5 (b) 5 : 4 (c) 1 : 1 (d) 8 : 9.

3. The digits 1, 9, 9 and 8 are placed on four cards. Two of the cards are selected at random. The probability that the sum of the numbers on the cards selected is a multiple 3 is:
(a) 1/4 (b) 1/3 (c) 1/2 (d) 2/3.

4. The surface areas of the six faces of a rectangular solid are 4, 4, 8, 8, 18 and 18 square centimetres. The volume of the solid, in cubic centimetres is
(a) 24 (b) 48 (c) 60 (d) 324.

5. The area of the small triangle in the diagram is 8 square units. The area of the large triangle, in square units, is:
(a) 18 (b) 20 (c) 24 (d) 30.

6. At 6 : 15 the hands of the clock from two positive angles with a sum of 360°. The difference of the degree measures of these two angles is:
(a) 165 (b) 170 (c) 175 (d) 185.

7. The last digit of the number 8^{26} is :
(a) 0 (b) 2 (c) 4 (d) 6.

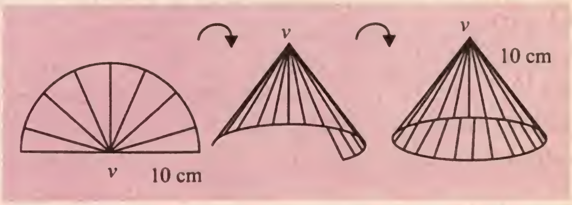
8. For the equation $\dfrac{A}{x+3}+\dfrac{B}{x-3}=\dfrac{-x+9}{x^2-9}$ to be true for all values of x for which the expressions in the equation make sense, the value of AB is:
(a) 2 (b) –1 (c) –2 (d) –3.

9. A hungry hunter came upon two shepherds, Joe and Frank. Joe had three small loaves of bread and Fank five loaves of the same size. The loaves were divided equally among the three people, and the hunter paid \$8 for his share. If the shepherds divide the money so that each gets an equitable share based on the amount of bread given to the hunter, the amount of money that Jose receives is:
(a) \$1 (b) \$1.50 (c) \$2 (d) \$2.50.

PART B

10. Four positive integers sum to 125. If the first of these numbers is increased by 4, the second is decreased by 4, the third is multiplied by 4 and the fourth is divided by 4, you produce four equal numbers. What are the four original numbers?

11. A semi-circular piece of paper of radius 10 cm is formed into a conical paper cup as shown (then cup is inverted in the diagram).



Find the height of the paper cup, that is, the depth of water in the cup when it is full.

12. In the diagram a quarter circle is inscribed in a square with side length 4, as shown. Find the radius of the small circle that is tangent to the quarter circle and two sides of the square.

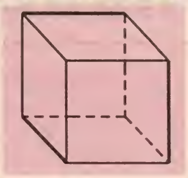
13. Using the digits 1, 9, 9 and 8 in that order create expressions equal to 1, 2, 3, 4, 5, 6, 7, 8, 9, and 10. You may use any of the four basic operations (+, –, ×, ÷), the square root symbol ($\sqrt{\ }$) and parenthesis, as necessary. For example, valid expressions for 25 and 36 would be $25 = -1 + 9 + 9 + 8$, $36 = 1 + 9 \times \sqrt{9} + 8$.

14. At 6 am one Saturday, you and a friend begin a recreational climb of Mt. Everest. Two hours into your climb, you are overtaken by some scouts. As they pass, they inform you that they are attempting to set a record for ascending and descending the mountain. At 10 am they pass you again on their way down, crowing that they had not stopped once to rest, not even at the top.

You finally reach the summit at noon. Assuming that both you and the scouts travelled at a constant vertical rate, both climbing and descending, when did the scouts reach the top of Mt. Everest.

SOLUTION

1. **(d) :** Suppose that every face of a cube has at least one black edge. Since every edge belongs to exactly two faces, and there are six faces, the cube has at least three black edges. on the other hand, three black edges suffice to satisfy the requirement, as we can see on the diagram. The black edges are represented by the thicker lines.



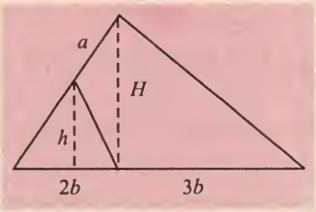
2. **(a) :** Suppose that AB has length 1. Then both semicircles lying above AE have radii of 1, while the semicircles below AE have radii of $1\frac{1}{2}$ and $\frac{1}{2}$. The ratio of the enclosed areas is

$$\left[\frac{1}{2}\pi(1)^2+\frac{1}{2}\pi(1)^2\right]\div\left[\frac{1}{2}\pi\left(1\frac{1}{2}\right)^2+\frac{1}{2}\pi\left(\frac{1}{2}\right)^2\right]=\frac{4}{5}.$$

3. **(b) :** Let a, b, c, d denote the cards with digits 1, 9, 9 and 8 respectively. There are six possible choices of two cards from the set of four. $\{a, b\}$, $\{a, c\}$, $\{a, d\}$, $\{b, c\}$, $\{b, d\}$, $\{c, d\}$. For exactly two of these, $\{b, c\}$ and $\{a, d\}$, the corresponding sums, 9 + 9 and 1 + 8, are divisible by 3. This gives the probability of $\frac{2}{6}=\frac{1}{3}$.

4. **(a) :** If the edges of a rectangular solid have lengths a, b and c, then the areas of its nonparallel faces are ab, bc and ac. Its volume $abc=\sqrt{(ab)(bc)(ac)}$. In our case $abc=\sqrt{4\cdot 8\cdot 18}=24$.

5. **(d) :** The length of the base of the larger triangle is $5b$, while the length of the base of the smaller triangle is $2b$. This gives the ratio of 5/2. Similarly, the ratio of the corresponding perpendicular heights, $H : h$, is $3a : 2a$ = 3/2. Hence, the area of the larger triangle is 5/2(3/2)(8) = 30.



6. **(a) :** At 6 : 15 the minute hand points at 3, while the hour hand is 1/4 of the way from 6 to 7. The smaller angle between the hands is $\left[90+\frac{1}{4}\left(\frac{360}{12}\right)\right]^\circ=97.5^\circ$, while the larger is $(360-97.5)^\circ=262.5^\circ$. This gives the difference of $(262.5-97.5)^\circ=165^\circ$.

7. **(c) :** By inspecting the last digit of the numbers in the sequence $8^1, 8^2, 8^3, 8^4, \ldots$ we discover a repeating pattern of length four : 8, 4, 2, 6. Since $8^{26}=8^{4(6)+2}$, we conclude that the last digit of 8^{26} is the same as the last digit of 8^2, that is 4.

8. **(c) :** The expression makes sense for all values of x, except ± 3. By multiplying both sides of the equation by the common denominator $x^2-9=(x-3)(x+3)$, we get $A(x-3)+B(x+3)=-x+9$. After multiplying out and collecting the like terms on the left hand side of this equation we get $(A+B)x+3B-3A=-x+9$. Clearly, the polynomials on both sides must be identical; therefore $A+B=-1$ and $3B-3A=9$. This system of two equations can be solved in any standard way. For example, we can find $B=3+A$ from the second equation and substitute this equation B in the first equation. In the way we find $A=-2$ and $B=1$.

9. **(a) :** Divide each loaf into 3 parts and distribute equally to each of the three persons. Each person receives

8 parts. The two shepherds start with 9 and 15 parts each, so (after removing their own 8 parts) they contribute 1 and 7 parts, respectively, to the hunter and should receive compensation from the hunter in that ratio. Thus the hunter who originally had 3 loaves should receive \$1.

10. The numbers are 16, 24, 5 and 80.
If x, y, z and w are the numbers then $x + y + z + w = 125$ and $x + 4 = y - 4 = 4z = w/4$. Hence, $y = x + 8$, $z = (x+4)/4$, $w = 4(x+4)$, By substituting these expressions to the first equation, we get

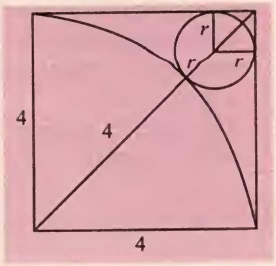
$$x + x + 8 + \frac{x+4}{4} + 4(x+4) = 125.$$

Thus, $x = 16$, and consequently, $y = 24$, $z = 5$, $w = 80$.

11. The height of the paper cup is $5\sqrt{3}$ cm.
The base of the conical paper cup is a circle with circumference equal to the length of the given semicircle. Thus, if r is the radius of the base then $2\pi r = \frac{1}{2}(2\pi 10)$. Hence, $r = 5$ cm. The side length of the cone s is the same as the radius of the semicircle; thus s = 10 cm. Finally, the height of the cone is

$$h = \sqrt{s^2 - r^2} = \sqrt{10^2 - 5^2} = 5\sqrt{3} \text{ cm.}$$

12. The radius of the small circle is $12 - 8\sqrt{2}$.
The Pythagorean Theorem implies that the diagonal of a square with side a has length $a\sqrt{2}$. Thus, the diagonal of the larger square has length $4\sqrt{2}$. It is equal to the sum of the radius of the larger circle, 4, the radius of the smaller circle, r, and the diagonal of the smaller square, $r\sqrt{2}$.



Hence

$$4\sqrt{2} = 4 + r + r\sqrt{2}.$$

This gives

$$r = \frac{4\sqrt{2}-4}{1+\sqrt{2}} = \frac{4\sqrt{2}-4}{1+\sqrt{2}}\left(\frac{1-\sqrt{2}}{1-\sqrt{2}}\right) = 12 - 8\sqrt{2}.$$

13. One of several possible solutions is :

$1 = -1 + \sqrt{9} - 9 + 8,$

$2 = 1 \times \sqrt{9} - 9 + 8,$

$3 = -1 + \sqrt{9} + 9 - 8,$

$4 = 1 \times \sqrt{9} + 9 - 8,$

$5 = 1 + \sqrt{9} + 9 - 8,$

$6 = -1 - 9 \div 9 + 8,$

$7 = -1 + 9 - 9 + 8,$

$8 = -1 + 9 \div 9 + 8,$

$9 = -1 + 9 + 9 - 8$

$10 = 1 + 9 \div 9 + 8.$

14. The scouts reached the top of Mt. Everest at 9.20 am. Suppose that during the time period from 8.00 am to 10.00 am you have travelled from point A to point B and you climbed a distance of x kilometeres. Then, since you have been climbing at a uniform rate and reached top at noon, the distance from B to top is also x kilometeres. During the two hours you climbed x kilometers from A to B, the scouts climbed the distance of $3x$ kilometres: x from A to B, x from B to the top, and x on the way back to B from the top. Since their pace was uniform, they needed 2/3 of an hour, that is 40 minutes, to get from the top to point B, where they met you at 10.00 am. This implies that they must have reached the top at 9 : 20 am.

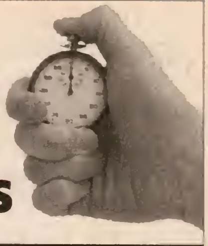
RAJASTHAN PLANS 30 ENGG COLLEGES

The Rajasthan government is mulling over giving the go ahead to 30 new engineering colleges from the next session, resulting an addition of around 5,000 seats to the existing 16,225 engineering seats in the state. At present, the state has 48 engineering colleges, including seven government-run colleges.

Rajasthan education minister Vasudev Devnani said that the government is serious about upgrading the technical education in the state. "We are working on fast track to approve the opening of new private colleges without compromising on the standard of education. We would try to make every engineering college a model college," he said.

Beat the time traps

– Shortcut techniques



SOME SHORT-CUT TECHNIQUES ON QUADRATIC EQUATIONS & EXPRESSIONS

Here, we are giving you a general quadratic equation its graph and properties of its roots. Go through these results very carefully and then answer the problems given below in the prescribed time.

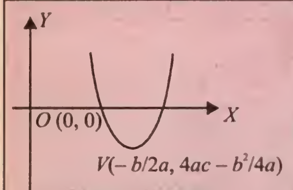
Consider $ax^2 + bx + c = 0$ be the given quadratic equation and its roots are α, β. Then, The roots of the above quadratic equation can be evaluated using Sridhracharya formula given as follows:

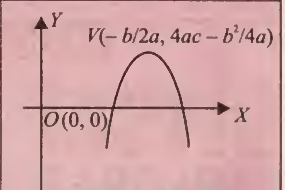
$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$

Where, the expression $b^2 - 4ac$ is known as discriminant and denoted by D. i.e. $D = b^2 - 4ac$.

GRAPH : The graph of a quadratic expression i.e. $y = ax^2 + bx + c$ is always parabolic such that its axis is parallel to y-axis. It can be drawn as follows:

TABLE I

<table>
<tr><th>$a > 0$</th><th>$a < 0$</th></tr>
<tr><td>Vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$</td><td>Vertex $\left(\frac{-b}{2a}, \frac{-D}{4a}\right)$</td></tr>
<tr><td>

TYPE I GRAPH
Its graph is a concave upwards parabola.</td><td>

TYPE II GRAPH
Its graph is a concave downwards parabola.</td></tr>
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NATURE OF ROOTS:

I. If $D = b^2 - 4ac > 0$, then roots of the equation $ax^2 + bx + c = 0$ are **real and distinct.**

II. If $D = 0$, then roots of the equation $ax^2 + bx + c = 0$ are **real and equal** and each one of them equals to $-\frac{b}{2a}$.

III. If $D < 0$, then roots of the equation $ax^2 + bx + c = 0$ are **imaginary**. (Note that imaginary roots of a quadratic equation with rational coefficients always occur in conjugates. i.e. if $p + iq$ is one root of the equation, then $p - iq$ will be the another root of the equation)

IV. If a, b, c are all rational numbers and D is a perfect square, then roots of the equation $ax^2 + bx + c = 0$ will be **rational** numbers.

V. If a, b, c are all rational numbers and $D > 0$ is not a perfect square, then roots of the equation $ax^2 + bx + c = 0$ will be **irrational** numbers. (Note that irrational roots of a quadratic equation with rational coefficients always occurs in conjugates i.e. if $p + \sqrt{q}$ is one of the root, then the other root will be $p - \sqrt{q}$)

GRAPHICAL INTERPRETATION OF THE ROOTS OF A QUADRATIC EQUATION :

I. If the graph of a quadratic expression $y = ax^2 + bx + c$ lies **completely above x-axis or completely below x-axis**, then the roots of the equation $ax^2 + bx + c = 0$ will be **imaginary**. (Note that root of an equation is the abscissae of that point where the graph intersects with the x-axis)

II. If the graph of a quadratic expression $y = ax^2 + bx + c$ **touches x-axis**, then the roots of the equation $ax^2 + bx + c = 0$ will be **real and equal**.

III. If the graph of a quadratic expression $y = ax^2 + bx + c$ **intersects x-axis in two points**, then the roots of the equation $ax^2 + bx + c = 0$ will be **real and distinct**.

VARIOUS CONDITIONS FOR ROOTS OF A QUADRATIC EQUATION :

Consider $ax^2 + bx + c = 0$ be the given quadratic equation and its roots are α, β. Then,

S-1: Sum of the roots $\alpha + \beta = -\frac{b}{a}$ and product of the roots $\alpha \cdot \beta = \frac{c}{a}$.

S-2: If both the roots are positive, then $\alpha + \beta > 0$ and $\alpha\beta > 0$.

S-3: If both the roots are negative, then $\alpha + \beta < 0$ and $\alpha\beta > 0$.

S-4: If its roots are of opposite sign, then $\alpha\beta < 0$.

S-5: If its both the roots are greater than k (a scalar), then

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CHINESE Olympiad Problems

SECTION 1

1. How many of the triangles whose vertices are chosen from the vertices of a cube are equilateral?
(a) 4 (b) 8 (c) 12 (d) 24

2. Let $\omega = \dfrac{-1+i\sqrt{3}}{2}$

The non-zero complex numbers a, b and c satisfy

$$\frac{a}{b} = \frac{b}{c} = \frac{c}{a}$$

What is the value of $\dfrac{a+b-c}{a-b+c}$?

(a) 1 (b) $\pm\omega$
(c) $1, \omega$ or ω^2 (d) $1, -\omega$ or $-\omega^2$

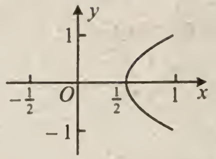
3. For how many positive integer $a < 100$ will $a^3 + 23$ be divisible by 24?
(a) 4 (b) 5 (c) 9 (d) 0

4. The function $f(x)$ has six distinct real roots, and $f(3+x) = f(3-x)$ for any real number x. What is the sum of the six roots of $f(x)$?
(a) 18 (b) 12 (c) 9 (d) 10

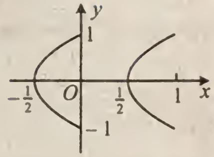
5. Let $S = \{(x, y) \mid x^2 - y^2 \text{ is odd, } x, y \in \mathbb{R}\}$
and $T = \{(x, y) \mid \sin(2\pi x^2) - \sin(2\pi y^2) = \cos(2\pi x^2) - \cos(2\pi y^2),\ x, y \in \mathbb{R}\}$.
Which of the following statements is true?
(a) $S \subset T$ (b) $T \subset S$ (c) $S = T$ (d) $S \cap T = \phi$

6. Which of the following diagrams represents the curve $|x - y^2| = 1 - |x|$?

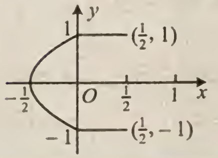
(a)



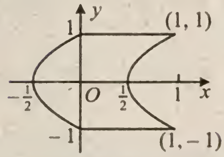
(b)



(c)



(d)



SECTION 2

7. What is the value of $\cos^2 10° + \cos^2 50° - \sin 40° \sin 80°$?

8. In triangle ABC, $\angle A$, $\angle B$ and $\angle C$ form an arithmetic progression. The length of the altitude from B to AC is equal to $AB - BC$. What is the value of $\sin\dfrac{C-A}{2}$?

9. A parabola has focus F and vertex V, where $VF = a$. A chord PQ of length b passes through F. Determine the area of triangle VPQ in terms of a and b.

10. Let a, x and y be real numbers such that $0 < a < 1$ and $x^2 + y = 0$. Prove that

$$\log_a(a^x + a^y) \le \log_a 2 + \frac{1}{8}.$$

Olympiad Enrichment Series-IX

useful for IIT-JEE 2008-09

This series is selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solution.

1. Evaluate
$$\binom{2000}{2}+\binom{2000}{5}+\binom{2000}{8}+\ldots+\binom{2000}{2000}.$$

2. Let x, y, z be positive real numbers such that
$$x^4+y^4+z^4=1.$$
Determine with proof the minimum value of
$$\frac{x^3}{1-x^8}+\frac{y^3}{1-y^8}+\frac{z^3}{1-z^8}.$$

3. Find all real solutions to the equation
$$2^x+3^x+6^x=x^2.$$

4. Let $\{a_n\}_{n\geq 1}$ be a sequence such that $a_1=2$ and
$$a_{n+1}=\frac{a_n}{2}+\frac{1}{a_n}$$
for all $n\in\mathbb{N}$. Find an explicit formula for a_n.

5. Let x, y, and z be positive real numbers. Prove that
$$\frac{x}{x+\sqrt{(x+y)(x+z)}}+\frac{y}{y+\sqrt{(y+z)(y+x)}}+\frac{z}{z+\sqrt{(z+x)(z+y)}}\leq 1.$$

SOLUTIONS

1. Let $f(x)=(1+x)^{2000}=\sum_{k=0}^{2000}\binom{2000}{k}x^k.$

Let $\omega=(-1+\sqrt{3}i)/2$. Then $\omega^3=1$ and $\omega^2+\omega+1=0$. Hence
$$3\left(\binom{2000}{2}+\binom{2000}{5}+\ldots+\binom{2000}{2000}\right)$$
$$=f(1)+\omega f(\omega)+\omega^2 f(\omega^2)$$
$$=2^{2000}+\omega(1+\omega)^{2000}+\omega^2(1+\omega^2)^{2000}$$
$$=2^{2000}+\omega(-\omega^2)^{2000}+\omega^2(-\omega)^{2000}$$
$$=2^{2000}+\omega^2+\omega=2^{2000}-1.$$

Thus the desired value is $\dfrac{2^{2000}-1}{3}$.

2. For $0<u<1$, let $f(u)=u(1-u^8)$. Let A be a positive real number. By the AM-GM inequality,
$$A(f(u))^8=Au^8(1-u^8)\ldots(1-u^8)\leq\left[\frac{Au^8+8(1-u^8)}{9}\right]^9$$

Setting $A=8$ in the above inequality yields
$$8(f(u))^8\leq\left(\frac{8}{9}\right)^9\quad\text{or}\quad f(u)\leq\frac{8}{\sqrt[4]{3^9}}$$

It follows that
$$\frac{x^3}{1-x^8}+\frac{y^3}{1-y^8}+\frac{z^3}{1-z^8}=\frac{x^4}{x(1-x^8)}+\frac{y^4}{y(1-y^8)}+\frac{z^4}{z(1-z^8)}$$
$$\geq\frac{(x^4+y^4+z^4)\sqrt[4]{3^9}}{8}=\frac{9\sqrt[4]{3}}{8}$$

with equality if and only if
$$x=y=z=\frac{1}{\sqrt[4]{3}}.$$

3. For $x<0$, the function $f(x)=2^x+3^x+6^x-x^2$ is increasing, so the equation $f(x)=0$ has the unique solution $x=-1$.

Assume that there is a solution $s\geq 0$. Then
$s^2=2^s+3^s+6^s\geq 3$,
so $s\geq\sqrt{3}$, and hence $\lfloor s\rfloor\geq 1$.
But then $s\geq\lfloor s\rfloor$ yields
$2^s\geq 2^{\lfloor s\rfloor}=(1+1)^{\lfloor s\rfloor}\geq 1+\lfloor s\rfloor\geq s,$
which in turn implies that
$6^s>4^s=(2^s)^2\geq s^2.$
So $2^s+3^s+6^s>s^2$, a contradiction.
Therefore $x=-1$ is the only solution to the equation.

4. Solving the equation
$$x=\frac{x}{2}+\frac{1}{x}$$
leads to $x=\pm\sqrt{2}$. Note that
$$\frac{a_{n+1}+\sqrt{2}}{a_{n+1}-\sqrt{2}}=\frac{a_n^2+2\sqrt{2}a_n+2}{a_n^2-2\sqrt{2}a_n+2}=\left(\frac{a_n+\sqrt{2}}{a_n-\sqrt{2}}\right)^2$$

Therefore, $\dfrac{a_n+\sqrt{2}}{a_n-\sqrt{2}}=\left(\dfrac{a_1+\sqrt{2}}{a_1-\sqrt{2}}\right)^{2^{n-1}}=(\sqrt{2}+1)^{2^n}$

and $a_n=\dfrac{\sqrt{2}[(\sqrt{2}+1)^{2^n}+1]}{(\sqrt{2}+1)^{2^n}-1}$

5. Note that

$$\sqrt{(x+y)(x+z)} \geq \sqrt{xy} + \sqrt{xz}$$

In fact, squaring both sides of the above inequality yields

$$x^2 + yz \geq 2x\sqrt{yz}$$

which is evident by the AM-GM inequality. Thus

$$\frac{x}{x+\sqrt{(x+y)(x+z)}} \leq \frac{x}{x+\sqrt{xy}+\sqrt{xz}} = \frac{\sqrt{x}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$$

Likewise, $\dfrac{y}{y+\sqrt{(y+z)(y+x)}} \leq \dfrac{\sqrt{y}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$,

and $\dfrac{z}{z+\sqrt{(z+x)(z+y)}} \leq \dfrac{\sqrt{z}}{\sqrt{x}+\sqrt{y}+\sqrt{z}}$

Adding the last three inequalities leads to the desired result. ■■

Training can boost intelligence

New Study Say IQ Not Inherited, Shows Methods To Increase Brainpower

A new study has found that it may be possible to train people to be more intelligent, increasing the brainpower they had at birth.

Until now, it had been widely assumed that the kind of mental ability that allows us to solve new problems without having any relevant previous experience – what psychologists call fluid intelligence – is innate and cannot be taught (though people can raise their grades on such tests by practicing).

But in the new study, researchers describe a method for improving this skill, along with experiments to prove it works.

The key, researchers found, was carefully structured training in working memory – the kind that allows memorization of a telephone number just long enough to dial it. This type of memory is related to fluid intelligence and appears to rely on the same brain circuitry. So the researchers reasoned that improving it might lead to improvements in fluid intelligence.

First they measured the fluid intelligence of four groups of volunteers using standard tests. Then they trained each in a memory task, a variation on Concentration, the child's card game, in which they memorized simultaneously presented auditory and visual stimuli that they had to recall later.

The game was set up so that as the participants succeeded, the tasks became harder, and as they failed, the tasks became easier.

This assured a high level of difficulty but not so high as to destroy motivation to keep working.

The four groups underwent a half-hour of training daily for 8, 12, 17 and 19 days, respectively. At the end of each training, researchers tested the participants' fluid intelligence again.

The results, published on Monday in the proceedings of the National Academy of Sciences, were striking.

Although the control groups also made gains, presumably because they had practice with the fluid intelligence tests, improvement in the trained groups was substantially greater. Moreover, the longer they trained, the higher their scores were.

"Intelligence has always been considered principally an immutable inherited trait," said Susanne Jaeggi, co-author of the paper. "Our results show you can increase your intelligence with appropriate training."

Why did the training work? The authors suggest several aspects of the exercise relevant to solving new problems: ignoring irrelevant items, monitoring ongoing performance, managing two tasks simultaneously and connecting related items to one another in space and time.

CHINESE Olympiad Problems

SOLUTIONS

SECTION 1

1. For each vertex of the cube, its three neighbours determine an equilateral triangle. In any equilateral triangle determined by three vertices of the cube, the sides are diagonals on three faces meeting at a vertex. Hence there is a one-to-one correspondence between the vertices of the cube and the equilateral triangles determined by the vertices of the cube. It follows that the number of such triangles is 8.

2. Let t denote the common values of $\frac{a}{b}, \frac{b}{c}$ and $\frac{c}{a}$. Then $t^3 = 1$ and $t = 1$, ω or ω^2. Note that

$$\frac{1}{t} = \frac{a}{c} = \frac{b}{a} = \frac{c}{b} = \frac{a+b-c}{c+a-b}$$

The value of the last expression is also 1, ω or ω^2.

3. Note that

$$a^3 + 23 = (a-1)(a(a+1)+1) + 24$$

and $a(a+1)+1$ is odd. If $a^3 + 23$ is to be divisible by 24, so must $a - 1$ be. Now exactly one of $a - 1$, a and $a + 1$ is divisible by 3. If it is not $a - 1$, then $a(a+1)+1$ will not be divisible by 3 either, nor will $a^3 + 23$. It follows that $a - 1$ must be divisible by 24, and if this is the case, then $a^3 + 23$ will also be divisible by 24. Hence $a = 24k + 1 < 100$ for some non-negative integer k, and we have $k < 5$. It follows that there are 5 such values of a, namely, 1, 25, 49, 73 and 97.

4. Let $3 - r$ be one of the roots. Then $f(3 + r) = f(3 - r) = 0$ so that $3 + r$ is also a root. The sum of this pair of roots is 6, as is that of each of the other two pairs. Hence the sum of all six roots is 18.

5. Let $x^2 - y^2$ be any integer k. Then

$\sin(2\pi x^2) - \sin(2\pi y^2)$
$= \sin(2\pi y^2)\cos(2\pi k) - \cos(2\pi x^2)\sin(2\pi k) - \sin(2\pi y^2)$
$= 0$

Similarly, $\cos(2\pi x^2) - \cos(2\pi y^2) = 0$. Hence every element in S is in T, but if $x^2 - y^2$ is even, then (x, y) is in T but not in S.

6. Note that $1 - |x| = |x - y^2| \geq 0$ or $|x| \leq 1$. Let $-1 \leq x < 0$. Then the equation becomes $y^2 - x = 1 + x$. Hence the curve is the parabola $y^2 = 2x + 1$ restricted to the interval $[-1, 0)$. Let $0 \leq x \leq 1$. If $x < y$, then we have $y^2 - x = 1 - x$. If $y^2 \leq x$, then $x - y^2 = 1 - x$. Hence the curve consists of the lines $y = \pm 1$ and the parabola $y^2 = 2x - 1$ restricted to the interval $[0, 1]$.

SECTION 2

7. The given expression is equal to

$\cos^2 10° + \cos^2 50° - \cos 50° \cos 10°$

$$= (\cos 10° - \cos 50°)^2 + \cos 10° \cos 50°$$

$$= (2\sin 20° \sin 60°)^2 + \frac{1}{2}(\cos 60° + \cos 40°)$$

$$= \frac{1}{2}(1 - \cos 40°) + \frac{1}{2}\left(\frac{1}{2} + \cos 40°\right) = \frac{3}{4}.$$

8. Since $3\angle B = \angle A + \angle C + \angle B = 180°$ $\angle A + \angle C = 120°$. Let h be the altitude from B. Then

$$h = AB - BC = \frac{h}{\sin A} - \frac{h}{\sin C}$$

which simplifies to $\sin C - \sin A = \sin A \sin C$. This may be rewritten as

$$2\sin\frac{C-A}{2}\cos\frac{A+C}{2} = \frac{1}{2}(\cos(C-A) - \cos(A+C))$$

or $$\sin\frac{C-A}{2} = \frac{1}{2}\left(1 - 2\sin^2\frac{C-A}{2}\right) + \frac{1}{4}.$$

From $$\sin^2\frac{C-A}{2} - \sin\frac{C-A}{2} + \frac{3}{4} = 0$$

we have $$\sin\frac{C-A}{2} = \frac{1}{2}$$

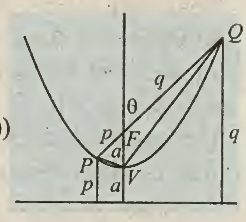
since the other root $-\frac{3}{2}$ must be rejected.

9. Let $\angle PFV = \theta$, and we may assume that $\theta < \frac{\pi}{2}$. Let $PF = p$ and $QF = q$. The vertical line in the diagram is the axis and the horizontal line is the directrix of the parabola. We have labelled various segments according to the definition that the parabola is the locus of a point equidistant from the focus F and the directrix. Consider now projections onto the axis. We have

$p + p\cos\theta = 2a = q - q\cos\theta$

Hence $$b = p + q = \frac{2a}{1+\cos\theta} + \frac{2a}{1-\cos\theta} = \frac{4a}{\sin^2\theta}$$

so that $\sin\theta = 2\sqrt{\frac{a}{b}}$



Now $[VPQ] = [VFP] + [VFQ]$

$$= \frac{1}{2}(ap\sin\theta + aq\sin(\pi - \theta))$$

$$= \frac{1}{2}ab\sin\theta = a\sqrt{ab}$$

10. Since $a > 0$, we have $a^x > 0$ and $a^y > 0$. By the Arithmetic Mean – Geometric Mean Inequality,

$$a^x + a^y \geq 2a^{\frac{x+y}{2}}$$

Since $0 < a < 1$, $\log_a(a^x + a^y) \leq \log_a(2a^{\frac{x+y}{2}})$

$$= \log_a 2 + \frac{x(1-x)}{2} \leq \log_a 2 + \frac{1}{8}$$

by the Arithmetic Mean – Geometric Mean Inequality again.

42. If Z is a complex number such that $Z=-\bar{Z}$, then
(a) Z is any complex number
(b) Real part of Z is the same as its imaginary part
(c) Z is purely real (d) Z is purely imaginary

43. The value of $\sum_{K=1}^{6}\left[\sin\frac{2K\pi}{7}-i\cos\frac{2K\pi}{7}\right]$ is
(a) $-i$ (b) -1 (c) i (d) 0

44. $\lim_{x\to\infty} x\sin\left(\frac{2}{x}\right)$ is equal to
(a) 2 (b) $\frac{1}{2}$ (c) ∞ (d) 0

45. A stone is thrown vertically upwards and the height x ft. reached by the stone in t seconds is given by $x = 80t - 16t^2$. The stone reaches the maximum height in
(a) 3 seconds (b) 1.5 seconds
(c) 2 seconds (d) 2.5 seconds

46. The genergal solution of $|\sin x| = \cos x$ is (when $n \in Z$) given by
(a) $n\pi\pm\frac{\pi}{4}$ (b) $n\pi-\frac{\pi}{4}$ (c) $n\pi+\frac{\pi}{4}$ (d) $2n\pi\pm\frac{\pi}{4}$

47. The real root of the equation $x^3 - 6x + 9 = 0$ is
(a) 6 (b) -3 (c) -6 (d) -9

48. The digit in the unit's place of 5^{834} is
(a) 3 (b) 5 (c) 0 (d) 1

49. The remainder when $3^{100} \times 2^{50}$ is divided by 5 is
(a) 3 (b) 4 (c) 1 (d) 2

50. $\int\frac{\sin x\cos x}{\sqrt{1-\sin^4 x}}dx=$
(a) $\tan^{-1}(\sin^2 x)+C$ (b) $\tan^{-1}(2\sin x)+C$
(c) $\frac{1}{2}\sin^{-1}(\sin^2 x)+C$ (d) $\frac{1}{2}\cos^{-1}(\sin^2 x)+C$

51. The maximum value of $\frac{\log x}{x}$ in $(2, \infty)$ is
(a) e (b) $\frac{1}{e}$ (c) 1 (d) $\frac{2}{e}$

52. If $f(x) = be^{ax} + ae^{bx}$, then $f''(0) =$
(a) $ab(a+b)$ (b) ab
(c) 0 (d) $2ab$

53. If $\sqrt{\frac{1+\cos A}{1-\cos A}}=\frac{x}{y}$, then the value of $\tan A =$
(a) $\frac{2xy}{x^2-y^2}$ (b) $\frac{2xy}{y^2-x^2}$ (c) $\frac{x^2+y^2}{x^2-y^2}$ (d) $\frac{2xy}{x^2+y^2}$

54. $\int\frac{\sec x}{\sec x+\tan x}dx=$
(a) $\sec x+\tan x+C$ (b) $\log\sin x+\log\cos x+C$
(c) $\tan x-\sec x+C$ (d) $\log(1+\sin x)+C$

55. If $\int f(x)dx = g(x)$, then $\int f(x)\, g(x)dx =$
(a) $\frac{1}{2}[g'(x)]^2$ (b) $f'(x)g(x)$
(c) $\frac{1}{2}f^2(x)$ (d) $\frac{1}{2}g^2(x)$

56. The value of $\int_{-2}^{2}(ax^3+bx+c)dx$ depends on the
(a) value of a (b) values of a and b
(c) value of b (d) value of c

57. The area of the region bounded by $y = 2x - x^2$ and the x-axis is
(a) $\frac{7}{3}$ sq. units (b) $\frac{2}{3}$ sq. units
(c) $\frac{8}{3}$ sq. units (d) $\frac{4}{3}$ sq. units

58. The differential equation $y\frac{dy}{dx}+x=c$ represents
(a) a family of parabolas
(b) a family of circles whose centres are on the x-axis
(c) a family of hyperbolas
(d) a family of circles whose centres are on the y-axis

59. If $f(x^5) = 5x^3$, then $f'(x) =$
(a) $\frac{3}{x}$ (b) $\sqrt[5]{x}$ (c) $\frac{3}{\sqrt[5]{x^2}}$ (d) $\frac{3}{\sqrt[5]{x}}$

60. $f(x) = 2a - x$ in $-a < x < a$
$= 3x - 2a$ in $a \le x$.
Then which of the following is true?
(a) $f(x)$ is differentiable at all $x \ge a$
(b) $f(x)$ is continuous at all $x < a$
(c) $f(x)$ is discontinuous at $x = a$
(d) $f(x)$ is not differentiable at $x = a$

ANSWER KEY

1.	(b)	2.	(a)	3.	(a)	4.	(b)	5.	(a)	6.	(a)
7.	(c)	8.	(b)	9.	(d)	10.	(c)	11.	(d)	12.	(c)
13.	(a)	14.	(b)	15.	(d)	16.	(c)	17.	(b)	18.	(d)
19.	(a)	20.	(d)	21.	(c)	22.	(b)	23.	(a)	24.	(a)
25.	(c)	26.	(b)	27.	(c)	28.	(c)	29.	(a)	30.	(d)
31.	(d)	32.	(b)	33.	(b)	34.	(c)	35.	(a)	36.	(d)
37.	(b)	38.	(d)	39.	(c)	40.	(a)	41.	(a)	42.	(d)
43.	(c)	44.	(a)	45.	(d)	46.	(d)	47.	(b)	48.	(b)
49.	(b)	50.	(c)	51.	(b)	52.	(a)	53.	(a)	54.	(c)
55.	(d)	56.	(d)	57.	(d)	58.	(b)	59.	(c)	60.	(d)

Math Archives

10 Best Problems

*Prof. Shyam Bhushan**

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. Discuss the monotonicity of the function g defined by $g(x) = f(x^2 - x - 10) + f(14 + x - x^2)$, $f''(x) > 0$ for all real numbers x except finite number of real numbers x, for which $f''(x) = 0$.

2. Suppose that f and g are non constant differentiable, real value functions on R. If for every $x, y \in R$, $f(x + y) = f(x) f(y) - g(x) g(y)$, $g(x + y) = g(x) f(y) + f(x) g(y)$ and $f'(0) = 0$ then prove that maximum and minimum value of the function $f^2(x) + g^2(x)$ are same for all $x \in R$.

3. Real valued function $f(x)$ satisfies the relation $f\left(\frac{x+y}{3}\right) = \frac{2f(x)+2f(y)-4}{6}\ \forall x, y \in R$. If $f'(0) = 2$, prove that $f(x)$ is an increasing function for all x.

4. Let $f(x) = -\frac{1}{2}(2\theta^2 - 4x - 2x^2)$, where '$\theta$' is a real parameter. Now let x_1, x_2 be the roots of $f(x)$ where $x_1 < x_2$. If $F(\theta) = \int_{x_1}^{x_2} f(x)dx$, find the minimum and the maximum value of $F(\theta)$ and the corresponding θ.

5. The function $f(x) = \sqrt{ax^3 + bx^2 + cx + d}$ has its non-zero local minimum and maximum values at -2 and 2 respectively. If 'a' is a root of $x^2 - x - 6 = 0$. Find possible values of a, b, c and d.

6. If $a = -1$, $b \geq 1$ and $f(x) = \frac{1}{|x|}$, show that the conditions of Lagrange's mean value theorem are not satisfied in the interval $[a, b]$, but the conclusion of the theorem is true if and only if $b > 1 + \sqrt{2}$.

7. If $f(x) = 2x^3 - 15x^2 + 24x$, and $g(a) = \int_0^a f(x)dx + \int_0^{5-a} f(x)dx, 0 < a < 5$. Find the interval in which $g(a)$ is increasing.

8. For what value of 'a' the point of local minima of $f(x) = x^3 - 3ax^2 + 3(a^2 - 1)x + 1$ is less than 4 and point of local maxima is greater than -2?

9. The equation $t^2 + 2xt + 4 = 0$ does not possess distinct real roots. Find the equation of the tangent of greatest slope to the curve $y = x^3 - 2x^2 + x$.

10. A point $P(x, y)$ moves on the curve $x^{2/3} + y^{2/3} = a^{2/3}$, $a > 0$. For each position (x, y) of P, perpendiculars are drawn from origin upon the tangent and normal at P, the length (absolute value) of them being $p_1(x)$ and $p_2(x)$ respectively. Prove that $\frac{dp_1}{dx} \cdot \frac{dp_2}{dx} < 0$.

Solutions

1. $f''(x) \geq 0 \Rightarrow f'(x)$ is an increasing function of x. [$f''(x) = 0$ at finitely many values of x does not affect the increasing ness of $f'(x)$]

Now $g'(x) = (2x - 1)[f'(x^2 - x - 10) - f'(14 + x - x^2)]$

Intervals of increase of g:

If $g(x)$ increases then $g'(x) \geq 0$

$\Rightarrow$ $2x - 1$ and $f'(x^2 - x - 10) - f'(14 + x - x^2)$ are of same sign.

Case I : $2x - 1 \geq 0$ and $f'(x^2 - x - 10) - f'(14 + x - x^2) \geq 0$

$\Rightarrow$ $x \geq \frac{1}{2}$ and $x^2 - x - 10 \geq 14 + x - x^2$, as f' is increasing function of x.

$\Rightarrow$ $x \geq \frac{1}{2}$ and $x^2 - x - 12 \geq 0 \Rightarrow x \geq 4$

Case II : $2x - 1 \leq 0$ and $f'(x^2 - x - 10) - f'(14 + x - x^2) \leq 0$

$\Rightarrow$ $x \leq \frac{1}{2}$ and $-3 \leq x \leq 4 \Rightarrow -3 \leq x \leq \frac{1}{2}$

Hence $g(x)$ increase for $x \in \left[-3, \frac{1}{2}\right] \cup [4, \infty)$

Similarly $g(x)$ decreases for $x \in (-\infty, -3] \cup \left[\frac{1}{2}, 4\right]$

2. We have $f(x + y) = f(x) f(y) - g(x) g(y)$

Differentiation both sides w.r.t x keeping y constant, we get $f'(x + y) = f'(x) f(y) - g'(x) g(y)$

Putting $x = 0$, we get $f'(y) = -g'(0) g(y)$ ... (1)

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as $f'(0) = 0$

We also have, $g(x + y) = g(x) f(y) + f(x)\, g(y)$

Differentiating both sides w.r.t. 'x' keeping 'y' constant, we get

$$g'(x) = g'(0) f(y) \qquad \dots (2)$$

as $f'(0) = 0$

from (1) $\times f(y)$ + (2) $\times g(y)$,

we get $f(y) f'(y) + g(y) g'(y) = 0$

$$\Rightarrow \frac{d}{dy}(f^2(y) + g^2(y)) = 0 \Rightarrow f^2(y) + g^2(y) = \lambda \text{(const)}$$

Now putting $x = y = 0$ in both the given functional equations we get;

$$f(0) = f^2(0) - g^2(0),\ g(0) = 2f(0)\, g(0)$$

$$\Rightarrow g(0) = 0 \text{ or } f(0) = \frac{1}{2}$$

But if $f(0) = \frac{1}{2}$, first equation gives $g^2(0) = -\frac{1}{4}$, which is not possible

Hence $g(0) = 0$ and $f(0) = 1 \Rightarrow \lambda = 1$

Hence $f^2(x) + g^2(x) = 1,\ \forall\, x \in R$.

$\Rightarrow$ Maximum and minimum value of $f^2(x) + g^2(x)$ are same for all $x \in R$.

3. For $x = 0, y = 0$, the given equation gives

$$f(0) = \frac{4f(0) - 4}{6} \Rightarrow f(0) = -2$$

Now,

$$f'(x) = \lim_{h\to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h\to 0} \frac{f\left(\frac{3x+3h}{3}\right) - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{\frac{2f(3x) + 2f(3h) - 4}{6} - f(x)}{h}$$

$$= \lim_{h\to 0} \frac{2f(3x) - 2f(3h) - 4 - 6f(x)}{6h}$$

for $y = 0$, the gievn relation yields

$$f\left(\frac{x}{3}\right) = \frac{2f(x) + 2f(0) - 4}{6}$$

$$\Rightarrow f(x) = \frac{2f(3x) - 4 - 4}{6} = \frac{f(3x) - 4}{3}$$

$\Rightarrow f(3x) = 3f(x) + 4$

$$\text{Hence } f'(x) = \lim_{h\to 0} \frac{6f(x) + 8 + 2f(3h) - 4 - 6f(x)}{6h}$$

$$= \lim_{h\to 0} \frac{f(3h) + 2}{3h}$$

$$\lim_{h\to 0} \frac{f(3h) - f(0)}{3h} = f'(0) = 2$$

$\Rightarrow f(x) = 2x + c$ at $x = 0$, $c = -2$

$\Rightarrow f(x) = 2x - 2$

$f'(x) = 2 > 0 \Rightarrow$ Always increasing.

4. Consider $g(x) = x^2 + 2x$ clearly $-\theta^2$ will be a negative number. If θ increases then $-\theta^2$ will decrease or graph of $g(x)$ will come down by the quantity $-\theta^2$. Also $F(\theta)$ is algebraic area bounded by x-axis and the curve and will be negative. So if we are increasing θ, $F(\theta)$ will decrease. Hence maximum value of $F(\theta)$ will be corresponding to $\theta = 0$ and this value is equal to

$$F(\theta)_{\max} = \int_{-2}^{0} (x^2 + 2x)dx = \left|\frac{x^3}{3} + x^2\right|_{-2}^{0} = \frac{-4}{3}$$

$\Rightarrow F(\theta)_{\max} = -\frac{4}{3}$ for $\theta = 0$ and clearly $F(\theta)$ min does not exist.

5. Since minimum occurs before maximum, so $a < 0$

Also 'a' is a root of $x^2 - x - 6 = 0 \Rightarrow a = -2$

Let $g(x) = ax^3 + bx^2 + cx + d = -2x^3 + bx^2 + cx + d$

$\Rightarrow g'(x) = -6x^2 + 2bx + c$

roots of $g'(x) = 0$ are -2 and 2

$\Rightarrow b = 0, c = 24$

Since minimum value is non-zero $g(-2) > 0$

$\Rightarrow d > 32$

so $a = -2, b = 0, c = 24, d > 32$.

6. Given, $f(x) = \frac{1}{|x|}, x \neq 0$

Let $f(0) = \lambda$, λ is definite real number.

Now $Rf'(0) = \lim_{h\to 0} \frac{f(0+h) - f(0)}{h}$

$$= \lim_{h\to 0}\frac{\frac{1}{|h|}-\lambda}{h} = \lim_{h\to 0}\frac{1-\lambda h}{h^2} = \infty$$

Hence $f(x)$ is not differentiable at $x = 0$. Thus the conditions of Lagrange's Mean Value theorem are not satisfied in the interval which includes the origin.

Conclusion of the Lagrange's Mean Value theorem:

$$\frac{f(b)-f(a)}{b-a} = f'(c), a<c<b$$

$$\frac{\frac{1}{|b|}-\frac{1}{|a|}}{b-a} = \frac{1}{|c|^2}$$

or $\frac{1}{|b|}-\frac{1}{|a|} = (b-a)\left(-\frac{1}{|c|^2}\right)$

or $\frac{1}{|b|}-1 = b+1\left(-\frac{1}{|c|^2}\right) = -\frac{b+1}{c^2}\ (\because a=-1)$

or $c^2 = \frac{b^2+b}{b-1}$ or $\frac{b^2+b}{b-1} < b^2\ (\text{as } b^2 > c^2)$

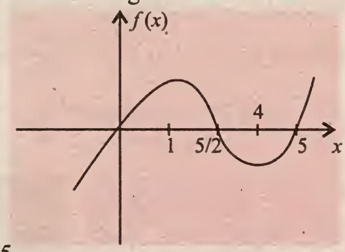
$\therefore\ \frac{b(b-1+\sqrt{2})(b-1-\sqrt{2})}{b-1} > 0$

$\Rightarrow\ b > 1+\sqrt{2}\quad (\because b \ge 1)$

Hence the conclusion of the L.M.V. theorem is true if $b > 1+\sqrt{2}$.

7. $f'(x) = 6x^2 - 30x + 24 = 6(x-4)(x-1)$

Graph of $f(x)$ will be as shown in fig.



$g'(a) = f(a) - f(5-a)$

if $a < 5 - a$

$\Rightarrow\ a < \frac{5}{2}$,

then from the graph

$f(a) > f(5-a)$

so $g'(a) > 0$ and if $a > \frac{5}{2}$ then $f(5-a) > f(a)$ so $g'(a) < 0$.

Hence $g(a)$ is increasing in $\left[0, \frac{5}{2}\right]$.

8. $f'(x) = 3(x^2 - 2ax + a^2 - 1)$ clearly roots of the equation $f'(x) = 0$ must be distinct and lie in the interval $(-2, 4)$

$\therefore\ \Delta > 0 \Rightarrow a \in R$ ...(1)

$f'(-2) > 2 \Rightarrow a^2 + 4a + 3 > 0$

$\Rightarrow a < -3$ or $a > -1$ ...(2)

$f'(4) > 0 \Rightarrow a^2 - 8a + 15 > 0$

$\Rightarrow a > 5$ or $a < 3$ ...(3)

and $-2 < \frac{-B}{2A} < 4 \Rightarrow -2 < a < 4$ ...(4)

from (1), (2), (3) and (4) $-1 < a < 3$.

9. Since $t^2 + 2xt + 4 = 0$ does not possess distinct real roots, $4x^2 - 16 < 0 \Rightarrow -2 \le x \le 2$.

Slope of the tangent at any point (x, y) is $\frac{dy}{dx} = 3x^2 - 4x + 1$

which has max. or min. $\frac{d^2y}{dx^2} = 0 \Rightarrow 6x - 4 = 0 \Rightarrow x = \frac{2}{3}$

Hence $\left(\frac{dy}{dx}\right)_{at\ x=-2} = 21, \left(\frac{dy}{dx}\right)_{x=2} = 5, \left(\frac{dy}{dx}\right)_{at\ x=2/3} = -\frac{1}{3}$

At $x = -2, y = -8 - 8 - 2 = -18$

10. Any point $P(x, y)$ on the curve can be represented by using parameter θ, as $(a\cos^3\theta, a\sin^3\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$\therefore$ equation of the tangent at P is

$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$

$\tan\theta\, x + y = a\sin^3\theta + a\cos^3\theta\tan\theta$

$$p_1 = \left|\frac{a\sin^3\theta + a\cos^3\theta\tan\theta}{\sec\theta}\right| = \frac{1}{2}|a\sin 2\theta|$$

Also equation of the normal at P is

$y - a\sin^3\theta = \cot\theta(x - a\cos^3\theta)$

or $x\cot\theta - y = a\cos^3\theta\cot\theta - a\sin^3\theta$

$$p_2 = \left|\frac{a\cos^3\theta\cdot\cot\theta - a\sin^3\theta}{\operatorname{cosec}\theta}\right| = |a\cos 2\theta|$$

$\Rightarrow\ 4p_1^2 + p_2^2 = a^2 = \text{constant } \forall \text{ point } P(x, y)$

$\Rightarrow$ If p_1 increases, p_2 decreases and commonly

$\Rightarrow\ \frac{dp_1}{dx}$ and $\frac{dp_2}{dx}$ are of opposite signs.

$\frac{dp_1}{dx}\cdot\frac{dp_2}{dx} < 0$

Math Musing Solution Senders

Set 65

1. Amitayu Banerjee, Kolkata
2. Kedar Nath Thakur, Darbhanga
3. Mohrin Shiraz, U.P.
4. Shirsho Mukherjee, Kolkata

Set 64

1. Satya Dev, Bangalore
2. Deboshish Sharma, Assam

Shitikanth has been a consistent performer. An alumnus of Patna's St Michael's High School, Shitikanth has been a consistent performer; he scored 93% in his Class X and 91% in Class XII exams, conducted by the CBSE. For the last two years, he had been staying at Kota in Rajasthan, where he had joined a coaching institute.`

Shitikanth with Anil Ahlawat, Editor of Mathematics Today.

Shiti is the second child of Dr. Arun Kumar Barnwal, posted as a surgeon in Gaya Medical College and Dr. Anita Kumari who is posted in Danapur. Right from his childhood days he was inclined towards Physics. His parents never pressurised him to become a doctor.

Overwhelmed by the success of the child, his mother said, "I knew even earlier that my son will succeed. My wish was that he would be at best in the top five but the happiness he gave us by coming first cannot be expressed in words. When early morning the father learnt about the news, he hugged so hard as if the son's success has given him all the happiness in the world."

Shiti who is interested in taking admission for computer engineering in Kanpur ultimately wants to do research in Physics. After completing engineering he would like to start working towards that. Right now, he is preparing to take part in the Physics Olympiad in Vietnam which is to take place between 20th and 29th of this month. He says he will do his best there and try hard to bring credit to the state and nation.

A fan of cricket, Shiti's ambition is to be famous like our respected earlier President and great scientist Dr. A.P.J. Abdul Kalam.

Shiti also said if any one wants to go to IIT, one should pay attention to the fundamentals. One should understand new situations and application of the concepts could give one success.

MAKING THIS YEAR' GRADE

ZONE	2006		2007		2008	
	A	Q	A	Q	A	Q
Mumbai	48,682	1,836	45,246	2640	61,396	2,551
Madras	40,071	1,266	38,192	2,021	49,831	2,237
Delhi	50,061	1,258	45,910	1,594	51,373	1,549
Guwahati	24,832	154	15,675	199	19,982	193
Kanpur	52,636	637	38,849	627	46,144	527
Kharagpur	39,773	679	32,693	979	40,915	767
Roorkee	31,509	513	26,464	575	41,617	828

A :Appeared Q :Qualified

EXAMINER's MIND

FUNCTIONS

by : R.K. Tyagi, HOD Maths,
Samarth Shiksha Samiti, New Delhi.

The questions given in this column has been prepared strictly on the basis of NCERT for XI[th] and XII[th] class. Over the last few years IIT-JEE/AIEEE has drawn its paper heavily from NCERT books.

Straight Objective Type I
Only one correct option

1. The range of $f(x)=\left[\dfrac{1}{\sin\{x\}}\right]$ (where [ ] & { } respectively denote the greatest integer and fractional part function) is
(a) I, the set of integers
(b) N, the set of natural numbers
(c) W, the set of whole numbers
(d) $\{2, 3, 4, ...\}$

2. A function $F(x)$ satisfies the equation $x^2F(x) + F(1-x) = 2x - x^4\ \forall\ x \in R$ then $f(x)$ equals ?
(a) x^2 (b) $1+x^2-x$
(c) $1-x^2$ (d) x^2+x+1

3. If x and y satisfy the equations $y = 2[x] + 2$ and $y = 3[x-1]$ simultaneously thus $[x+y]$ is, (where [ ] is greatest integer function).
(a) 5 (b) 4
(c) 17 (d) None of these

4. The maximum value of x^2y subject to the constraints $x+y+\sqrt{2x^2+2xy+3y^2}=k$ (constant), $x, y \geq 0$ is
(a) $\dfrac{k^2}{(2+\sqrt{15})^2}$ (b) $\dfrac{4k^3+k^2}{(3+\sqrt{15})^3}$
(c) $\dfrac{4k^3}{(3+\sqrt{15})^3}$ (d) None of these

5. The range of $f(x) = 5|\sin x| - 3|\cos x|$ is
(a) $[3, -5]$ (b) $[-\sqrt{34}, \sqrt{34}]$
(c) $[5, \sqrt{34}]$ (d) $[-3, 5]$

6. If $f(2x+3y, 2x-7y) = 20x$ then $f(x, y)$ equals
(a) $7x-3y$ (b) $7x+3y$ (c) $3x-7y$ (d) $x-y$

7. Let $f(x)=\begin{cases}1+[x] & , x<-2\\ |x| & , x\geq -2\end{cases}$ where [ ] denotes the greatest integer function then $f(f(-2.5))$ equals
(a) -3 (b) -2 (c) 2 (d) 3

8. If $f(x) = ax + b$, $g(x) = cx + d$, $c \neq 0$ then, $f(g(x)) = g(x)$ if and only if
(a) $f(x)=x$ (b) $f(a)=f(c)$
(c) $f(b)=g(b)$ (d) $f(d)=f(b)$

9. Let $f(x) = 2x^n + \lambda$, $\lambda \in R$, $f(4) = 133$ and $f(5) = 255$ then sum of the factor of $f(3) - f(2)$ is
(a) 20 (b) 60
(c) 21 (d) None of these

10. Lets $f(x) = [16^x - 4^x + 1]\ \forall\ x \in (-\infty, 1)$, (where [ ] denote the greatest integer) then range of $f(x)$ is
(a) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
(b) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
(c) $\{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$
(d) $\{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12, 13\}$

11. The range of $f(x)=\cos^{-1}\left[x^2+\dfrac{1}{2}\right]+\sin^{-1}\left[x^2-\dfrac{1}{2}\right]$ (where [ ] denotes the greatest integer function) is
(a) $\left\{\dfrac{\pi}{2}\right\}$ (b) $\left\{-\dfrac{\pi}{2}\right\}$ (c) $\{0\}$ (c) $\{\pi\}$

12. Let f: **R** → **R** be given by $f(x)=\dfrac{ax+b}{cx+d}$, (a, b, c, d being non zero real numbers) the condition for which $f(x)$ is inverse of itself, is
(a) $a+b=0$ (b) $a+d=0$
(c) $a+c=0$ (d) $b+c=0$

13. The domain of $f(x)=\sqrt{\log_3 \cos(\sin x)}$ is
(a) $x=\dfrac{n\pi}{2}, n\in I$ (b) $x=2n\pi, n\in I$
(c) $x=n\pi, n\in I$ (d) $x=\phi$

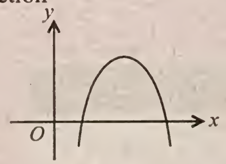
14. The period of the function $f(x) = \sin^3x + \cos^3x$ is
(a) 2π (b) π
(c) $\dfrac{2\pi}{3}$ (d) None of these

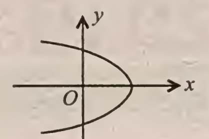
15. Let $f: \mathbf{R} \to \mathbf{R}$ be defined by $f(x) = \dfrac{e^x - e^{-x}}{2}$ then $f(x)$ is
(a) many-one-onto
(b) neither one-one nor onto
(c) one-one onto
(d) one-one but not onto

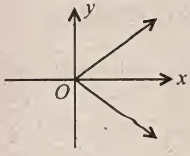
16. Let $f: [-10, 10] \to R$ where $f(x) = \sin x + \left[\dfrac{x^2}{a}\right]$ be an odd function, then set of values of parameter a is/are
(a) $(-10, 10) - \{0\}$
(b) $(0, 10)$
(c) $(100, \infty)$
(d) $(-100, 100)$

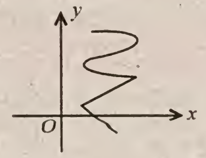
17. If $f(x)$ and $g(x)$ are periodic functions with period 9 and 5 then period of $t(x) = f\left(\dfrac{x}{2}\right)g(x) + g\left(\dfrac{x}{2}\right)f(x)$ is
(a) 18 (b) 45 (c) 8100 (d) 90

18. Which one of the following graph is graph of a function

(a)


(b)


(c)


(d)


19. The range of the function
$f(x) = {}^{16-x}C_{2x-1} + {}^{18-2x}P_{4x-5}$ is
(a) $\{f(2), f(4)\}$
(b) $\{f(2), f(3)\}$
(c) $\{f(2), f(8)\}$
(d) $\{f(1), f(5)\}$

20. The domains of function $f(x) = \log\left(\log \dfrac{x}{\{x\}}\right)$ is, (where { } denote the fractional part function).
(a) $(-\infty, \infty)$
(b) $R - I$ (set of integers)
(b) $(1, \infty) - I$ (set of integers)
(d) $(-\infty, 1) - I$ (set of integers)

Straight objective type II
One or more than one correct answers

In each of the questions four choices are given of which one or more than one are correct. You have to select the correct choices accordingly.

21. Let f, g be two function such that
(a) If f is even, g is even $\Rightarrow f \circ g$ is an even function
(b) f is odd, g is odd $\Rightarrow f \circ g$ is an odd function
(c) f is even, g is odd $\Rightarrow f \circ g$ is an even function
(d) f is odd, g is even $\Rightarrow f \circ g$ is an even function

22. If f is an even function defined on the interval $[-5, 5]$, then real value(s) of x satisfies the equation $f(x) = f\left(\dfrac{x+1}{x+2}\right)$ is/are
(a) $\dfrac{\sqrt{5}-1}{2}$
(b) $\dfrac{-(3+\sqrt{5})}{2}$
(c) $\dfrac{-3+\sqrt{5}}{2}$
(d) $\dfrac{-(\sqrt{5}+1)}{2}$

23. Which of the following functions are non periodic
(a) $f(x) = x + \sin x$
(b) $f(x) = \cos x + \{x\}$ (where $\{x\}$ is functional part of x)
(c) $f(x) = \cos x^2$
(c) $f(x) = \cos\{(x+3) - [x+3]\}$

24. Which of the following functions have symmetrical graph about y-axis.
(a) $f(x) = -|x|$
(b) $f(x) = 1/x^2, x \neq 0$
(c) $f(x) = x^2 - 2|x|$
(d) $f(x) = x^2 - x$

25. Let $f: \mathbf{R} \to \mathbf{R}$, $g: \mathbf{R} \to \mathbf{R}$ such that $f(x) = e^x$, $g(x) = 3x - 2$, then which of the following is true ?
(a) $(fog)(x) = e^{3x-2}$
(b) $(gof)(x) = 3e^x - 2$
(c) Domain of $(fog)^{-1}$ is $(0, \infty)$
(d) Domain of $(gof)^{-1}$ is $(-2, \infty)$

26. If $f(x) = \sin[\pi^2]x + \sin[-\pi^2]x$, [ ] denotes the greatest integer function then which of the following is true ?
(a) period of $f(x)$ is 2π
(b) $f\left(\dfrac{\pi}{2}\right) = 1$
(b) $f'(\pi) = -19$
(d) $f(-x) = -f(x)$

27. Let $f: \mathbf{R} \to \mathbf{R}$, $g: \mathbf{R} \to \mathbf{R}$ be two one-one and onto functions such that they are mirror images of each other about the line $y = \lambda$. If $h(x) = f(x) + g(x)$, then which of the following is not true for $h(x)$.
(a) one-one onto
(b) not a constant function
(c) many-one into
(d) many-one onto

28. Let n be a positive integer with $f(x) = 1! + 2! + 3! + \dots + x!$ and $P(x)$, $Q(x)$ be polynomials in x such that $f(x+2) = P(x)f(x+1) + Q(x)f(x)\ \forall\ x \geq 1$ then
(a) $P(x) = x + 3$
(b) $P(x) = -x - 2$
(c) $Q(x) = -x - 2$
(d) $Q(x) = x + 3$

29. Let $f: \left[-\dfrac{\pi}{3}, \dfrac{2\pi}{3}\right] \to [0, 4]$ be a function defined as $f(x) = \sqrt{3}\sin x - \cos x + 2$, then f^{-1} equals
(a) $\sin^{-1}\left(\dfrac{x-2}{2}\right) - \dfrac{\pi}{6}$
(b) $\sin^{-1}\left(\dfrac{x-2}{2}\right) + \dfrac{\pi}{6}$
(c) $\dfrac{2\pi}{3} - \cos^{-1}\left(\dfrac{x-2}{2}\right)$
(d) None of these

30. Let $f(x) = \max\{1 + \sin x, 1, 1 - \cos x\}, x \in [0, 2\pi]$ and $g(x) = \max\{1, |x - 1|\}\ \forall\ x \in R$ then,
(a) $g(f(0)) = 1$
(b) $g(f(1)) = 1$
(c) $f(g(1)) = 1$
(d) $f(g(0)) = \sin 1$

31. Let $f(x) = |1 - x| + |1 + x|\ \forall -2 \leq x \leq 2$, then

(a) $f(x) = -2x \ \forall\ -2 \le x \le -1$
(b) $f(x) =$ constant function $\forall\ |x| \le 1$
(c) $f(x) = 2x \ \forall\ 1 \le x \le 2$
(d) graph of $f(x)$ lies in the Ist and IInd quadrant

Linked COMPREHENSION Type Questions

In these questions a passage has been given followed by questions based on the passage. You have to answer the questions based on the given passage.

Passage 1 : Let $f(x) = x^2 - 2x - 3$, $g(x) = f(|x|)$, $h(x) = |g(x)|$ are three functions.
On the basis of above informations answer the following questions

(i) The interval in which $f(x) > 0$ is
(a) $R - [-1, 3]$ (b) $[-1, 3]$
(c) $(-\infty, \infty)$ (d) None of these

(ii) The minimum value of $f(x)$ occurs at
(a) $x = 2$ (b) $x = 4$ (c) $x = 1$ (d) $x = -1$

(iii) $f(x)$ is increasing in the interval
(a) $(-\infty, -1)$ (b) $(-\infty, 1)$ (c) $(-\infty, \infty)$ (d) $(1, \infty)$

(iv) The zeros of $f(x)$ are respresented by the set
(a) $\{-1, 3\}$ (b) $\{1, -3\}$
(c) $\{-1, -3\}$ (d) None of these

(v) Number of solutions of $g(x) = 0$ is/are
(a) 4 (b) 3 (c) 2 (d) 0

(vi) The value of λ for which the equation $g(x) - \lambda = 0$ has exactly three real roots which are distinct is
(a) 2 (b) 3
(c) –3 (d) None of these

(vii) The curve $h(x)$ meets the positive direction of y-axis at the point
(a) $(1, 4)$ (b) $(-1, 4)$ (c) $(0, 4)$ (d) $(0, 3)$

Passage 2 : Let $f : X \to Y$ be a bijection. We define $g : Y \to X$ such that $f(x) = y \Leftrightarrow g(y) = x$, $x \in X$, $y \in Y$, then g is called the inverse of f and is denoted by f^{-1}.
On the basis of above information answer the following questions.

(i) Let $f : (-\infty, 1) \to (1, \infty)$ such that $f(x) = x(2 - x)$ then $f^{-1}(x)$ equals
(a) $1 + \sqrt{1 - x}$ (b) $1 - \sqrt{1 - x}$
(c) $\sqrt{1 - x}$ (d) None of these

(ii) If the function f: $(1, \infty) \to (1, \infty)$ is defined by $f(x) = 2^{x(x-1)}$ then $f^{-1}(x)$ equals
(a) $\left(\frac{1}{2}\right)^{x(x-1)}$ (b) $\frac{1}{2}\left(1 + \sqrt{4\log_2 x + 1}\right)$
(c) $\frac{1}{2}\left(1 - \sqrt{4\log_2 x + 1}\right)$ (d) not defined

(iii) If the function $f : \mathbf{R} \to \mathbf{R} \ni f(x) = x - [x]$, (where $[y]$ denotes greatest integer $\le y$) then $f^{-1}(x)$ equals,
(a) $\frac{1}{x - [x]}$ (b) $[x] - x$
(b) not defined (d) none of these

(iv) If $f : \mathbf{R} \to (-\infty, 1)$ such that $f(x) = 1 - 2^{-x}$ then $f^{-1}(x)$ is
(a) $1 + \log_2(-x)$ (b) $1 - \log_2(-x)$
(c) $\log_2(1 - x)$ (d) $-\log_2(1 - x)$

(v) The number of roots of the equation
$$1 + \log_2(1 - x) = 2^{-x} \text{ is}$$
(a) 0 (b) 1 (c) 2 (d) ∞

Passage 3 : Let f and g be two real functions with domain D_1 and D_2 respectively then the domain of $f \pm g : D_1 \cap D_2 \to R \ni (f \pm g)(x) = f(x) \pm g(x)\ \forall\ x \in D_1 \cap D_2$ and $\frac{f}{g} : D_1 \cap D_2 - \{x \mid g(x) = 0\} \to R$ such that
$$(f/g)(x) = \frac{f(x)}{g(x)}\ \forall\ x \in D_1 \cap D_2 - \{x \mid g(x) = 0\}$$
Let $p(x) = \frac{\sin^{-1} x}{[x]}$, $q(x) = 2^{\sin^{-1} x}$,
$$R(x) = \sin^{-1}\left(\log_2\left(\frac{x^2}{2}\right)\right), f(x) = \frac{\cos^{-1} x}{[x]}$$
On the basis of above informations answer the following questions.

(i) The domain of $p(x) + f(x)$ is
(a) $[-1, 1]$ (b) $(-1, 1)$
(c) $[-1, 0) \cup \{1\}$ (d) $R - [-1, 1]$

(ii) Domain of $p(x) + q(x)$ is
(a) $[-1, 0) \cup \{1\}$ (b) $(0, 1)$
(c) $[-1, 1]$ (d) R

(iii) Domain of $R(x) + q(x)$ is
(a) $[-2, -1]$ (b) $[1, 2]$ (c) $(-1, 1)$ (d) $\{-1, 1\}$

(iv) Domain of $p(x) + q(x) + R(x) + f(x)$ *i.e.*, Domain of $\{\underbrace{p(x) + f(x)} + \underbrace{q(x) + R(x)}\}$ is equal to
(a) $[-1, 1]$ (b) $(-1, 1)$ (c) $\{-1, 1\}$ (d) $(0, 1)$

Matrix-Match type questions

Given below are Matrix-match type questions, with two columns (each having same items). Each item of Column I has to be matched with the item of Column II. It is to be noted that an item of Column I can be match with more than one item of Column II. All items of Column II must be matched.

1.

	Column I	Column II
(A)	Range of $\sin x + \cos x$	(P) $\left(-\frac{3\pi}{2}, \frac{\pi}{2}\right)$

(B)	Range of $\dfrac{1}{2+\cos 3x}$	(Q) $R-\left[\frac{1}{3},3\right]$
(C)	Range of $\tan^{-1}x-\cot^{-1}x$	(R) $[-\sqrt{2},\sqrt{2}]$
(D)	Range of $\dfrac{\sin x\cos 3x}{\sin 3x\cos x}$	(S) $\left[\frac{1}{3},1\right]$
		(T) $\left[-\frac{3\pi}{2},\frac{\pi}{2}\right]$

(a) (A) → (R) (B) → (S) (C) → (T) (D) → (P)
(b) (A) → (R) (B) → (S) (C) → (P) (D) → (Q)
(c) (A) → (R) (B) → (S) (C) → (P) (D) → (T)
(d) (A) → (R) (B) → (S) (C) → (Q) (D) → (T)

2.

	Column I	Column II
(A)	The function $\sin(7x+5)$ is	(P) non periodic
(B)	The function $\log_a(x-\sqrt{x^2+1})$ is ($a>0$, $a\neq 1$ assume to be into)	(Q) one-one onto
(C)	Let $f:\mathbf{R}\to\mathbf{R}$ defined by $f(x)=7x-5$ is	(R) invertible
		(S) many one
		(T) periodic

(a) A → S, T B → P, Q, R C → P, R, T
(b) A → S, R B → P, Q, R C → P, Q, R
(c) A → S, T B → P, Q, R C → P, Q, R
(d) A → S, T B → S, P, R C → P, Q, R

Fill in the blanks

(1) The range of $f(x)=\sin(\sin^{-1}\{x\})$ (where { } is fractional part function is ....

(2) The inverse of $\sin(\tan^{-1}x)$ is .... or ....

(3) Let $f(x)=\log\left(\dfrac{1+x}{1-x}\right)\ \forall x\in(-1,1)$ and $g(x)=\dfrac{3x+x^3}{1+3x^2}$, then $(fog)(0)$ equals ....

(4) Let $f(x)=1+x^3$ and $g(x)=x-x^3$ then $(gof)(x)$ for $x=1$ is ....

(5) The range of $f(x)=\sqrt{x-2}+\sqrt{4-x}$ is ....

(6) $sgn(x)+x^{2008}$ is .... function.

(7) If $g(x)=\cos^{-1}\left(\dfrac{6-3x}{4}\right)+\operatorname{cosec}^{-1}\left(\dfrac{x-1}{2}\right)$, then $D_g=$ ....

(8) The function is defined for all $x\in R$.
If $f(a+b)=f(ab)\ \forall\ a$ and b and $f\left(-\frac{1}{4}\right)=-\frac{1}{4}$, then $f(2008)$ equals ....

(9) If ${}^{10}C_{x-1}>2\ {}^{10}C_x$ and domain of x is $(a, b, c)\ \ni a<b<c$ then $a=$ ...., $b=$ ...., $c=$ ....

(10) The set of values of 'a' for which $x^2-ax+\sin^{-1}(\sin 4)>0\ \forall\ x\in R$ is ....

True / false

(1) If $f(x)$ be defined in $[-3, 3]$ then domain of the definition $f(|x|)+1$ is $[-4, 2]$

(2) The function $f:\mathbf{R}\to\mathbf{R}$ defined by $f(x)=(x-\alpha)(x-\beta)(x-\gamma)$ is many-one onto function.

(3) The range of the function $\dfrac{x^2+x+2}{x^2+x+1}\ \forall\ x\in R$ is $\left(1,\dfrac{7}{3}\right]$

(4) Let $f:[-1,1]-\{0\}\to R$ such that $f(x)=\cot(\sin x)+\left[\dfrac{x^2}{a}\right]$, (where $[\cdot]$ denotes greatest integer function) is an odd function then a must be greater than one.

(5) If $f(x)=\sqrt{\sec^{-1}\left(\dfrac{2-|x|}{4}\right)}$ then domain of $f(x)$ is $(-6, 6)$.

Assertion & Reason

Each question contains Statement - 1 (Assertion A) and Statement-2 (Reason R). Each question has four choices (a), (b), (c) and (d) out of which only one is correct.

Choice (a) : Both assertion (A) & Reason (R) are correct & Reason (R) is also the feasible explanation of assertion (A)

Choice (b) : Both assertion (A) & Reason (R) are correct but Reason (R) is not proper explanation of assertion (A).

Choice (c) : Assertion (A) is correct but Reason (R) is not correct

Choice (d) : Assertion (A) is not correct but Reason (R) is correct.

(1) Statement 1: The period of $f(x)=\sin 2x\cos[2x]-\cos 2x\sin[2x]$ (where $[\cdot]$ is greatest integer function) is 1/2
Statement 2: Period of $\{x\}$ is 1 (where $\{x\}$ denotes the fractional part function)

(2) Statement 1: $f(x)=|x-2|+|x-3|+|x-5|$ is an odd function for all value of x lies between 3 and 5
Statement 2: For odd function $f(-x)=-f(x)$

(3) Statement 1: $f:\mathbf{R}\to\mathbf{R}$ is a function defined by $f(x)=\dfrac{7x+8}{4}$ then $f^{-1}(x)=\dfrac{4(x-2)}{7}$

Statement 2 : $f(x)$ is not bijective

(4) **Statement 1 :** $f : \mathbf{R} \to \mathbf{R}$ defined by $f(x) = \cos x$, possess its inverse.
Statement 2 : f is both one-one and onto

(5) **Statement 1 :** The domain of the function
$f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ is $[-1, 1]$
Statement 2 : $\sin^{-1}x$ and $\cos^{-1}x$ is defined for $\forall -1 \le x \le 1$ and $\tan^{-1}$ is defined $\forall\, x \in R$

ANSWER

1. **(b) :** $\because\ x \in (0, \sin 1)$, $f(x)$ is defined if $\sin\{x\} \ne 0$

$\therefore\ \frac{1}{\sin\{x\}} \in \left(\frac{1}{\sin 1}, \infty\right)$

$\therefore\ \left[\frac{1}{\sin\{x\}}\right] \in \left[\left(\frac{1}{\sin 1}, \infty\right)\right]$

$\Rightarrow \left[\frac{1}{\sin\{x\}}\right] \in [(1.155, \infty)]$ $\left(\because \begin{array}{c} 1 < \pi/3 \\ \sin 1 < \sin\frac{\pi}{3}\end{array}\right)$

$\left[\frac{1}{\sin\{x\}}\right] \in \{1, 2, 3, ...\}$ = set of natural numbers

Hence choice (b) is correct

2. **(c) :** $x^2F(x) + F(1-x) = 2x - x$ ......(i)

Replacing $x \to 1 - x^2$

$\therefore\ (1-x)^2 F(1-x) + F(x) = 2(1-x) - (1-x)^4$ .....(ii)

Eliminating $F(1-x)$ from (i) and (ii) we get

$F(x) = 1 - x^2$

$\therefore$ Choice (c) is correct answer.

3. **(c) ;** $2[x] + 2 = 3[x-1]$

$\Rightarrow 2[x] + 2 = 3[x] - 3 \qquad \Rightarrow [x] = 5$

$\therefore\ y = 2[x] + 2 = 12$

$[x + y] = [x + 12] = [x] + 12 = 17$

Hence choice (c) is correct.

4. **(c) :** $x + y = \frac{x}{2} + \frac{x}{2} + y$

Using AM $\ge$ GM we have,

$x + y = \frac{x}{2} + \frac{x}{2} + y \ge 3\left(\frac{x^2y}{4}\right)^{1/3}$

Equality hold if and only if $\frac{x}{2} = y \Rightarrow x = 2y$

Also, $2x^2 + 2xy + 3y^2 =$

$$\underbrace{\frac{2x^2}{8} + \frac{2x^2}{8} + ... + \frac{2x^2}{8}}_{8\text{ times}} + \underbrace{\frac{2xy}{4} + ... + \frac{2xy}{4}}_{4\text{ times}} + y^2 + y^2 + y^2$$

$$\therefore\ 2x^2 + 2xy + 3y^2 \ge 15\left[\left(\frac{2x^2}{8}\right)^8\left(\frac{2xy}{8}\right)^4(y^2)^3\right]^{\frac{1}{15}}$$

Equality hold $\Leftrightarrow \frac{2x^2}{8} = \frac{2xy}{4} = y^2$ or $x = 2y$

$$\therefore\ k = x + y + (2x^2 + 2xy + y^2)^{1/2} \ge (3 + \sqrt{15})\left(\frac{x^2y}{4}\right)^{1/3}$$

$$\therefore\ k^3 \ge (3 + \sqrt{15})^3\,\frac{x^2y}{4}$$

$$\Rightarrow \frac{4k^3}{(3+\sqrt{15})^3} \ge x^2y \Rightarrow x^2y \le \frac{4k^3}{(3+\sqrt{15})^3}$$

$\Rightarrow$ maximum value of $x^2y = \frac{4k^3}{(3+\sqrt{15})^3}$

Hence choice (c) is correct.

5. **(d) :** Since $f(x)$ is continuous function and $|\sin x|$ and $|\cos x|$ both are always > 0

Hence minimum when $|\sin x| = 0$ and $|\cos x| = 1$ (when $x = 0$) and maximum when $|\sin x| = 1$ and $|\cos x| = 0$ when $x = \frac{\pi}{2}$

$\therefore$ Required range $[-3, 5]$

Hence (d) is the correct choice.

6. **(b) :** $f(2x + 3y, 2x - 7y) = 20x$

$\Rightarrow f(2x + 3y, 2x - 7y) = 7(2x + 3y) + 3(2x - 7y)$

$\therefore\ f(x, y) = 7x + 3y$

Hence (b) is correct answer.

7. **(c) :** $f(-2.5) = 1 + [-2.5] = 1 - 3 = -2$

$\therefore\ f(f(-2.5)) = f(-2) = |-2| = 2$

Hence (c) is correct answer

8. **(a) :** $g(x) = f(g(x))$

$\Rightarrow cx + d = f(cx + d) \qquad \Rightarrow cx + d = a(cx + d) + b$

$\Rightarrow cx + d = acx + ad + b \qquad \Rightarrow ac = x$ and $ad + b = d$.

$\Rightarrow a = 1 \qquad \therefore\ b = 0$

$\therefore\ f(x) = ax + b = x$

Hence (a) is correct choice.

9. **(b) :** $f(x) = 2x^n + \lambda, f(4) = 133, f(5) = 255$

$\therefore\ f(4) = 2\cdot 4^n + \lambda = 133$ .....(i)

$f(5) = 2\cdot 5^n + \lambda = 255$ .....(ii)

$\therefore\ f(5) - f(4) = 2(5^n - 4^n) = 122$

$\Rightarrow 5^n - 4^n = 5^3 - 4^3 \qquad \Rightarrow n = 3$

$\therefore\ f(4) = 133 = 2 \times 4^3 + \lambda \qquad \therefore\ \lambda = 5$

$\therefore\ f(x) = 2x^3 + 5$

$\therefore\ f(3) = 59$ and $f(2) = 21$

$\therefore\ f(3) - f(2) = 38 = 2 \times 19 = 2^a 19^b, (a = 1; b = 1)$

Sum of factors $f(3) - f(2) = \frac{2^{a+1}-1}{2-1}\cdot\frac{19^{b+1}-1}{19-1}$

$= \frac{3 \times 360}{18} = 60$

Hence choice (b) is correct.

10. **(b) :** $16^x - 4^x + 1 = (4^x)^2 - 4^x + 1 = \left(4^x - \frac{1}{2}\right)^2 + \frac{3}{4}$

$\Rightarrow 16^x - 4^x + 1 \ge \frac{3}{4}$

$\Rightarrow \frac{3}{4} \le 16^x - 4^x + 1 < \frac{52}{4}$ ($\because x \in (-\infty, 1)$ & maximum value occurs at $x = 1$)

$\Rightarrow [16^x - 4^x + 1] \in \left[\frac{3}{4}, \frac{52}{4}\right]$

$\Rightarrow [16^x - 4^x + 1] = \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12\}$
Hence choice (b) is correct answer.

11. (c) : $\left[x^2 - \frac{1}{2}\right] = \left[x^2 + \frac{1}{2} - 1\right] = \left[x^2 + \frac{1}{2}\right] - 1$

and $x^2 + \frac{1}{2} \geq \frac{1}{2}$ $\quad\therefore \left[x^2 + \frac{1}{2}\right] = 0$ or 1

now $\cos^{-1}\left[x^2 + \frac{1}{2}\right] + \sin^{-1}\left[x^2 - \frac{1}{2}\right]$

$= \cos^{-1}\left[x^2 + \frac{1}{2}\right] + \sin^{-1}\left(\left[x^2 + \frac{1}{2}\right] - 1\right)$

$$= \begin{cases} \cos^{-1} 0 + \sin^{-1}(-1) & \text{if } \left[x^2 + \frac{1}{2}\right] = 0 \\ \cos^{-1} + \sin^{-1} 0 & \text{if } \left[x^2 + \frac{1}{2}\right] = 1 \end{cases} = \begin{cases} \frac{\pi}{2} - \frac{\pi}{2} \\ 0 + 0 \end{cases}$$

$= 0 \;\forall \left[x^2 + \frac{1}{2}\right] = 0$ or 1

$\therefore$ Range of $f(x)$ is $\{0\}$
Hence choice (c) is correct answer.

12. (b) : $f(x) = y = \dfrac{ax + b}{cx + d}$

$\Rightarrow x = f^{-1}(x) = \dfrac{b - yd}{yc - a} \quad \Rightarrow f^{-1}(x) = \dfrac{b - xd}{xc - a}$

But $f^{-1}(x) = f(x)$

$\Rightarrow \dfrac{b - xd}{xc - a} = \dfrac{ax + b}{cx + d}$

$\Rightarrow c(a + d)x^2 - (a + d)(a - d)x - b(a + d) = 0$
$\Rightarrow (a + d)\{(x^2 - (a - d)x - b\} = 0$
$\Rightarrow a + d = 0$
Hence choice (b) is correct.

13. (c) : $f(x)$ to be defined if $\log_3 \cos(\sin x) \geq 0$
$\Rightarrow \cos(\sin x) \geq 3 \qquad \Rightarrow \cos(\sin x) \geq 1$
$\Rightarrow \cos(\sin x) = 1$ (as maximum of $\cos\theta = 1$
i.e, $-1 \leq \cos\theta \leq 1$
$\Rightarrow \sin x = \cos^{-1}(\cos 0) \qquad \Rightarrow \sin x = 0$
$\Rightarrow x = n\pi, n \in I$
Hence choice (c) is correct

14. (a) : $\sin^3 x = \dfrac{3\sin x - \sin 3x}{4}$, $\cos^3 x = \dfrac{3\cos x + \cos 3x}{4}$

Period of $\sin x$ is 2π, period of $\sin 3x$ is $\frac{2\pi}{3}$ and period of $\cos x$ is 2π, period of $\cos 3x$ is $\frac{2\pi}{3}$

Now, $\sin^3 x + \cos^3 x = \frac{1}{4}\{\sin x + \cos x + \cos 3x - \sin 3x\}$

Period of $\sin^3 x + \cos^3 x$ is LCM of the periods of ($\sin x$, $\cos x$, $\sin 3x$, $\cos 3x$)

$\Rightarrow$ LCM of $\left\{\dfrac{2\pi}{1}, \dfrac{2\pi}{3}\right\} = \dfrac{\text{LCM of }(2\pi, 2\pi)}{\text{HCF of }(1,3)} = 2\pi$

Hence choice (a) is correct answer.

15. (c) : In such type of problem we need to check whether the function is
(a) One-one (if f is not one-one then it is many one)
(b) onto (if it not onto it must be into)
Now for 1 – 1, let $x_1, x_2 \in R \ni x_1 < x_2$
$\Rightarrow e^{x_1} < e^{x_2}$ (as base is e) .....(i)
Also $x_1 < x_2 \Rightarrow -x_2 < -x_1$
$e^{-x_2} < e^{-x_1}$ (as base is e) .....(ii)
Adding (i) and (ii) we get
$e^{x_1} + e^{-x_2} < e^{x_2} + e^{-x_1}$
$\Rightarrow e^{x_1} - e^{-x_1} < e^{x_2} - e^{-x_2}$

$\Rightarrow \dfrac{e^{x_1} - e^{-x_1}}{2} < \dfrac{e^{x_2} - e^{-x_2}}{2} \Rightarrow f(x_1) < f(x_2)$

$\Rightarrow f(x)$ is an increasing function
$\Rightarrow f(x)$ is one-one so it cannot be many one which discard the choices (a) and (b)
Again for onto

$$\left.\begin{array}{r}\text{as } x \to \infty,\ f(x) \to \infty \\ \text{similarly as } x \to -\infty,\ f(x) \to -\infty\end{array}\right\} *$$

$\therefore$ by $*$, $-\infty < f(x) < \infty$ so long as $x \in (-\infty, \infty)$
$\Rightarrow$ codomian of f = range of f
$\Rightarrow f(x)$ is onto which discard the choice (d)
Hence we left with the choice (c) which is to prove that f is one-one onto.
Hence choice (c) is correct.

16. (c) : Since $f(x)$ is an odd function *i.e.* $f(-x) = -f(x)$

$\therefore \left[\dfrac{x^2}{a}\right] = 0 \;\forall\; x \in [-10, 10]$

$\Rightarrow 0 \leq \dfrac{x^2}{a} < 1 \;\forall\; x \in [-10, 10]$ and $0 \leq x^2 \leq 100$

$\Rightarrow a > 100$
Hence choice (c) is correct answer.

17. (d) : As the period of $f(x)$ and $g(x)$ are 9 and 5 respectively

$\therefore$ Period of $f\left(\dfrac{x}{2}\right) = \dfrac{9}{1/2} = 18$

$\therefore$ Period of $g\left(\dfrac{x}{2}\right) = \dfrac{5}{1/2} = 10$

$\therefore$ period of $f(x/2)g(x)$ = L.C.M. {of the period of $f(x/2)$ and $g(x)$} = 18×5
and period of $f(x) \cdot g(x/2) = 9 \times 10$
$\therefore$ period of $f(x)$ = L.C.M. of (90, 90) = 90
Hence (d) is correct answer.

18. (a) : If any line drawn paralleld to y-axis meet the graph more than one point then graph does not represent the fucntion. In the given graph, a line drawn parallel to y-axis meet the graph (b), (c), (d) more than one point in chioices (b), (c), (d) and in choice (a) the line drawn parallel to y-axis cuts the graph exactly at one point.
Hence choice (a) is the correct answer.

19. (b) : For ${}^{16-x}C_{2x-1}$

(i) $16 - x \geq 2x - 1 \quad \therefore\ x \leq \dfrac{17}{3}$

(ii) $2x - 1 \geq 0 \qquad \therefore\ x \geq \dfrac{1}{2}$

(iii) $16 - x > 0 \qquad \therefore\ x < 16$

$\therefore\ x = 1, 2, 3, 4, 5$ by taking intersection of (i), (ii) and (iii) ...(A)

Again $18 - 2x > 0 \qquad \therefore\ x < 9$

$4x - 5 \geq 0 \qquad \therefore\ x \geq 5/4$

$18 - 2x \geq 4x - 5 \qquad \therefore\ x \leq 23/6$

$\therefore\ x = 2, 3$ by taking intersection of

$x < 9, x \geq 5/4, x \leq 23/6$ ...(B)

$\therefore$ Domain of $f(x)$ is $\{2, 3\}$

$\therefore$ Range of $f(x)$ is $\{f(2), f(3)\}$

20. (c) : $\log\left(\log\dfrac{x}{\{x\}}\right)$ is defined if

if $\log\left(\dfrac{x}{\{x\}}\right) > 0$

if $\dfrac{x}{\{x\}} > 1 \qquad (\because\ \log_e x > 0 \Rightarrow x > 1)$

$\Rightarrow \dfrac{[x]}{\{x\}} > 1 \qquad (\because\ x - \{x\} = [x])$

$\Rightarrow [x] > 0 \Rightarrow x \geq 1 \Rightarrow x \in (1, \infty) - I$

Hence choice (c) is correct.

21. (a, b, c, d) : (a) f is even, g is even

$\therefore\ f(-x) = f(x), g(-x) = g(x)$

$\therefore\ (fog)(-x) = f(g(-x)) = f(g(x)) = (fog)(x)$ (even)

Choice (a) is correct.

(b) f is odd, g is odd

$\therefore\ (fog)(-x) = f(g(-x)) = f(-g(x)) = -(fog)(x)\ \forall\ x$

$\therefore\ fog$ is an odd function

Choice (b) is correct.

(c) f is even, g is odd

$\therefore\ (fog)(-x) = f(g(-x)) = f(-g(x))$

$= f(g(x))$ (as $g(-x) = g(x)$) $= fog(x)\ \forall\ x$

$\therefore\ fog$ is an even function

$\therefore$ Choice (c) is correct.

(d) $(fog)(-x) = f(g(-x)) = f(g(x))$ (as g is even)

$= (fog)(x)$

$\therefore\ fog(x)$ is on even function

$\therefore$ choice (d) is correct.

Hence all choices (a), (b), (c) and (d) are correct.

22. (a, b, c,d) : As $f(x)$ is an even function defined on $[-5, 5]$

$\therefore\ f(-x) = f(x)\ \forall\ x \in [-5, 5]$

$\therefore\quad x = \dfrac{x+1}{x+2} \quad$ or $\quad x = -\left(\dfrac{x+1}{x+2}\right)$

$\Rightarrow x^2 + x - 1 = 0 \quad$ or $\quad x^2 + 3x + 1 = 0$

$\Rightarrow\quad x = \dfrac{-1 \pm \sqrt{5}}{2} \quad$ or $\quad x = \dfrac{-3 \pm \sqrt{5}}{2}$

$\therefore\quad x = \dfrac{\sqrt{5} - 1}{2} \quad$ or $\quad -\left(\dfrac{\sqrt{5} + 1}{2}\right)$

or $\dfrac{-3 + \sqrt{5}}{2}$ or $\dfrac{-3 - \sqrt{5}}{2}$

$\therefore$ all the choices (a), (b), (c) and (d) are correct.

23. (a, b, c) : (a) $f(x) = x + \sin x$

$\sin x$ is periodic function with period 2π where as x is a non periodic function

$\therefore\ f(x) = x + \sin x$ is a non periodic function

(b) $f(x) = \cos x + \{x\} = \cos x + x - [x]$

The period of $\cos x$ is 2π and period of $x - [x]$ is 1

$\therefore$ Period of $f(x)$ = LCM of periods of $\cos x$ and $\{x\}$ = LCM of $(2\pi, 1)$,

which does not exist, as 2π is an irrational number and 1 is rational and LCM of a rational number and irrational number is not possible.

$\therefore\ f(x) = \cos x + \{x\}$ is non periodic.

(c) $f(x) = \cos x^2$

Let $f(x)$ be a periodic with period T

$\therefore\ f(x + T) = f(x)\ \forall\ x \in R$

$\Rightarrow \cos(x + T)^2 = \cos x^2\ \forall\ x \in R$

$\Rightarrow (x + T)^2 = 2n\pi \pm x^2\ \forall\ x \in R$

$\Rightarrow (x + T)^2 \pm x^2 = 2n\pi\ \forall\ x \in R$

which is not possible because R.H.S. is an integral multiple of 2π where as LHS is a quadratic function of x.

$\therefore\ f(x) = \cos x^2$ is non periodic function.

(d) $f(x) = \cos\{(x + 3) - [x + 3]\}$

We know that $x - [x]$ is periodic with period 1.

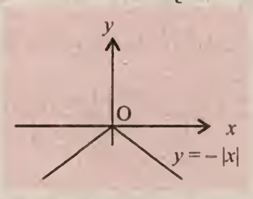
$\therefore\ (x + 3) - [x + 3]$ is a periodic function with period 1.

$\therefore\ \cos\{(x + 3) - [x + 3]\}$ is a periodic function with period 1.

$f(x)$ is periodic function.

$\therefore$ functions in options (a), (b) and (c) are non period functions.

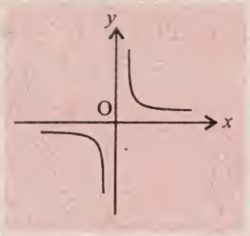
24. (a, c) : (a) $f(x) = -|x| = \begin{cases} -x \text{ if } x > 0 \\ 0 \text{ if } x = 0 \\ x \text{ if } x < 0 \end{cases}$



which is symmetrical about y-axis or $x = 0$

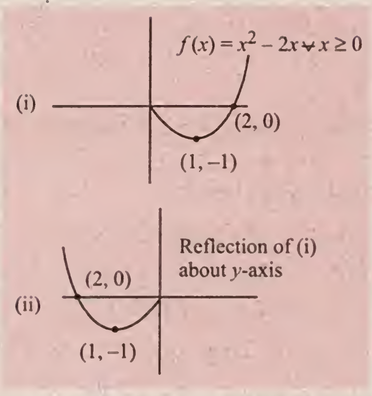
(b) $f(x) = 1/x$

Symmetrical in the opposite quadrants

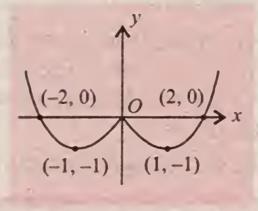


(c) $f(x) = x^2 - 2|x| = |x|^2 - 2|x| = f(|x|)$
where $f(x) = x^2 - 2x$

Now draw the graph of $f(x) = \begin{cases} x^2 - 2x \ \forall \ x \geq 0 \\ x^2 - 2x \ \forall \ x < 0 \end{cases}$

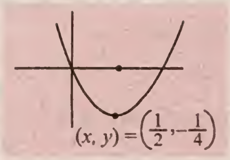


Now join the graph (i) and (ii) for the graph of $x^2 - 2|x|$



which is symmetrical about y-axis

(d) $f(x) = x^2 - x = x(x - 1)$, to draw the graph note the following
(i) x-Intercept (0, 0), (1, 0)
(ii) y-Intercept (0, 0)
(iii) $f'(x) = 0 \Rightarrow \quad x = \frac{1}{2}$ and $\frac{d^2y}{dx^2} = 2 > 0$
So point $x = 1/2$ is point of local minima.
(iv) $x < 1/2$ $f(x)$ decreases and $x > 1/2$, $f(x)$ increases



$\therefore$ graph is symmetrical about the line $x = 1/2$ but not symmetrical about y-axis

$\therefore$ Group of options (a) and (c) are symmetrical about y-axis
So, (a) and (c) are correct.

25. (a, b, c, d) : $f(x) = e^x$, $g(x) = 3x - 2$
(a) $\therefore$ $(fog)(x) = f(g(x)) = f(3x - 2) = e^{3x-2}$
(b) $(gof)(x) = g(f(x)) = g(e^x) = 3e^x - 2$
(c) let $(fog)(x) = e^{3x-2} = y \quad \Rightarrow \quad 3x - 2 = \log y$

$$\therefore \quad (fog)^{-1}(y) = \frac{2 + \log y}{3}$$

$\Rightarrow$ Domain of $(fog)^{-1}$ is $(0, \infty)$ as $\log y$ is defined only for $y > 0$

(d) $(gof)(x) = 3e^x - 2 = z$ (say)

$$\Rightarrow \quad e^x = \frac{z+2}{3} \qquad \therefore \quad x = \log\left(\frac{z+2}{3}\right)$$

as $x \in R$ and $\log\left(\frac{z+2}{3}\right)$ is defined only if $z > -2$

$\therefore$ Domain of $(gof)^{-1} = (-2, \infty)$
Hence all choices (a), (b), (c) and (d) are correct.

26. (a, b, c, d) : $f(x) = \sin[\pi^2]x + \sin[-\pi^2]x$
$f(x) = \sin[(3.1428)^2]x + \sin[-(3.1428)^2]x$
$f(x) = \sin 9x - \sin 10x$
$\therefore$ $f(-x) = -f(x)$ $\quad\therefore$ Choice (d) is correct

Again $f\left(\frac{\pi}{2}\right) = \sin\frac{9\pi}{2} - \sin 5\pi = \sin\left(4\pi + \frac{\pi}{2}\right) = 1$

$\therefore$ choice (b) is correct
Again $f'(x) = 9\cos 9x - 10\cos 10x$
$\therefore$ $f'(\pi) = 9\cos 9\pi - 10\cos 10\pi$
$= 9(-1) - 10(1) \ (\because \ \cos n\pi = (-1)^n)$
$= -19$ $\therefore$ choice (c) is correct

and period of $\sin 9x = \frac{2\pi}{9}$, period of $\sin 10x = \frac{\pi}{5}$

$\therefore$ Period of $\sin 9x - \sin 10x$ = LCM of $\left\{\frac{2\pi}{9}, \frac{\pi}{5}\right\}$

$$= \frac{\text{LCM of } (2\pi, \pi)}{\text{HCF of } (9, 5)} = 2\pi$$

$\therefore$ choice (a) is correct.
$\therefore$ All the choices (a), (b), (c), (d) are correct.

27. (a, b, d) : As $f(x)$, $g(x)$ are mirror images of each other about the line $y = a$, $f(x)$ and $g(x)$ are equidistant from the line $y = \lambda$. Let for some particular value of x say $x = x_0$
$f(x_0) = \lambda + k$, $g(x_0) = \lambda - k$
Now $h(x_0) = f(x_0) + g(x_0) = 2\lambda$
$\therefore$ $h(x) = 2\lambda \ \forall \ x \in R$
$\Rightarrow$ $h(x)$ is a constant function $\therefore$ choice (b) is false
By the fact that every constant function is many one function, and many one function cannot be one-one. Again $h(x)$ is constant function and co-domain of

$h(x) \neq$ Range of $h(x)$. So $h(x)$, cannot be onto and even function which is not onto, must be into function.
Hence (a), (b) and (d) are correct choices.

28. (a, b) : $f(x+2) = P(x)f(x+1) + Q(x)f(x)$ (given) ...(A)

$f(x) = 1! + 2! + ... + x!$

$\therefore\ f(x+2) - f(x+1) = (x+2)! = (x+2)(x+1)!$

$= (x+2)\{f(x+1) - f(x)\}$

$\therefore\ f(x+2) = (x+3)f(x+1) - (x+2)f(x)$ .... (B)

from (A) and (B) we have

$P(x) = x+3$ and $Q(x) = -(x+2)$

Hence choices (a) and (b) are correct.

29. (b, c) : $f(x) = \sqrt{3}\sin x - \cos x + 2 = 2\sin\left(x - \dfrac{\pi}{6}\right) + 2$

let $x_1, x_2 \in x$

$\therefore\ f(x_1) = f(x_2) \Rightarrow x_1 = x_2 \Rightarrow f$ is one-one and co-domain of f = range of f

$\therefore\ f(x)$ is one-one and onto so f^{-1} exist

$\therefore\ fof^{-1} = x$

$$\Rightarrow\quad 2\sin\left(f^{-1}(x) - \frac{\pi}{6}\right) + 2 = x$$

$$\Rightarrow\quad \sin\left(f^{-1}(x) - \frac{\pi}{6}\right) = \frac{x-2}{2} = \frac{x}{2} - 1$$

$$\Rightarrow\quad f^{-1}(x) = \sin^{-1}\left(\frac{x}{2} - 1\right) + \frac{\pi}{6}$$

$$\left(\because\ \left|\frac{x}{2} - 1\right| \leq 1 \ \forall\ x \in [0,4]\right)$$

$\therefore$ choice (b) is correct

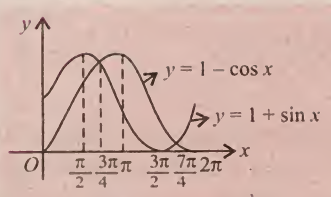
$$= \frac{\pi}{2} - \cos^{-1}\left(\frac{x}{2} - 1\right) - \frac{\pi}{6} = \frac{2\pi}{3} - \cos^{-1}\left(\frac{x}{2} - 1\right)$$

i.e., choice (c) is correct

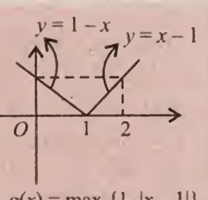
Hence (b) and (c) are correct answer.

30. (a, b) : $f(x) = \max\ \{1 + \sin x, 1, 1 - \cos x\}$

$$= \begin{cases} 1 + \sin x, & 0 \leq x \leq \dfrac{3\pi}{4} \\ 1 - \cos x, & \dfrac{3\pi}{4} \leq x \leq \dfrac{3\pi}{2} \\ 1, & \dfrac{3\pi}{2} \leq x \leq 2\pi \end{cases}$$



$f(x)$



$g(x) = \max\ \{1, |x-1|\}$

$$g(x) = \max\ \{1, |x-1|\} = \begin{cases} 1 - x, & \text{if } x \leq 0 \\ 1, & \text{if } 0 \leq x \leq 2 \\ x - 1, & \text{if } x \geq 2 \end{cases}$$

$\therefore\ f(0) = 1 \qquad \Rightarrow (gof)(0) = 1.$

and $f(1) = 1 + \sin 1. \qquad \left(\because\ 0 < 1 < \dfrac{3\pi}{4}\right)$

$\Rightarrow g(f(1)) = 1 \qquad (\because\ 1 < 1 + \sin 1 < 2)$

$\therefore$ choice (a) is correct

Further $g(1) = 1 \qquad \Rightarrow f(g(1)) = 1 + \sin 1$

and $g(0) = 1 \qquad \Rightarrow f(g(0)) = 1 + \sin 1$

$\therefore$ choice (b) correct.

Hence (a) and (b) are correct choices.

31. (a, b, c, d) : $f(x) = |1 - x| + |1 + x|\ \forall\ -2 \leq x \leq 2$

Putting $1 - x = 0 \qquad 1 + x = 0$

$\therefore\quad x = -1, 1$

Now divide the interval $-2 \leq x \leq 2$ keeping in mind the points $x = -2, -1, 1\ 2$

$\therefore\ f(x) = \{|1 - x| + |1 + x|$

$$= \begin{cases} (1-x) - (1+x) \text{ if } -2 \leq x \leq -1 \\ (1-x) + (1+x) \text{ if } -1 \leq x \leq 1 \\ -(1-x) + 1 + x \text{ if } 1 \leq x \leq 2 \end{cases}$$

$$\therefore\ f(x) = \begin{cases} -2x, & \text{if } -2 \leq x \leq -1 \\ 2(\text{constant}), & -1 \leq x \leq 1 \\ 2x, & \text{if } 1 \leq x \leq 2 \end{cases}$$

Graph of $f(x)$ lies in the first and second quadrant because $f(x)$ is the sum of two modulus functions, the range of $f(x)$ is set of $R^+ \cup \{0\}$ which lies in the first and second quadrants, $\forall$ value of $x \in R$

Hence choices (a), (b), (c) and (d) are correct.

Linked Comprehension type

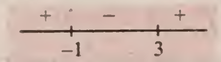
Passage - 1.

$f(x) = x^2 - 2x - 3$

$\Rightarrow f(|x|) = |x|^2 - 2|x| - 3 = x^2 - 2|x| - 3 = g(x)$

$\Rightarrow |f(|x|)| = |x^2 - 2|x| - 3| = h(x)$

(i) (a): $f(x) = x^2 - 2x - 3 = (x+1)(x-3) > 0$



$\Rightarrow f(x) > 0\ \forall\ (-\infty, -1) \cup (3, \infty)$

$\Rightarrow R - [-1, 3]$

Hence (a) is correct answer

(ii) (c): $f(x) = x^2 - 2x - 3$

minimum value occurs at vertex $V\left(-\dfrac{b}{2a}, -\dfrac{D}{4a}\right)$

$\therefore$ point of minima $= -\dfrac{b}{2a} = \dfrac{2}{2} = 1$

Contd. on page no. 75

equal to

(a) $\dfrac{\vec{b}' \times \vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$ (b) $-\dfrac{\vec{b}' \times \vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']}$

(c) $\dfrac{\vec{b}' \times \vec{c}'}{[\vec{a}'\ \vec{b}'\ \vec{c}']^2}$ (d) none of these

$M_{17\text{-}19}$: Paragraph for question No. 17 to 19

Let ABC be a triangle. R be the circumradius of the triangle. Also given $R^2 = \frac{1}{8}(a^2+b^2+c^2)$. Then,.

17. Which of the following is/are true?

(a) $\sum \cos 2A = -1$ (b) $\sum \cos 2A = 1$
(c) $\sum \sin 2A = 1$ (d) $\sum \sin 2A = -1$

18. Hence the triangle ABC can be

(a) equilateral (b) isosceles
(c) scalene (d) none of these

19. Further, we have

(a) $r + 2R = 2s$ (b) $r - 2R = s$
(c) $r + 2R = s$ (d) none of these

SECTION - IV

Matrix-Match type

This section contains 3 question. Each question contains statements given in two columns which have to be matched. Statements (p, q, r, s) in Columns II have to be matched with statements (A, B, C, D) in Column I.

20.

	Column I	Column II
(A)	The no. of solutions of the equation $\sin(e^x) = 5^x + 5^{-x}$ is	(p) 8
(B)	The number of solutions of the equation $\tan x + \sec x = 2\cos x$ lying in the interval $[0, \pi]$ is	(q) 6
(C)	The no. of values of x in the interval $[0, 5\pi]$ satistying the equation $3\sin^2 x - 7\sin x + 2 = 0$ is	(r) 2
(D)	The number of integral values of K, for which the equation $7\cos x + 5\sin x = 2K + 1$ has a solution is	(s) 0

21.

	Column I	Column II
(A)	If $f(x) = \sin^2 x + \sin^2\left(x+\frac{\pi}{3}\right) + \cos x \cos\left(x+\frac{\pi}{3}\right)$ and $g\left(\frac{5}{4}\right) = 1$ then $gof\left(\frac{\pi}{8}\right) =$	(p) 3
(B)	Let $f_n(\theta) = \tan\frac{\theta}{2}(1+\sec\theta)(1+\sec 2\theta)(1+\sec 4\theta)\ldots(1+\sec 2^n\theta)$ then $f_2\left(\frac{\pi}{16}\right) f_3\left(\frac{\pi}{32}\right) =$	(q) $-\sqrt{\frac{3}{2}}$
(C)	If $\cot(\theta-\alpha)$, $3\cot\theta$, $\cot(\theta+\alpha)$ are in A.P. and θ is not an integral multiple of $\pi/2$, then $\sin\theta \operatorname{cosec}\alpha$ is equal to	(r) 1
(D)	Number of ordered pairs (a, x) satisyfing the equation $\sec^2(a+2)x + a^2 - 1 = 0, -\pi < x < \pi$ is	(s) $\sqrt{\frac{3}{2}}$

22.

	Column I	Column II
(A)	If $\vec{a}, \vec{b}, \vec{c}$ are non coplanar unit vectors such that $\vec{a} \times (\vec{b} \times \vec{c}) = \frac{\vec{b}+\vec{c}}{\sqrt{2}}$, then the angle between $\vec{a}$ and $\vec{b}$ is	(p) $\frac{\pi}{3}$
(B)	Four vectors $\vec{a}, \vec{b}, \vec{c}, \vec{d}$ such that $(\vec{a}\times\vec{b})\times(\vec{c}\times\vec{d}) = 0$. Let P_1 and P_2 be planes determined by the pairs of vectors $\vec{a}, \vec{b}$ and $\vec{c}, \vec{d}$ respectively, then the angles between P_1 and P_2 is	(q) $\frac{3\pi}{4}$
(C)	If $\vec{a}$ and $\vec{b}$ are two unit vectors such that $\vec{a} + 2\vec{b}$ and $5\vec{a} - 4\vec{b}$ are perpendicular to each other then the angles between $\vec{a}$ and $\vec{b}$ is	(r) $\frac{\pi}{6}$
(D)	If $\lvert\vec{a}\rvert = 3, \lvert\vec{b}\rvert = 5, \lvert\vec{c}\rvert = 7$ and $\vec{a}+\vec{b}+\vec{c} = 0$. The angle between $\vec{a}$ and $\vec{b}$ is	(s) 0

ANSWER

Paper I

1. (d) 2. (a) 3. (b) 4. (a) 5. (a) 6. (d) 7. (a)
8. (a) 9. (a) 10. (c) 11. (d) 12. (c) 13. (a) 14. (d)
15. (c) 16. (b) 17. (b), (c) 18. (d) 19. (a)
20. (A) → (r), (B) → (p), (C) → (q), (D) → (s)
21. (A) → (q), (B) → (r), (C) → (s), (D) → (p)
22. (A) → (s), (B) → (p), (C) → (q), (D) → (r)

Paper II

1. (a) 2. (a) 3. (a) 4. (d) 5. (d) 6. (c) 7. (b)
8. (d) 9. (c) 10. (c) 11. (a) 12. (a) 13. (d) 14. (d)
15. (d) 16. (a) 17. (a) 18. (b), (c) 19. (c)
20. (A) → (s), (B) → (r), (C) → (q), (D) → (p)
21. (A) → (r), (B) → (r), (C) → (q), (s), (D) → (p)
22. (A) → (q), (B) → (s), (C) → (p), (D) → (p)

CHINESE Olympiad Problems

SECTION I

1. For each positive integer n, the parabola $y=(n^2+n)x^2-(2n+1)x+1$ cuts the x-axis at the points A_n and B_n. What is the value of

$$\sum_{n=1}^{1992} A_nB_n\ ?$$

(a) $\frac{1991}{1992}$ (b) $\frac{1992}{1993}$ (c) $\frac{1991}{1993}$ (d) $\frac{1993}{1992}$

2. What is the equation of the curve which consists of the unit circle minus the part in the second quadrant ?

(a) $(x+\sqrt{1-y^2})(y+\sqrt{1-x^2})=0.$

(b) $(x-\sqrt{1-y^2})(y-\sqrt{1-x^2})=0.$

(c) $(x+\sqrt{1-y^2})(y-\sqrt{1-x^2})=0.$

(d) $(x-\sqrt{1-y^2})(y+\sqrt{1-x^2})=0.$

3. Let S be the largest of the areas of the four faces of a tetrahedron and T be the total surface area of the tetrahedron. Which of the following statements is true ?

(a) $2<\frac{T}{S}\le 4.$ (b) $3<\frac{T}{S}<4.$

(c) $\frac{5}{2}<\frac{T}{S}\le\frac{9}{2}.$ (d) $\frac{7}{2}<\frac{T}{S}<\frac{11}{2}.$

4. In triangle ABC, $CA=b\neq 1$. Both $\frac{C}{A}$ and $\frac{\sin B}{\sin A}$ are roots of $\log_{\sqrt{b}}x=\log_b(4x-4)$. Which of the following statements is true ?

(a) ABC is isosceles but not a right triangle.

(b) ABC is a right triangle but not isosceles.

(c) ABC is an isosceles right triangle.

(d) ABC is not isosceles and not a right triangle.

5. In the complex plane, O is the origin, and points A and B are represented by the complex numbers z and ω respectively. If $|z|=4$ and $4z^2-2z\omega+\omega^2=0$, what is the area of triangle OAB ?

(a) $8\sqrt{3}$ (b) $4\sqrt{3}$ (c) $6\sqrt{3}$ (d) $12\sqrt{3}$

6. The function $f(x)$ satisfies $f(10+x)=f(10-x)$ and $f(20-x)=-f(20+x)$. Which of the following statements about $f(x)$ is true ?

(a) It is a periodic even function.

(b) It is an even function but not periodic.

(c) It is a periodic odd function.

(d) It is an odd function but not periodic.

SECTION II

7. Let x, y and z be real numbers such that $3x$, $4y$ and $5z$ form a geometric progression while $1/x$, $1/y$ and $1/z$ form an arithmetic progression. What is the value of

$$\frac{x}{z}+\frac{z}{x}\ ?$$

8. How many roots of the equation $\cos 7x=\cos 5x$ lie in the interval $[0,\pi]$?

9. From the sides and face diagonals of a cube, at most how many can be chosen so that every two of the chosen ones are a pair of skew lines ?

10. Let z_1 and z_2 be complex numbers such that $|z_1|=3$, $|z_2|=5$ and $|z_1+z_2|=7$. What is the value of arg $\frac{z_2^3}{z_1^3}$?

CHINESE Olympiad Problems

SOLUTIONS

SECTION I

1. **(b)** : The roots of
$(n^2+n)x^2-(2n+1)x+1=0$
are $\frac{1}{n}$ and $\frac{1}{n+1}$. Hence
$$\sum_{n=1}^{1992} A_nB_n=\sum_{n=1}^{1992}\left(\frac{1}{n}-\frac{1}{n+1}\right)=1-\frac{1}{1993}=\frac{1992}{1993}.$$

2. **(d)** : The curve is the union of the right half and bottom half of the unit circle. Their respective equations are $x-\sqrt{1-y^2}=0$ and $y+\sqrt{1-x^2}=0$. Hence the equation of the curve is
$$(x-\sqrt{1-y^2})(y+\sqrt{1-x^2})=0.$$

3. **(a)** : Clearly, $T\le 4S$ or $\frac{T}{S}\le 4$, and equality holds for the regular tetrahedron. On the other hand, since the sum of the areas of any three faces of a tetrahedron is greater than that of the fourth, $\frac{T}{S}>2$. To show that this cannot be improved, consider a tetrahedron $ABCD$ in which the projection of A onto BCD is inside that triangle and the length of the altitude from A to BCD is arbitrarily small. Then the area of the face BCD is S and the total area of the other three faces is arbitrarily close to S. For such a tetrahedron, $\frac{T}{S}$ is arbitrarily close to 2.

4. **(b)** : The equation
$$\log_b x^2=\log_{\sqrt{b}} x=\log_b(4x-4)$$
simplifies to $x^2-4x+4=0$, which has double roots $x=2$. Hence $C=2A$ and $\sin B=2\sin A$. Now
$$3A+B=A+B+C=180°.$$
It follows that
$$2\sin A=\sin B=\sin 3A=3\sin A-4\sin^3 A.$$
This may be rewritten as
$$\sin A(1-2\sin A)(1+2\sin A)=0.$$
Since $\sin A>0$, we have $\sin A=\frac{1}{2}$. It follows that $A=30°$, $C=60°$ and $B=90°$.

5. **(a)** : The vector AB is represented by the complex number $\omega-z$. We have $(\omega-z)^2+3z^2=0$ so that $\omega-z=\pm\sqrt{3}zi$. It follows that AB and OA are perpendicular to each other. Hence
$$\text{area of }\Delta OAB=\frac{1}{2}|OA||AB|=\frac{1}{2}|z|\left|\sqrt{3}z\right|=8\sqrt{3}.$$

6. **(c)** : From the first given condition, we have
$$f(10+(10-x))=f(10-(10-x))$$
or $\quad f(20-x)=f(x)$
Similarly, we have
$$-f(20+x)=f(x)$$
from the second. Hence
$f(x)=-f(20+x)=f(20+(20+x))=f(40+x)$,
and $f(x)$ is periodic. Moreover,
$$f(-x)=f(20+x)=-f(x),$$
and $f(x)$ is an odd function.

SECTION II

7. From
$$\frac{2}{y}=\frac{1}{x}+\frac{1}{z},$$
we have
$$y=\frac{2xz}{x+z}$$
Substituting into $16y^2=15xz$, we have
$$\frac{64xz}{(x+z)^2}=15.$$
Hence
$$\frac{x}{z}+\frac{z}{x}=\frac{(x+z)^2}{xz}-2=\frac{34}{15}.$$

8. The equation may be rewritten as
$0=\cos 7x-\cos 5x=-2\sin 6x\sin x.$
Now $\sin x=0$ if $\sin 6x=0$. Hence the general solution is $x=n\frac{\pi}{6}$ for any integer n. Within $[0,\pi]$, there are 7 solutions corresponding to $0\le n\le 6$.

9. Skew lines do not intersect. Hence the 8 vertices of the cube can determine at most 4 mutually skew lines. On each of two opposite faces, take a diagonal such that the two are skew lines. The other 4 vertices of the cube determine any such pair of face diagonals. It is easy to see that these are 4 mutually skew lines.

10. Let O, Z_1, Z_2 and Z_3 represent 0, z_1, z_2 and z_1+z_2 on the complex plane. By the Cosine Formula,
$$\cos OZ_2Z_3=\frac{3^2+5^2-7^2}{2\cdot 3\cdot 5}=-\frac{1}{2}.$$
Hence
$$\angle OZ_2Z_3=\frac{2\pi}{3}.$$
Now
$$\arg\frac{z_2}{z_1}=\angle Z_2OZ_1=\pi-\angle OZ_2Z_3=\frac{\pi}{3}\text{ or }\frac{5\pi}{3}.$$
It follows that
$$\arg\frac{z_2^3}{z_1^3}=3\arg\frac{z_2}{z_1}=\pi.$$

10 BEST PROBLEMS

Definite Integrals

1. If $f(x) = f(a+b-x)$, then what is the value of $\int_a^b x f(x)\, dx$?

2. $\int_2^5 ([x]+[-x])\, dx = ?$

3. If $f: R \to R$ is continuous and differentiable function such that

$$\int_{-1}^{x} f(t)\, dt + f'''(3)\int_x^0 dt = \int_1^x t^3\, dt - f'(1)\int_0^x t^2 dt + f''(2)\cdot\int_x^3 t\, dt,$$

then find the value of $f'(4)$.

4. Evaluate $\int_0^{2\pi} (\cos x)^{2001} dx$.

5. If $f(5-x) = f(x)$, then evaluate $\int_2^3 x\, f(x)\, dx$.

6. Evaluate $\int_5^{10} (x-4)\, dx$.

7. If $g''(x)$ is continuous for all 'x', $g(0) = g'(1) = 1$ and $\int_0^1 x\, g''(x)\, dx$ vanishes, then find $g(1)$.

8. Evaluate $\int_{-3}^{7} \frac{\lfloor x \rfloor}{x}\, dx$.

9. If 'a' is a positive integer, then find the number of solution of inequation

$$\int_0^{\frac{\pi}{2}} \left\{a^2\left[\frac{\cos(3x)}{4}+\frac{3}{4}\cos x\right] + a\sin x - 20\cos x\right\} dx \le \frac{-a^2}{3}.$$

10. If $f(x)$ satisfies $f'(x) = f(x)$, $f(0) = 1$ and $f(x) + g(x) = x^2$, then find

$$\int_0^1 f(x)\, g(x)\, dx.$$

SOLUTIONS

1. Let $I = \int_a^b x f(x)\, dx$ ...(1)

Again $I = \int_a^b (a+b-x)\, f(a+b-x)dx$

$= \int_a^b (a+b-x)\, f(x)dx$ ...(2)

Adding (1) & (2) $[\because$ given $f(x) = f(a+b-x)]$

$2I = \int_a^b (a+b)\, f(x)\, dx$

$\therefore\ I = \frac{a+b}{2}\int_a^b f(x)\, dx$

this can be used as a standard result

e.g. (i) $\int_0^\pi x\sin^3 x\, dx = \frac{\pi}{2}\cdot\int_0^\pi \sin^3 x\, dx = \frac{\pi}{2}\cdot\frac{4}{3} = \frac{2\pi}{3}.$

[Because here $f(x) = \sin^3 x$

$a = 0, b = \pi$

$f(a+b-x) = f(\pi - x) = \sin^3(\pi - x) = \sin^3 x = f(x)$]

2. Given integral

$$= \int_2^3 [x]dx + \int_3^4 [x]dx + \int_4^5 [x]dx + \int_2^3 [-x]dx + \int_3^4 [-x]dx + \int_4^5 [-x]dx$$

$$= \int_2^3 2\, dx + \int_3^4 3\, dx + \int_4^5 4\, dx + \int_2^3 -3\, dx + \int_3^4 -4\, dx + \int_4^5 -5\, dx$$

$$= \int_2^3 (-1)\, dx + \int_3^4 (-1)\, dx + \int_4^5 (-1)\, dx$$

$= -3.$

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3. Diff. the given equation by Leibnitz's rule.

We get $f(x)-f'''(3)=x^3-f'(1)\cdot x^2-f''(2)\cdot x$

Again differentiating

$f'(x)-0=3x^2-2x f'(1)-f''(2)$

$\therefore\ f'(4)=48-8f'(1)-f''(2).$

4. $I=\int_0^{2\pi}(\cos x)^{2001}\,dx$

$=2\int_0^{\pi}(\cos x)^{2001}\,dx \quad [\because f(2\pi-x)=f(x)]$

$=2\cdot 0=0 \quad [\because f(2\tfrac{\pi}{2}-x)=-f(x)]$

using properties

$$\int_0^{2a} f(x)\,dx=\begin{cases}2\cdot\int_0^a f(x)\,dx & \text{if } f(2a-x)=f(x)\\ 0 & \text{if } f(2a-x)=-f(x)\end{cases}$$

Now from this, a standard result can be written as

$\int_0^{2\pi}(\cos x)^p\,dx=0$, p is positive odd integer

and $\int_0^{2\pi}(\sin x)^p\,dx=0$, p is positive odd integer

e.g. $\int_0^{2\pi}\cos^5 x\,dx=0$ and $\int_0^{2\pi}\sin^{1001} x\,dx=0$

5. $I=\int_2^3 x\,f(x)\,dx \quad \text{...(1)}$

Again $I=\int_2^3 (2+3-x)\,f(2+3-x)\,dx$

$\left[\text{using} \int_a^b f(x)\,dx=\int_a^b f(a+b-x)\,dx\right]$

$I=\int_2^3 (5-x)\,f(5-x)\,dx$

$=\int_2^3 (5-x)\,f(x)\,dx$ (given $f(5-x)=f(x)$)

$=\int_2^3 (5-x)\,f(x)\,dx \quad \text{...(2)}$

Adding (1) and (2)

$2I=\int_2^3 5\,f(x)\,dx=5\int_2^3 f(x)\,dx$

$I=\frac{5}{2}\int_2^3 f(x)\,dx$

Aliter : using property derived in question (1)

$a=2,\ b=3,$

$\therefore\ I=\frac{2+3}{2}\int_2^3 f(x)\,dx=\frac{5}{2}\int_2^3 f(x)\,dx$

6. $\int_5^{10}(x-4)\,dx=\left[\frac{(x-4)^2}{2}\right]_5^{10}=\frac{35}{2}$

Aliter : short cut for such type of questions.

$\int_{a+c}^{b+c} f(x-c)\,dx=\int_a^b f(x)\,dx$

Here $I=\int_5^{10}(x-4)\,dx=\int_{1+4}^{6+4}(x-4)\,dx$

$\left[\because \text{ Here } f(x-c)=x-4 \Rightarrow c=4,\ a=1,\ b=6\right]$

$\Rightarrow\ I=\int_1^6 x\,dx=\left[\frac{x^2}{2}\right]_1^6=\frac{35}{2}$

Similarly $\int_3^6 \frac{1}{x-2}\,dx=\int_1^4 \frac{1}{x}\,dx=[\log|x|]_1^4$

$=\log 4$

Similarly $\int_{\frac{\pi}{2}}^{\pi}\cos\left(x-\frac{\pi}{4}\right)dx$

$=\left[\sin\left(x-\frac{\pi}{4}\right)\right]_{\frac{\pi}{2}}^{\pi}=\sin\left(\frac{3\pi}{4}\right)-\sin\left(\frac{\pi}{4}\right)=0$

using above property

Contd. on page no. 89

Contd. from page no. 18

$$c = \frac{\pi}{4} \quad \therefore \quad I = \int_{\frac{\pi}{4}+\frac{\pi}{4}}^{\frac{3\pi}{4}+\frac{\pi}{4}} \cos\left(x - \frac{\pi}{4}\right) dx$$

$$= \int_{\frac{\pi}{4}}^{3\pi/4} \cos x \, dx = [\sin x]_{\frac{\pi}{4}}^{3\pi/4}$$

$$= \sin\left(\frac{3\pi}{4}\right) - \sin\frac{\pi}{4} = 0$$

7. $\int_0^1 x\, g''(x)\, dx = [x\, g'(x)]_0^1 - \int_0^1 1 \cdot g'(x)\, dx$

$= [1 \cdot g'(1) - 0] - [g(x)]_0^1 = g'(1) - g(1) + g(0)$

Also given

$g'(1) - g(1) + g(0) = 0 \quad \therefore \quad g(1) = g'(1) + g(0)$

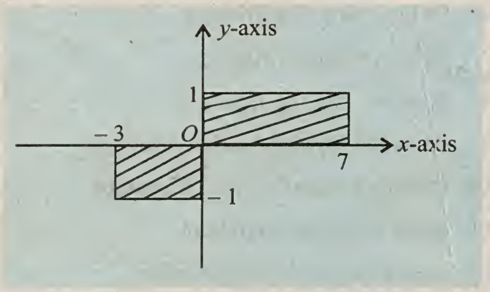
or, $g(1) = 1 + 1 = 2$

8. $I = \int_{-3}^{0} \frac{\lfloor x \rfloor}{x} dx + \int_0^7 \frac{\lfloor x \rfloor}{x} dx$

$= \int_{-3}^{0} \frac{-x}{x} dx + \int_0^7 \frac{x}{x} dx = \int_{-3}^{0} -1\, dx + \int_0^7 1\, dx$

$= [-x]_{-3}^{0} + [x]_0^7 = -3 + 7 = 4$ units.

Aliter



$$\text{sgn}(x) = \frac{\lfloor x \rfloor}{x} = \begin{cases} 1, & x > 0 \\ 0, & x = 0 \\ -1, & x < 0 \end{cases}$$

Reqd. solution $= (1 \times 7) + (3 \times -1) = 4$

9. $\int_0^{\frac{\pi}{2}} \left\{ a^2 \left[\frac{\cos 3x}{4} + \frac{3}{4}\cos x \right] + a \sin x - 20 \cos x \right\} dx \le -\frac{a^2}{3}$

$\Rightarrow \quad a^2 \left[\frac{\sin(3x)}{12} + \frac{3}{4} \sin x \right]_0^{\frac{\pi}{2}}$

$- a\,[\cos x]_0^{\frac{\pi}{2}} - 20\,[\sin x]_0^{\frac{\pi}{2}} \le -\frac{a^2}{3}$

$\Rightarrow \quad a^2 \left[-\frac{1}{12} + \frac{3}{4} - 0 \right] - a[0 - 1]$

$-20[1 - 0] \le -\frac{a^2}{3}$

$\Rightarrow a^2 + a - 20 \le 0$

$\Rightarrow (a + 5)(a - 4) \le 0$

$-5 \le a \le 4$

But 'a' should be a positive integer

$\therefore \quad a = 1, 2, 3, 4$

$\therefore$ a has four solutions.

10. Clearly $f(x) = e^x$ satisfies the given conditions

$\therefore \quad I = \int_0^1 e^x g(x)\, dx = \int_0^1 e^x [x^2 - e^x]\, dx$

$= e - \frac{e^2}{2} - \frac{3}{2}$

Aliter : given $f(x) = f'(x) \Rightarrow \frac{f'(x)}{f(x)} = 1$

$\int \frac{f'(x)}{f(x)} dx = \int 1\, dx$

$\Rightarrow \log |f(x)| = x + c$

$f(x) = e^{x+c} = e^x \cdot e^c$

Again $f(0) = 1$

$\Rightarrow 1 = e^0 \cdot e^c \Rightarrow e^c = 1$

$\therefore \; f(x) = e^x \cdot 1 = e^x$.

MATHS MUSING

Solution Sets 67

1. **(d) :** $A = \pm I, \pm \begin{bmatrix} 1 & 0 \\ c & -1 \end{bmatrix}, \begin{bmatrix} 1 & b \\ \frac{1}{b} & 0 \end{bmatrix}, \begin{bmatrix} a & b \\ \frac{1-a^2}{b} & -a \end{bmatrix}$

where a, b, c are arbitrary and $h \neq 0$.

2. **(d) :** $(2s - 3a)(2s - 3b)(2s - 3c) = 0$

Since $abc = 4Rrs$ and

$$\sum ab = s^2 + r^2 + 4Rs,$$
$$s^2 = 18Rr - 9r^2$$

3. **(c) :** $A = \sqrt{2} + \sqrt{3},\ B = \dfrac{24}{\sqrt{3} - \sqrt{2}}$

$x^2 + 24x + c = 0$

$\alpha - \beta = 8,\ \alpha + \beta = -24 \to \alpha\beta = 2^7$

4. **(b) :** The coefficient of x^8 in the expansion of $(1 + x)^{-12} + 6(1 + x)^{-10}(8x^7 - 9x^8)$. . . . is 75048 $= 2^3 \times 3 \times 53 \times 59$.

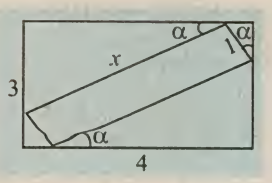
5. **(a) :**

$x \cos\alpha + \sin\alpha = 4$

$x \sin\alpha + \cos\alpha = 3$

Eliminating α,

$x^4 - 27x^2 + 48x - 24 = 0$



6. **(d) :** $\text{Area} = 8\left[\dfrac{1}{2} + \displaystyle\int_2^{\sqrt{5}} \sqrt{5 - x^2}\, dx\right]$

$= 5\left(\pi - \tan^{-1}\dfrac{24}{7}\right) - 4$

7. **(b) :** $\dbinom{5}{3} + \dbinom{6}{3} = 30$

8. **(a) :** 5! (without zero) + (5 ! – 4!) (with zero) = 216

9. **(d) :** $6! - 5! = 600$

10. **(a) – (r), (b) – (s), (c) – (s), (d) – (s)**

Let $x = 4\sin^2\theta - 1$

(a) $\displaystyle\int_0^{\pi/2} 2\, d\theta = \pi$

(b) $\displaystyle\int_0^{\pi/2} 8\sin^2\theta\, d\theta = 2\pi$

(c) $\displaystyle\int_0^{\pi/2} 8\cos^2\theta\, d\theta = 2\pi$

(d) $\displaystyle\int_0^{\pi/2} 32\sin^2\theta\cos^2\theta\, d\theta = 2\pi$

❖❖❖

SOLUTION SENDERS OF MATH MUSING – SET 67

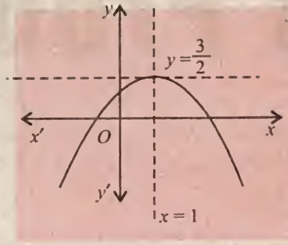
1. Shirsho Mukherjee, Kolkata
2. Kedar Nath Thakur, Darbhanga
3. Kumar Rana, Paschim Medinipur
4. Debashish Sharma, Silchar
5. Anup Kumar Kuila, Chakradharpur
6. Deepraj Jha, Darbhanga
7. Pawan Kumar, Bhatinda
8. Manisha Mishra, Balasore
9. Gottipalli Naveen, Visakhapatnam
10. Soumik Bhattacharyya, Durgapur
11. Vivek Goenka, Jhunjhunu
12. Arka Bhattacharya, Burdwan

∴ Statement 1 and Statement 2 both are true but Statement 2 is not correct explanation of Statement 1..

52. (c) : $y=-\frac{x^2}{2}+x+1$ or $2y=-x^2+2x+2$

or $2y-3=-x^2+2x-1$ or $2\left(y-\frac{3}{2}\right)=-(x-1)^2$

curve is symmetrical about $x=1$.
That is, about its axis.
Statement 1 and Statement 2 both are correct and Statement 2 is correct explanation for Statement 1.



53. (a) : Statement 1:
$\frac{PR}{RQ}=\frac{2\sqrt{2}}{\sqrt{5}}$

OR is the internal angle bisector of $\angle POQ$.

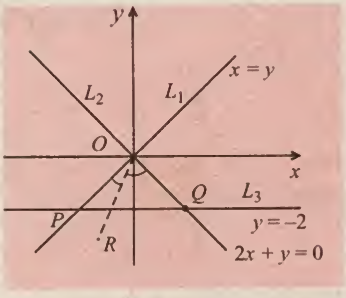
$\therefore \frac{OP}{OQ}=\frac{PR}{RQ}$

$P(-2,-2)$, $Q(1,-2)$

$\therefore OP=2\sqrt{2}$

$OQ=\sqrt{5}$

$\therefore \frac{PR}{RQ}=\frac{2\sqrt{2}}{\sqrt{5}}$



⇒ Statement 1 is correct

Statement 2 : In any triangle, bisector of an angle divides the triangle into two similar triangles. Consider ΔOPR and ΔORQ. For similarity of triangles, all the angles of ΔOPR must equal the angles of ΔORQ. But this is not the case. Hence, statement 2 is false.

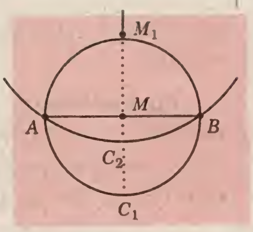
54. (c) : Area of the triangle made by the intersection points of tangents at points $A(t_1)$, $B(t_2)$ and $C(t_3)$ is

$\frac{1}{2}|t_1-t_2||t_2-t_3||t_3-t_1|\neq 0$

Hence, Assertion is wrong. Reason is correct

55. (c) : The ΔPQR form a right angled triangle, right angled at $P(0, 0)$ and QR is hypotenuse. So $P(0, 0)$ is orthocentre and the mid. point S of QR, i.e., $(1, 1)$ is circumcentre.

56. (d) : Let C_1 be a circle which passes through A and B whose diameter is AB and C_2 be another circle which passes through A and B, then centres of C_1 and C_2 must lie on perpendicular bisector of AB.



Indeed the centre of C_1 is mid-point M of AB and centre of any other circle lies somewhere else on the bisector. Then, $M_1A > AM$ (hypotenuse of right-angled triangle AMM_1)

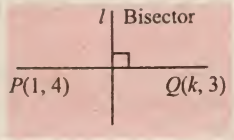
⇒ radius of $C_2 > \frac{1}{2}AB$.

⇒ C_1 is the circle whose radius is the least. Thus Statement 1 is true but does not actually follow from Statement 2 which is certainly true.

57. (a) : The slope of

$l=-\frac{1}{\text{the slope of the original line } PQ}$

$=-\frac{1}{\frac{3-4}{k-1}}=(k-1)$



The midpoint $=\left(\frac{k+1}{2},\frac{7}{2}\right)$

The equation to the bisector l is

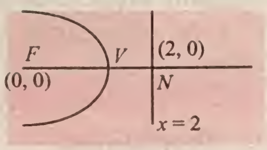
$\left(y-\frac{7}{2}\right)=(k-1)\left(x-\frac{k+1}{2}\right)$

As $x=0$, $y=-4$ satisfies it, we have

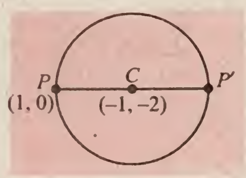
$\left(-4-\frac{7}{2}\right)=(k-1)\left(0-\frac{k+1}{2}\right) \Rightarrow -\frac{15}{2}=-\frac{k^2-1}{2}$

$\Rightarrow k^2-1=15 \Rightarrow k^2=16 \therefore k=\pm 4$.

58. (c) : The vertex is the mid point of FN, that is, vertex = (1, 0)



59. (d) : The centre C of the circle is seen to be (– 1, – 2). As C is the mid point of P and P', the coordinate of P' is given by



$P' \equiv (2\times -1-1, 2\times -2-0)$
$\equiv(-3,-4)$

60. (b) : Obviously the major axis is along the x-axis The distance between the focus and the corresponding

directrix $=\left|\frac{a}{e}-ae\right|=4$

$\Rightarrow \frac{a}{e}-ae=4$ (note that $\frac{a}{e}>ae$)

$\Rightarrow a\left(\frac{1}{e}-e\right)=4 \Rightarrow a\left(2-\frac{1}{2}\right)=4$

$\Rightarrow a\cdot\frac{3}{2}=4 \therefore a=\frac{8}{3}$

CHINESE Olympiad Problems

1. Let $a_1, a_2, \ldots, a_n$ be real numbers. Prove that the following two statements are equivalent :

$a_i + a_j \geq 0$ for any i and j with $i \neq j$;

$a_1x_1 + a_2x_2 + \ldots + a_xx_n \geq a_1x_1^2 + a_2x_2^2 + \ldots + a_nx_n^2$,

where $x_1, x_2, \ldots, x_n$ are any non-negative real numbers satisfying $x_1 + x_2 + \ldots + x_n = 1$.

2. Let $z_1, z_2, \ldots, z_n$ be complex numbers such that $|z_1| + |z_2| + \ldots + |z_n| = 1$. Prove that the sum s of a subset of $\{z_1, z_2, \ldots z_n\}$ satisfies $|s| \geq \frac{1}{6}$.

3. Is there a permutation of the numbers 1, 1, 2, 2, 3, 3, . . ., 1986, 1986 such that for any k, there are exactly k other numbers between the two k's ?

4. Each point on a plane is coloured black or white arbitrarily. Prove that there exists an equilateral triangle of side 1 or $\sqrt{3}$ such that all three vertices have the same colour.

SOLUTIONS

1. First Solution :

We have

$$\sum_{k=1}^{n} a_kx_k = \sum_{k=1}^{n} a_kx_k \sum_{k=1}^{n} x_k = \sum_{k=1}^{n} a_kx_k^2 + \sum_{i \neq j} (a_i + a_j)\, x_1x_j.$$

Assuming the first statement, we have

$$\sum_{i \neq j} (a_i + a_j)\, x_ix_j \geq 0.$$

It follows that $\sum_{k=1}^{n} a_kx_k \geq \sum_{k=1}^{n} a_kx_k^2$.

We now assume the second statement. For $1 \leq i < j \leq n$, let $x_i = x_j = \frac{1}{2}$ and $x_k = 0$ for $k \neq i, j$. It follows that

$$\frac{a_i + a_j}{2} \geq \frac{a_i + a_j}{4} \quad \text{or} \quad a_i + a_j \geq 0.$$

Second solution :

We use the same argument in the first solution to prove that the second statement implies the first. We now assume the first statement, and prove the second by induction on n. For $n = 2$, $x_1 = 1 - x_2$ and $x_2 = 1 - x_1$. Hence

$$(a_1x_1 + a_2x_2) - (a_1x_1^2 + a_2x_2^2) = a_1x_1\,(1 - x_1) + a_2x_2\,(1 - x_2)$$

$$= (a_1 + a_2)\, x_1x_2 \geq 0$$

Suppose the result holds for some $n \geq 2$.

Let $x_1, x_2, \ldots, x_{n+1}$ be non-negative real numbers satisfying

$$\sum_{k=1}^{n+1} x_k = 1.$$

If $x_{n+1} = 1$, then $x_k = 0$ for $1 \leq k \leq n$, and the desired conclusion follows from $a_{n+1} \geq a_{n+1}$. If $x_{n+1} < 1$, then

$$\sum_{k=1}^{n} \frac{x_k}{1 - x_{n+1}} = 1.$$

By the induction hypothesis,

$$\sum_{k=1}^{n} \frac{a_kx_k}{1 - x_{n+1}} \geq \sum_{k=1}^{n} a_k \left(\frac{x_k}{1 - x_{n+1}}\right)^2,$$

or equivalently,

$$(1 - x_{n+1}) \sum_{k=1}^{n} a_kx_k \geq \sum_{k=1}^{n} a_kx_k^2.$$

Hence

$$\sum_{k=1}^{n+1} a_kx_k = (1 - x_{n+1}) \sum_{k=1}^{n} a_kx_k + x_{n+1} \sum_{k=1}^{n} a_kx_k + a_{n+1}\, x_{n+1}\,(1 - x_{n+1}) + a_{n+1}\, x_{n+1}^2$$

$$\geq \sum_{k=1}^{n+1} a_kx_k^2 + x_{n+1} \sum_{k=1}^{n} a_kx_k + a_{n+1}\, x_{n+1} \sum_{k=1}^{n} x_k$$

$$\geq \sum_{k=1}^{n+1} a_kx_k^2$$

since $(a_k + a_{n+1})\, x_kx_{n+1} \geq 0$. This completes the induction argument.

2. First solution :

For $1 \leq k \leq n$, we may assume that $z_k \neq 0$. Then

$z_k = r_k\,(\cos\theta_k + i\sin\theta_k)$

with $r_k > 0$. We divide these n complex numbers into

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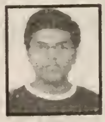

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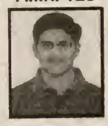

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three subsets, consisting respectively of those satisfying $60° < \theta_k \le 180°$, $-60° < \theta_k \le 60°$ and $-180° < \theta \le -60°$

It follows from the Pigeonhole Principle that for at least one of these subsets, the sum of the moduli of the complex numbers is at least 1/3. Using rotation about the origin if necessary, we may assume that this subset S consists of those satisfying $-60 < \theta_k \le 60°$. Let the complex numbers in S be $z_1, z_2, \ldots, z_m$. Then

$$|z_1 + z_2 + \ldots + z_m| \ge r_1 \cos\theta_1 + r_2 \cos\theta_2 + \ldots + r_m \cos\theta_m$$
$$\ge \frac{1}{2}(r_1 + r_2 + \ldots + r_m)$$
$$\ge \frac{1}{2} \times \frac{1}{3}$$
$$\ge \frac{1}{6}.$$

Second solution :

For $1 \le k \le n$, let $z_k = x_k + iy_k$. Then

$$1 = \sum_{k=1}^{n} |z_n|$$
$$\le \sum_{k=1}^{n} (|x_k| + |y_k|)$$
$$= \sum_{x_k \ge 0} |x_k| + \sum_{x_k < 0} |x_k| + \sum_{y_k \ge 0} |y_k| + \sum_{y_k < 0} |y_k|$$

By the Pigeonhole Principle, at least one summation in the last expression is not less than $\frac{1}{4}$. We may assume that $\sum_{x_k < 0} |x_k| \ge \frac{1}{4}$. Since all terms have the same sign,

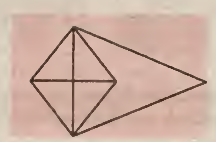
$$\left|\sum_{x_k < 0} z_k\right| \ge \left|\sum_{x_k < 0} x_k\right| = \sum_{x_k < 0} |x_k| \ge \frac{1}{4} > \frac{1}{6}.$$

3. First Solution : Suppose such a permutation exists. Paint the 3972 spaces alternately black and white. The two copies of k has exactly k spaces between them. If k is even, they will occupy spaces of different colours. Since there are 993 even pairs, they will occupy an odd number of black spaces. If k is odd, the two copies of k will occupy spaces of the same colour. Hence the 993 odd pairs will occupy an even number of black spaces. It follows that the total number of black spaces must be odd. However, this number is 1986, and we have a contradiction.

Second solution :

Suppose such a permutation exists. If a copy of a number x lies between the two copies of another number y, we say that x is pinched by y. We may have both x's pinched by the y's, both y's pinched by the x's, one x pinched by the y's and one y pinched by the x's, or no pinches among the x's and the y's. Hence the total number of pinches arising from any two pairs of numbers must be even. On the other hand, the two 1's make 1 pinch, the two 2's make 2 pinches, and so on. Thus the total number of pinches is $1 + 2 + \ldots + 1986 = 1987 \times 993$, which is odd. We have a contradiction.

4. If every two points at a distance 1 apart have the same colour, we will have an equilateral triangle of side 1 with the desired property. Hence we may assume that there exist two points A and B of different colours such that $AB = 1$. Let C be a point such that $AC = 2 = BC$. Then C is different in colour from either A or B.



We may therefore assume that there is a black point D and a white point F such that $DF = 2$. Let E be the midpoint of DF. By symmetry, we may assume that E is black. Complete the equilateral triangles DEG and DEH. If either G or H is black, we have an equilateral triangle of side 1 with three black vertices. If not, then FGH is an equilateral triangle of side $\sqrt{3}$ with three white vertices.

Olympiad Enrichment Series XI

useful for IIT-JEE 2009-10

This series is selected for their motivating, interesting and stimulating sets of quality problems, with a lucid expository style in their solution.

1. Find all functions $f: Q \to Q$ such that
$$f(x+y)+f(x-y)=2f(x)+2f(y)$$
for all $x, y \in Q$.

2. Let $\frac{3}{4}<a<1$. Prove that the equation
$x^3(x+1)=(x+a)(2x+a)$ has four distinct real solutions and find these solutions in explicit form.

3. Let a, b, and c be positive real numbers such that $abc=1$.
Prove that
$$\frac{1}{a+b+1}+\frac{1}{b+c+1}+\frac{1}{c+a+1}\le 1.$$

4. Find all functions f, defined on the set of ordered pairs of positive integers, satisfying the following properties :
$$f(x,x)=x, f(x,y)=f(y,x), (x+y)f(x,y)=yf(x,x+y).$$

SOLUTIONS

1. The only such functions are $f(x)=kx^2$ for rational k. Any such function works, since
$$f(x+y)+f(x-y)=k(x+y)^2+k(x-y)^2$$
$$=kx^2+2kxy+ky^2+kx^2-2kxy+ky^2$$
$$=2kx^2+2ky^2=2f(x)+2f(y).$$
Now suppose f is any function satisfying
$$f(x+y)+f(x-y)=2f(x)+2f(y).$$
Then letting $x=y=0$ gives $2f(0)=4f(0)$, so $f(0)=0$.
We will prove by induction that $f(nz)=n^2f(z)$ for any positive integer n and any rational number z.
The claim holds for $n=0$ and $n=1$; let $n\ge 2$ and suppose the claim holds for $n-1$ and $n-2$.
Then letting $x=(n-1)z$, $y=z$ in the given equation we obtain
$$f(nz)+f((n-2)z)=f((n-1)z+z)+f((n-1)z-z)$$
$$=2f((n-1)z)+2f(z)$$
so
$$f(nz)=2f((n-1)z)+2f(z)-f((n-2)z)$$
$$=2(n-1)^2f(z)+2f(z)-(n-2)^2f(z)$$
$$=(2n^2-4n+2+2-n^2+4n-4)f(z)=n^2f(z)$$
and the claim holds by induction.
Letting $x=0$ in the given equation gives
$$f(y)+f(-y)=2f(0)+2f(y)=2f(y),$$
so $f(-y)=f(y)$ for all rational y; thus $f(nz)=n^2f(z)$ for all integers n.
Now let $k=f(1)$; then for any rational number $x=p/q$,
$$q^2f(x)=f(qx)=f(p)=p^2f(1)=kp^2$$
so $f(x)=kp^2/q^2=kx^2$.
Thus the functions $f(x)=kx^2$, $k\in Q$, are the only solutions.

2. Look at the given equation as a quadratic equation in a:
$a^2+3xa+2x^2-x^3-x^4=0$.
The discriminant of this equation is
$9x^2-8x^2+4x^3+4x^4=(x+2x^2)^2$.
Thus
$$a=\frac{-3x\pm(x+2x^2)}{2}$$
The first choice $a=-x+x^2$ yields the quadratic equation $x^2-x-a=0$, whose solutions are
$$x=\frac{(1\pm\sqrt{1+4a})}{2}$$
The second choice $a=-2x-x^2$ yields the quadratic equation
$$x^2+2x+a=0, \text{ whose solutions are } -1\pm\sqrt{1-a}$$
The inequalities
$$-1-\sqrt{1-a}<-1+\sqrt{1-a}<\frac{1-\sqrt{1+4a}}{2}<\frac{1+\sqrt{1+4a}}{2}$$
show that the four solutions are distinct.
Indeed
$$-1+\sqrt{1-a}<\frac{1-\sqrt{1+4a}}{2}$$
reduces to
$$2\sqrt{1-a}<3-\sqrt{1+4a}$$
which is equivalent to $6\sqrt{1+4a}<6+8a$,
or $3a<4a^2$, which is evident.

3. **Alternative 1**
Setting $x=a+b$, $y=b+c$ and $z=c+a$, the inequality becomes
$$\frac{1}{x+1}+\frac{1}{y+1}+\frac{1}{z+1}\le 1,$$
i.e. $\frac{1}{y+1}+\frac{1}{z+1}\le\frac{x}{x+1}$, *i.e.* $\frac{y+z+2}{(y+1)(z+1)}\le\frac{x}{x+1}$,

i.e. $xy + xz + 2x + y + z + 2 \le xyz + xy + xz + x$,

i.e. $x + y + z + 2 \le xyz$,

i.e. $2(a + b + c) + 2 \le (a + b)(b + c)(c + a)$,

i.e. $2(a + b + c) \le a^2b + ab^2 + b^2c + bc^2 + c^2a + ca^2$.

By the AM-GM inequality,

$$(a^2b + a^2c + 1) \ge 3\sqrt[3]{a^4bc} = 3a$$

Likewise, $(b^2c + b^2a + 1) \ge 3b$

and $(c^2a + c^2b + 1) \ge 3c$.

Therefore we only need to prove that

$$2(a + b + c) + 3 \le 3(a + b + c),$$

i.e. $3 \le a + b + c$

which is evident from AM-GM inequality and $abc = 1$.

Alternative 2

Let $a = a_1^3, b = b_1^3, c = c_1^3$. Then $a_1b_1c_1 = 1$. Note that

$$a_1^3 + b_1^3 - a_1^2b_1 - a_1b_1^2 = (a_1 - b_1)(a_1^2 - b_1^2) \ge 0,$$

which implies that

$$a_1^3 + b_1^3 \ge a_1b_1(a_1 + b_1)$$

Therefore, $\dfrac{1}{a+b+1} = \dfrac{1}{a_1^3 + b_1^3 + a_1b_1c_1}$

$$\le \frac{1}{a_1b_1(a_1 + b_1) + a_1b_1c_1}$$

$$= \frac{a_1b_1c_1}{a_1b_1(a_1 + b_1 + c_1)} = \frac{c_1}{a_1 + b_1 + c_1}$$

Likewise,

$$\frac{1}{b+c+1} \le \frac{a_1}{a_1 + b_1 + c_1}$$

and $\dfrac{1}{c+a+1} \le \dfrac{b_1}{a_1 + b_1 + c_1}$

Adding the three inequalities yields the desired result.

4. We claim that $f(x, y) = \text{lcm}(x, y)$, the least common multiple of x and y. It is clear that

$$\text{lcm}(x, x) = x \quad\text{and}\quad \text{lcm}(x,y) = \text{lcm}(y, x).$$

Note that $\text{lcm}(x, y) = \dfrac{xy}{\gcd(x, y)}$

and $\gcd(x,y) = \gcd(x, x + y)$,

where $\gcd(u, v)$ denotes the greatest common divisor of u and v. Then

$$(x + y)\,\text{lcm}(x,y) = (x + y)\cdot\frac{xy}{\gcd(x,y)}$$

$$= y\cdot\frac{x(x+y)}{\gcd(x, x+y)} = y\,\text{lcm}(x, x + y).$$

Now we prove that there is only one function satisfying the conditions of the problem.

For the sake of contradiction, assume that there is another function $g(x, y)$ also satisfying the given conditions.

Let S be the set of all pairs of positive integers (x, y) such that $f(x, y) \ne g(x, y)$, and let (m, n) be such a pair with minimal sum $m + n$. It is clear that $m \ne n$, otherwise $f(m,n) = f(m,m) = m = g(m, m) = g(m, n)$.

By symmetry $f(x, y) = f(y, x)$, we can assume that $n - m > 0$.

Note that

$$nf(m, n - m) = [m + (n - m)]\,f(m, n - m)$$

$$= (n - m)\,f(m, m + (n - m)) = (n - m)\,f(m, n)$$

or $f(m, n - m) = \dfrac{n-m}{n}\cdot f(m, n)$

Likewise, $g(m, n - m) = \dfrac{n-m}{n}\cdot g(m, n)$.

Since $f(m, n) \ne g(m, n), f(m, n - m) \ne g(m, n - m)$.

Thus $(m, n - m) \in S$.

But $(m, n - m)$ has a smaller sum $m + (n - m) = n$, a contradiction. Therefore our assumption is wrong and $f(x, y) = \text{lcm}(x, y)$ is the only solution.

Trigonometric Functions

GRAPHS OF SINE AND COSINE FUNCTIONS

Basic sine and cosine curves

In this section you will study techniques for sketching the graphs of the sine and cosine functions. The graph of the sine function is a **sine curve**. In figure 1, the black portion of the graph represents one period of the function and is called **one cycle** of the sine curve.

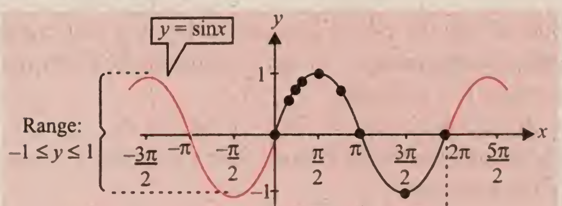


Figure : 1

The coloured portion of the graph indicates that the basic sine wave repeats indefinitely to the right and left. The graph of the cosine function is as shown in figure 2.

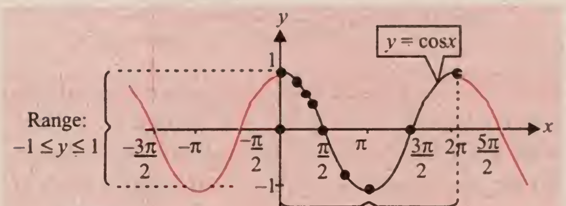


Figure : 2

x	$\sin x$	$\cos x$
0	0	1
$\pi/6$	$1/2$	$\sqrt{3}/2$
$\pi/4$	$\sqrt{2}/2$	$\sqrt{2}/2$
$\pi/3$	$\sqrt{3}/2$	$1/2$
$\pi/2$	1	0
$3\pi/4$	$\sqrt{2}/2$	$-\sqrt{2}/2$
π	0	-1
$3\pi/2$	-1	0
2π	0	1

Domain of the sine and cosine functions is the set of all real numbers. Moreover, the range of each function is the interval $[-1, 1]$, and each function has a period of 2π. The sine graph is symmetric with respect to the origin, whereas the cosine graph is symmetric with respect to the y-axis. These properties of symmetry follow from the fact that the sine function is odd whereas the cosine function is even.

Example 1 : Sketch the graph of $y = 2\sin x$ on the interval $[-\pi, 4\pi]$

Soln.: Note that $y = 2\sin x = 2(\sin x)$ indicates that the y-values for the key points will have twice the magnitude of the graph of $y = \sin x$. Divide the period 2π into four equal parts to get the following key points for $y = 2\sin x$.

$$(0, 0), \left(\frac{\pi}{2}, 2\right), (\pi, 0), \left(\frac{3\pi}{2}, -2\right) \text{ and } (2\pi, 0)$$

By connecting these key points with a smooth curve and extending the curves in both directions over the interval $[-\pi, 4\pi]$, you obtain the graph shown in figure 3.

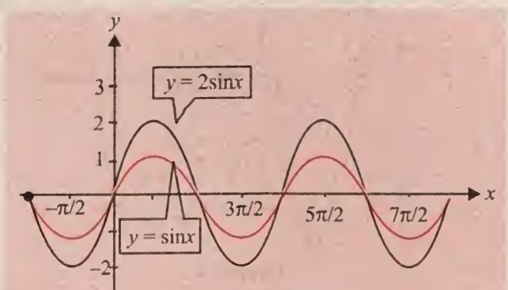


Figure : 3

Amplitude and period of sine and cosine curves

In the rest of this section you will study the graphic effect of each of the constants a, b, c, and d in equations of the forms

$$y = d + a\sin(bx - c) \text{ and } y = d + a\cos(bx - c).$$

The constant factor a in $y = a\sin x$ acts as a *scaling factor* - a *vertical stretch* or *vertical shrink* of the basic sine curve. If $|a| > 1$, the basic sine curve is stretched, and if $|a| < 1$, the basic sine curve is shrunk. The result is that the graph of $y = a\sin x$ ranges between $-a$ and a instead of between -1 and 1. The absolute value of a is the **amplitude** of the function $y = a\sin x$. The range of the function $y = a\sin x$ is $-a \le y \le a$.

Definition of amplitude of sine and cosine curves : The amplitude of $y = a\sin x$ and $y = a\cos x$ is the largest value of y and is given by

Amplitude $= |a|$.

Example 2 : On the same coordinate axes, sketch the graphs of $y=\frac{1}{2}\cos x$ and $y=3\cos x$.

Soln.: Because the amplitude of $y=\frac{1}{2}\cos x$ is $\frac{1}{2}$, the maximum value is 1/2 and the minimum value is $-1/2$. Divide one cycle, $0 \le x \le 2\pi$, into four equal parts to get the key points.

$$\left(0, \frac{1}{2}\right), \left(\frac{\pi}{2}, 0\right), \left(\pi, -\frac{1}{2}\right), \left(\frac{3\pi}{2}, 0\right) \text{ and } \left(2\pi, \frac{1}{2}\right)$$

A similar analysis shows that the amplitude of $y = 3\cos x$ is 3, and the key points are

$$(0, 3), \left(\frac{\pi}{2}, 0\right), (\pi, -3), \left(\frac{3\pi}{2}, 0\right) \text{ and } (2\pi, 3)$$

The graphs of these two functions are shown in figure 4.

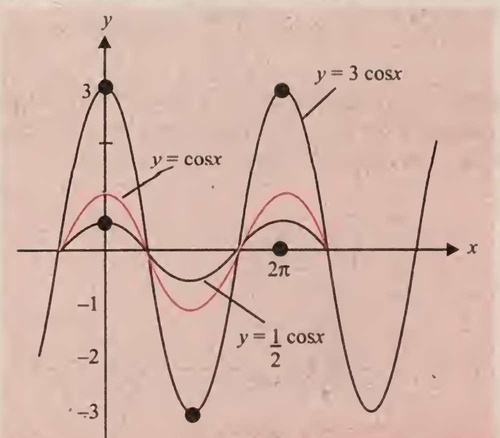


Figure : 4

The graph of $y=-f(x)$ is a reflection in the x-axis of the graph of $y=f(x)$. For instance, the graph of $y = -3\cos x$ is a reflection of the graph of $y = 3\cos x$, as shown in figure 5.

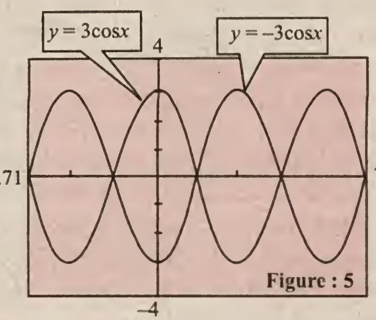


Figure : 5

Period of sine and cosine functions

Let b be a positive real number. The period of $y = a\sin bx$ and $y = a\cos bx$ is $2\pi/b$.

Note that if $0 < b < 1$, the period of $y = a\sin bx$ is greater than 2π and represents a horizontal stretching of the graph of $y = a\sin x$. Similarly, if $b > 1$, the period of $y = a\sin bx$ is less than 2π and represents a horizontal shrinking of the graph of $y = a\sin x$. If b is negative, we use the identities $\sin(-x) = -\sin x$ and $\cos(-x) = \cos x$ to rewrite the function.

Scaling - Horizontal stretching:

Example 3 : Sketch the graph of $y = \sin(x/2)$.

Soln.: The amplitude is 1. Moreover, because $b = 1/2$, the period is $\frac{2\pi}{b}=\frac{2\pi}{1/2}=4\pi$.

Now, divide the period-interval $[0, 4\pi]$ into four equal parts with the values π, 2π and 3π to obtain the following key points on the graph.

$(0, 0)$, $(\pi, 1)$, $(2\pi, 0)$, $(3\pi, -1)$ and $(4\pi, 0)$.

The graph is shown in figure 6. Use a graphing utility to confirm this graph.

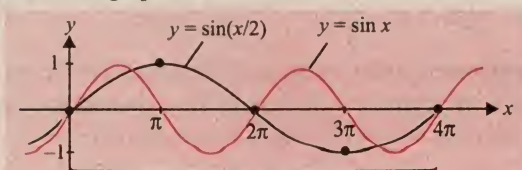


Figure : 6

Study tip : In general, to divide a period-interval into four equal parts, successively add "period/4", starting with the left end point of the interval. For instance, for the period-interval $[-\pi/6, \pi/2]$ of length $2\pi/3$, you would successively add $\frac{2\pi/3}{4}=\frac{\pi}{6}$ to get

$$-\frac{\pi}{6}, 0, \frac{\pi}{6}, \frac{\pi}{3} \text{ and } \frac{\pi}{2}.$$

Translations of sine and cosine curves

The constant c in the general equations

$$y = a\sin(bx - c) \text{ and } y = a\cos(bx - c)$$

creates *horizontal translations* (shifts) of the basic sine and cosine curves.

Comparing $y = a\sin bx$ with $y = a\sin(bx - c)$, we find that the graph of $y = a\sin(bx - c)$ completes one cycle from $bx - c = 0$ to $bx - c = 2\pi$. By solving for x, we find the interval for one cycle to be

$$\overbrace{\frac{c}{b}}^{\text{Left end point}} \le x \le \overbrace{\frac{c}{b}+\underbrace{\frac{2\pi}{b}}_{\text{Period}}}^{\text{Right end point}}$$

This implies that the period of $y = a\sin(bx - c)$ is $2\pi/b$, and graph of $y = a\sin bx$ is shifted by an amount c/b. The number c/b is the **phase shift**.

Graphs of sine and cosine functions

The graphs of $y = a\sin(bx - c)$ and $y = a\cos(bx - c)$ have the following characteristics. (Assume $b > 0$).

Amplitude $= |a|$, Period $= 2\pi/b$.

The left and right endpoints of a one-cycle interval can be determined by solving the equations $bx - c = 0$ and $bx - c = 2\pi$.

Horizontal translation:

Example 4 : Sketch the graph of $y = \frac{1}{2}\sin\left(x - \frac{\pi}{3}\right)$

Soln.: The amplitude is 1/2 and the period is 2π. By solving the equations,

$$x - \frac{\pi}{3} = 0 \quad \text{and} \quad x - \frac{\pi}{3} = 2\pi$$

$$x = \frac{\pi}{3} \qquad\qquad x = \frac{7\pi}{3}.$$

you see that the interval $[\pi/3, 7\pi/3]$ corresponds to one cycle of the graph. Dividing this interval into four equal parts produces the following key points.

$$\left(\frac{\pi}{3}, 0\right), \left(\frac{5\pi}{6}, \frac{1}{2}\right), \left(\frac{4\pi}{3}, 0\right), \left(\frac{11\pi}{6}, -\frac{1}{2}\right) \text{ and } \left(\frac{7\pi}{3}, 0\right)$$

The graph is shown in figure 7.

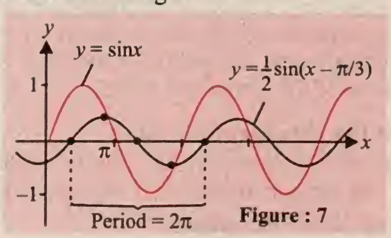


Figure : 7

Horizontal translation:

Example 5 : Analyse the graph of $y = -3\cos(2\pi x + 4\pi)$.

Soln.: The amplitude is 3 and the period is $2\pi/2\pi = 1$. By solving the equations

$$2\pi x + 4\pi = 0 \quad \text{and} \quad 2\pi x + 4\pi = 2\pi$$

$$2\pi x = -4\pi \qquad\qquad 2\pi x = -2\pi$$

$$x = -2 \qquad\qquad x = -1$$

You see that the interval $[-2, -1]$ corresponds to one cycle of the graph. Dividing this interval into four equal parts produces the following key points.

$$(-2, -3), \left(-\frac{7}{4}, 0\right), \left(-\frac{3}{2}, 3\right), \left(-\frac{5}{4}, 0\right) \text{ and } (-1, -3)$$

The graph is shown in figure 8.

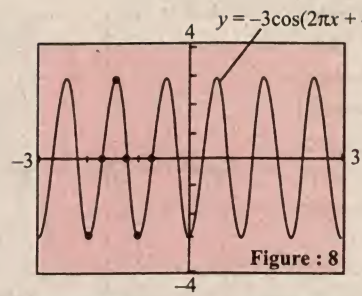


Figure : 8

The final type of transformations is the *vertical translation* caused by the constant d in the equations

$$y = d + a\sin(bx - c) \quad \text{and} \quad y = d + a\cos(bx - c)$$

The shift is d units upward for $d > 0$ and downward for $d < 0$. In other words, the graph oscillates about the horizontal line $y = d$ instead of the x-axis.

Vertical translation:

Example 6 : Analyze the graph of $y = 2 + 3\cos 2x$.

Soln.: The amplitude is 3 and the period is π. The key points over the interval $[0, \pi]$ are

$$(0, 5), \left(\frac{\pi}{4}, 2\right), \left(\frac{\pi}{2}, -1\right), \left(\frac{3\pi}{4}, 2\right) \text{ and } (\pi, 5).$$

The graph is shown in figure 9. Compared with the graph of $f(x) = 3\cos 2x$, the graph of $y = 2 + 3\cos 2x$ is shifted upward two units.

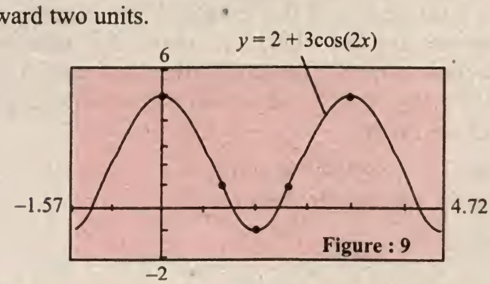


Figure : 9

GRAPHS OF OTHER TRIGNO. FUNCTIONS

Graph of the tangent function

The tangent function is odd. That is, $\tan(-x) = -\tan x$. Consequently, the graph of $y = \tan x$ is symmetric with respect to the origin. You also know from the identity $\tan x = \frac{\sin x}{\cos x}$ that the tangent is undefined when $\cos x = 0$. Two such values are $x = \pm\,\pi/2 \approx \pm\,1.5708$.

x	$\frac{-\pi}{2}$	−1.57	−1.5	−1	0	1	1.5	1.57	$\frac{\pi}{2}$
tan$x$	Undef.	−1255.8	−14.1	−1.56	0	1.56	14.1	1255.8	Undef.

tan$x$ approaches $-\infty$ as x approaches $-\pi/2$ from the right

tan$x$ approaches ∞ as x approaches $\pi/2$ from the left

As indicated in the table, tan$x$ increases without bound as x approaches $\pi/2$ from the left, and decreases without bound as x approaches $-\pi/2$ from the right.

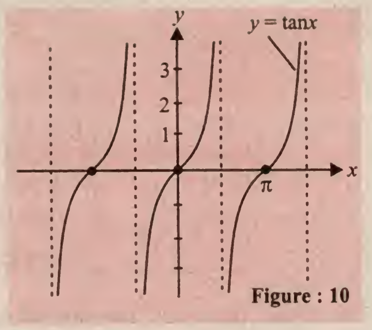


Figure : 10

Thus, the graph of $y = \tan x$ has vertical asymptotes at $x = \pi/2$ and $-\pi/2$, as shown in figure 10.

Moreover, because the period of the tangent function is π, vertical asymptotes also occur when $x = \pi/2 + n\pi$, where n is an integer. The domain of the tangent function is the set of all real numbers other than $x = \pi/2 + n\pi$, and the range is the set of all real numbers.

Sketching the graph of a function of the form $y = a\tan(bx - c)$ is similar to sketching the graph of $y = a\sin(bx - c)$ in that you locate key points that ideitify the intercepts and asymptotes. Two consecutive asymptotes can be found by solving the equations.

$$bx - c = -\pi/2 \quad \text{and} \quad bx - c = \pi/2$$

The midpoint between two consecutive asymptotes is an x-intercept of the graph. After ploting the asymptotes and the x-intercept, plot a few additional points between the two asymptotes and sketch one cycle. Finally, sketch one or two additional cycles to the left and right.

Example 7 : Sketch the graph of $y = \tan(x/2)$.

Soln.: By solving the equations,

$$\frac{x}{2} = -\frac{\pi}{2} \quad \text{and} \quad \frac{x}{2} = \frac{\pi}{2}$$

$$x = -\pi \qquad\qquad x = \pi$$

you can see that two consecutive asymptotes occur at $x = -\pi$ and $x = \pi$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table.

x	$-\pi$	$-\pi/2$	0	$\pi/2$	π
$\tan(x/2)$	Undef.	-1	0	1	Undef.

Three cycles of the graph are shown in figure 11.

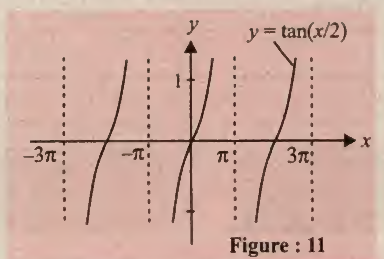


Figure : 11

Example 8 : Sketch the graph of $y = -3\tan 2x$.

Soln.: By solving the equations

$$2x = -\frac{\pi}{2} \quad \text{and} \quad 2x = \frac{\pi}{2}$$

$$x = -\frac{\pi}{4} \qquad\qquad x = \frac{\pi}{4}$$

you can see that two consecutive asymptotes occur at $x = -\pi/4$ and $x = \pi/4$. Between these two asymptotes, plot a few points, including the x-intercept, as shown in the table.

x	$-\pi/4$	$-\pi/8$	0	$\pi/8$	$\pi/4$
$-3\tan 2x$	Undef.	3	0	-3	Undef.

Three complete cycles of the graph are shown in figure 12.

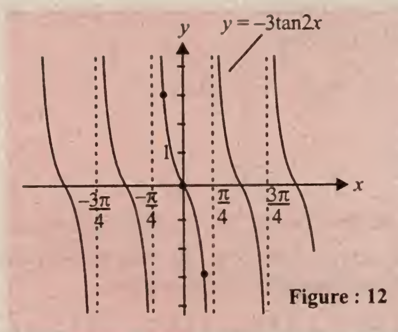


Figure : 12

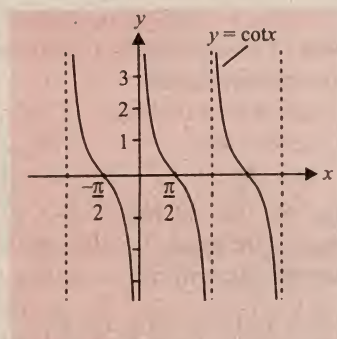
By comparing the graphs in examples 7 and 8, you can see that the graph of $y = a\tan(bx - c)$ is increasing between consecutive vertical asymptotes if $a > 0$, and decreasing between consecutive vertical asymptotes if $a < 0$. In other words, the graph for $a < 0$ is a reflection in the x-axis of the graph for $a > 0$.

Graph of the cotangent function

The graph of the cotangent function is similar to the graph of the tangent function. It also has a period of π. However, from the identity

$$y = \cot x = \frac{\cos x}{\sin x}$$

you can see that the cotangent function has vertical asymptotes $x = n\pi$, where n is an integer, because $\sin x$ is zero at these x-values. The graph of the cotangent function is shown in figure 13.



Example 9 : Sketch the graph of $y = 2\cot(x/3)$.

Soln.: To locate two consecutive vertical asymptotes

of the graph, solve the equations $x/3 = 0$ and $x/3 = \pi$, as follows.

$$\frac{x}{3}=0 \quad \text{and} \quad \frac{x}{3}=\pi$$
$$x=0 \qquad\qquad x=3\pi$$

Then, between these two asymptotes, plot a few points, including the x-intercept, as shown in the table.

x	0	$\frac{3\pi}{4}$	$\frac{3\pi}{2}$	$\frac{9\pi}{4}$	3π
$2\cot\frac{x}{3}$	Undef.	2	0	−2	Undef.

Three cycles of the graph are shown in figure 14.

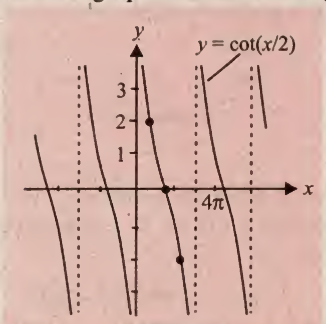


Figure : 14

Graphs of reciprocal functions

The graphs of the two remaining trigonometric functions can be obtained from the graphs of the sine and cosine functions using the reciprocal identities

$$\operatorname{cosec}x=\frac{1}{\sin x} \quad \text{and} \quad \sec x=\frac{1}{\cos x}$$

For instance, at a given value of x, the y-coordinate for $\sec x$ is the reciprocal of the y-coordinate for $\cos x$. Of course, when $\cos x = 0$, the reciprocal does not exist. Near such values of x, the behaviour of the secant function is similar to that of the tangent function. In other words, the graphs of

$$\tan x=\frac{\sin x}{\cos x} \quad \text{and} \quad \sec x=\frac{1}{\cos x}$$

have vertical asymptotes at $x = \pi/2 + n\pi$, where n is an integer and the cosine is zero at these x-values. Similarly,

$$\cot x=\frac{\cos x}{\sin x} \quad \text{and} \quad \operatorname{cosec} x=\frac{1}{\sin x}$$

have vertical asymptotes where $\sin x = 0$ - that is, $x = n\pi$.

To sketch the graph of a secant or cosecant function, we suggest that you first make a sketch of its reciprocal function. For instance, to sketch the graph of $y = \operatorname{cosec}x$, first sketch the graph of $y = \sin x$. Then take reciprocals of the y-coordinates to obtain points on the graph of $y = \csc x$. You can use this procedure to obtain the graph shown in figure 15.

Period: 2π	Period : 2π
Domain: all $x \neq n\pi$	Domain : all $x\neq\frac{\pi}{2}+n\pi$
Range: all y not in (−1 1)	Range: all y not in (−1, 1)
Vertical asymptotes: $x = n\pi$	Vertical asymptotes: $x=\frac{\pi}{2}+n\pi$
Symmetry: origin	Symmetry: y-axis.

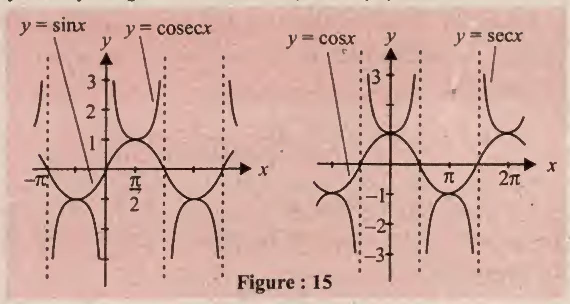


Figure : 15

In comparing the graphs of the secant and cosecant functions with those of the sine and cosine functions, note that the "hills" and "valleys" are interchanged. For example, a hill (or maximum point) on the sine curve corresponds to a valley (a local minimum) on the cosecant curve. Similarly, a valley (or minimum point) on the sine curve corresponds to a hill (a local maximum) on the cosecant curve, as shown in figure 16.

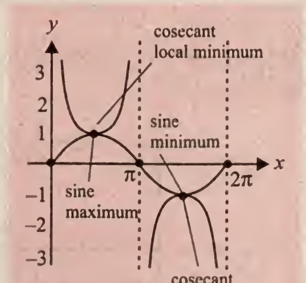


Figure : 16

Comparing trigonometric graphs:

Example 10 : Compare the graphs of

$$y=2\sin\left(x+\frac{\pi}{4}\right) \quad \text{and} \quad y=2\csc\left(x+\frac{\pi}{4}\right).$$

Soln.: The two graphs are shown in figure 17.

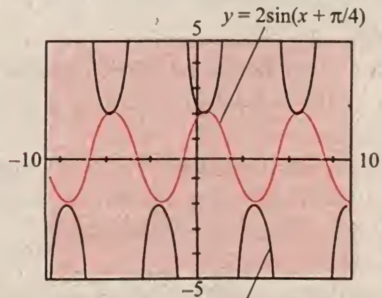


Figure : 17

Note how the "hills" and "valleys" of the graphs are related. For the function $y = 2\sin[x + (\pi/4)]$, the amplitude is 2 and the period is 2π. By solving the double inequality

$$0 < x + \frac{\pi}{4} < 2\pi \;\Rightarrow\; -\frac{\pi}{4} < x < \frac{7\pi}{4}$$

you can see that one cycle of the sine fuction corresponds to the interval from $x = -\pi/4$ to $x = 7\pi/4$. The graph of this sine function is represented by the light curve in figure 17. Because the sine function is zero at the end points of this interval, the corresponding cosecant function

$$y = 2\operatorname{cosec}\left(x + \frac{\pi}{4}\right) = 2\left(\frac{1}{\sin[x + (\pi/4)]}\right)$$

has vertical asymptotes at $x = -\pi/4, 3\pi/4, 7\pi/4$ etc. The graph of the cosecant function is represented by the black curve.

Example 11 : Compare the graphs of $y = \cos 2x$ and $y = \sec 2x$.

Soln.: Begin by graphing the two functions, as shown in figure 18.

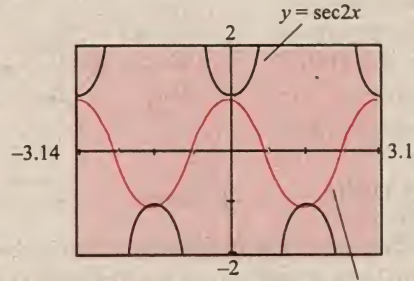


Figure : 18

Note that the x-intercepts of $y = \cos 2x$.

$$\left(\frac{\pi}{4}, 0\right), \left(\frac{3\pi}{4}, 0\right), \left(\frac{5\pi}{4}, 0\right), \ldots.$$

correspond to the vertical asymptotes

$$x = \frac{\pi}{4},\; x = \frac{3\pi}{4},\; x = \frac{5\pi}{4},\; \ldots.$$

of the graph of $y = \sec 2x$.

Damped trigonometric graphs

A product of two functions can be graphed using properties of the individual functions. For instance, consider the function $f(x) = x \sin x$

as the product of the functions $y = x$ and $y = \sin x$. Using properties of absolute value and the fact that $|\sin x| \le 1$, we have $0 \le |x|\,|\sin x| \le |x|$. Consequently,

$$-|x| \le x \sin x \le |x|$$

which means that the graph of $f(x) = x \sin x$ lies between the lines $y = -x$ and $y = x$. Furthermore, because

$$f(x) = x\sin x = \pm x \text{ at } x = \frac{\pi}{2} + n\pi$$

$$f(x) = x\sin x = 0 \text{ at } x = n\pi$$

the graph of f touches the line $y = -x$ or the line $y = x$ at $x = \pi/2 + n\pi$ and has x-intercepts at $x = n\pi$. A sketch of f is shown in figure 19. In the function $f(x) = x \sin x$, the factor x is called the damping factor.

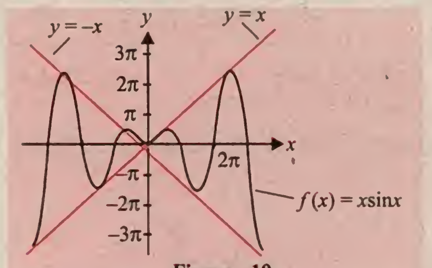


Figure : 19

Example 12 : Analyse the graph of $f(x) = e^{-x} \sin 3x$.

Soln.: Consider $f(x)$ as the product of the two functions

$$y = e^{-x} \text{ and } y = \sin 3x$$

each of which has the set of reat numbers as its domain. For any real number x, you know that $e^{-x} \ge 0$ and $|\sin 3x| \le 1$. Therefore, $|e^{-x}|\,|\sin 3x| \le e^{-x}$, which means that

$$-e^{-x} \le e^{-x} \sin 3x \le e^{-x}.$$

Furthermore, because

$$f(x) = e^{-x}\sin 3x = \pm e^{-x} \text{ at } x = \frac{\pi}{6} + \frac{n\pi}{3}$$

and $f(x) = e^{-x}\sin 3x = 0$ at $x = \frac{n\pi}{3}$

the graph of f touches the curves $y = -e^{-x}$ and $y = e^{-x}$ at $x = \frac{\pi}{6} + \frac{n\pi}{3}$ and has intercepts at $x = n\pi/3$. The graph is shown in figure 20.

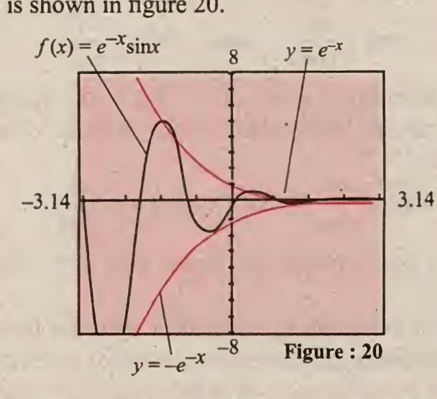


Figure : 20

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. If [.] stands for the greatest integer function, $\int_1^2 [3x]\,dx$ is equal to
   (a) 3 (b) 4 (c) 5 (d) 6

2. If $\int_{\ln 2}^{x} \frac{du}{\sqrt{e^u-1}} = \frac{\pi}{6}$, then the value of x is
   (a) 4 (b) ln 8
   (c) ln 4 (d) None of these

3. The equation of a curve is $y = f(x)$. The tangents at $(1, f(1))$, $(2, f(2))$ and $(3, f(3))$ make angles $\pi/6$, $\pi/3$ and $\pi/4$ respectively with positive direction of x-axis, then the value of
$$\int_2^3 f'(x)f''(x)dx + \int_1^3 f''(x)dx \text{ is equal to}$$
   (a) $-\frac{1}{\sqrt{3}}$ (b) $\frac{1}{\sqrt{3}}$
   (c) 0 (d) None of these

4. Let $f(x)$ be a function defined by
$$f(x) = \int_1^x t(t^2-3t+2)dt,\ 1 \le x \le 3.$$ Then the range of $f(x)$ is
   (a) [0, 2] (b) [–1/4, 4]
   (c) [–1/4, 2] (d) None of these

5. All the values of 'a' for which
$$\int_1^2 (a^2+(4-4a)x+4x^3)dx \le 12 \text{ are given by}$$
   (a) $a = 3$ (b) $a \le 4$
   (c) $0 \le a < 3$ (d) None of these

6. Evalute $\int \frac{(1+x)\sin x}{(x^2+2x)\cos^2 x-(1+x)\sin 2x}dx$

7. Show that the value of $\int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x}{\sin\left(\frac{x}{2}\right)} dx,\ n \in N,$ is independent of n. Find the value of the integral.

8. Find the orthogonal trajectories of the family of circles having their centres on the y-axis and touching the x-axis.

9. Find the area bounded by $x = \cos^{-1} y$, x-axis and the lines $|x| = 1$.

10. A family of curves intersect the hyperbola $y = 16/x$ at an angle $\pi/2$. Find the equation of the curves.

Solutions

1. **(b)** :
$$\int_1^2 [3x]dx = \int_1^{4/3} [3x]dx + \int_{4/3}^{5/3} [3x]dx + \int_{5/3}^{2} [3x]dx$$
$$= \int_1^{4/3} 3dx + \int_{4/3}^{5/3} 4dx + \int_{5/3}^{2} 5dx = 4$$

2. **(c)** : Put $e^u - 1 = t^2$. Then $e^u du = 2t\,dt$, so that
$$\frac{\pi}{6} = \int_1^{\sqrt{e^x-1}} \frac{2tdt}{t(t^2+1)} = 2\left[\tan^{-1} t\right]_1^{\sqrt{e^x-1}} = 2\tan^{-1}\sqrt{e^x-1} - \frac{\pi}{2}$$
$$\Rightarrow \tan^{-1}\sqrt{e^x-1} = \frac{\pi}{3} \Rightarrow \sqrt{e^x-1} = \tan\frac{\pi}{3}$$
$$= \sqrt{3} \Rightarrow e^x = 4 \therefore x = \ln 4.$$

3. **(a)** : Here $f'(1) = \tan\frac{\pi}{6} = \frac{1}{\sqrt{3}}, f'(2)$
$$= \tan\frac{\pi}{3} = \sqrt{3}, f'(3) = \tan\frac{\pi}{4} = 1$$
Now, $\int_2^3 f'(x).f''(x)dx = \left[\frac{1}{2}\{f'(x)^2\}\right]_2^3$

By : **Prof. Shyam Bhushan**, Director, Narayana Institute, Jamshedpur. Mobile: 09334870021

$$= \frac{1}{2}\left[\{f'(3)^2\} - \{f'(2)^2\}\right]$$

and $\int_1^3 f''(x)dx = [f'(x)]_1^3 = f'(3) - f'(1)$

$\therefore$ value $= \frac{1}{2}[1^2 - (\sqrt{3})^2] + 1 - \frac{1}{\sqrt{3}} = -\frac{1}{\sqrt{3}}$

4. **(c) :** $f'(x) = x(x^2 - 3x + 2) = x(x-1)(x-2)$.
The sign scheme for $f'(x)$ is as below.

$$\begin{array}{ccccccc} - & & + & & - & & + \\ \hline & 0 & & 1 & & 2 & \end{array}$$

$\therefore$ $f'(x) \le 0$ in $1 \le x \le 2$ and $f'(x) \ge 0$ in $2 \le x \le 3$.

$\therefore$ $f'(x)$ is decreasing in [1, 2] and increasing in [2,3]

$\therefore$ $\min f(x) = f(2) = \int_1^2 x(x^2 - 3x + 2)dx$

$$= \left[\frac{x^4}{4} - x^3 + x^2\right]_1^2 = -\frac{1}{4}$$

max. $f(x)$ = the greatest among $\{f(1), f(3)\}$.

$f(1) = \int_1^1 x(x^2 - 3x + 2)dx = 0,\ f(3) = \int_1^3 x(x^2 - 3x + 2)dx = 2$

$\therefore$ $\max f(x) = 2$, so the range $[-1/4, 2]$

5. **(a) :** $\int_1^2 (a^2 + (4-4a)x + 4x^3)dx$

$= a^2[x]_1^2 + (2-2a)[x^2]_1^2 + [x^4]_1^2$

$= a^2 + (2-2a)(3) + 15$

Given $a^2 - 6a + 21 \le 12$

$= a^2 - 6a + 9 \le 0 \Rightarrow (a-3)^2 \le 0 \Rightarrow (a-3)^2 = 0 \Rightarrow a = 3$

6. $\int(1+x)\sin x\, dx = \sin x - (1+x)\cos x + C$

$\therefore$ $I = \int \frac{(1+x)\sin x}{[\sin x - (1+x)\cos x]^2 - 1}dx$

$$= \frac{1}{2}\ln\left|\frac{\sin x - (1+x)\cos x - 1}{\sin x - (1+x)\cos x + 1}\right| + C$$

7. Let $I_n = \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x\, dx}{\sin\frac{x}{2}}$

Then $I_n - I_{n+1} = \int_0^{\pi} \frac{\sin\left(n+\frac{1}{2}\right)x - \sin\left(n+\frac{3}{2}\right)x}{\sin\frac{x}{2}}dx$

$$= -2\int_0^{\pi}\cos(n+1)x\, dx = -\frac{2}{n+1}.[\sin(n+1)x]_0^{\pi} = 0$$

$\Rightarrow I_n = I_{n+1}$ Hence $I_0 = I_1 = I_2 = I_3 = ...$

Thus $I_n = I_0 = \int_0^{\pi} dx = \pi$.

8. The family of circles is given by $x^2 + y^2 - ay = 0$, $a \in R$ .... (1)

Differentiating w.r.t.x, we have

$$2x + 2y\frac{dy}{dx} - a\frac{dy}{dx} = 0 \Rightarrow a = 2\left(x\frac{dx}{dy} + y\right)$$

Putting this value in (1), we have

$$x^2 + y^2 - 2\left(x\frac{dx}{dy} + y\right)y = 0 \Rightarrow x^2 - y^2 - 2xy\frac{dx}{dy} = 0$$

For the orthogonal trajectories we replace

$\left(\frac{dy}{dx}\right)$ by $\left(-\frac{dx}{dy}\right)$, to get

$$x^2 - y^2 + 2xy\frac{dy}{dx} = 0 \Rightarrow \frac{dy}{dx} = \frac{y^2 - x^2}{2xy}$$

This is a homogeneous differential equation in x and y.

Putting $y = Vx$, we have

$$V + x\frac{dV}{dx} = \frac{V^2x^2 - x^2}{2x \cdot Vx} = \frac{V^2 - 1}{2V}$$

$$\Rightarrow x\frac{dV}{dx} = \frac{V^2 - 1 - 2V^2}{2V} = -\frac{1 + V^2}{2V}$$

$$\Rightarrow \frac{dx}{x} = -\frac{2V}{1 + V^2}dV$$

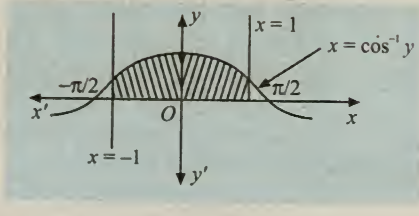
$\Rightarrow$ $\log|x| + \log(1 + V^2) = \text{constant}$

$\Rightarrow$ $|x|(1 + y^2/x^2) = \text{constant} = b$ (say)

which is required orthogonal trajectories.

9. $x = \cos^{-1} y \Rightarrow y = \cos x, x \in [0, \pi]$.

The required area (shaded portion) is shown in the adjacent figure.



Required area

$$= 2\int_0^1 \cos x\, dx = 2\sin x\Big|_0^1 = 2\sin 1 \text{ sq. units.}$$

10. For the required family of curves, let $m_1 = \frac{dy}{dx}$

and for the curve $y = 16/x$,

m_2 = value of $\frac{dy}{dx}$

$$\therefore m_2 = \frac{dy}{dx} = \frac{-16}{x^2}$$

As the family of curves intersect the hyperbola at $\frac{\pi}{2}$,

$m_1 \times m_2 = -1$

$$\Rightarrow \frac{dy}{dx}\cdot\left(\frac{-16}{x^2}\right) = -1 \Rightarrow \frac{dy}{dx} = \frac{x^2}{16}$$

$$\Rightarrow \frac{dy}{dx} = \frac{1}{16}x^2\, dx$$

Integrating both sides

$$\int dy = \frac{1}{16}\int x^2\, dx \Rightarrow y = \frac{x^3}{48} + C$$

$\therefore y = \frac{x^2}{48} + C$ is the required family of curves.

Choice of Toppers : IIT-Bombay

Delhi Retains No. 2 Slot, Madras Follows.

The cream of the country's young brains continues to hanker after IIT-Bombay. The institution has retained its position as the most sought-after IIT in the country, with Delhi and Madras a distant second and third respectively.

A number of factors have been responsible for this: from geography to gastronomy and placement records to what coaching classes tell students.

Of the top 100 JEE-2008 rankers who have been admitted to the IITs this year, more than 50% preferred IIT-B over any other IIT (see box). This was followed by Delhi – where 27 of the top 100-have been admitted. While Bombay and Delhi have maintained their positions over the years, IIT-Madras has overtaken Kanpur this year.

IIT-Centre	2008	2007	2006	2005
Bombay	54	50	46	52
Delhi	27	29	28	21
Madras	10	5	6	7
Kanpur	9	15	20	17
Kharagpur	0	1	0	3
Guwahati	0	0	0	0

There was a time when up to 30% of the top rankers chose Madras but "food" became an issue for students. "Students have often said that IIT-M does not have the kind of food that Bombay or Delhi have. But all our students are good, whether they are in the top 100 or in the ranks below," said IIT-M director M S Ananth. Food for thought, that.

Source : Joint Entrance Exam Cell, IIT

49. The length of the chord joining the points $(4\cos\theta, 4\sin\theta)$ and $(4\cos(\theta+60°), 4\sin(\theta+60°))$ of the circle $x^2+y^2=16$ is
(a) 16 (b) 2
(c) 4 (d) 8

50. The number of common tangents to the circles $x^2+y^2-y=0$ and $x^2+y^2+y=0$ is
(a) 0 (b) 1
(c) 2 (d) 3

51. The co-ordinates of the centre of the smallest circle passing through the origin and having $y=x+1$ as a diameter are
(a) $(-1, 0)$ (b) $\left(\frac{-1}{2}, \frac{1}{2}\right)$
(c) $\left(\frac{1}{2}, \frac{-1}{2}\right)$ (d) $\left(\frac{1}{2}, \frac{1}{3}\right)$

52. The length of the diameter of the circle which cuts three circles
$x^2+y^2-x-y-14=0;$
$x^2+y^2+3x-5y-10=0;$
$x^2+y^2-2x+3y-27=0$
orthogonally, is
(a) 4 (b) 2
(c) 8 (d) 6

53. For the parabola $y^2=4x$, the point P whose focal distance is 17, is
(a) (2, 8) or (2, −8) (b) (16, 8) or (16, −8)
(c) (8, 8) or (8, −8) (d) (4, 8) or (4, −8)

54. The angle between the tangents drawn to the parabola $y^2=12x$ from the point (−3, 2) is
(a) 30° (b) 45°
(c) 90° (d) 60°

55. The number of values of 'c' such that the line $y=4x+c$ touches the curve $\frac{x^2}{4}+y^2=1$ is
(a) infinite (b) 0
(c) 1 (d) 2

56. If the circle $x^2+y^2=a^2$ intersects the hyperbola $xy=c^2$ in four points $P(x_1, y_1)$, $Q(x_2, y_2)$, $R(x_3, y_3)$ and $S(x_4, y_4)$, then
(a) $y_1y_2y_3y_4=2c^4$ (b) $x_1+x_2+x_3+x_4=0$
(c) $y_1+y_2+y_3+y_4=2$ (d) $x_1x_2x_3x_4=2c^4$

57. The foot of the perpendicular from the point (2, 4) upon $x+y=4$ is
(a) (1, 3) (b) (3, −1)
(c) (2, 2) (d) (4, 0)

58. The vertices of triangle are (6, 0), (0, 6) and (6, 6). The distance between its circumcentre and centroid is
(a) 1 (b) $2\sqrt{2}$
(c) 2 (d) $\sqrt{2}$

59. The angle between the pair of lines $x^2+2xy-y^2=0$ is
(a) 0 (b) $\pi/3$
(c) $\pi/6$ (d) $\pi/2$

60. $\lim\limits_{n\to\infty}\dfrac{3\cdot 2^{n+1}-4\cdot 5^{n+1}}{5\cdot 2^n+7\cdot 5^n}=$
(a) $\frac{-20}{7}$ (b) 0
(c) $\frac{3}{5}$ (d) $\frac{-4}{7}$

SOLUTIONS

1. (d)	2. (a)	3. (a)	4. (b)	5. (b)	6. (d)
7. (a)	8. (a)	9. (c)	10. (c)	11. (a)	12. (d)
13. (c)	14. (a)	15. (c)	16. (d)	17. (a)	
18. Question is incompleted					
19. (d)	20. (b)	21. (b)	22. (c)	23. (b)	24. (c)
25. (d)	26. (b, d)	27. (c)	28. (d)	29. (d)	30. (b)
31. (b)	32. (d)	33. (b)	34. (a)	35. (d)	36. (c)
37. (c)	38. (c)	39. (a)	40. (c)	41. (c)	42. (c)
43. (a)	44. (b)	45. (d)	46. (b)	47. (b)	48. (d)
49. (c)	50. (d)	51. (b)	52. (a)	53. (b)	54. (c)
55. (d)	56. (b)	57. (a)	58. (d)	59. (d)	60. (a)

Maths Musing Solution Sender

PROBLEM Set 78

- Abhishek Maji, Burdwan (W.B.)
- Subhajit Jana, Howrah (W.B.)
- Tushant Jha, New Delhi
- Amit Kumar Tripathi, Gorakhpur (U.P.)
- Prithvijit Chakrborty, Birbhum
- Neelalja Chatterjee, Jamluk, (W.B.)
- J. Suruli Vel, Tirupati, (A.P.)
- A. Rajeev Bhatt, Warangal (A.P.)
- Ranu Vikram, Ranchi
- Satya Dav, Bangalore

OLYMPIAD CORNER
International Olympiad Problems

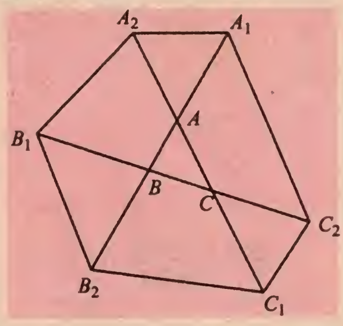
Challenging problems for Olympiads, IIT-JEE and other contests.

1. Find all polynomials $f(x) = x^n + a_1x^{n-1} + ... + a_n$ with the following properties :
(i) all the coefficients $a_1, a_2, ..., a_n$ belong to the set $\{-1, 1\}$;
(ii) all the roots of the equation $f(x) = 0$ are real.

2. Let Z denote the set of all integers. Consider a function $f: Z \to Z$ with the properties :
$$f(92+x) = f(92-x)$$
$$f(19{\cdot}92+x) = f(19{\cdot}92-x) \quad (19{\cdot}92 = 1748)$$
$$f(1992+x) = f(1992-x)$$
for all $x \in Z$. Is it possible that all positive divisors of 92 occur as values of f?

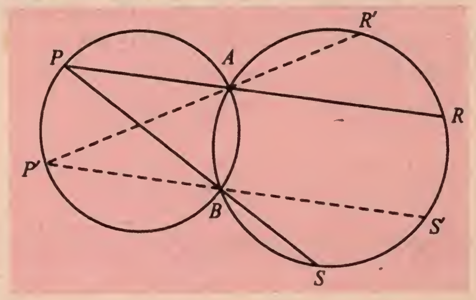
3. Prove or disprove that
$$\sqrt{5+\sqrt{21}}+\sqrt{8+\sqrt{55}} = \sqrt{7+\sqrt{33}}+\sqrt{6+\sqrt{35}}.$$

4. From the triangle T with vertices A, B and C, the hexagon H with vertices A_1, A_2, B_1, B_2, C_1, C_2 is constructed, as shown in the figure. Show that the area of H is at least thirteen times the area of T.



5. Let (a_n) and (b_n) be two sequences of integer numbers which satisfy the following conditions :
(i) $a_0 = 0, b_0 = 8$;
(ii) $a_{n+2} = 2a_{n+1} - a_n + 2$; $b_{n+2} = 2b_{n+1} - b_n$;
(iii) $a_n^2 + b_n^2$ is a perfect square, for all n.
Determine at least two possible values for the pair (a_{1992}, b_{1992}).

6. Two circles intersect at A and B. P is any point on an arc AB of one circle. The lines PA, PB intersect the other circle at R and S, as shown below. If P' is any other point on the same arc of the first circle and if R', S' are the points in which the lines $P'A$, $P'B$ intersect the other circle, prove that the arcs RS and $R'S'$ are equal.



7. Suppose that the 4th-degree polynomial $p(x)$ has three local extrema, at $x = x_0$, x_1 and x_2, so that $p(x_0) = p(x_2) = m$ and $p(x_1) = M$, where $m < M$. Let A be the area of the region bounded by $y = m$ and $y = p(x)$, and let B be the area of the region bounded by $y = p(x)$ and $y = M$. Find B/A.

8. Let ABC be a triangle and let D, E, F be points on sides BC, CA, AB respectively (different from A, B, C). If $AFDE$ is inscribable in a circle show that :
$$\frac{4ar(DEF)}{ar(ABC)} \le \left(\frac{EF}{AD}\right)^2$$

9. Prove that for each positive integer n
$(2n^2 + 3n + 1)^n \ge 6^n(n!)^2$.

10. If $a = \sin 10°$, $b = \sin 50°$, $c = \sin 70°$. Prove that
(i) $a + b = c$
(ii) $a^{-1} + b^{-1} = c^{-1} + 6$
(iii) $8abc = 1$.

SOLUTIONS

1. Let $\gamma_1, \gamma_2, ..., \gamma_n$ represent the roots of $f(x)$. Then
$$\sum_{i=1}^{n}\gamma_i^2 = \left(\sum_{i=1}^{n}\gamma_i\right)^2 - 2\sum_{i<j}\gamma_i\gamma_j = a_1^2 - 2a_2 = 1 - 2a_2. \quad ...(1)$$
By the Arithmetic Mean-Geometric Mean inequality we have
$$\frac{\Sigma\gamma_i^2}{n} \ge (\Pi\gamma_i^2)^{1/n} = (a_n^2)^{1/n} = 1$$
with equality if and only if $\gamma_i^2 = 1$ for all i.
From (1) we obtain

$1 - 2a_2 \geq n.$

This implies

1. $n \leq 3$.
2. If $n = 3, \gamma_i = \pm 1$ for all i.
3. If $n = 2, 3, a_2 = -1$.

This gives the following polynomials:

$n=1$ $\quad f(x) = x+1$ or $f(x) = x-1$

$n=2$ $\quad f(x) = x^2+x-1$ or $f(x) = x^2-x-1$

$n=3$ $\quad f(x) = (x-1)^2(x+1) = x^3+x^2-x-1$

or $f(x) = (x+1)^2(x-1) = x^3-x^2-x+1$.

The set of polynomials is

$\{x-1, x+1, x^2-x-1, x^2+x-1, x^3+x^2-x-1,$
$x^3-x^2-x+1\}$.

2. $f: Z \to Z$ satisfies

1. $f(92+x) = f(92-x)$
2. $f(1748+x) = f(1748-x)$, and
3. $f(1992+x) = f(1992-x)$.

Then, we have, $f(488+x) = f(244+244+x)$

$= f(1992-1748+244+x)$

$= f(1992+1748-244-x)$, by (3)

$= f(1748+1992-244-x)$

$= f(1748-1992+244+x)$, by (2)

$= f(x)$ ....(4)

Then, we have, $f(40+x) = f(1992-4\cdot 448+x)$

$= f(1992+x)$, by repeated application of (4),

$= f(1992-x)$, by (2)

$= f(1992-4\cdot 448-x)$, by repeated application of (4)

$= f(40-x)$ ....(5)

So, $f(104+x) = f(52+52+x) = f(92-40+52+x)$

$= f(92+40-52-x)$ by (1)

$= f(40+92-52-x)$

$= f(40-92+52+x)$ by (5)

$= f(x)$ ....(6)

Now $8 = 3 \times 488 - 14 \times 104$. Therefore

$f(8+x) = f(3 \times 488 - 14 \times 104 + x)$

$= f(-14 \times 104 + x)$, by repeated application of (4)

$= f(x)$, by repeated application of (6).

This shows that f is periodic, and all the possible values of f are in the list $f(0), f(1), f(2), \ldots, f(7)$. Finally

$f(4+x) = f(92-8\times 11+x) = f(92+x)$, by periodicity

$= f(92-x)$ by (1)

$= f(92-8\times 11+x) = f(4-x)$.

In particular $f(7) = f(1), f(6) = f(2), f(5) = f(3)$. Hence all the possible values of f are $f(0), f(1), f(2), f(3)$ and $f(4)$. In particular, f assumes no more than 5 function values. However, 92 has 6 positive divisors, namely 1, 7, 4, 23, 46 and 92.

Hence, the answer is No.

3. In order to simplify the radicals, the radicands should be forced to equal square numbers (*e.g.*, $7+\sqrt{33}$ should be a square of some number). Numbers whose squares have a rational and radical part are usually in the form $a+b$.

So let $\sqrt{7+\sqrt{33}} = a+b = \sqrt{(a+b)^2} = \sqrt{a^2+b^2+2ab}$, and set

$$a^2+b^2 = 7 \text{ and } 2ab = \sqrt{33}, \text{ i.e. } b = \frac{\sqrt{33}}{2a}.$$

Thus, $a^2 + \left(\dfrac{\sqrt{33}}{2a}\right)^2 = 7$

which multiplying by $4a^2$ gives

$$(2a^2-3)(2a^2-11) = 4a^4+33-28a^2 = 0$$

So $2a^2 = 3$ or $2a^2 = 11$, *i.e.*

$$a = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}, b = \frac{\sqrt{33}}{\sqrt{6}} = \frac{\sqrt{22}}{2}$$

or $\quad a = \sqrt{\dfrac{11}{2}} = \dfrac{\sqrt{22}}{2}, b = \dfrac{\sqrt{6}}{2}$,

and so $\sqrt{7+\sqrt{33}} = a+b = \dfrac{\sqrt{6}+\sqrt{22}}{2}$.

Using the same process for the other radicals, we get

$$\sqrt{6+\sqrt{35}} = \frac{\sqrt{10}+\sqrt{14}}{2},$$

so

$$\sqrt{7+\sqrt{33}} + \sqrt{6+\sqrt{35}} = \frac{\sqrt{6}+\sqrt{22}+\sqrt{10}+\sqrt{14}}{2}; \quad \ldots(1)$$

and $\sqrt{8+\sqrt{55}} = \dfrac{\sqrt{10}+\sqrt{22}}{2}, \sqrt{5+\sqrt{21}} = \dfrac{\sqrt{6}+\sqrt{14}}{2}$,

so

$$\sqrt{8+\sqrt{55}} + \sqrt{5+\sqrt{21}} = \frac{\sqrt{10}+\sqrt{22}+\sqrt{6}+\sqrt{14}}{2} \quad \ldots(2)$$

from (1) and (2)

$$\sqrt{7+\sqrt{33}} + \sqrt{6+\sqrt{35}} = \sqrt{8+\sqrt{55}} + \sqrt{5+\sqrt{21}}.$$

4. Note that ΔCC_1C_2 and ΔCB_1A_2 are both isosceles triangles and are thus similar. Also the area of $\Delta CB_1A_2 + \Delta CC_1C_2$ is equal to

$$\frac{1}{2}c^2\sin\gamma + \frac{1}{2}(a+b)^2\sin\gamma = \frac{1}{2}[c^2+(a+b)^2]\frac{2|T|}{ab},$$

since we know that the area of T,

$$|T| = \frac{1}{2}ab\sin\gamma$$

Similarly $\Delta BB_1B_2 + \Delta BA_1C_2 = [b^2+(a+c)^2]\dfrac{|T|}{ac}$

and $\Delta AA_1A_2 + \Delta AC_1B_2 = [a^2+(b+c)^2]\dfrac{|T|}{bc}$.

So all together

$$\Delta CB_1A_2 + \Delta CC_1C_2 + \Delta BD_1B_2 + \Delta BA_1C_2 + \Delta AA_1A_2 + \Delta AC_1B_2 = |H| + 2|T| + |T| \left[\frac{c^2+(a+b)^2}{ab} + \frac{b^2+(a+c)^2}{ac} + \frac{a^2+(b+c)^2}{bc}\right]$$

So

$$|H|+2|T| = |T|\left[\frac{c^2+a^2+b^2}{ab}+2+\frac{b^2+a^2+c^2}{ac}+2+\frac{a^2+b^2+c^2}{bc}+2\right]$$

$$=|T|\left[(a^2+b^2+c^2)\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}+6\right)\right].$$

Note that by the arithmetic mean - Geometric Mean inequality

$$(a^2+b^2+c^2) \geq 3(abc)^{2/3}$$

and that $\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc} \geq 3\left(\frac{1}{abc}\right)^{2/3}$

with equality when $a = b = c$ in both cases, so we have

$$|T|\left[(a^2+b^2+c^2)\left(\frac{1}{ab}+\frac{1}{ac}+\frac{1}{bc}\right)+6\right] \geq 15|T|$$

which implies $|H| \geq 13|T|$ as required, with equality when ΔABC is equilateral.

5. The recurrence relation $U_{n+2} = 2U_{n+1} + U_n = 0$ has general solution $U_n = An + B$, for constants A and B. The recurrence relation $U_{n+2} = 2U_{n+1} + U_n = 2$ has the particular solution $U_n = n^2$. Thus $a_n = n^2 + An + B$ and $b_n = Cn + D$.

$a_0 = 0, a_1, = \alpha$ (say)

$\Rightarrow \quad B = 0, A = \alpha - 1$

$b_0 = 8, b_1 = \beta$ (say)

$\Rightarrow \quad D = 8, C = \beta - 8,$

and we have, $a_n = n^2 + An \quad (A = a_1 - 1)$

$b_n = Bn + 8 \quad (B = b_1 - 8)$

and $a_n^2 + b_n^2 = n^4 + 2An^3 + (A^2+B^2)n^2 + 16Bn + 64$

$= (n^2 + An + 8)^2$ iff $A = B = \pm 4$.

(I) $A = B = 4$ gives $a_1 = 5$, $b_1 = 12$ and $5^2 + 12^2 = 13^2$ checks, whence $a_n = n^2 + 4n$, $b_n = 4n + 8$ giving

$(a_{1992}, b_{1992}) = (3976032, 7976)$.

(II) $A = B = -4$ gives $a_1 = -3$, $b_1 = 4$ and $(-3)^2 + 4^2 = 5^2$ checks; whence $a_n = n^2 - 4n$, $b_n = -4n + 8$ giving

$(a_{1992}, b_{1992}) = (3960096, -7960)$

6. Because opposite angles of a cyclic quadrilateral are supplementary we have that $\angle PBA = \pi - \angle ABS = \angle ARS$. Similarly $\angle PAB = \angle BSR$. Thus ΔPAB and ΔPSR are similar, from which

$$\frac{PA}{PS} = \frac{PB}{PR} = \frac{AB}{RS}$$

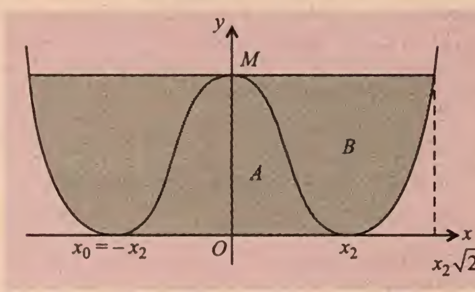
(Notice that this also gives the 'power of the point' result for P, $PA \cdot PR = PB \cdot PS$)

Similarly, $\frac{P'A}{P'S} = \frac{P'B}{P'R'} = \frac{AB}{R'S'}$

Consider now triangles APS and $AP'S'$. We have $\angle APS = \angle APB = \angle AP'B = \angle AP'S'$, because P, P' lie on the same arc of chord AB of the one circle. From the fact that S and S' lie on the same arc of chord AB of the second circle $\angle AS'P' = \angle ASP$. But then ΔAPS and $\Delta AP'S'$ are similar. Thus

$$\frac{PA}{PS} = \frac{P'A}{P'S'}, \quad \text{so} \quad \frac{AB}{RS} = \frac{AB}{R'S'}.$$

Thus $RS = R'S'$ and the arcs are equal.

7. 

Since A and B are unaffected by translation in the x or y direction, we may assume without loss of generality that $x_1 = 0$ and $m = 0$. Using Taylor's formula, the conditions $p(x_0) = p'(x_0) = 0$ and $p(x_2) = p'(x_2) = 0$ imply that $p(x)$ is divisible by $(x - x_0)^2$ and $(x - x_2)^2$. Since the degree of $p(x)$ is 4, we must have $p(x) = a(x - x_0)^2(x - x_2)^2$, for some $a \neq 0$. Then, the condition $p'(0) = 0$ implies that $-2ax_0x_2(x_0 + x_2) = 0$, *i.e.*, $x_0 = -x_2$. Using the condition $p(0) = M$, we get

$$p(x) = a(x^4 - 2x_2^2x^2 + x_2^4) = ax^4 - 2ax_2^2x^2 + M$$

[so $M = ax_2^4$]. This last expression implies that $p(x) = M$ when $x = 0$, $\pm x_2\sqrt{2}$ (see the figure). Finally, since $p(x)$ is symmetric with respect to the y-axis,

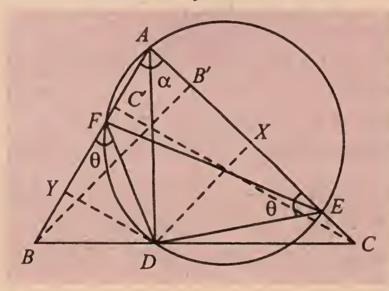
$$A = 2\int_0^{x_2} p(x)dx = 2\left(\frac{ax_2^5}{5} - \frac{2ax_2^2x_2^3}{3} + ax_2^4x_2\right) = \frac{16ax_2^5}{15},$$

$$B = 2\int_0^{x_2\sqrt{2}} (M - p(x))dx = 2a\int_0^{x_2\sqrt{2}} (2x_2^2x^2 - x^4)dx$$

$$= 2a\left(\frac{2x_2^2x_2^3 2\sqrt{2}}{3} - \frac{x_2^5 4\sqrt{2}}{5}\right) = \frac{16ax_2^5\sqrt{2}}{15},$$

so, $\frac{B}{A} = \sqrt{2}$.

8. As A, F, D, E are concyclic we have



$$\angle EDF + \angle EAF = \pi,$$

so $\sin\angle EDF = \sin\angle EAF = \sin\angle BAC$.

Therefore

$$\frac{ar(DEF)}{ar(ABC)}=\frac{\frac{1}{2}DE\cdot DF\cdot\sin\angle EDF}{\frac{1}{2}AB\cdot AC\cdot\sin\angle BAC}=\frac{DE\cdot DF}{AB\cdot AC}\quad ...(1)$$

Let B', X be the feet of the perpendiculars from B, D to AC, respectively, and let C', Y be the feet of the perpendiculars from C, D to AB. Because BB' is parallel to DX, we get

$$\frac{DX}{BB'}=\frac{DC}{BC}\text{ and }\frac{DY}{CC'}=\frac{BD}{BC},$$

thus $$\frac{DX}{BB'}\cdot\frac{DY}{CC'}=\frac{DC}{BC}\cdot\frac{BD}{BC}=\frac{BD\cdot DC}{BC^2}.\quad ...(2)$$

As $BC^2=(BD+DC)^2=(BD-DC)^2+4BD\cdot DC\geq 4BD\cdot DC$, we have from (2)

$$\frac{DX\cdot DY}{BB'\cdot CC'}\leq\frac{1}{4}.\quad ...(3)$$

We put $\angle EAF=\alpha$, and $\angle DEA=\angle DFB=\theta$, and we denote the circumradius of $AFDE$ by R, then we have $EF=2R\sin\alpha$ and $AD=2R\sin\theta$, so that

$$\frac{EF}{AD}=\frac{\sin\alpha}{\sin\theta}\quad ...(4)$$

Because $BB'=AB\sin\alpha$, $CC'=AC\sin\alpha$, $DX=DE\sin\theta$, and $DY=DF\sin\theta$ we get, $\frac{DX\cdot DY}{BB'\cdot CC'}=\frac{DE\cdot DF\sin^2\theta}{AB\cdot AC\sin^2\alpha}$

Hence we have from (3), $\frac{DE\cdot DF\sin^2\theta}{AB\cdot AC\sin^2\alpha}\leq\frac{1}{4}$,

i.e. $$\frac{4DE\cdot DF}{AB\cdot AC}\leq\frac{\sin^2\alpha}{\sin^2\theta}\quad ...(5)$$

Thus we have from (1), (4) and (5) $\frac{4ar(DEF)}{ar(ABC)}\leq\left(\frac{EF}{AD}\right)^2$.

9. By the Arithmetic Mean-Geometric Mean inequality we have $\frac{n(n+1)(2n+1)}{6n}$

$$=\frac{1^2+2^2+...+n^2}{n}\geq(1^2 2^2...n^2)^{1/n}=(n!)^{2/n}$$

or $(n+1)(2n+1)\geq 6(n!)^{2/n}$

or $(2n^2+3n+1)^n>6n(n!)^2$

with equality if and only if $n=1$.

Next we give Wildhagen's version.

For $1\leq k\leq n$ we have $k(n+1-k)\leq((n+1)/2)^2$, hence

$$(n!)^2=\prod_{k=1}^{n}k(n+1-k)\leq\left(\frac{n+1}{2}\right)^{2n}.$$

Therefore it suffices to show that

$$(2n+1)^n(n+1)^n\geq 6^n\cdot\left(\frac{n+1}{2}\right)^{2n}$$

or $(2n+1)(n+1)\geq 6\left(\frac{n+1}{2}\right)^2$

or $2n+1\geq\frac{3}{2}(n+1)$.

This last inequality holds trivially (and is strict unless $n=1$).

10. Since $\sin 30°=\sin 150°=-\sin 210°=\frac{1}{2}$, we have

$$\sin(3\times 10)°=\sin(3\times 50)°=\sin(3\times -70)°=\frac{1}{2}\quad ...(1)$$

But $\sin^3 x=\frac{3}{4}\sin x-\frac{1}{4}\sin 3x$,

so that by (1)

$$\sin^3 10°=\frac{3}{4}\sin 10°-\frac{1}{8},\quad \sin^3 50°=\frac{3}{4}\sin 50°-\frac{1}{8}$$

and $\sin^3(-70)°=\frac{3}{4}\sin(-70)°-\frac{1}{8}$,

i.e. $a, b, -c$ are the three roots of

$$f(t)=t^3-\frac{3}{4}t+\frac{1}{8}.$$

Therefore

(i) $a+b+(-c)=0$

(iii) $ab(-c)=-\frac{1}{8}$, and

(ii) $ab+b(-c)+(-c)a=-\frac{3}{4}$

Dividing by abc and using (iii) this leads to

$$c^{-1}-a^{-1}-b^{-1}=\left(-\frac{3}{4}\right)\cdot 8\Rightarrow a^{-1}+b^{-1}=c^{-1}+6$$

Math Archives 10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. $\tan^2\frac{\pi}{16}+\tan^2\frac{2\pi}{16}+\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16}+\tan^2\frac{6\pi}{16}+\tan^2\frac{7\pi}{16}$

is equal to

(a) 24 (b) 34
(c) 44 (d) none of these

2. If $f(x)=e^{\sin(x-[x])\cos\pi x}$, then $f(x)$ is ($[x]$ denotes the greatest integer function)

(a) non-periodic
(b) periodic with no fundamental period
(c) periodic with period 2
(d) periodic with period π

3. Which of the following homogeneous functions are of degree zero?

(a) $\frac{x}{y}\ln\frac{y}{x}+\frac{y}{x}\ln\frac{x}{y}$ (b) $\frac{x(x-y)}{y(x+y)}$
(c) $\frac{xy}{x^2+y^2}$ (d) all of the these

4. If θ is small and positive number then which of the following is/are correct ?

(a) $\frac{\sin\theta}{\theta}=1$ (b) $\theta<\sin\theta<\tan\theta$
(c) $\frac{\tan\theta}{\theta}>\frac{\sin\theta}{\theta}$ (d) none of these

5. Match the column. (Each entry of column I matches with exactly one entry of column II)

Column I		Column II	
A	$\lim\limits_{x\to 0}\log_{\sin x}\sin 2x$	p	-1
B	$\lim\limits_{x\to 0}\frac{\ln(\cos x)}{x}$	q	$-1/2$
C	$\lim\limits_{x\to 0}\left(\frac{1}{\ln x}-\frac{x}{\ln x}\right)$	r	1
D	$\lim\limits_{x\to 0}\frac{x-\sin x}{x-\tan x}$	s	0

(a) A → s, B → p, C → q, D → r
(b) A → p, B → q, C → r, D → s
(c) A → r, B → s, C → p, D → q
(d) A → q, B → r, C → s, D → p

6. If $y=\cos^{-1}\sqrt{\frac{\cos 3x}{\cos^3 x}}$ then prove that

$$\frac{dy}{dx}=\sqrt{\frac{3}{\cos x.\cos 3x}}.$$

7. Let $a\in R$, then prove that a function $f: R\to R$ is differentiable at 'a' if a function $\phi: R\to R$ satisfies $f(x)-f(a)=\phi(x)(x-a)\ \forall\ x\in R$ and ϕ is continuous at 'a'.

8. If $\beta,\gamma\in(0,\pi)$ such that

$\cos\alpha+\cos(\alpha+\beta)+\cos(\alpha+\beta+\gamma)=0$ and

$\sin\alpha+\sin(\alpha+\beta)+\sin(\alpha+\beta+\gamma)=0$.

Then evaluate $f'(\beta)$ and $\lim\limits_{x\to\gamma}g(x)$

where $f(x)=\sin 2x(1+\cos 2x)^{-1}$ and

$g(x)=\frac{1+\sin x-\cos x}{1+\sin x+\cos x}$

(Here $f'(x)$ denotes derivative of f with respect to x.)

9. Find area of the triangle formed with vertices

$$(0,0),\left(\lim_{x\to\frac{\pi}{2}}\left[\frac{x-\frac{\pi}{2}}{\cos x}\right],0\right)\text{ and }\left(0,\lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}\right)$$

where [.] denotes the greatest integer function.

By : Prof. Shyam Bhushan, Director, Narayana Institute, Jamshedpur. Mobile : 09334870021

10. Prove that the straight lines whose direction cosines are given by the relations $al + bm + cn = 0$ and $fmn + gnl + hlm = 0$ are perpendicular if $\frac{f}{a}+\frac{g}{b}+\frac{h}{c}=0$

SOLUTIONS

1. **(b)** : Let $\theta = \frac{\pi}{16} \Rightarrow 8\theta = \pi/2$,

$\therefore \quad \tan^2\frac{\pi}{16} = \tan^2\theta$

and $\tan^2\frac{7\pi}{16} = \tan^2\left(\frac{\pi}{2}-\frac{\pi}{16}\right) = \tan^2\left(\frac{\pi}{2}-\theta\right) = \cot^2\theta$

$\therefore \quad \tan^2\frac{\pi}{16}+\tan^2\frac{7\pi}{16}$

$= \tan^2\theta + \cot^2\theta = \dfrac{8}{1-\cos 4\theta} - 2$

$\therefore \quad \tan^2\frac{\pi}{16}+\tan^2\frac{7\pi}{16} = \dfrac{8}{1-\cos 4\left(\frac{\pi}{16}\right)} - 2 = 14+8\sqrt{2}$

Applying similar process with other terms, we get

$\tan^2\frac{2\pi}{16}+\tan^2\frac{6\pi}{16} = 6$

and $\tan^2\frac{3\pi}{16}+\tan^2\frac{5\pi}{16} = 14-8\sqrt{2}$

2. **(c)** : $f(x) = e^{\sin(x-[x])\cos \pi x}$

$\sin(x-[x]) = \sin\{x\}$ which has period 1

Period of $\cos \pi x$ is $\frac{2\pi}{\pi} = 2$

Hence period of $f(x)$ is 2.

3. **(d)** : $f(x, y)$ is homogeneous function of degree $n \in R$ in x, y if $f(kx, ky) = k^n f(x, y)$; where $k > 0$

4. **(c)**

5. **(c)** :

(A) $\lim\limits_{x\to 0}\dfrac{\ln\sin 2x}{\ln\sin x} = \lim\limits_{x\to 0}\dfrac{\ln 2+\ln\cos x+\ln\sin x}{\ln\sin x}$

$= \lim\limits_{x\to 0}\dfrac{\ln 2}{\ln x} + \lim\limits_{x\to 0}\dfrac{\ln\cos x}{\ln\sin x} + \lim\limits_{x\to 0}\dfrac{\ln\sin x}{\ln\sin x}$

$= \dfrac{\ln 2}{\infty} + \dfrac{\ln\cos 0}{\infty} + 1 = 0 + 0 + 1$

$\Rightarrow$ A $\to$ r

(B) $\lim\limits_{x\to 0}\ln\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{x}}$

$= \lim\limits_{x\to 0}\ln\left[\left(1-2\sin^2\frac{x}{2}\right)^{\frac{-1}{2\sin^2\frac{x}{2}}}\right]^{\frac{-2\sin^2\frac{x}{2}}{x}}$

$= \ln\lim\limits_{x\to 0}\left[\left(1-2\sin^2\frac{x}{2}\right)^{\frac{1}{2\sin^2\frac{x}{2}}}\right]^{-2\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2\frac{x}{4}}$

$= \ln e^{\lim\limits_{x\to 0} -\frac{x}{2}\left(\frac{\sin\frac{x}{2}}{\frac{x}{2}}\right)^2}$ $\quad\left[\lim\limits_{x\to a}\{1+f(x)\}^{\frac{1}{f(x)}} = e,\right.$ where $\left.\lim\limits_{x\to a} f(x) = 0\right]$

$= \lim\limits_{x\to 0}\ln e^0 = \ln 1 = 0$

$\Rightarrow$ B $\to$ s

(C) $\lim\limits_{x\to 0}\dfrac{1-x}{\ln x} = \lim\limits_{h\to 0}\dfrac{h}{\ln(1-h)}$ let $x = 1-h$

$= \lim\limits_{h\to 0}\dfrac{h}{\left(-h-\frac{h^2}{2}-\frac{h^3}{3}-\ldots\ldots\right)}$

$= \lim\limits_{h\to 0}\dfrac{1}{\left(-1-\frac{h}{2}-\frac{h^2}{3}-\ldots\ldots\right)} = -1$

$\Rightarrow$ C $\to$ p

(D) $\lim\limits_{x\to 0}\dfrac{\frac{x-\sin x}{x^3}}{\frac{x-\tan x}{x^3}}$

$= \lim\limits_{x\to 0}\dfrac{\frac{x-\left(x-\frac{x^3}{\lfloor 3}+\frac{x^5}{\lfloor 5}\ldots\right)}{x^3}}{\frac{x-\left(x+\frac{x^3}{3}+\frac{2x^5}{15}\ldots\right)}{x^3}} = \lim\limits_{x\to 0}\dfrac{\frac{1}{\lfloor 3}-\frac{x^2}{\lfloor 5}+\ldots}{-\frac{1}{3}-\frac{2x^2}{15}-\ldots}$

$= \dfrac{\frac{1}{6}}{-\frac{1}{3}} = -\dfrac{1}{2}$

$\Rightarrow$ D $\to$ q

6. $\cos y = \sqrt{\dfrac{\cos 3x}{\cos^3 x}} = \sqrt{\dfrac{4\cos^3 x - 3\cos x}{\cos^3 x}}$ ...(1)

$\Rightarrow\ 1-\sin^2 y = \sqrt{4-3\sec^2 x}$

$\Rightarrow \cos^2 y = 4-3\sec^2 x$

$= 4-3\left(1+\tan^2 x\right) = 1-3\tan^2 x$

$\Rightarrow\ \sin^2 y = 3\tan^2 x \Rightarrow \sin y = \sqrt{3}\tan x$

$\Rightarrow\ \cos y\dfrac{dy}{dx} = \sqrt{3}\sec^2 x$

$\therefore \dfrac{dy}{dx} = \sqrt{\dfrac{3}{\cos x . \cos 3x}}$ $\quad$ [$\because$ from (1)]

7. $\because \phi : R \to R$ is continuous at $x = a$ and satisfies

$$f(x) - f(a) = \phi(x)(x-a) \ \forall \ x \in R$$

$$\Rightarrow \frac{f(x)-f(a)}{x-a} = \phi(x)$$

$$\Rightarrow \lim_{x\to a} \frac{f(x)-f(a)}{x-a} = \lim_{x\to a} \phi(x)$$

$$\Rightarrow f'(a) = \phi(a) \qquad \because \lim_{x\to a}\phi(x) = \phi(a)$$

$\Rightarrow$ f is differentiable at $x = a$.

8. Given

$$\cos\alpha + \cos(\alpha+\beta) + \cos(\alpha+\beta+\gamma) = 0$$

$$\sin\alpha + \sin(\alpha+\beta) + \sin(\alpha+\beta+\gamma) = 0$$

Where $\beta, \gamma \in (0, \pi)$

$$\cos^2(\alpha+\beta+\gamma) + \sin^2(\alpha+\beta+\gamma) = 1$$

$$\Rightarrow \left[\cos\alpha + \cos(\alpha+\beta)\right]^2 + \left[\sin\alpha + \sin(\alpha+\beta)\right]^2 = 1$$

$$\Rightarrow 2 + 2\left[\cos(\beta)\right] = 1$$

$$\therefore \cos\beta = -\frac{1}{2}$$

Similarly $\cos\gamma = -\dfrac{1}{2}$

$$\therefore \beta = \gamma = \frac{2\pi}{3}$$

But $f(x) = \dfrac{\sin 2x}{1+\cos 2x} = \tan x$ and $g(x) = \tan\dfrac{x}{2}$

$$\therefore f'\left(\frac{2\pi}{3}\right) = \sec^2\frac{2\pi}{3} = 4$$

and $\displaystyle\lim_{x\to\frac{2\pi}{3}} g(x) = \tan\frac{\pi}{3} = \sqrt{3}$

9. Let $O = (0, 0)$

$$A = \left(\lim_{x\to\frac{\pi}{2}}\left[\frac{x-\frac{\pi}{2}}{\cos x}\right], 0\right) \qquad (\because \ [-x] = -1-[x] \ x \notin I)$$

Let $x - \dfrac{\pi}{2} = t$

$$= \left(\lim_{t\to 0}\left[\frac{-t}{\sin t}\right],\ 0\right) = \left(-1 - \lim_{t\to 0}\left[\frac{t}{\sin t}\right],\ 0\right)$$

$$= (-2, 0)$$

$$B = \left(0, \lim_{x\to 0}\left(\frac{\tan x}{x}\right)^{\frac{1}{x}}\right) = (0,1)$$

$\therefore$ Area of $\Delta OAB = \dfrac{1}{2}\left|-2-0\right| = 1$ square units.

10. $al + bm + cn = 0 \qquad ....(1)$

$fmn + gnl + hlm = 0 \qquad ....(2)$

Eliminate n

$$\Rightarrow ag\left(\frac{l}{m}\right)^2 + (af + bg - ch)\left(\frac{l}{m}\right) + bf = 0 \qquad .....(3)$$

Now, if l_1, m_1, n_1 and l_2, m_2, n_2 are d.c's of two lines then roots of (3) are $\dfrac{l_1}{m_1}$ and $\dfrac{l_2}{m_2}$

$\therefore$ Product of the roots $= \dfrac{l_1}{m_1}\cdot\dfrac{l_2}{m_2} = \dfrac{bf}{ag}$

$$\therefore \frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b}$$

$$\therefore \frac{l_1 l_2}{f/a} = \frac{m_1 m_2}{g/b} = \frac{n_1 n_2}{h/c}$$

$\because$ Lines are perpendicular

$$\therefore l_1 l_2 + m_1 m_2 + n_1 n_2 = 0$$

$$\Rightarrow \frac{f}{a} + \frac{g}{b} + \frac{h}{c} = 0$$

CONCEPTUAL PROBLEMS

Functions

1. Range of $f(x)=\sqrt{\sin^{-1}(2x)-\frac{\pi}{6}}$:

(a) $\left[0, \frac{\pi}{3}\right]$ (b) $\left[\frac{\pi}{6}, \frac{\pi}{3}\right]$

(c) $\left[\sqrt{\frac{\pi}{6}}, \sqrt{\frac{\pi}{3}}\right]$ (d) $\left[0, \sqrt{\frac{\pi}{3}}\right]$

2. If $x^2 f(x)=\sqrt{1+\cos 2x}+|f(x)|$, where $\frac{\pi}{2}<x<\pi$ and $f(x)=\frac{l\cos x}{x^2-1}$. Then l is :

(a) 1 (b) $-\sqrt{2}$

(c) $\sqrt{2}$ (d) 1/2

3. Let $f:\left[\frac{1}{2}, 1\right]\to[-1, 1]$ is given by $f(x)=4x^3-3x$; then $f^{-1}(x)$ is given by :

(a) $\cos\left(\frac{1}{3}\cos^{-1}x\right)$ (b) $3\cos\left(\cos^{-1}x\right)$

(c) $3\sin^{-1}(\sin x)$ (d) Not defined

4. Number of solution of the equation $f(x-1)+f(x+1)=\sin\alpha,\ 0<\alpha<\frac{\pi}{2}$ where

$$f(x)=\begin{cases}1-|x| & ; \ |x|\le 1\\ 0 & ; \ |x|>1\end{cases}$$

(a) 0 (b) 2

(c) 4 (d) infinite

5. Range of $f(x)=\log_2(|\sin x|+|\cos x|)$ is :

(a) $[-1, 0]$ (b) $\left[-\frac{1}{2}, 0\right]$

(c) $[0, 1]$ (d) $\left[0, \frac{1}{2}\right]$

6. If $f: R\to R$ be defined by $f(x)=x+\{x\}^2$, (where $\{.\}$ denotes fractional part) is :

(a) one-one, into function

(b) one-one, onto function

(c) many-one, into function

(d) many-one, onto function

7. If $f: R\to R$ where $f(x)=ax+\cos|x|$ is an invertible function then :

(a) $a\in(-2, 1]\cup[1, 2)$

(b) $a\in[-2, 2]$

(c) $a\in[-\infty, -1]\cup[1, \infty)$

(d) $a\in[-1, 1]$

8. Range of $f(x)$ satisfying $2^{|\sin x|}+2^{f(x)}=2$ is :

(a) $(-\infty, 2]$ (b) $(-\infty, 1]$

(c) $[-1, 1]$ (d) $[2, \infty)$

9. Domain of $f(x)=\sqrt{\cos^{-1}(\cos x)-[x]}$, where [.] denotes the greatest integer function, is :

(a) $(-\infty, 2\pi+3]$ (b) $(-\infty, \pi-3]$

(c) $(-\infty, \pi+3]$ (d) $(-\infty, 2\pi-3]$

10. Range of $f(x)=[\cos^{-1}\{x\}]$, where [.] and {.} denote the greatest integer function and fractional part respectively, is :

(a) $\{0, 1\}$ (b) $\{0, 1, 2\}$

(c) $\{0, 1, 2, 3\}$ (d) $\{0\}$

SOLUTIONS

1. **(d) :** $f(x)=\sqrt{\sin^{-1}(2x)-\frac{\pi}{6}}$,

let $h(x)=\sin^{-1}(2x)-\frac{\pi}{6}$

For domain

$\sin^{-1}(2x)-\frac{\pi}{6}\ge 0$ and $-1\le 2x\le 1$

$\Rightarrow \sin^{-1}(2x)\ge\frac{\pi}{6}$ and $-\frac{1}{2}\le x\le\frac{1}{2}$

$\Rightarrow \sin^{-1}2x\ge\sin^{-1}\frac{1}{2}$ and $\frac{-1}{2}\le x\le\frac{1}{2}$

$\Rightarrow 2x\ge\frac{1}{2}$ and $\frac{-1}{2}\le x\le\frac{1}{2}$

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$x \geq \frac{1}{4}$ and $\frac{-1}{2} \leq x \leq \frac{1}{2}$

Now we take common and get

$\frac{1}{4} \leq x \leq \frac{1}{2}$

So domain $\in \left[\frac{1}{4}, \frac{1}{2}\right]$

Let $h(x) = \sin^{-1} 2x - \frac{\pi}{6}$

Now we find maximum and minimum value of $h(x)$ (*i.e.* range of $h(x)$ for $\frac{1}{4} \leq x \leq \frac{1}{2}$)

$$\therefore \quad h_{\max} = \sqrt{\sin^{-1}\left(2 \cdot \frac{1}{2}\right) - \frac{\pi}{6}} = \sqrt{\sin^{-1}(1) - \frac{\pi}{6}}$$

$$= \sqrt{\frac{\pi}{2} - \frac{\pi}{6}} = \sqrt{\frac{\pi}{3}}$$

$$h_{\min} = \sqrt{\sin^{-1}\left(2 \cdot \frac{1}{4}\right) - \frac{\pi}{6}} = \sqrt{\frac{\pi}{6} - \frac{\pi}{6}} = 0$$

So range of $f(x) = \left[0, \sqrt{\frac{\pi}{3}}\right]$

2. **(b)** : $x^2 f(x) = \sqrt{1 + 2\cos^2 x - 1} + |f(x)|$

$x^2 f(x) = \sqrt{2}\,|\cos x| + |f(x)|$ ..... (i)

In equation (i), R.H.S. is positive so L.H.S. must be positive $\therefore f(x) \geq 0$

$x^2 f(x) = \sqrt{2}\,|\cos x| + f(x)$

$\Rightarrow \quad (x^2 - 1)\, f(x) = \sqrt{2}\,|\cos x|$

$$\Rightarrow \quad f(x) = \frac{\sqrt{2}\,|\cos x|}{x^2 - 1}$$

$$\Rightarrow \quad f(x) = \frac{-\sqrt{2}\cos x}{x^2 - 1} \qquad \left(\because \frac{\pi}{2} < x < \pi\right)$$

$\therefore \quad l = -\sqrt{2}$

3. **(a)** : $f(x) = 4x^3 - 3x$ now put $x = \cos\theta$

$\Rightarrow \quad \theta = \cos^{-1} x$ ....(i)

$= 4\cos^3\theta - 3\cos\theta \qquad \because \frac{1}{2} \leq x \leq 1$

$= \cos 3\theta$

Let $y = \cos 3\theta \qquad \frac{1}{2} \leq \cos\theta \leq 1$

$\Rightarrow 0 \leq \theta \leq \frac{\pi}{3}$

$\therefore 0 \leq 3\theta \leq \pi$

$R_f = [-1, 1]$, so function is one-one onto so above function will be invertible

$\therefore 3\theta = \cos^{-1} y$ or $\theta = \frac{1}{3}\cos^{-1} y$

Now put value of θ from equation (i)

$\cos^{-1} x = \frac{1}{3}\cos^{-1} y$

$\therefore \quad x = \cos\frac{1}{3}\left(\cos^{-1} y\right) \quad \therefore \quad f^{-1}(x) = \cos\left(\frac{1}{3}\cos^{-1} x\right)$

4. **(c)** : $f(x) = \begin{cases} 1 - x \ ; & 0 \leq x \leq 1 \\ 1 + x \ ; & -1 \leq x < 0 \\ 0 \ ; & x < -1 \text{ or } x > 1 \end{cases}$

then $f(x-1) = \begin{cases} 1 - (x-1) \ ; & 0 \leq (x-1) \leq 1 \\ 1 + (x-1) \ ; & -1 \leq x - 1 < 0 \\ 0 \ ; & x - 1 < -1 \text{ or } x - 1 > 1 \end{cases}$

$= \begin{cases} 2 - x \ ; & 1 \leq x \leq 2 \\ x \ ; & 0 \leq x < 1 \\ 0 \ ; & x < 0 \text{ or } x > 2 \end{cases}$ .... (i)

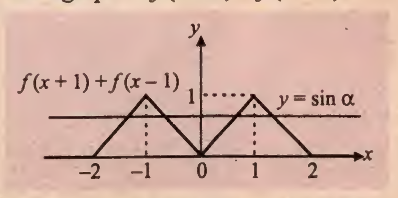
Now $f(x+1) = \begin{cases} 1 - (x+1) \ ; & 0 \leq x + 1 \leq 1 \\ 1 + (x+1) \ ; & -1 \leq 1 + x < 0 \\ 0 \ ; & 1 + x < -1 \text{ or } x + 1 > 1 \end{cases}$

$\therefore f(x+1) = \begin{cases} -x \ ; & -1 \leq x \leq 0 \\ x + 2 \ ; & -2 \leq x < -1 \\ 0 \ ; & x < -2 \text{ or } x > 0 \end{cases}$ .... (ii)

$\therefore$ Adding (i) & (ii)

$f(x-1) + f(x+1) = \begin{cases} 2 - x \ ; & 1 \leq x \leq 2 \\ x \ ; & 0 \leq x < 1 \\ 0 \ ; & x > 2 \\ -x \ ; & -1 \leq x \leq 0 \\ x + 2 \ ; & -2 \leq x < -1 \\ 0 \ ; & x < -2 \end{cases}$

Now draw the graph of $f(x-1) + f(x+1)$



Now $0 < \sin\alpha < 1$ so intersection of graph on four points only.

5. **(d)** : $f(x)$ is defined for all real x, but $f(x)$ is periodic with period $\pi/2$.

Let $h(x) = |\sin x| + |\cos x|$, $h(x)$ is also periodic with period $\pi/2$.

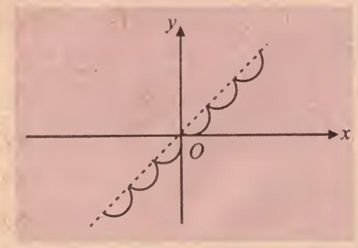
$h(x) = \sin x + \cos x \ ; \ 0 \leq x \leq \frac{\pi}{2}$

$h(x) = \sqrt{2}\sin\left(x + \frac{\pi}{4}\right)$

So range of $h(x)$ will be $[1, \sqrt{2}]$

$\therefore$ Range of $f(x) = \left[0, \frac{1}{2}\right]$

6. **(b) :**



Graph of $f(x)$ from graph function is one-one and range of functions is R so this is equal to co-domain of function so function will be one-one onto.

7. **(c) :** The function $f(x)$ is clearly onto if $a \neq 0$

$\because$ $(\cos|x| = \cos x)$

Now, $f'(x) = a - \sin x$, $f'(x) \geq 0\ \forall x \in R$

$\Rightarrow a \geq \sin x\ \forall\ x \in R\ \Rightarrow a \geq 1$

For $f'(x) \leq 0\ \ \forall x \in R$

We get $a \leq \sin x \forall\ x \in R\ \Rightarrow a \leq -1$

$\therefore a \in (-\infty, -1] \cup [1, \infty)$

8. **(b) :** $2^{|\sin x|} + 2^{f(x)} = 2\ \Rightarrow 2^{f(x)} = 2 - 2^{|\sin x|}$

taking log both side

$f(x) = \log_2(2 - 2^{|\sin x|})$

$(f(x)$ is periodic function with period $\pi)$

$f(x)$ is real valued when $2 - 2^{|\sin x|} > 0$

$\Rightarrow\ 2^{|\sin x|} < 2 \Rightarrow\ \ |\sin x| < 1\ \Rightarrow\ -1 < \sin x < 1$

$\therefore$ Let $h(x) = 2 - 2^{|\sin x|}$

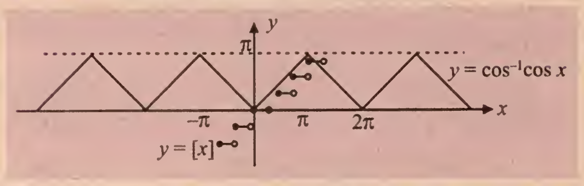
$h(x) = 2 - 2^{\sin x}\quad \left(0 \leq x < \frac{\pi}{2}\text{ or }\frac{\pi}{2} < x \leq \pi\right)$

$\therefore$ Maximum values of $h(x) = 1$

Minimum value of $h(x) = 0$

Range of $f(x)$ is $(-\infty, 0]$

9. **(d) :** We must have $\cos^{-1}(\cos x) \geq [x]$



Above figure represents the graphs of $\cos^{-1}(\cos x)$ and $[x]$. It clearly indicates that graph of $\cos^{-1}(\cos x)$ either coincident or lies above the graph of $[x]$ for all $x \leq 3$ and these two intersect some where in $(\pi, 4)$ for this point $2\pi - x = 3\ \Rightarrow x = 2\pi - 3$. Thus domain is $(-\infty, 2\pi - 3]$

10. **(a) :** Range of $\cos^{-1}\{x\}$ is $\left(0, \frac{\pi}{2}\right]$

Range of $[\cos^{-}\{x\}]$ is $\{0, 1\}$

$\Rightarrow \quad 1 - y^4 - 4y^2 = 0$

$\Rightarrow \quad y^4 + 4y^2 - 1 = 0$

$$\Rightarrow \quad y^2 = \frac{-4 \pm \sqrt{16+4}}{2} = \frac{-4 \pm \sqrt{20}}{2} = -2 \pm \sqrt{5}$$

$$y^2 = -2 + \sqrt{5} \text{ and } -2 - \sqrt{5}$$

but $y^2 \neq -2 - \sqrt{5}$ $\qquad (\because y^2 > 0)$

$$\Rightarrow \quad y^2 = \tan^2(x/2) = -2 + \sqrt{5}$$

59. (c) : Given system of equations

$$\left.\begin{aligned} x\sin 3\theta - y + z &= 0 \\ x\cos 2\theta + 4y + 3z &= 0 \\ 2x + 7y + 7z &= 0 \end{aligned}\right] \text{are homogeneous system of linear equations}$$

Since system has non trivial solution

$$\therefore \quad \begin{vmatrix} \sin 3\theta & -1 & 1 \\ \cos 2\theta & 4 & 3 \\ 2 & 7 & 7 \end{vmatrix} = 0$$

$\Rightarrow \quad \sin 3\theta[28 - 21] + 1[7\cos 2\theta - 6] + [7\cos 2\theta - 8] = 0$

$\Rightarrow \quad 7\sin 3\theta + 7\cos 2\theta + 7\cos 2\theta - 14 = 0$

$\Rightarrow \quad 3\sin\theta - 4\sin^3\theta + 2(1 - 2\sin^2\theta) - 2 = 0$

$\Rightarrow \quad \sin\theta(4\sin^2 + 4\sin\theta - 3) = 0$

either $\sin\theta = 0$ or $4\sin^2\theta + 6\sin\theta - 2\sin\theta - 3 = 0$

$\Rightarrow \quad (2\sin\theta - 1)(2\sin\theta + 3) = 0$

$$\therefore \quad \sin\theta = \frac{1}{2}, \sin\theta \neq -\frac{3}{2} \qquad (\because \sin\theta > -1)$$

$$\therefore \quad \theta = n\pi \text{ or } \theta = n\pi + (-1)^n \frac{\pi}{6}$$

$$\Rightarrow \quad \theta = \pi\left[n + \frac{(-1)^n}{6}\right]$$

60. (c) : Since $\dfrac{1+\cos 2\theta}{2} = \cos^2\theta$

$$\Rightarrow \quad \cos^4\frac{\pi}{8} = \left(\cos^2\frac{\pi}{8}\right)^2$$

$$= \left[\frac{1+\cos(2\pi/8)}{2}\right]^2 = \left[\frac{1+\cos(\pi/4)}{2}\right]^2$$

Similarly $\cos^4\dfrac{3\pi}{8} = \left[\dfrac{1+\cos(3\pi/4)}{2}\right]^2$

$$\cos^4\frac{5\pi}{8} = \left[\frac{1+\cos(5\pi/4)}{2}\right]^2$$

$$\cos^4\frac{7\pi}{8} = \left[\frac{1+\cos(7\pi/4)}{2}\right]^2$$

$\therefore$ Given expression will become

$$\frac{1}{4}\left[2\left(1+\frac{1}{\sqrt{2}}\right)^2 + 2\left(1-\frac{1}{\sqrt{2}}\right)^2\right]$$

$$= \frac{1}{4} \times 2[2+1] = \frac{1}{4} \times 2 \times 3 = \frac{3}{2}$$

$$\left(\because \cos\frac{7\pi}{4} = -\cos\frac{3\pi}{4}, \quad \cos\frac{5\pi}{4} = -\cos\frac{\pi}{4}\right)$$

concept BOOSTERS

Class XII

Functions

– MTG Editorial Board

This column is especially aimed at Class XII so that they can prepare for competitive exams such as IIT, AIEEE, DCE, etc and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

PART - A

General understanding

1. (i) Let f and g be defined as

$$f(x)=\begin{cases}x+1, & x\le 1\\ 2x+1, & 1<x\le 2\end{cases},\quad g(x)=\begin{cases}x^2, & -1\le x<2\\ x+2, & 2\le x\le 3\end{cases}$$

Find fog and gof.

(ii) A function $f: R\to R$ is defined by $f(x)=\dfrac{\alpha x^2+6x-8}{\alpha+6x-8x^2}$. Find the interval of value of α for which the function is onto. Is the function one-one for $\alpha=3$?

(iii) If A, B are finite non empty set of coordinates m, n respectively, show that the number of functions from A to B is n^m. How many of these are ONTO?

(iv) If A, B are finite non empty sets of coordinates m, n respectively, show that the number of ONTO functions from A to B is

$n^m-{}^nC_1(n-1)^m+{}^nC_2(n-2)^m-...+(-1)^{n-1}\,{}^nC_{n-1}1^m$.

2. (i) Decide about the following functions being even or odd.

(a) $f(x)=\ln(x+\sqrt{1+x^2})$

(b) $f(x)=\dfrac{x}{e^x-1}+\dfrac{x}{2}+1$

(c) $f(x+y)=f(x)+f(y)\ \forall x, y\in R$

(ii) Decide about the following functions being even or odd.

(a) $f(x)=\sqrt{1+x+x^2}-\sqrt{1-x+x^2}$

(b) $f(x)=|x|-\tan x$

(c) Let $f(x)=x^2+x+\sin x-\cos x$, $x\in[0, 1]$. Find the odd and even extensions of $f(x)$ in the interval $[-1, 1]$.

(d) $f(x)=\dfrac{e^x-1}{e^x+1}$

Domain and Range

3. (i) The function $f(x)$ is defined on $[0, 1]$. Find the domain of definition of $f(\tan x)$, $f(\sin x)$ and $f(2x+3)$.

(ii) Find the domain of the following functions

(a) $f(x)=\sqrt{\log_{1/2}(2x-3)}$

(b) $f(x)=\sqrt{\dfrac{(x-1)(x-2)}{(x-4)(x+8)}}$

(c) $f(x)=\sin^{-1}(x^2-6x+9)$

(d) $f(x)=\cos^{-1}(2x^2-3)$

(e) $f(x)=\sqrt{[x]-1}+\sqrt{4-[x]}$, where $[x]$ is the greatest integer function

(f) $f(x)=\sin^{-1}\left(\dfrac{2x-1}{(3x-2)}\right)+\cos^{-1}\left(\dfrac{3x-4}{5}\right)$

4. (i) Find the range of the following function

(a) $f(x)=3\sin x+4\cos(x+\pi/3)+5$

(b) $f(x)=\sin^{-1}\left(\dfrac{1}{2}+x^2\right)$

(c) $f(x)=\log((\cos x)^{\cos x}+1)$, $x\in(0, \pi/2)$

(d) $f(x)=\sqrt{6-x}+\sqrt{x-2}$

(e) $f(x)=x^3+3x^2+4x+7$

(ii) Find the range of the function

$$f(x)=\frac{\sin^2 x+\sin x-1}{\sin^2 x-\sin x+2}$$

Periodicity

5. (i) (a) Find the period of the function $f(x)=\cos(\sin x)+\cos(\cos x)$

(b) If $f(x)=\sin x+\cos ax$ is a periodic function, then show that a is a rational number.

(ii) (a) $f(x-1)+f(x+1)=\sqrt{3}f(x)$, then find the period of f.

(b) $f(x-1)+f(x+1)=\sqrt{2}f(x)$, then find the period of f.

(c) $f(x)=\dfrac{\sqrt{3}x-1}{\sqrt{3}+x}$, then find the period of f.

6. (i) Let $f(x, y)$ be a function satisfying $f(x, y)=f(2x+2y, 2y-2x)$ for all x, y. Define $g(x)=f(2^x, 0)$. Show that g is periodic having period 12.

(ii) Let f be a real valued function defined by $f(x+p)=\dfrac{1}{2}+\sqrt{f(x)-\{f(x)\}^2}$, $p>0$. Prove that f is periodic

○ Functional Equation

7. (i) If m and n are positive integers, f is a function defined on positive numbers and takes only positive values such that $f(x)f(y)=x^m y^n$, prove that $n=m^2$

(ii) Determine all functions f satisfying the functional relation.

$$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)} \text{ where } x\in R-\{0, 1\}$$

8. (i) Let f be a function from set of positive integers to the set of real numbers. $f: N\to R$, such that $f(1)=1$ and $f(1)+2f(2)+3f(3)+\ldots+nf(n)=n(n+1)f(n)$, for $n\geq 2$. Find $f(2010)$.

(ii) Let $f(x)=ax^2+bx+c$. Such that $|f(0)|\leq 1$, $|(f(1)|\leq 1$ and $|f(-1)|\leq 1$. Prove that $|f(x)|\leq 5/4$ $\forall x\in[-1, 1]$

○ Miscellaneous

9. (i) Show that $f(x)=\dfrac{2x(\sin x+\tan x)}{2\left[\dfrac{x+2\pi}{\pi}\right]-3}$ is an odd function.

(ii) If $f(x+y)=f(x)\cdot f(y)$ $\forall x, y\in R$ and $f(0)\neq 0$. Then prove that $g(x)=\dfrac{f(x)}{1+\{f(x)\}^2}$ is an even function.

(iii) If f satisfies $f(x+y)+f(x-y)=2f(x)f(y)$ $\forall x, y\in R, f(0)\neq 0$, then $f(x)$ is an even function.

10. (i) Let $f_1(x)=\dfrac{x}{3}+10$, $\forall x\in R$ and $f_n(x)=f_1(f_{n-1}(x))$, $n\geq 2$. Find $f_n(x)$.

(ii) Let $f(x)=\dfrac{4^x}{4^x+2}$. Find the value of

$$f\left(\frac{1}{2010}\right)+f\left(\frac{2}{2010}\right)+f\left(\frac{3}{2010}\right)+\ldots+f\left(\frac{2009}{2010}\right)$$

(iii) Solve the equation

$$x^2-x+1=\frac{1}{2}+\sqrt{x-\frac{3}{4}}$$

PART - B

Multiple Choice Questions

Single Option Correct

1. The domain of the function $f(x)=\log_{10}\log_{10}(1+x^3)$ is
(a) $(-1, \infty)$ (b) $(0, \infty)$
(c) $[0, \infty)$ (d) $(-1, \infty)$

2. The domain of $f(x)=\sqrt{\log_{1/4}\left(\dfrac{5x-x^2}{4}\right)}+{}^{10}C_x$ is
(a) $\{1, 4\}$ (b) $(1, 4)$
(c) $(0, 1)\cup(4, 5)$ (d) $(0, 5)$

3. If $f(x)$ is defined on $[0, 1]$ the $f(2\sin x)$ is defined on
(a) $\bigcup\limits_{n\in z}\left\{\left[2n\pi, 2n\pi-\dfrac{\pi}{6}\right]\cup\left[2n\pi+\dfrac{5\pi}{6}, (2n+1)\pi\right]\right\}$
(b) $\bigcup\limits_{n\in z}\left\{\left[2n\pi, 2n\pi+\dfrac{\pi}{3}\right]\cup\left[2n\pi+\dfrac{2\pi}{3}, (2n+1)\pi\right]\right\}$
(c) $\bigcup\limits_{n\in z}\left\{\left[2n\pi, 2n\pi-\dfrac{\pi}{3}\right]\cup\left[2n\pi-\dfrac{2\pi}{3}, (2n+1)\pi\right]\right\}$
(d) $\bigcup\limits_{n\in z}\left\{\left[2n\pi, 2n\pi+\dfrac{\pi}{6}\right]\cup\left[2n\pi+\dfrac{5\pi}{6}, (2n+1)\pi\right]\right\}$

4. The range of $\sin^{-1}\left(\dfrac{x^2+1}{x^2+2}\right)$ is
(a) $\left[\dfrac{\pi}{6}, \dfrac{\pi}{2}\right)$ (b) $\left(\dfrac{\pi}{3}, \dfrac{\pi}{2}\right)$
(c) $\left(\dfrac{\pi}{6}, \dfrac{\pi}{2}\right)$ (d) $\left(\dfrac{\pi}{7}, \dfrac{\pi}{2}\right)$

5. The domain of $f(x)=\log_2\log_3\log_4\log_5 x$ is
(a) $(5, \infty)$ (b) $(125, \infty)$
(c) $(25, \infty)$ (d) $(625, \infty)$

More than One Option Correct

6. Consider the equation $\dfrac{1}{[x]}+\dfrac{1}{[2x]}=\{x\}+3$. About the solution of this equation which statement (s) is/are correct ?
(a) The number of solutions is 3
(b) The equation does not possess any integral solution.
(c) The equation has three positive integral solution
(d) The equation doesn't possess any negative real number as its solution.

7. Let $f(x)=|x-1|+|x-2|+|x-3|+|x-4|$. Which of the following statements are correct?
(a) The least value of f is 4.
(b) The least value is not attained at a unique point.

(c) The number of integral solutions of $f(x) = 4$ is 2.

(d) The value of $\dfrac{f(\pi-1)+f(e)}{2f(2.4)}$ is 1.

8. Let $f(x)$ be an invertible function and $f^{-1}(x)$ be its inverse. Suppose $f(f^{-1}(x)) = f^{-1}(x)$ has two real roots α and β, which are in the domain of f, then

(a) $f(x) = x$ also have the same two real roots

(b) $f^{-1}(x) = x$ also have the same two real roots

(c) $f(x) = f^{-1}(x)$ also have the same two real roots

(d) The points $(0, 0)$, $(\alpha, f(\alpha))$, $(\beta, f(\beta))$ determine a triangle.

Assertion and Reason Type

Directions : Question numbers 9 and 10 are Assertion-Reason type questions. Each of these questions contains two statements :

Statement - 1 (Assertion) and Statement - 2 (Reason).

Each of these questions also has four alternative choices, only one of which is the correct answer. You have to select the correct choice.

(a) Statement-1 is true, Statement-2 is true; Statement-2 is correct explanation for Statement-1

(b) Statement-1 is true, Statement-2 is true; Statement-2 is not a correct explanation for Statement-1

(c) Statement- 1 is true, Statement-2 is false

(d) Statement-1 is false, Statement-2 is true

9. **Statement-1 :** The function $y = \dfrac{ax+b}{cx+d}$, $(ad - bc \neq 0)$ can't attain the value a/c

Statement-2 : The domain of the function $g(y) = \dfrac{b-dy}{cy-a}$ is all real except $\dfrac{a}{c}$.

10. **Statement-1 :** The domain of the function $\sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ is $[-1, 1]$

Statement-2 : $\sin^{-1}x$, $\cos^{-1}x$ are defined for $|x| \leq 1$ and $\tan^{-1}x$ is defined for all x.

Linked Paragraph Type

A rational function is defined as a quotient of two polynomials $p(x)$ and $q(x)$. The domain of the rational function is all reals except the roots of $q(x) = 0$. The range of rational function can be found by finding minimum and maximum values of the function. In case $p(x) = 0$ and $q(x) = 0$, have a common factor $x - \beta$, then after cancelling the common factor, the rational function must assume a value at $x = p$, which has to be deleted from the range otherwise found, since β is not there in the domain of the rational function.

11. The range of the rational function $f(x) = \dfrac{5x+1}{7x-3}$ is

(a) $R - (5/7)$ (b) $R - \left\{\dfrac{3}{7}, \dfrac{5}{7}\right\}$

(c) $R - \left\{\dfrac{7}{5}\right\}$ (d) $R - \left\{\dfrac{5}{7}\right\}$

12. The range of the rational function

$$f(x) = \frac{2x+1}{2x^2+5x+2} \text{ is}$$

(a) $R - \left(0, \dfrac{2}{3}\right)$ (b) $R - \left\{0, \dfrac{2}{3}\right\}$

(c) $R - \left\{\dfrac{2}{3}\right\}$ (d) $R - \{0\}$

13. The range of the function $f(x) = \dfrac{x^2-x+1}{x^2+x+1}$ is

(a) $\left[\dfrac{1}{3}, 3\right]$ (b) $\left(\dfrac{1}{3}, 3\right)$

(c) $R - \left[\dfrac{1}{3}, \dfrac{1}{3}\right]$ (d) $R - \left(\dfrac{1}{3}, \dfrac{1}{3}\right)$

Matrix-Match Type

14. Match the function in column I with their inverse in column II.

Column - I		Column - II	
(A)	$f(x) = x^2 - x + 1$, $\left(-\infty, \dfrac{1}{2}\right] \to \left[\dfrac{3}{4}, \infty\right)$	(p)	$f^{-1}(x) = \dfrac{-1+\sqrt{4x-3}}{2}$
(B)	$f(x) = x^2 - x + 1$ $\left[\dfrac{1}{2}, \infty\right) \to \left[\dfrac{3}{4}, \infty\right)$	(q)	$f^{-1}(x) = \dfrac{-1-\sqrt{4x-3}}{2}$
(C)	$f(x) = x^2 + x + 1$ $\left[\dfrac{-1}{2}, \infty\right) \to \left[\dfrac{3}{4}, \infty\right)$	(r)	$f^{-1}(x) = \dfrac{1+\sqrt{4x-3}}{2}$
(D)	$f(x) = x^2 + x + 1$, $\left(-\infty, \dfrac{-1}{2}\right] \to \left[\dfrac{3}{4}, \infty\right)$	(s)	$f^{-1}(x) = \dfrac{1-\sqrt{4x-3}}{2}$

Subjective Type

15. The period of the function

$$\sin \pi x + \cos\frac{\pi x}{2} + \tan\frac{\pi x}{3} + \cot\frac{\pi x}{4} + \sec\frac{\pi x}{5} + \text{cosec}\frac{\pi x}{6} \text{ is}$$

16. Let $f(x) = \sin^2 x + \sin^2\left(x + \dfrac{\pi}{3}\right) + \cos x \cos\left(x + \dfrac{\pi}{3}\right)$ and $g\left(\dfrac{5}{4}\right) = 1$ then $g(f(x))$ equals

SOLUTIONS

PART - A

1. (i) :

$$fog(x)=f(g(x))=\begin{cases}g(x)+1, & g(x)\le 1\\ 2g(x)+1, & 1<g(x)\le 2\end{cases}$$

Consider $g(x)\le 1$

Then we have two cases

(a) $x^2\le 1, -1\le x<2$

$\Rightarrow$ $-1\le x\le 1, -1\le x<2$ whose intersection gives $-1\le x\le 1$

(b) $x+2\le 1, 2\le x\le 3$

$\Rightarrow$ $x\le -1, 2\le x\le 3$ whose intersection gives $x\in\phi$

Again consider $1<g(x)\le 2$, Then two cases arise

(a) $1<x^2\le 2, -1\le x<2$

$\Rightarrow$ $1<x\le \pm\sqrt{2}, -1\le x<2$

$\Rightarrow$ $x\in[-\sqrt{2},-1)\cup(1,\sqrt{2}],\ -1\le x<2$

whose intersection gives

$1<x\le\sqrt{2}$

(b) $1<x+2\le 2, 2\le x\le 3$

$\Rightarrow$ $-1<x\le 0, 2\le x\le 3$, whose intersection gives $x\in\phi$

Thus we have
$$f(g(x))=\begin{cases}x^2+1, & -1\le x\le 1\\ 2x^2+1, & 1<x\le\sqrt{2}\end{cases}$$

Now
$$gof(x)=g(f(x))=\begin{cases}f^2(x), & -1\le f(x)<2\\ f(x)+2, & 2\le f(x)\le 3\end{cases}$$

Consider $-1\le f(x)<2$, two cases arise

(a) $-1\le x+1<2, x\le 1$

$\Rightarrow$ $-2\le x<1, x\le 1$ whose intersection gives $-2\le x<1$

(b) $-1\le 2x+1<2, 1<x\le 2$

$-2\le 2x<1, 1<x\le 2$

$-1\le x<1/2, 1<x\le 2$ whose intersection gives $x\in\phi$

Again consider $2\le f(x)\le 3$ two case arise

(a) $2\le x+1\le 3, x\le 1$

$\Rightarrow$ $1\le x\le 2, x\le 1$ whose intersection gives $x=1$

(b) $2\le 2x+1\le 3, 1<x\le 2$

$\Rightarrow$ $1\le 2x\le 2, 1<x\le 2$, whose intersection gives $x\in\phi$

Thus $g(f(x))=\begin{cases}(x+1)^2, & -2\le x<1\\ x+3, & x=1\end{cases}$

As at $x=1$, the two branches yield the same result.

We can also write $g(f(x))=(x+1)^2, -2\le x\le 1$

(ii) Let $y=\dfrac{\alpha x^2+6x-8}{\alpha+6x-8x^2}$

The function $y=f(x)$ will be onto if y assumes all real values $\forall x\in R$.

$\Rightarrow$ $\alpha y+6xy-8x^2y=\alpha x^2+6x-8$

$\Rightarrow$ $(8y+\alpha)x^2-6x(y-1)-(\alpha y+8)=0$

As $x\in R$, Discriminant ≥ 0

$\Rightarrow$ $36(y-1)^2+4(8y+\alpha)\cdot(\alpha y+8)\ge 0$

$\Rightarrow$ $9(y^2-2y+1)+(8\alpha y^2+64y+\alpha^2y+8\alpha)\ge 0$

$\Rightarrow$ $(8\alpha+9)y^2+(\alpha^2+46)y+(8\alpha+9)\ge 0$

For this to happen $8\alpha+9>0$ and discriminant <0

$\Rightarrow$ $(\alpha^2+46)^2-4(8\alpha+9)^2<0$

$\Rightarrow$ $(\alpha^2+46+16\alpha+18)(\alpha^2+46-16\alpha-18)<0$

$\Rightarrow$ $(\alpha^2+16\alpha+64)(\alpha^2-16\alpha+28)<0$

$\Rightarrow$ $(\alpha+8)^2(\alpha-14)(\alpha-2)<0$

$\Rightarrow$ $\alpha\in(2, 14)$

For this range $8\alpha+9>0$

Thus the function is onto iff $\alpha\in(2, 14)$

Again when $\alpha=3$
$$f(x)=\frac{3x^2+6x-8}{3+6x-8x^2}$$

In this case $f(x)=0$

$\Rightarrow$ $3x^2+6x-8=0$ has real roots given by
$$x=\frac{-6\pm\sqrt{36+96}}{2\cdot 3}=\frac{-6\pm\sqrt{132}}{6}=\frac{-3\pm\sqrt{33}}{3}$$

As $f\left(\dfrac{-3+\sqrt{33}}{3}\right)=f\left(\dfrac{-3-\sqrt{33}}{3}\right)=0$

we have $f(x_1)=f(x_2)$ but $x_1\ne x_2$. Hence the function is not one-one.

(iii) Let A consist of $\{a_1, a_2, ..., a_m\}$ and B consist of $\{b_1, b_2, ... b_n\}$. So to form a function from A to B, amounts to choose a sequence $\{b_1, b_2, ..., b_n\}$ of n elements and setting $f(a_1)=b_1, f(a_2)=b_2 ... f(a_m)=b_m$. Since each of $b_1, b_2, ... b_m$ can be chosen in n ways, it follows that there are n^m functions from A into B.

Next to form one-one function from A into B, amounts to choosing a sequence $\{b_1, b_2, ..., b_m\}$ of m distinct elements of B. This is impossible if $m>n$, so that in this case there are no one-one functions from A into B. Suppose now that $m\le n$. Then b_1 can be chosen in n ways, having chosen b_1, b_2 can be choose in $(n-1)$ ways and so on. It follows that there are $n(n-1) ... (n-m+1)$ one-one function from A to B.

Note that the integer $n(n-1) ... (n-m+1)$ is zero when $m>n$. Therefore, in all cases, the number of one-one functions from A to B is $n(n-1) ... (n-m+1)$.

(iv) Let $A=\{a_1, a_2, ..., a_m\}, B=\{b_1, b_2, ... b_n\}$.

For $1\le i\le n$, define f_i the set of those function from A into B which never take the values b_i. Thus f_i is the set of function from A into $B-\{b_i\}$, then $f=f_1\cup f_2\cup .... \cup f_n$ is the set of those functions from A into B which are not ONTO. The coordinates $N(f)$ of f is given by
$$N(f)=\sum_i N(f_i)-\sum_{i<j}N(f_i\cap f_j)+...+(-1)^{n-1}N(f_1\cap f_2\cap ... f_n)$$

The 1st sum has ${}^nC_1$ terms, each equal to $(n-1)^m$, The second sum has ${}^nC_2$ term, each equal to $(n-2)^m$, etc. Thus

$$N(f) = {}^nC_1(n-1)^m - {}^nC_2(n-2)^m + \ldots + (-1)^{n-2}\,{}^nC_{n-1}1^m$$

And hence the number of surjective functions is

$$n^m - N(f) = n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m + \ldots + (-1)^{n-1}\,{}^nC_{n-1}1^m.$$

If $m < m_1$ then an no surjective functions is

$$n^m - n(f) = n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m + \ldots + (-1)^{n-1}\,{}^nC_{n-1}1^m.$$

If $m < m_1$ there are no surjective functions from A into B, hence the above sum is zero. Suppose that $m = n$. Now it means that the function which is surjective becomes one-one functions. We already know that this value is $\lfloor n$.

Hence

$$n^m - {}^nC_1(n-1)^m + {}^nC_2(n-2)^m + \ldots + (-1)^{n-1}\,{}^nC_{n-1}1^m = \lfloor n$$

2. **(i) (a) 1st Solution :**

$$f(x) = \ln(x + \sqrt{1+x^2})$$

$$f(-x) = \ln(-x + \sqrt{1+x^2})$$

Now $f(x) + f(-x) = \ln(x+\sqrt{1+x^2}) + \ln(\sqrt{1+x^2} - x)$
$= \ln 1 = 0$

$\Rightarrow\ f(-x) = -f(x)$

Thus f is odd.

2nd Solution :

$$f(x) = \ln(x+\sqrt{1+x^2})$$

$$f'(x) = \frac{1 + \dfrac{2x}{2\sqrt{1+x^2}}}{x + \sqrt{1+x^2}} = \frac{1}{\sqrt{1+x^2}} = f'(-x)$$

which gives $f'(x)$ is an even function.

Hence its anti derivative $f(x)$ is odd.

(b) 1st Solution :

$$f(x) = \frac{x}{e^x - 1} + \frac{x}{2} + 1$$

$$f(-x) = \frac{-x}{e^{-x}-1} - \frac{x}{2} + 1$$

$$= -\frac{x}{\dfrac{1}{e^x} - 1} - \frac{x}{2} + 1$$

$$= -\frac{xe^x}{1-e^x} - \frac{x}{2} + 1$$

$$= \frac{xe^x}{e^x - 1} - \frac{x}{2} + 1$$

We have $f(x) - f(-x) = \dfrac{x}{e^x-1} + \dfrac{x}{2} + 1 - \dfrac{xe^x}{e^x-1} + \dfrac{x}{2} - 1$

$$= \frac{x(1-e^x)}{e^x - 1} + x = -x + x = 0$$

Thus $f(-x) = f(x)$. Hence f is even.

2nd Solution :

Rewrite $f(x) = \dfrac{x}{e^x-1} + \dfrac{x}{2} + 1$

$$= \frac{x\{2 + e^x - 1\}}{2(e^x-1)} + 1 = \frac{x}{2}\left(\frac{e^x+1}{e^x-1}\right) + 1$$

$$= \frac{x}{2}\left(\frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}}\right) + 1$$

$$f(-x) = \left(\frac{-x}{2}\right)\cdot\left(\frac{e^{-x/2} + e^{x/2}}{e^{-x/2} - e^{x/2}}\right) + 1$$

$$= \frac{x}{2}\left(\frac{e^{x/2} + e^{-x/2}}{e^{x/2} - e^{-x/2}}\right) + 1$$

As $f(-x) = f(x)$ we find that f is even.

(c) $f(x+y) = f(x) + f(y)\ \forall x, y \in R$

Replacing x, y by zero we get

$$f(0) = f(0) + f(0)$$

$\Rightarrow\ f(0) = 0$

Putting $-x$ in place of y we have

$$f(x-x) = f(x) + f(-x)$$

$\Rightarrow\ f(x) + f(-x) = 0$

then $f(-x) = -f(x)$

Hence f is an odd function.

(ii) (a) $f(x) = \sqrt{1+x+x^2} - \sqrt{1-x+x^2}$

$$f(-x) = \sqrt{1-x+x^2} - \sqrt{1+x+x^2}$$

As $f(-x) = -f(-x)$, f is odd.

(b) $f(x) = |x| - \tan x$

$$f(-x) = f(x) - \tan(-x) = |x| + \tan x$$

Thus f is neither even nor odd because $f(-x)$ is neither $f(x)$ nor $-f(x)$.

(c) To make f an odd function on $[-1, 1]$, we define f as follows

$$f(x) = \begin{cases} f(x), & 0 \le x \le 1 \\ -f(-x), & -1 \le x < 0 \end{cases}$$

$$= \begin{cases} x^2 + x + \sin x - \cos x, & 0 \le x \le 1 \\ -x^2 + x + \sin x + \cos x, & -1 \le x < 0 \end{cases}$$

To make f an even function on $[-1, 0]$ we define f as

$$f(x) = \begin{cases} f(x), & 0 \le x \le 1 \\ f(-x) & -1 \le x < 0 \end{cases}$$

$$= \begin{cases} x^2 + x + \sin x - \cos x, & 0 \le x \le 1 \\ x^2 - x - \sin x - \cos x, & -1 \le x < 0 \end{cases}$$

(d) 1st Solution :

$$f(x) = \frac{e^x - 1}{e^x + 1}$$

$$f(-x) = \frac{e^{-x}-1}{e^{-x}+1} = \frac{1-e^x}{1+e^x} = -\frac{e^x - 1}{e^x + 1} = -f(x)$$

Thus f is an odd function

2nd Solution :

Rewrite $f(x)=\dfrac{e^x-1}{e^x+1}=\dfrac{\dfrac{e^x-1}{e^{x/2}}}{\dfrac{e^x+1}{e^{x/2}}}=\dfrac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}}$

$$f(-x)=\frac{e^{-x/2}-e^{x/2}}{e^{-x/2}+e^{x/2}}=-\frac{e^{x/2}-e^{-x/2}}{e^{x/2}+e^{-x/2}}=-f(x)$$

Thus f is an odd function.

3. (i) If $f(x)$ is defined on [0, 1], then $f(g(x))$ will be defined when $0 \le g(x) \le 1$

Then $f(\tan x)$ is defined when $0 \le \tan x \le 1$

$\Rightarrow \tan 0 \le \tan x \le \tan \pi/4$

Then $n\pi \le x \le n\pi + \pi/4$

Again $f(\sin x)$ is defined when $0 \le \sin x \le 1$

$\Rightarrow \sin 0 \le \sin x \le \sin \pi/2$

$\Rightarrow n\pi \le x \le 2n\pi + \pi/2$

Also $f(2x+3)$ is defined when

$0 \le 2x+3 \le 1$

$\Rightarrow -3 \le 2x \le -2$

$\Rightarrow -3/2 \le x \le -1$

Thus the domain is $[-3/2, -1]$

(ii) (a) $f(x)=\sqrt{\log_{1/2}(2x-3)}$

For root to be defined $\log_{1/2}(2x-3) \ge 0$

$\Rightarrow 2x-3 \le \left(\dfrac{1}{2}\right)^0$

$\Rightarrow 2x-3 \le 1$

$\Rightarrow 2x \le 4$

$\Rightarrow x \le 2$

Again for log function to be defined, $2x-3>0$

$\Rightarrow x > 3/2$

Thus the domain is $\left(\dfrac{3}{2}, 2\right]$

(b) $f(x)=\sqrt{\dfrac{(x-1)(x-2)}{(x-4)(x+8)}}$

For root to be defined $\dfrac{(x-1)(x-2)}{(x-4)(x+8)} \ge 0$

Also Denominator should not be zero

i.e. $(x-4)(x+8) \ne 0$

from the wavy curve method we have the sign scheme for the rational expression $\dfrac{(x-1)(x-2)}{(x-4)(x+8)}$ as

$$\begin{array}{ccccccccc} + & & - & & + & & - & & + \\ & -8 & & 1 & & 2 & & 4 & \end{array}$$

Thus the domain is $(-\infty, -8) \cup [1, 2] \cup (4, \infty)$

(c) $f(x)=\sin^{-1}(x^2-6x+9)$

As $\sin^{-1} t$ is defined only for $-1 \le t \le 1$ we have

$-1 \le x^2-6x+9 \le 1$

$\Rightarrow -1 \le (x-3)^2 \le 1$

$\Rightarrow (x-3)^2 \le 1$

$\Rightarrow -1 \le x-3 \le 1$

$\Rightarrow 2 \le x \le 4$

Thus the domain is [2, 4]

(d) $f(x)=\cos^{-1}[2x^2-3]$

for f to be defined $-1 \le [2x^2-3] \le 1$

$\Rightarrow -1 \le 2x^2-3 < 2$

$\Rightarrow 2 \le 2x^2 < 5$

$\Rightarrow 1 \le x^2 < 5/2$

$x^2 \ge 1$ givens $x \in (-\infty, -1] \cup [1, \infty)$

$x^2 < 5/2$ given $x \in \left(-\sqrt{\dfrac{5}{2}}, \sqrt{\dfrac{5}{2}}\right)$

Taking intersection we have $x \in \left(-\sqrt{\dfrac{5}{2}}, -1\right] \cup \left[1, \sqrt{\dfrac{5}{2}}\right)$

(e) $f(x)=\sqrt{[x]-1}+\sqrt{4-[x]}$

f is defined when $[x]-1 \ge 0$ and $4-[x] \ge 0$

Thus $1 \le [x] \le 4 \quad \Rightarrow 1 \le x < 5$

Hence domain = [1, 5)

(f) $f(x)=\sin^{-1}\left(\dfrac{2x-1}{3x-2}\right)+\cos^{-1}\left(\dfrac{3x-4}{5}\right)$

We have $-1 \le \dfrac{2x-1}{3x-2} \le 1$

$\Rightarrow \left|\dfrac{2x-1}{3x-2}\right| \le 1$

$\Rightarrow |2x-1| \le |3x-2|$

$\Rightarrow |2x-1|^2 \le |3x-2|^2$

$\Rightarrow (2x-1)^2-(3x-2)^2 \le 0$

$\Rightarrow (5x-3)(-x+1) \le 0$

$\Rightarrow (x-1)(5x-3) \ge 0$

thus $x \in \left(-\infty, \dfrac{3}{5}\right] \cup [1, \infty)$

Also for $\cos^{-1}\left(\dfrac{3x-4}{5}\right)$ to be defined

$-1 \le \dfrac{3x-4}{5} \le 1$

$\Rightarrow -5 \le 3x-4 \le 5 \qquad \Rightarrow -1 \le 3x \le 9$

$\Rightarrow -\dfrac{1}{3} \le x \le 3$

Thus $x \in \left[\dfrac{-1}{3}, 3\right]$

Taking intersection, the domain is $\left[-\dfrac{1}{3}, \dfrac{3}{5}\right] \cup [1, 3]$

4. (i) (a) $f(x)=3\sin x+4\cos\left(x+\dfrac{\pi}{3}\right)+5$

$=3\sin x+4\left\{\dfrac{1}{2}\cos x-\dfrac{\sqrt{3}}{2}\sin x\right\}+5$

$=3\sin x+2\cos x-2\sqrt{3}\sin x+5$

$$y=(3-2\sqrt{3})\sin x+2\cos x+5$$

Let $P=(3-2\sqrt{3})\sin x+2\cos x$

$$-\sqrt{((3-2\sqrt{3})^2+2^2)}\le P\le\sqrt{(3-2\sqrt{3})^2+2^2}$$

$$5-\sqrt{(3-2\sqrt{3})^2+2^2}\le P+5\le\sqrt{(3-2\sqrt{3})^2+2^2}+5$$

The greatest value of y is $\sqrt{(3-2\sqrt{3})^2+2^2}+5$

$$=\sqrt{25-12\sqrt{3}}+5$$

and the least value is $5-\sqrt{25-12\sqrt{3}}$

Then the range is

$$\left[5-\sqrt{25-12\sqrt{3}},5+\sqrt{25-12\sqrt{3}}\right]$$

(b) $f(x)=\sin^{-1}\left[\frac{1}{2}+x^2\right]$

For any values of x, $\left[\frac{1}{2}+x^2\right]$ is a non-negative integer and $\sin^{-1}x$ is defined only for two non-negative integers 0 and 1.

i.e. $\left[\frac{1}{2}+x^2\right]=0\Rightarrow 0\le\frac{1}{2}+x^2<1$

$\Rightarrow\ x^2<\frac{1}{2}$

Also $\left[\frac{1}{2}+x^2\right]=1\quad\Rightarrow\ 1\le\frac{1}{2}+x^2<2$

$\Rightarrow\ -\frac{1}{2}\le x^2<\frac{3}{2}$

Thus the domain is $0\le x^2\le 1$

Thus the range of f is $\{0, \pi/2\}$, the range consists of just two elements.

(c) $f(x)=\ln((\cos x)^{\cos x}+1), x\in(0,\pi/2)$

As $x\in(0,\pi/2)$, $\quad\Rightarrow 0<\cos x<1$

Range of $\ln\{(\cos x)^{\cos x}+1\}$ in $(0,\pi/2)$ = range of $\ln(x^x+1)$ in $(0, 1)$

Let $y=x^x+1$

$$\frac{dy}{dx}=x^x(1+\ln x)$$

$$\frac{dy}{dx}>0\Rightarrow x>\frac{1}{e}$$

and $\frac{dy}{dx}<0\Rightarrow x<\frac{1}{e}$

Thus y(x) is minimum at $x=1/e$

Minimum value of $y(x)$ is $y(1/e)=(1/e)^{1/e}+1$

And maximum value of $y(x)$ is $y(1)=1+1=2$

Thus $\ln\left(1+\left(\frac{1}{e}\right)^{1/e}\right)<\ln(1+x^x)<\ln 2$

Hence range of $f(x)$ is $\left(\ln\left(1+\left(\frac{1}{e}\right)^{1/e}\right),\ln 2\right)$

(d) $f(x)=\sqrt{6-x}+\sqrt{x-2}$

1st Solution

Let $y=\sqrt{6-x}+\sqrt{x-2}$

the domain is $2\le x\le 6$

Now squaring

$$y^2=6-x+x-2+2\sqrt{(6-x)(x-2)}$$

$$y^2=4+2\sqrt{(6-x)(x-2)}$$

As $\sqrt{(6-x)(x-2)}\ge 0$ we have $y_m=2$ (note that y is always non-negative)

Again $y^2-4=2\sqrt{(6-x)(x-2)}$

$\Rightarrow\ y^4-8y^2+16=4(6-x)(x-2)$

Writing in the form of quadratic in x,

$$4x^2-32x+y^4-8y^2+64=0$$

Its discriminant ≥ 0

$\Rightarrow\ (32)^2\ge 4.4\{y^4-8y^2+64)$

$\Rightarrow\ y^4-8y^2\le 0$

$\Rightarrow\ y^2\le 8$

$\Rightarrow\ y\le\sqrt{8}$

Hence the range is $[2, 2\sqrt{2}]$

2nd Solution :

After $y=4+2\sqrt{(6-x)(x-2)}$

we have AM & GM inequality

$$\frac{(6-x)+(x-2)}{2}\ge\sqrt{(6-x)(x-2)}$$

$\Rightarrow\ 2\ge\sqrt{(6-x)(x-2)}$

$\Rightarrow\ (6-x)(x-2)\le 4$

Now $y^2\le 8$

$\Rightarrow\ y\le 2\sqrt{2}\quad(\because y \text{ is} +ve)$

Hence the range is $[2, 2\sqrt{2}]$

(e) Here we present a different argument.

Note that f is an unbounded function.

As $x\to\infty$, $f(x)\to\infty$

Also as $x\to-\infty\ f(x)\to-\infty$

When x increases, f keeps on increasing in unbounded manner.

x decreases, $f(x)$ keeps on decreasing in unbounded manner.

As f is a polynomial function it is continuous and hence will vary continuously. Thus $f: R\to R$ *i.e.* the range of f is $(-\infty,\infty)$.

(ii) $f(x)=\frac{\sin^2x+\sin x-1}{\sin^2x-\sin x+2}$

Let $\sin x=t$, so that $t\in[-1, 1]$

The range of $f(x)$ is equivalent to the range of $g(t)$ defined by

$$g(t)=\frac{t^2+t-1}{t^2-t+2},t\in[-1,1]$$

let $y = \dfrac{t^2+t-1}{t^2-t+2}$

$\Rightarrow\ (y-1)t^2-(y+1)t+(2y+1)=0$

For $t \in R$ we have $(y+1)^2-4(y-1)(2y+1) \geq 0$

$\Rightarrow\ -7y^2+6y+5 \geq 0$

$\Rightarrow\ 7y^2-6y-5 \leq 0$

$\Rightarrow\ \dfrac{3-2\sqrt{11}}{7} \leq y \leq \dfrac{3+2\sqrt{11}}{7}$

But only this is not sufficient, t must be lying between $[-1, 1]$

Three cases arise :

I : Both roots are greater than 1

this mean

$t_1+t_2 > 2$ and $(t_1-1)(t_2-1) > 0$

$\Rightarrow\ \dfrac{y+1}{y-1} > 2$ and $\dfrac{2y+1}{y-1}-\dfrac{y+1}{y-1}+1 > 0$

$\Rightarrow\ \dfrac{y-3}{y-1} < 0$ and $\dfrac{2y-1}{y-1} > 0$

$\Rightarrow\ 1 < y < 3$ and $y > 1$ or $y < \dfrac{1}{2}$

Taking intersection $y \in (1, 3)$

II : Both roots are less than -1, then $t_1 < -1$ and $t_2 < -1$

$\Rightarrow\ t_1+t_2 < -1$ and $(t_1+1)(t_2+1) < 0$

$\Rightarrow\ \dfrac{y+1}{y-1}+2 < 0$ and $\dfrac{2y+1}{y-1}+\dfrac{y+1}{y-1}+1 > 0$

and $\dfrac{3y-1}{y-1} < 0$ and $\dfrac{4y+1}{y-1} > 0$

$\Rightarrow\ \dfrac{1}{3} < y < 1$ and $y > 1$ or $y < -\dfrac{1}{4}$

Taking intersection we get $y \in \phi$.

III : If the one root < 1 and the other root > 1 then $t_1 < -1$ and $t_2 > 1$

$\Rightarrow\ t_1 < -1 < t_2$ and $t_1 < 1 < t_2$

$\Rightarrow\ t_1 < -1$ and $t_2 > -1$ and $t_1 < 1$ and $t_2 > 1$

$\Rightarrow\ (t_1+1)(t_2+1) < 0$ and $(t_1-1)(t_2-1) < 0$

$\Rightarrow\ \dfrac{2y+1}{y-1}+\dfrac{y+1}{y-1}+1 < 0$ and $\dfrac{2y+1}{y-1}-\dfrac{y+1}{y-1}+1 < 0$

$\Rightarrow\ \dfrac{4y+1}{y-1} < 0$ and $\dfrac{2y-1}{y-1} < 0$

$\Rightarrow\ -\dfrac{1}{4} < y < 1$ and $\dfrac{1}{2} < y < 1$

Thus $\dfrac{1}{2} < y < 1$

The range of $f(x)$ is

$$\left(\frac{3-2\sqrt{11}}{7}, \frac{3+2\sqrt{11}}{7}\right) - \left\{\left(\frac{1}{2}, 1\right) \cup (1, 3)\right\}$$

$$= \left(\frac{3-2\sqrt{11}}{7}, \frac{1}{2}\right) \cup \left(3, \frac{3+2\sqrt{11}}{7}\right) \cup \{1\}$$

5. (i) (a) $f(x) = \cos(\sin x) + \cos(\cos x)$

$$f\left(x+\frac{\pi}{2}\right) = \cos\left(\sin\left(\frac{\pi}{2}+x\right)\right) + \cos\left\{\cos\left(\frac{\pi}{2}+x\right)\right\}$$

$= \cos(\sin x) + \cos(-\sin x)$

$= \cos(\sin x) - \cos(\sin x)$

Thus f is periodic of period $\pi/2$.

(b) $f(x) = \sin x + \cos ax$

period of $\sin x = 2\pi$, period of $\cos ax = 2\pi/a$

Thus period of

$f(x) =$ LCM of $\dfrac{2\pi}{1}$ and $\dfrac{2\pi}{a}$

$= \dfrac{\text{LCM of } 2\pi \text{ and } 2\pi}{\text{HCF of 1 and } a} = \dfrac{2\pi}{k}$

$\dfrac{1}{k} =$ integer $= q$ (say $\neq 0$)

and $\dfrac{a}{k} =$ integer $= p$

Hence $\dfrac{a/k}{1/k} = \dfrac{p}{q} \Rightarrow a = \dfrac{p}{q}$

Thus a is a rational number.

(ii) (a) $f(x-1)+f(x+1)=\sqrt{3}f(x)$

Changing $x-1$ to x we have

$f(x)+f(x+2)=\sqrt{3}f(x+1)$ ...(1)

Again changing x to $x+2$

$f(x+2)+f(x+4)=\sqrt{3}f(x+3)$ ...(2)

Adding (1) and (2)

$f(x)+2f(x+2)+f(x+4)=\sqrt{3}\cdot(f(x+1)+f((x+3))$

$f(x)+2f(x+2)+f(x+4)=\sqrt{3}.\sqrt{3}f(x+2)$

$= 3f(x+2)$

$\Rightarrow\ f(x)-f(x+2)+f(x+4)=0$ ...(3)

Again changing x to $x+2$

$f(x+2)-f(x+4)+f(x+6)=0$ ...(4)

Adding (3) and (4)

$f(x)+f(x+6)=0$ ...(A)

Again changing x to $x+6$

$f(x+6)+f(x+12)=0$ ...(B)

Subtracting (B) from (A)

$f(x)-f(x+12)=0$

$f(x)=f(x+12)$

Thus the period is 12.

(b) $f(x-1)+f(x+1)=\sqrt{2}f(x)$

Changing x to $x+1$, we have

$f(x)+f(x+2)=\sqrt{2}f(x+1)$ ...(1)

Again replace x with $x+2$

$f(x+2)+f(x+4)=\sqrt{2}f(x+3)$ ...(2)

Adding (1) and (2)

$f(x)+2f(x+2)+f(x+4)=\sqrt{2}(f(x+1)+f(x+3))$

$=\sqrt{2}\cdot\sqrt{2}f(x+2)=2f(x+2)$

Thus $f(x)+f(x+4)=0$ ...(A)

Changing x to $x+4$,

$$f(x+4)+f(x+8)=0 \qquad ...(B)$$

Subtracting (B) from (A)

$$f(x)-f(x+8)=0$$

$$\Rightarrow f(x)=f(x+8)$$

Thus the period is 8.

1st Solution :

(c) $f(x)=\dfrac{\sqrt{3}x-1}{\sqrt{3}+x}$

$$f(f(x))=\frac{\dfrac{\sqrt{3}(\sqrt{3}x-1)}{\sqrt{3}+x}-1}{\sqrt{3}+\dfrac{\sqrt{3}x-1}{\sqrt{3}+x}}=\frac{3x-\sqrt{3}-\sqrt{3}-x}{3+\sqrt{3}x+\sqrt{3}x-1}$$

$$\frac{2x-2\sqrt{3}}{2\sqrt{3}x+2}=\frac{x-\sqrt{3}}{\sqrt{3}x+1}$$

$$f(f(f(x)))=\frac{\dfrac{\sqrt{3}(x-\sqrt{3})}{\sqrt{3}x+1}-1}{\sqrt{3}+\dfrac{x-\sqrt{3}}{\sqrt{3}x+1}}=\frac{\sqrt{3}x-3-\sqrt{3}x-1}{3x+\sqrt{3}+x-\sqrt{3}}$$

$$=\frac{-4}{4x}=-\frac{1}{x}$$

Let $g(x)=f(f(f(x)=f^3(x)$, say where $f^n(x)$ denotes the composition f applied 3 times.

we have $g(g(x)=f^6(x)=x$

Thus the period is 6.

2nd Solution :

Rewrite $f(x)=\dfrac{x-\dfrac{1}{\sqrt{3}}}{1+\dfrac{1}{\sqrt{3}}x}$

let $x=\tan\theta$

$$f(x)=\frac{x-\tan\dfrac{\pi}{6}}{1+x\tan\dfrac{\pi}{6}}=\tan\left(\theta-\frac{\pi}{6}\right)$$

$$f^2(x)=\tan(\theta-2\pi/6)$$

$$f^3(x)=\tan(\theta-3\pi/6)$$

In this chain we find

$$f^6(x)=\tan\theta-\frac{6\pi}{6}=\tan(\theta-\pi)=\tan\theta$$

Thus f is periodic of period 6.

6. (i) $f(x,y)=f(2x+2y,2y-2x)$

Applying this again and again

$f(2x+2y,2y-2x)$

$=f(2(2x+2y)+2(2y-2x),2(2y-2x)-2(2x+2y))$

$=f(8y,-8x)$

Again $f(x,y)=f(8y,-8x)$

$=f(8(-8x),-8(8y))$

$=f(-64x,-64y)$

So $f(x,y)=f(-64,-64y)$

$f(-64x,-64y)=f(64^2x,64^2y)$

$=f(2^{12}x,2^{12}y)$

$f(x,0)=f(2^{12}x,0)$

So $f(2^x,0)=f(2^{x+12},0)$

Thus $g(x)=g(x+12)$

Hence g is periodic of period 12.

(ii) 1st solution :

$$f(x+p)=\frac{1}{2}+\sqrt{f(x)-f^2(x)}$$

$$f(x+2p)=\frac{1}{2}+\sqrt{f(x+p)-f^2(x+p)}$$

$$=\frac{1}{2}+\sqrt{f(x+p)\{1-f(x+p)\}}$$

$$=\frac{1}{2}+\sqrt{\left\{\frac{1}{2}+\sqrt{f(x)-f^2(x)}\right\}\left\{\frac{1}{2}-\sqrt{f(x)-f^2(x)}\right\}}$$

$$=\frac{1}{2}+\sqrt{\frac{1}{4}-f(x)+f^2(x)}$$

$$=\frac{1}{2}+\sqrt{\left(f(x)-\frac{1}{2}\right)^2}$$

$$=\frac{1}{2}+\left|f(x)-\frac{1}{2}\right|$$

$$=\frac{1}{2}+f(x)-\frac{1}{2}=f(x)$$

Thus f is periodic with period $2p$.

2nd Solution :

Rewrite $f(x+p)=\dfrac{1}{2}+\sqrt{f(x)-f^2(x)}$

$$\text{as }\left\{f(x+p)-\frac{1}{2}\right\}^2=f(x)-f^2(x)$$

$$\Rightarrow\ f^2(x+p)-f(x+p)+\frac{1}{4}=f(x)-f^2(x)$$

$$\Rightarrow\ \left\{f(x+p)-\frac{1}{2}\right\}^2=\left\{f(x)-\frac{1}{2}\right\}^2+\frac{1}{4}$$

$$\Rightarrow\ \left\{f(x)-\frac{1}{2}\right\}^2+\left\{f(x+p)-\frac{1}{2}\right\}^2=\frac{1}{4}$$

Set $g(x)=f(x)-\dfrac{1}{2}$

then we have

$$g(x)+g(x+p)=\frac{1}{4}$$

Changing x to $x+p$ we have

$$g(x+p)+g(x+2p)=\frac{1}{4}$$

Subtraction gives $g(x)-g(x+2p)=0$

Then $g(x)=g(x+2p)$

$$\Rightarrow \quad f(x)-\frac{1}{2}=f(x+2p)-\frac{1}{2}$$

$\therefore \quad f(x)=f(x+2p)$

The period of f is $2p$, as before.

7. (i) $f(xf(y))=x^m y^n$

$\Rightarrow \quad \{f(xf(y))\}^{1/m}=xy^{n/m}$

$$\Rightarrow \quad x=\frac{\{f(xf(y))\}^{1/m}}{y^{n/m}} \quad \text{...(A)}$$

Let there be x, y such that $xf(y)=1 \Rightarrow \quad x=\dfrac{1}{f(y)}$

from (A)

$$f(y)=\frac{y^{n/m}}{\{f(1)\}^{1/m}}$$

Now

$$f(1)=\frac{1}{\{f(1)\}^{1/n}} \quad \Rightarrow f(1)=1$$

Thus $f(y)=y^{n/m}$

Hence $f(xy^{n/m})=x^m y^n$

Put $y^{n/m}=t$ so that $f(xt)=(xt)^m$

$\Rightarrow \quad f(\lambda)=\lambda^m$

on comparision $m=n/m \Rightarrow n=m^2$

(ii) $$f(x)+f\left(\frac{1}{1-x}\right)=\frac{2(1-2x)}{x(1-x)}=\frac{2}{x}-\frac{2}{1-x} \quad \text{...(A)}$$

Replacing x by $\dfrac{1}{1-x}$,

$$f\left(\frac{1}{1-x}\right)+f\left(\frac{1}{1-\frac{1}{1-x}}\right)=2(1-x)-\frac{2}{1-\frac{1}{1-x}}$$

$$\Rightarrow \quad f\left(\frac{1}{1-x}\right)+f\left(1-\frac{1}{x}\right)=2(1-x)-2\left(1-\frac{1}{x}\right) \quad \text{...(B)}$$

$$=-2x+\frac{2}{x}$$

Again replacing x by $1-\dfrac{1}{x}$, we obtain

$$f\left(1-\frac{1}{x}\right)+f\left(\frac{1}{1-\left(1-\frac{1}{x}\right)}\right)=\frac{2}{1-\frac{1}{x}}-\frac{2}{1-\left(1-\frac{1}{x}\right)}$$

$$\Rightarrow \quad f\left(1-\frac{1}{x}\right)+f(x)=\frac{2x}{x-1}-2x \quad \text{...(C)}$$

Subtracting (B) from (A),

$$f(x)-f\left(1-\frac{1}{x}\right)=2x-\frac{2}{1-x} \quad \text{...(D)}$$

Adding (C) and (D) we have

$$2f(x)=\frac{2x}{x-1}-\frac{2}{1-x}$$

thus $$f(x)=\frac{x+1}{x-1}$$

8. (i) Given

$f(1)+2f(2)+3f(3)+\ldots+nf(n)=n(n+1)f(n)$ ...(A)

Replacing n by $(n+1)$

$$f(1)+2f(2)+3f(3)+\ldots+(n+1)f(n+1)$$
$$=(n+1)(n+2)f(n+1) \quad \text{...(B)}$$

Subtracting (A) from (B)

$(n+1)f(n+1)=(n+1)(n+2)f(n+1)-n(n+1)f(n)$

$\Rightarrow \quad nf(n)=(n+1)f(n+1)$

from which we have the chain of equalities

$2f(2)=3f(3)=4f(4)=\ldots=nf(n)$

According to question

$f(1)+(n-1)nf(n)=n(n+1)f(n)$

$\Rightarrow \quad f(1)=2nf(n)$

$$\Rightarrow \quad f(n)=\frac{f(1)}{2n}=\frac{1}{2n} \text{ then } f(2010)=\frac{1}{4020}$$

(ii) $f(x)=ax^2+bx+c$

$f(0)=c$

$f(1)=a+b+c$

$f(-1)=a-b+c$

Solving for a, b and c

$$a=\frac{f(-1)+f(1)-2f(0)}{2}$$

$$b=\frac{f(1)-f(-1)}{2}$$

$c=f(0)$

Thus $f(x)=ax^2+bx+c$

$$=\frac{f(-1)+f(1)-2f(0)}{2}x^2+\frac{f(1)-f(-1)}{2}x+f(0)$$

$$=\frac{(x^2-x)+f(-1)+(x^2+x)f(1)+2(1-x^2)f(0)}{2}$$

$\Rightarrow \quad 2f(x)=(x^2-x)f(-1)+(x^2+x)f(1)+2(1-x^2)f(0)$

$|2f(x)| \le |x^2-x||f(-1)|+|x^2+x||f(1)|+2|1-x^2|\,|f(0)|$

$\le |x^2-x|+|x^2+x|+2|1-x^2|$

$\le -|x|(x-1)+|x|(x+1)+2(1-x^2)$

$=2|x|+2-2x^2$

Thus $|f(x)| \le |x|+1-x^2$

$=-\{|x|^2-|x|-1\}$

$$=-\left\{\left(|x|-\frac{1}{2}\right)^2-\frac{5}{4}\right\}=\frac{5}{4}-\left(|x|-\frac{1}{2}\right)^2$$

Thus $|f(x)| \le \dfrac{5}{4}$

9. (i) $$f(x)=\frac{2x(\sin x+\tan x)}{2\left[\frac{x+2\pi}{\pi}\right]-3}$$

$$=\frac{2x(\sin x+\tan x)}{2\left(\frac{x}{\pi}+2\right)-3}$$

$$=\frac{2x(\sin x+\tan x)}{2\left(\left(\frac{x}{\pi}\right)+2\right)-3}=\frac{2x(\sin x+\tan x)}{2\left(\frac{x}{\pi}\right)+1}$$

$$= \frac{x(\sin x + \tan x)}{\left(\frac{x}{\pi}\right) + \frac{1}{2}}$$

$$f(-x) = \frac{(-x)(\sin(-x) + \tan(-x))}{\left(-\frac{x}{\pi}\right) + \frac{1}{2}}$$

$$= \frac{x(\sin x + \tan x)}{\left(-\frac{x}{\pi}\right) + \frac{1}{2}}$$

Now two cases arise

When $x = n\pi$, $n \in z$, then $\left[-\frac{x}{\pi}\right] = -\left[\frac{x}{\pi}\right]$

The $f(x) = 0$, also $f(-x) = 0$

When $x \neq n\pi$, $n \in z$

then $\left[-\frac{x}{\pi}\right] = -\left[\frac{x}{\pi}\right] - 1$

$$f(-x) = \frac{x(\sin x + \tan x)}{-\left(\frac{x}{\pi}\right) - 1 + \frac{1}{2}} = -f(x)$$

Hence f is an odd function

(ii) $f(x+y) = f(x)f(y)$ ...(1)

Let $x = y = 0$

$\Rightarrow \{f(0)\}^2 = f(0)$

$\therefore f(0)\{f(0) - 1\} = 0$

Put $f(0) \neq 0$ (given). Thus $f(0) = 1$

Set $y = -x$ in (1), $f(0) = f(x)f(-x)$

$$\Rightarrow f(-x) = \frac{1}{f(x)}$$

Now $g(x) = \dfrac{f(x)}{1 + \{f(x)\}^2}$

$$g(-x) = \frac{f(-x)}{1 + \{f(-x)\}^2} = \frac{\frac{1}{f(x)}}{1 + \frac{1}{\{f(x)\}^2}} = \frac{f(x)}{1 + \{f(x)\}^2}$$

As $g(x) = g(-x)$, we conclude that g is an even function

(iii) $f(x+y) + f(x-y) = 2f(x)f(y)$ ...(A)

Replace x by y and y by x

$f(x+y) + f(y-x) = 2f(y)f(x)$ ...(B)

From (A) and (B) on subtraction

$f(x-y) - f(y-x) = 0$

$\Rightarrow f(y-x) = f(x-y)$

Set $x - y = t \in R$, so that

$f(-t) = f(t)$

Hence f is an even function.

10. (i) $f_2(x) = f_1(f_1(x)) = f(x/3 + 10)$

$$= \frac{x/3 + 10}{3} + 10 = \frac{x}{3^2} + \frac{10}{3} + 10$$

$$f_3(x) = f_1(f_2(x)) = f_1\left(\frac{x}{3^2} + \frac{10}{3} + 10\right)$$

$$= \frac{\frac{x}{3^2} + \frac{10}{3} + 10}{3} + 10$$

$$= \frac{x}{3^3} + \frac{10}{3^2} + \frac{10}{3} + 10$$

$$f_4(x) = \frac{x}{3^4} + \frac{10}{3^3} + \frac{10}{3^2} + \frac{10}{3} + 10$$

Thus continuity in the same fashion

$$f_n(x) = \frac{x}{3^n} + \frac{10}{3^{n-1}} + \frac{10}{3^{n-2}} + \ldots + \frac{10}{3} + 10$$

$$= \frac{x}{3^n} + 10\left\{1 + \frac{1}{3} + \ldots + \frac{1}{3^{n-1}}\right\}$$

$$= \frac{x}{3^n} + 10 \cdot \frac{1 - \left(\frac{1}{3}\right)^n}{1 - \frac{1}{3}}$$

$$= \frac{x}{3^n} + 15\left\{1 - \left(\frac{1}{3}\right)^n\right\}$$

$$= \left(\frac{x - 15}{3^n}\right) + 15$$

(ii) $f(x) = \dfrac{4^x}{4^x + 2}$

we have

$$f(1-x) = \frac{4^{1-x}}{4^{1-x} + 2} = \frac{\frac{4}{4^x}}{\frac{4}{4^x} + 2} = \frac{4}{4 + 2 \cdot 4^x} = \frac{2}{4^x + 2}$$

so $f(x) + f(1-x) = 1$

$$f\left(\frac{1}{2010}\right) + f\left(\frac{2009}{2010}\right) = 1$$

$$f\left(\frac{2}{2010}\right) + f\left(\frac{2008}{2010}\right) = 1$$

....

$$f\left(\frac{1004}{2010}\right) + f\left(\frac{1006}{2010}\right) = 1$$

$$f\left(\frac{1005}{2010}\right) = \frac{1}{2}$$

Adding we have

$$f\left(\frac{1}{2010}\right) + f\left(\frac{2}{2010}\right) + f\left(\frac{3}{2010}\right) + \ldots + f\left(\frac{2009}{2010}\right) = 1004.5$$

(iii) let $f(x) = x^2 - x + 1$

$$f : \left[\frac{1}{2}, \infty\right) \to \left[\frac{3}{4}, \infty\right)$$

Suppose $y = x^2 - x + 1$

on the given domain and co-domain the function f is

invertible, we have as a quadratic in x

$$x^2 - x + 1 - y = 0$$

$$x = \frac{1 \pm \sqrt{1-4(1-y)}}{2} = \frac{1 \pm \sqrt{4y-3}}{2}$$

As $x \in [1/2, \infty)$ we have $x = \frac{1}{2} + \sqrt{y - \frac{3}{4}}$

Thus the inverse function is

$$f^{-1}(x) = \frac{1}{2} + \sqrt{x - \frac{3}{4}}$$

we observe that the equation to be solved is

$$f(x) = f^{-1}(x)$$

we already know that when a function and its inverse meet, $y = x$

$\Rightarrow \; x^2 - x + 1 = x$

$\Rightarrow \; x^2 - 2x + 1 = 0$

$\Rightarrow \; (x-1)^2 = 0 \quad \therefore x = 1$

PART - B

1. **(b) :** For $f(x)$ to be defined, $\log_{10}(1 + x^3) > 0$

$\Rightarrow \; 1 + x^3 > 10^0 \quad \Rightarrow \quad x^3 > 0$

$\Rightarrow \; x \in (0, \infty)$

2. **(a) :** For

$$\sqrt{\log_{1/4} \frac{5x - x^2}{4}} \text{ to be defined}$$

$$\frac{5x-x^2}{4} > 0 \text{ and } \frac{5x-x^2}{4} \le 1$$

$\Rightarrow \; 0 < \frac{5x - x^2}{4} \le 1 \;\Rightarrow\; 0 < 5x - x^2 \le 4$

which gives $x(x-5) < 0$ and $x^2 - 5x + 4 \ge 0$

$\Rightarrow \; x \in (0, 5)$ and $x \in (-\infty, 1] \cup [4, \infty)$

Thus intersection is $x \in (0, 1] \cup [4, 5)$

$\log_x$ is defined for $x \in \{0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10\}$

Thus the given function is defined for the intersection, that is $x \in \{1, 4\}$

3. **(d) :**

$f(x)$ is defined on $[0, 1] \Rightarrow 0 \le x \le 1$

Now $f(2\sin x)$ shall be defined, if $0 \le 2\sin x \le 1$

$\Rightarrow \; 0 \le \sin x \le \frac{1}{2}$

which means $\left[0, \frac{\pi}{6}\right] \cup \left[\frac{5\pi}{6}, \pi\right]$

The general solution is

$$\bigcup_{n \in z} \left\{\left[2n\pi, 2n\pi + \frac{\pi}{6}\right] \cup \left[2n\pi + \frac{5\pi}{6}, 2n\pi + \pi\right]\right\}$$

4. **(a) :**

Let $t = \frac{x^2+1}{x^2+2} = 1 - \frac{1}{1+x^2}$

As $x \to \infty \; t \to 1$

Also $t \ge 1 - \frac{1}{2} = \frac{1}{2}$

so $t \in \left[\frac{1}{2}, 1\right)$, note that 1 is not in the range of t.

Now $\sin^{-1} t$ takes the value $[\pi/6, \pi/2)$

Notice that $\pi/2$ is not included because for no value of x does $\sin^{-1} \frac{x^2+1}{x^2+2}$ becomes $\pi/2$.

5. **(d) :** In these type of problems we should start with outermost log function

$$\log_3 \log_4 \log_5 x > 0$$

$\Rightarrow \; \log_4 \log_5 x > 3^0 \quad \Rightarrow \quad \log_4 \log_5 x > 1$

$\Rightarrow \; \log_5 x > 4^1 \quad \Rightarrow \quad x > 5^4$

$\Rightarrow \; x > 625$

Thus $x \in (625, \infty)$

6. We have $\frac{1}{[x]} + \frac{1}{[2x]} = \{x\} + \frac{1}{3}$

As $\{x\} \ge 0$ we have RHS always positive. So the solution must satisfy $x > 0$.

Let $x = n + f$ where $n = [x]$ and $f = \{x\}$

$$\Rightarrow \; \frac{1}{n} + \frac{1}{2n + [2f]} = f + \frac{1}{3}$$

Case I :

$0 \le 2f < 1 \quad \Rightarrow \quad 0 \le f < 1/2$

which gives

$$\frac{1}{n} + \frac{1}{2n} = f + \frac{1}{3} \;\Rightarrow\; \frac{3}{2n} = f + \frac{1}{3}$$

As $0 \le f < 1/2$ we have $0 \le \frac{3}{2n} - \frac{1}{3} < \frac{1}{2}$

$\Rightarrow \; \frac{1}{3} \le \frac{3}{2n} < \frac{5}{6} \quad \Rightarrow \quad \frac{1}{9} \le \frac{1}{2n} < \frac{5}{18}$

$\Rightarrow \; 9 \ge 2n > \frac{18}{5} \;\Rightarrow\; \frac{18}{5} < 2n \le 9$

$\Rightarrow \; \frac{9}{5} < n \le \frac{9}{2}$

Thus $n = 2, 3, 4$

for $n = 2, f = 5/12$

$n = 3, f = 1/6$

$n = 4, f = 1/24$ and $x = \frac{29}{12}, \frac{19}{6}, \frac{97}{24}$

Again $1 \le 2f < 2$

so that the equation becomes

$$\frac{1}{n} + \frac{1}{2n+1} = f + \frac{1}{3}$$

$n = 1 \;\Rightarrow\; f = 1$

$n = 2 \;\Rightarrow\; f = 11/30$

$n = 3 \;\Rightarrow\; f = 1/7$

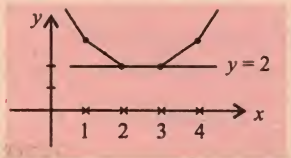
$n = 4 \;\Rightarrow\; f = 1/36$

$n = 5 \;\Rightarrow\; f < 0$

no value of 'f' is in the interval $\left[\frac{1}{2}, 1\right)$

Then the only valid solutions are $\frac{29}{12}, \frac{19}{6}, \frac{97}{24}$

7. **(a,b,c,d) :** $f(x) = |(x-1)| + |x-2| + |x-3| + |x-4|$



The least value of $f(x)$ is 2 attained at all $x \in [2, 3]$
$f(2) = 1 + 0 + 1 + 2 = 4$
The number of integral solution of $f(x) = 4$ is just two, viz $\{2, 3\}$
As $\pi - 1, e \in [2, 3]$ we have $f(\pi - 1) = f(e) = k$ (say)
Also $2 - 4 \in [2, 3]$ we have $f(2-4) = k$

Thus $\frac{f(\pi-1)+f(e)}{2f(2-4)} = 1$

8. **(a, b, c):** As $y = f(x)$ and $y = f^{-1}(x)$ intersect at $y = x$, we have that the reflection of $y = f(x)$ in the line $y = x$ gives $f^{-1}(x)$

$f(f^{-1}(x)) = f^{-1}(x) \quad \Rightarrow x = f^{-1}(x)$
and $f(x) = x \quad \Rightarrow \quad f^{-1}(x) = f(x)$.
Thus all have same roots α and β. Also $(0, 0)$, $(\alpha, f(\alpha))$, $(\beta, f(\beta))$ are collinear and so they don't form a triangle.

9. **(a) :** $y = \frac{ax+b}{cx+d}$

$\Rightarrow \; cxy + dy = ax + b \quad \Rightarrow \; x(cy - a) = b - dy$

$\Rightarrow \; x = \frac{b-dy}{cy-a}$

Thus y can't take the value a/c. Indeed the range of y is $R - \left\{\frac{a}{c}\right\}$.

Thus statement 1 is true and statement 2 explains it also.

10. **(a) :** Obviously $x \in [-1, 1]$ if $\sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ is to be defined then both statement 1 and statement 2 are correct and statement 2 implies statement 1.

11. **(d) :** $y = \frac{5x+1}{7x-3}$

$\Rightarrow \; 7xy - 3y = 5x + 1 \quad \Rightarrow \quad x(7y - 5) = 1 + 3y$

$\Rightarrow \; x = \frac{1+3y}{7y-5}$

As $7y - 5 \neq 0 \quad \Rightarrow \quad y \neq 5/7$
then the range is $R - \{5/7\}$

12. **(b) :** $y = \frac{2x+1}{2x^2+5x+2} = \frac{(2x+1)}{(2x+1)(x+2)}$

Thus $y = \frac{1}{x+2}, x \neq -\frac{1}{2}$

Range of y is $R - \{0\}$
Also since $x \neq -1/2$ we have $y \neq \frac{1}{2-\frac{1}{2}}$ *i.e.* $y \neq \frac{2}{3}$
The range is $R - \{0, 2/3\}$

13. **(a) :** Let $y = \frac{x^2-x+1}{x^2+x+1}$

$\Rightarrow \; yx^2 + yx + y = x^2 - x + 1$
$\Rightarrow \; (y-1)x^2 + (y+1)x + y - 1 = 0$
As $x \in R \Rightarrow (y+1)^2 - 4(y-1)^2 \geq 0$
$\Rightarrow \; (y + 1 + 2y - 2)(y + 1 - 2y + 2) \geq 0$
$\Rightarrow \; (3y-1)(-y+3) \geq 0 \qquad \Rightarrow \; (3y-1)(y-3) \leq 0$
Thus $y \in [1/3, 3]$

14. (A) $\to$ (s), (B) $\to$ (r), (C) $\to$ (p), (D) $\to$ (q)
$y = x^2 - x + 1$
$\Rightarrow \; x^2 - x + (1 - y) = 0$

$\Rightarrow \; x = \frac{1 \pm \sqrt{1-4(1-y)}}{2} = \frac{1 \pm \sqrt{4y-3}}{2}$

(A) $f: (-\infty, 1/2] \to [3/4, \infty)$

$f^{-1}(x) = \frac{1-\sqrt{4x-3}}{2}$

(B) $f: [1/2, \infty) \to [3/4, \infty)$

$f^{-1}(x) = \frac{1+\sqrt{4x-3}}{2}$

Again $y = x^2 + x + 1$
$\Rightarrow \; x^2 + x + (1 - y) = 0$

$x = \frac{-1 \pm \sqrt{1-4(1-y)}}{2} = \frac{-1 \pm \sqrt{4y-3}}{2}$

$f: [-\infty, -1/2] \to [3/4, \infty), \; f^{-1}(x) = \frac{-1-\sqrt{4x-3}}{2}$

$f: [-1/2, \infty) \to [3/4, \infty), \quad f^{-1}(x) = \frac{-1+\sqrt{4x-3}}{2}$

15. The period is the LCM of

$\frac{2\pi}{\pi}, \frac{2\pi}{(\pi/2)}, \frac{2\pi}{(\pi/3)}, \frac{2\pi}{(\pi/4)}, \frac{2\pi}{(\pi/5)}, \frac{2\pi}{(\pi/6)}$

$\Rightarrow \; \frac{2\pi}{\pi}, \frac{4\pi}{\pi}, \frac{6\pi}{\pi}, \frac{8\pi}{\pi}, \frac{10\pi}{\pi}, \frac{12\pi}{\pi}$

i.e. the LCM of (2, 4, 6, 8, 10, 12) *i.e.* 60

16. $f(x) = \sin^2 x + \sin^2(x + \pi/3) + \cos x \cos(x + \pi/3)$

$= \frac{1}{2}\left[1 - \cos 2x + 1 - \cos\left(2x + \frac{2\pi}{3}\right) + \cos\frac{\pi}{3} + \cos\left(2x + \frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\left[2 - \cos 2x + \frac{1}{2} + \cos\left(2x + \frac{\pi}{3}\right) + \cos\left(2x + \frac{\pi}{3}\right)\right]$

$= \frac{1}{2}\left[\frac{5}{2} - \cos 2x + 2\cos 2x \cdot \cos\frac{\pi}{3}\right]$

$= \frac{1}{2}\left(\frac{5}{2} - \cos 2x + \cos 2x\right) = \frac{5}{4} = \text{constant}$

$g(f(x)) = g(5/4) = 1$

ASSERTION AND REASON

Direction : In each of the following questions, a statement of assertion is given and a corresponding statement of reason is given just below it. Of the statement, mark the correct answer as -

(a) If both assertion and reason are true and reason is the correct explanation of assertion
(b) If both assertion and reason are true but reason is not the correct explanation of assertion
(c) If assertion is true but reason is false
(d) If assertion is false and reason is true.

41. Consider the equation $x^4 - 5x^3 + 56|x-3| = 0$
Assertion : The number 13 is not a root of the equation.
Reason : A polynomial equation with integral coefficients can have only the divisor of constant term as it's integral root.

42. Assertion : If z satisfy $|z| = 1$ and $z = 2 - \bar{z}$ then z is purely real.
Reason : Principal argument of z is 0.

43. Assertion : Period of
$f(x) = \sin\frac{\pi x}{(n-1)!} + \cos\frac{\pi x}{n!}$ is $2(n)!$
Reason : Period of $|\cos x| + |\sin x| + 3$ is π

44. Assertion : Let $\vec{a}, \vec{b}, \vec{c}$ be three non-coplaner vectors, then $(\vec{b}-\vec{c}).[(\vec{c}-\vec{a})\times(\vec{a}-\vec{b})] = 0$. $\vec{b}-\vec{c}$ can be expressed as linear combination of $\vec{c}-\vec{a}$ and $\vec{a}-\vec{b}$.
Reason : Given non-coplanar vectors one vector can be expressed as a linear combination of the other two.

45. Assertion : If $x + y + z = xyz$, then at most one of the numbers could be negative.
Reason : In a triangle ABC,
$\tan A + \tan B + \tan C = \tan A \tan B \tan C$ and there can be at most one obtuse angle in a triangle.

46. Assertion : Let $L_1 : a_1x + b_1y + c_1 = 0$,
$L_2 : a_2x + b_2y + c_2 = 0$ and $L_3 : a_3x + b_3y + c_3 = 0$ are three concurrent lines, then
$$\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0.$$
Reason : If $\begin{vmatrix} a_1 & b_1 & c_1 \\ a_2 & b_2 & c_2 \\ a_3 & b_3 & c_3 \end{vmatrix} = 0$, then the lines L_1, L_2 and L_3 must be concurrent.

47. $f(x)$ is polynomial of degree 3 passing through origin having local extrema at $x = \pm a$.
Assertion : Ratio of areas in which $f(x)$ cuts the circle $x^2 + y^2 = 36$ is 1 :1.
Reason : Both $y = f(x)$ and the circle are symmetric about origin.

48. Let $f(x) = \sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ and $x \in [-1,1]$.
Assertion : Range of $f(x)$ is $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$.
Reason : $f(x)$ is an increasing function.

49. Assertion : If a normal drawn at a point P on the parabola $y^2 = 4ax$ meets the curve again at Q, then the least distance of the point Q from the axis of the parabola is $4\sqrt{2}a$.
Reason : If normal drawn at point $P(at^2, 2at)$ on parabola $y^2 = 4ax$ meets the curve again at $Q(at_1^2, 2at_1)$, then $t_1 = t + \frac{2}{t} \Rightarrow$ minimum value of $t_1 = 2\sqrt{2}$.

50. Assertion : $\lim_{x\to 0}\frac{x}{a}\left(\frac{1}{x}\right)$ does not exist, (where [.] denotes the greatest integer function)
Reason : $\lim_{x\to 0}\left(\frac{1}{x}\right)$ does not exist.

ANSWER KEY

1.	(d)	2.	(b)	3.	(b)	4.	(b)	5.	(a)	6.	(d)
7.	(a)	8.	(b)	9.	(c)	10.	(b)	11.	(d)	12.	(b)
13.	(d)	14.	(d)	15.	(c)	16.	(b)	17.	(a)	18.	(d)
19.	(d)	20.	(b)	21.	(c)	22.	(a)	23.	(c)	24.	(b)
25.	(a)	26	(b)	27.	(d)	28.	(a)	29.	(b)	30.	(b)
31.	(a)	32.	(a)	33.	(b)	34.	(b)	35.	(b)	36.	(b)
37.	(a)	38.	(b)	39.	(d)	40.	(c)	41.	(a)	42.	(b)
43.	(c)	44.	(c)	45.	(d)	46.	(c)	47.	(a)	48.	(a)
49.	(c)	50.	(d)								

OLYMPIAD CORNER

International Olympiad Problems

Challenging problems for Olympiads, IIT-JEE and other contests.

1. Show that if $x, y, z > 0$, then
$$(xy+yz+zx)\left(\frac{1}{(x+y)^2}+\frac{1}{(y+z)^2}+\frac{1}{(z+x)^2}\right)\ge\frac{9}{4}.$$

2. (a) Find all positive integers $p \le q \le r$ satisfying the equation
$$p+q+r+pq+qr+rp = pqr+1.$$
(b) For each such solution (p, q, r), evaluate
$$\tan^{-1}(1/p)+\tan^{-1}(1/q)+\tan^{-1}(1/r).$$

3. If m_a, m_b, m_c are the medians of a triangle with sides a, b, c, prove that
$$m_a(bc-a^2)+m_b(ca-b^2)+m_c(ab-c^2)\ge 0.$$

4. Let n be a fixed positive integer. Show that for any non negative integer k, the diophantine equation
$$x_1^3+x_2^3+\ldots+x_n^3=y^{3k+2}$$
has infinitely many solutions in positive integers x_i and y.

5. Determine all sequences $a_1 \le a_2 \le \ldots \le a_n$ of positive real numbers such that
$$\sum_{i=1}^{n} a_i = 96, \sum_{i=1}^{n} a_i^2 = 144 \text{ and } \sum_{i=1}^{n} a_i^3 = 216.$$

6. Find all polynomials $P(x)$ that satisfy the equation
$$P(x^2)+2x^2+10x = 2xP(x+1)+3.$$

7. Prove or disprove that
$$2<\frac{\sin A}{A}+\frac{\sin B}{B}+\frac{\sin C}{C}\le\frac{9\sqrt{3}}{2\pi},$$
where A, B, C are the angles (in radians) of a triangle.

8. $ABCD$ is a convex quadrilateral, and O is the intersection of its diagonals. Suppose that the area of the (nonconvex) pentagon $ABOCD$ is equal to the area of triangle OBC. Let P and Q be the points on BC such that $OP\|AB$ and $OQ\|DC$. Prove that
$$[OAB]+[OCD]=2[OPQ],$$
where [XYZ] denotes the area of triangle XYZ.

9. Are there any non constant differentiable functions $f: R \to R$ such that
$$f(f(f(x))) = f(x) \ge 0$$
for all $x \in R$?

10. Let D, E, F be points on the sides BC, CA, AB respectively of triangle ABC, and let R be the circumradius of ΔABC. Prove that
$$\left(\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}\right)(DE+EF+FD)\ge\frac{AB+BC+CA}{R}.$$

SOLUTIONS

1. Writing $y+z=a$, $z+x=b$ and $x+y=c$, we have
$$x=\frac{b+c-a}{2}, \text{ etc.,}$$
thus $$yz=\frac{a^2-(b-c)^2}{4}=\frac{a^2-b^2-c^2+2bc}{4}, \text{ etc.,}$$
and hence, in cyclic sum notation,
$$\sum yz=\frac{1}{4}\sum(2bc-a^2).$$
Now assume $a \ge b \ge c$ without loss of generality; then
$$2bc-a^2 \le 2ca-b^2 \le 2ab-c^2$$
and therefore, by Chebyshev's inequality and the A.M. - G.M. inequality,
$$\begin{aligned}\frac{4}{9}(\Sigma yz)\left(\Sigma\frac{1}{(y+z)^2}\right)&=\frac{1}{3}\Sigma(2bc-a^2)\cdot\frac{1}{3}\Sigma\frac{1}{a^2}\\ &\ge\frac{1}{3}\Sigma\left((2bc-a^2)\cdot\frac{1}{a^2}\right)\\ &=\frac{1}{3}\left(\Sigma\frac{2bc}{a^2}\right)-1\\ &\ge\left(\Pi\frac{2bc}{a^2}\right)^{1/3}-1=2-1=1.\end{aligned}$$

2. (a) There are exactly 3 solutions, given by
$$(p, q, r) = (2, 4, 13), (2, 5, 8) \text{ and } (3, 3, 7).$$
Note first that
$$(p-1)(q-1)(r-1) = pqr-(pq+qr+rp)+(p+q+r)-1$$
which becomes, using the given equation,
$$(p-1)(q-1)(r-1) = 2(p+q+r-1) \quad \ldots(1)$$
If $p \ge 4$, then $4 \le p \le q \le r$ implies
$$(p-1)(q-1)(r-1) \ge 9(r-1) \text{ and}$$
$$2(p+q+r-1) \le 2(3r-1); \text{ and since}$$
$$9(r-1)-2(3r-1) = 3r-7 > 0,$$
(1) cannot hold in this case. Thus $p < 4$. Since $p = 1$ clearly does not satisfy (1), we have $p = 2$ or 3. When $p = 2$, (1) becomes $(q-1)(r-1) = 2(q+r+1)$ or $(q-3)(r-3) = 10$. Thus $q-3 = 1, r-3 = 10$ or $q-3=2$, $r-3 = 5$, yielding two solutions: (2, 4, 13) and (2, 5, 8). When $p = 3$, (1) becomes $(q-1)(r-1) = q+r+2$ or

$(q-2)(r-2) = 5$ which yields the third solution : $(3, 3, 7)$.

(b) The value is $\pi/4$ in all cases. To see this, set $A = \tan^{-1}(1/p)$, $B = \tan^{-1}(1/q)$ and $C = \tan^{-1}(1/r)$. Since

$$0 < \frac{1}{r} \le \frac{1}{q} \le \frac{1}{p} < 1,$$

we have $0 < C \le B \le A < \pi/4$ and thus $0 < A + B + C < 3\pi/4$. From the well known formula for $\tan(x+y)$ one easily deduces that for all x, y, z with $x + y + z \ne k\pi + \pi/2$ (where k denotes an integer),

$$\tan(x+y+z) = \frac{\tan x + \tan y + \tan z - \tan x \tan y \tan z}{1 - (\tan x \tan y + \tan y \tan z + \tan z \tan x)}.$$

Thus

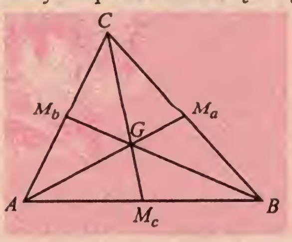
$$\tan(A+B+C) = \frac{\frac{1}{p} + \frac{1}{q} + \frac{1}{r} - \frac{1}{pqr}}{1 - \left(\frac{1}{pq} + \frac{1}{qr} + \frac{1}{rp}\right)}$$

$$= \frac{pq + qr + rp - 1}{pqr - (p+q+r)} = 1.$$

Hence $A + B + C = \pi/4$, that is,

$$\tan^{-1}\left(\frac{1}{p}\right) + \tan^{-1}\left(\frac{1}{q}\right) + \tan^{-1}\left(\frac{1}{r}\right) = \frac{\pi}{4}.$$

3. Let G be the center of gravity and M_a, M_b, M_c be the midpoints of the sides. We now apply the Mobius-Neuberg inequality to quadrilateral M_cBM_aG and get



$$\overline{BG} \cdot \overline{M_aM_c} \le \overline{M_cB} \cdot \overline{M_aG} + \overline{BM_a} \cdot \overline{GM_c},$$

i.e., $\dfrac{2m_b}{3} \cdot \dfrac{b}{2} \le \dfrac{c}{2} \cdot \dfrac{m_a}{3} + \dfrac{a}{2} \cdot \dfrac{m_c}{3}$,

i.e., $2bm_b \le cm_a + am_c$. ...(1)

Thus $b^2 m_b \le \frac{1}{2}(abm_c + bcm_a)$,

and cyclic permutation yields

$$c^2 m_c \le \frac{1}{2}(bcm_a + cam_b) \text{ and}$$

$$a^2 m_a \le \frac{1}{2}(cam_b + abm_c).$$

Now adding, we obtain the claimed inequality.

4. Since $1^3 + 2^3 + \ldots + n^3 = \left(\dfrac{n(n+1)}{2}\right)^2$ we see that when $k = 0$,

$$(x_1, x_2, \ldots, x_n; y) = \left(1, 2, \ldots, n; \frac{n(n+1)}{2}\right)$$

is a solution. To see that we can generate infinitely many solutions in general, set $c = \dfrac{(n(n-1))}{2}$ and notice that for all positive integers q, we have :

$$(c^k q^{3k+2})^3 + (2c^k q^{3k+2})^3 + \ldots + (nc^k q^{3k+2})^3$$

$$= c^{3k} q^{3(3k+2)} (1^3 + 2^3 + \ldots + n^3)$$

$$= c^{3k} q^{3(3k+2)} \left(\frac{n(n+1)}{2}\right)^2$$

$$= c^{3k+2} q^{3(3k+2)} = (cq^3)^{3k+2}.$$

That is, $(x_1, x_2, \ldots, x_n; y) = (c^k q^{3k+2}, 2c^k q^{3k+2}, \ldots, nc^k q^{3k+2}; cq^3)$ is a solution. This completes the proof.

5. According to Cauchy's inequality, we have

$$96 \times 216 = (a_1 + a_2 + \ldots + a_n)(a_1^3 + a_2^3 + \ldots + a_n^3)$$

$$= ((a_1^{1/2})^2 + (a_2^{1/2})^2 + \ldots + (a_n^{1/2})^2)((a_1^{3/2})^2 + \ldots + (a_n^{3/2})^2)$$

$$\ge (a_1^{1/2} a_1^{3/2} + a_2^{1/2} a_2^{3/2} + \ldots + a_n^{1/2} a_n^{3/2})^2$$

$$= (a_1^2 + a_2^2 + \ldots + a_n^2)^2 = 144^2,$$

where equality holds if and only if $a_i^{3/2} = \lambda a_i^{1/2}$ for all $1 \le i \le n$. We can see that $144^2 = 96 \times 216$, thus $a_i = \lambda$ for all $1 \le i \le n$, which implies

$$\lambda n = 96, \ \lambda^2 n = 144, \ \lambda^3 n = 216.$$

The first two equations imply $n = 96^2/144 = 64$. This gives $\lambda = 96/64 = 3/2$, and this (λ, n) satisfies $\lambda^3 n = 216$ too. Hence the only solution is

$$(a_i) = \overbrace{\left(\frac{3}{2}, \frac{3}{2}, \ldots, \frac{3}{2}\right)}^{64 \text{ times}}.$$

2nd Solution : Let $a_i = \frac{3}{2} b_i, \quad i = 1, 2, \ldots, n.$

Then we must find all sequences $0 < b_1 \le b_2 \le \ldots \le b_n$ such that

$$\sum_{i=1}^{n} b_i = \frac{2}{3}(96) = 64, \quad \sum_{i=1}^{n} b_i^2 = \frac{4}{9}(144) = 64,$$

$$\sum_{i=1}^{n} b_i^3 = \frac{8}{27}(216) = 64.$$

We can see that

$$\sum_{i=1}^{n} b_i(b_i - 1)^2 = \sum (b_i^3 - 2b_i^2 + b_i)$$

$$= 64 - 2 \cdot 64 + 64 = 0$$

It is obvious that since $b_i > 0$ it must be that $b_i = 1$ for all i, and thus $n = 64$. So the only sequence is $a_i = 3/2$, $i = 1, 2, \ldots, 64$.

6. If $P(x)$ is a polynomial of degree n, then $P(x^2)$ has degree $2n$, while $2xP(x+1)$ has degree $n + 1$. Thus $n = 1$, *i.e.* $P(x)$ is linear, say $P(x) = a + bx$.

From $x = 0$, $P(0) = 3 = a$, and from $x = -1$, $P(1) + 2 - 10 = -2P(0) + 3 = -3$.

So $P(1) = 5 = a + b$. Thus $b = 2$ and $P(x) = 2x + 3$.

7. First note that $\Sigma \dfrac{\sin A}{A} = \displaystyle\int_0^1 \Sigma \cos tA\, dt$,

where the sums are cyclic over A, B, C. We know that

$$3\sin^2\frac{t\pi}{3} \le \Sigma\sin^2 tA < \sin^2 t\pi \text{ for } 0 < t \le \frac{1}{2}$$

Using $\sin^2\frac{t\pi}{3} = \frac{1}{2}\left(1-\cos\frac{2t\pi}{3}\right)$, etc.,

and replacing t by $t/2$, we get

$$2+\cos t\pi < \Sigma\cos tA \le 3\cos\frac{t\pi}{3} \text{ for } 0 < t \le 1.$$

Thus

$$2 = \int_0^1 (2+\cos t\pi)dt < \int_0^1 \Sigma\cos tA dt \le \int_0^1 3\cos\frac{t\pi}{3}dt = \frac{9\sqrt{3}}{2\pi},$$

and the result follows.

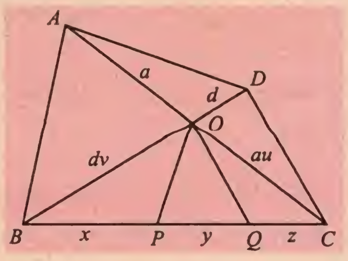
8. Let $[DOC] = u[AOD]$ and $[BOA] = v[AOD]$; then $[BCO] = uv[AOD]$, and the condition of the problem implies

$$1+u+v = uv. \qquad \ldots(1)$$

Furthermore,

$$\frac{AD}{OC} = \frac{1}{u} = \frac{BP}{PC} \text{ and } \frac{DO}{OB} = \frac{1}{v} = \frac{CQ}{QB}.$$

Let $BP = x$, $PQ = y$, and $QC = z$. Then, since $OP \parallel AB$ and $OQ \parallel DC$,



$$\frac{x}{y+z} = \frac{1}{u} \text{ and } \frac{z}{x+y} = \frac{1}{v},$$

so $\quad z = ux - y$ and $x = vz - y$.

We may let $y = 1$, which implies $x = v(ux-1) - 1$, so

$$x = \frac{v+1}{uv-1}.$$

Similarly $z = \frac{u+1}{uv-1}$, and thus by (1)

$$x+y+z = \frac{v+1+uv-1+u+1}{uv-1} = \frac{2uv}{u+v}.$$

Now $[BPO] : [PQO] : [QCO] = x : y : z$, and so

$$[OPQ] = \frac{y}{x+y+z}[BCO] = \frac{u+v}{2uv}\cdot uv[AOD]$$

$$= \frac{u+v}{2}[AOD] = \frac{[DOC]+[BOA]}{2},$$

which implies the result.

9. Applying f to both sides of the functional equation

$$f(f(f(x))) = f(x) \ge 0 \qquad \ldots(1)$$

gives $g(g(x)) = g(x)$, where $g(x) = f(f(x))$ for all $x \in R$. Of course g is also a differentiable function on R and $g(x) \ge 0$ for all $x \in R$. Then the range $T = g(R)$ of g is an interval contained in $[0, +\infty)$. Let a be the infimum of T. Since $g(t) = t$ for all $t \in T$ and g is continuous, it follows that $g(a) = a$. Assuming that T has more than one element, choose $\delta > 0$ such that $(a, a+\delta) \subseteq T$.

Then $x \in (a-\delta, a)$ implies $g(x) \ge g(a)\ (= a)$, hence

$$\frac{g(x)-g(a)}{x-a} \le 0.$$

Therefore

$$g'(a^-) = \lim_{x\to a^-}\frac{g(x)-g(a)}{x-a} \le 0. \qquad \ldots(2)$$

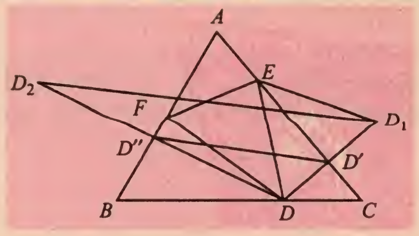
For $x \in (a, a+\delta)$ we have

$$\frac{g(x)-g(a)}{x-a} = 1,$$

hence $g'(a^+) = \lim_{x\to a^+}\frac{g(x)-g(a)}{x-a} = 1 \qquad \ldots(3)$

(2) and (3) are contradictory since g is differentiable at a. We are led to the conclusion that T is a single point, *i.e.*, g is a constant function, say $g(x) = c$ for all $x \in R$. This gives, using (1), that $f(c) = f(x)$ for all $x \in R$, showing that f is a constant function. Thus there is no non constant differentiable function satisfying (1).

10.



Let $s = \frac{DE+EF+FD}{2}$.

Reflect D in AC to get D_1, with DD_1 intersecting AC at D'. Reflect D in AB to get D_2, with DD_2 intersecting AB at D''. Then

$DE + EF + FD = D_2F + FE + ED_1 \ge D_2D_1 = 2D''D'$,

so $\quad s \ge D''D' \qquad \ldots(1)$

Now $AD'DD''$ is concyclic with AD as diameter, so

$$\sin A = \frac{D''D'}{AD}$$

Putting (1) and (2) together, $\frac{s}{AD} \ge \frac{D''D'}{AD} = \sin A$

Likewise,

$$\frac{s}{BE} \ge \sin B \text{ and } \frac{s}{CF} = \sin C.$$

Finally, adding these inequalities,

$$\left(\frac{1}{AD}+\frac{1}{BE}+\frac{1}{CF}\right)s \ge \sin A + \sin B + \sin C$$

$$= \frac{AB+BC+CA}{2R}$$

and the required inequality follows.

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers' comments and suggestions regarding the problems and solutions offered are always welcome.

1. If $f(x)=\begin{cases}|x|, & \text{when } x\le 2\\ [x], & \text{when } x>2\end{cases}$, then

(a) $\underset{x\to 2^-}{Lt} f(x)=-2$

(b) $\underset{x\to 2^+}{Lt} f(x)=-2$

(c) $\underset{x\to 2^+}{Lt} f(x)=f(2)$

(d) $\underset{x\to 2}{Lt} f(x)$ does not exist

2. If $n\in N$, then $\underset{x\to\infty}{Lt}\dfrac{x^n}{e^x}=0$.

(a) when n is even only (b) for no value of n
(c) for all values of n (d) when n is odd only

3. If $y=\sin^{-1}\left[\dfrac{1-x^2}{1+x^2}\right]$, then $\dfrac{dy}{dx}=$

(a) $\dfrac{-2}{1+x^2}$ (b) $\dfrac{2}{1+x^2}$

(c) $\dfrac{1}{2+x^2}$ (d) $\dfrac{2}{2-x^2}$

4. The derivative of $\tan^{-1}\left(\dfrac{\sqrt{1+x^2}-1}{x}\right)$ with respect to $\tan^{-1}\left(\dfrac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ at $x=0$ is

(a) $\dfrac{1}{8}$ (b) $\dfrac{1}{4}$ (c) $\dfrac{1}{2}$ (d) 1

5. If $y=\tan^{-1}\left(\dfrac{\log(e/x^2)}{\log(ex^2)}\right)+\tan^{-1}\left(\dfrac{3+2\log x}{1-6\log x}\right)$, then $\dfrac{d^2y}{dx^2}$ is

(a) 2 (b) 1 (c) 0 (d) −1

6. Let $f(x)=\begin{cases}xe^{-\left(\frac{1}{|x|}+\frac{1}{x}\right)} & ;x\ne 0\\ 0 & ;x=0\end{cases}$. Test whether $f(x)$ is differentiable at $x=0$. Is it continuous at $x=0$? Justify

7. A triangle ABC, right angle at C, with $CA=b$ and $CB=a$, moves such that the angular points A and B slide along x-axis and y-axis respectively. Find locus of C.

8. Prove that $\sin\theta\cdot\sec 3\theta=\dfrac{1}{2}(\tan 3\theta-\tan\theta)$ and hence find the sum to 'n' terms of the series
$\sin\theta\cdot\sec 3\theta+\sin 3\theta\cdot\sec 3^2\theta+\sin 3^2\theta\cdot\sec 3^3\theta+\ldots$

9. Consider a real valued function $f(x)$ satisfying $2f(xy)=(f(x))^y+(f(y))^x\ \forall\ x,y\in R$ and $f(1)=p$ where $p\ne 1$, then find $(p-1)\sum_{r=1}^{n}f(r)$.

10. A line makes angles $\alpha,\beta,\gamma,\delta$ with four diagonals of a cube. Prove that $\sum_{r\in\{\alpha,\beta,\gamma,\delta\}}\cos^2(r)=\dfrac{4}{3}$.

SOLUTIONS

1. (c) : $\lim_{x\to 2^+} f(x)=\lim_{h\to 0}[2+h]=2=f(2)$

$\lim_{x\to 2^-} f(x)=\lim_{h\to 0}|2-h|=2$

Hence limit exists

2. (c) :

$$\lim_{x\to\infty}\frac{x^n}{e^x}\quad\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{x\to\infty}\frac{\frac{d}{dx}x^n}{\frac{d}{dx}e^x}\quad\text{(Apply L'Hospital Rule)}$$

$$=\lim_{x\to\infty}\frac{nx^{n-1}}{e^x}\quad\left(\frac{\infty}{\infty}\text{ form}\right)$$

$$=\lim_{x\to\infty}\frac{n(n-1)x^{n-2}}{e^x}\quad\text{(Apply L'Hospital Rule)}$$

Apply L'Hospital Rule until we get $\left[\dfrac{0}{0}\text{ or }\dfrac{\infty}{\infty}\text{ form}\right]$

By : Prof. Shyam Bhushan, Director, Narayana Institute, Jamshedpur.

$$= \lim_{x\to\infty} \frac{n(n-1)(n-2)\ldots 4\cdot 3\cdot 2\cdot 1}{e^x}$$

$$= \lim_{x\to\infty} \frac{n!}{e^x} = \frac{n!}{e^{\infty}} = \frac{n!}{\infty} = 0$$

3. **(a) :** Put $x = \tan\theta$

$\therefore\ y = \sin^{-1}\left[\frac{1-\tan^2\theta}{1+\tan^2\theta}\right] = \sin^{-1}(\cos 2\theta)$

$= \sin^{-1}\sin\left(\frac{\pi}{2} - 2\theta\right) = \frac{\pi}{2} - 2\theta$

$\Rightarrow y = \frac{\pi}{2} - 2\tan^{-1}x$

$\frac{dy}{dx} = \frac{-2}{1+x^2}$

4. **(b) :** $P = \tan^{-1}\left(\frac{\sqrt{1+x^2}-1}{x}\right)$ put $x = \tan\theta$

$\therefore\ P = \tan^{-1}\left(\frac{\sqrt{1+\tan^2\theta}-1}{\tan\theta}\right) = \tan^{-1}\left(\frac{\sec\theta - 1}{\tan\theta}\right)$

$= \tan^{-1}\left(\frac{1-\cos\theta}{\sin\theta}\right) = \tan^{-1}\left(\frac{1-1+2\sin^2(\theta/2)}{2\sin(\theta/2)\cos(\theta/2)}\right)$

$= \tan^{-1}\left(\tan\frac{\theta}{2}\right)$

$\therefore\ P = \frac{\theta}{2} = \frac{\tan^{-1}x}{2}$

Hence

$\frac{dP}{dx} = \frac{1}{2(1+x^2)}$ ...(1)

$Q = \tan^{-1}\left(\frac{2x\sqrt{1-x^2}}{1-2x^2}\right)$ put $x = \sin\phi$

$= \tan^{-1}\left(\frac{2\sin\phi\sqrt{1-\sin^2\phi}}{1-2\sin^2\phi}\right)$

$= \tan^{-1}\left(\frac{2\sin\phi\cos\phi}{\cos 2\phi}\right) = \tan^{-1}(\tan 2\phi)$

$\therefore\ Q = 2\phi$

$\frac{dQ}{dx} = \frac{2}{\sqrt{1-x^2}}$ ...(2)

from (1) & (2)

$$\frac{dP}{dQ} = \frac{\frac{1}{2(1+x^2)}}{\frac{2}{\sqrt{1-x^2}}}$$

Put $x = 0$

$\left.\frac{dP}{dQ}\right|_{x=0} = \frac{1}{4}$

5. **(c) :** $y = \tan^{-1}\left(\frac{\log(e/x^2)}{\log(ex^2)}\right) + \tan^{-1}\left(\frac{3+2\log x}{1-6\log x}\right)$

In given function

Put $\log x^2 = \tan\theta$

$y = \tan^{-1}\left(\frac{1-\tan\theta}{1+\tan\theta}\right) + \tan^{-1}\left(\frac{3+\tan\theta}{1-3\tan\theta}\right)$ $[\log e = 1]$

$= \tan^{-1}\left[\tan\left(\frac{\pi}{4} - \theta\right)\right] + \tan^{-1}3 + \tan^{-1}(\tan\theta)$

$= \frac{\pi}{4} - \theta + \tan^{-1}3 + \theta = \frac{\pi}{4} + \tan^{-1}3$

$\therefore\ \frac{dy}{dx} = 0$

Then $\frac{d^2y}{dx^2} = 0.$

6. $f(x) = \begin{cases} xe^{-2/x} & ;x > 0 \\ 0 & ;x = 0 \\ x & ;x < 0 \end{cases}$

$\therefore\ f(0) = 0$

L.H.D.

$f'(0^-) = \underset{h\to 0}{Lt} \frac{f(0-h) - f(0)}{-h} = \underset{h\to 0}{Lt} \frac{-h}{-h} = 1$

R.H.D.

$f'(0^+) = \underset{h\to 0}{Lt} \frac{f(0+h) - f(0)}{h} = \underset{h\to 0}{Lt} \frac{he^{-2/h}}{h}$

$= e^{-\infty} = 0$

$\because$ L.H.D. $\neq$ R.H.D.

$\therefore$ f is not differentiable at $x = 0$

L.H.D. at $x = 0$

$\lim_{x\to 0^-} f(x) = \lim_{h\to 0}(0-h) = \lim_{h\to 0}(-h) = 0$

R.H.L. at $x = 0$

$\lim_{x\to 0^+} f(x) = \lim_{h\to 0}(0+h)e^{-2/(0+h)} = \lim_{h\to 0} he^{-2/h} = 0$

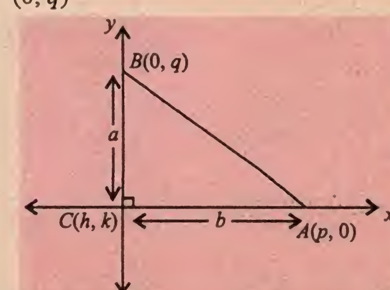
$f(0) = 0$ (Given)

$\therefore$ L.H.L. (at $x = 0$) = R.H.L. (at $x = 0$) = $f(0)$

$\therefore$ $f(x)$ is continuous

7. Let $A = (p, 0)$

$B = (0, q)$



Let $C(h, k)$ be the any point on the locus

$CB = a = \sqrt{h^2 + (k-q)^2}$ ...(i)

$CA = b = \sqrt{(h-p)^2 + k^2}$ ...(ii)

$$AB = \sqrt{p^2 + q^2} \quad \text{...(iii)}$$

$\because \quad \angle C = 90°$

$\therefore \quad AB^2 = AC^2 + BC^2$

$\Rightarrow \quad p^2 + q^2 = a^2 + b^2 \quad \text{...(iv)}$

from (i) and (ii)

$$q = k \pm \sqrt{a^2 - h^2}\ ,\ p = h \pm \sqrt{b^2 - k^2}$$

$\therefore$ from (iv), $p^2 + q^2 = a^2 + b^2$

or $(h \pm \sqrt{b^2 - k^2})^2 + (k \pm \sqrt{a^2 - h^2})^2 = a^2 + b^2$

$\Rightarrow \quad 2h\sqrt{b^2 - k^2} = 2k\sqrt{a^2 - h^2}$

$\Rightarrow \quad h^2b^2 - h^2k^2 = k^2a^2 - h^2k^2$ (squaring both sides)

$\Rightarrow \quad h^2b^2 = k^2a^2 \quad \Rightarrow \quad hb = \pm ka \quad \Rightarrow \quad bx = \pm ay$

Locus is $bx \pm ay = 0$

8. $$\frac{\sin\theta}{\cos 3\theta} = \frac{\cos\theta.\sin\theta}{\cos\theta.\cos 3\theta} = \frac{\sin 2\theta}{2\cos\theta\cos 3\theta}$$

$$= \frac{1}{2}\frac{\sin(3\theta - \theta)}{\cos\theta.\cos 3\theta} = \frac{1}{2}(\tan 3\theta - \tan\theta)$$

$$\therefore \sum_{r=1}^{n} \sin 3^{r-1}\theta.\sec 3^r\theta = \sum_{r=1}^{n} \frac{\sin 3^{r-1}\theta}{\cos 3^r\theta}$$

$$= \frac{1}{2}\sum_{r=1}^{n} \frac{\sin(3^r\theta - 3^{r-1}\theta)}{\cos 3^r\theta.\cos 3^{r-1}\theta} = \frac{1}{2}\sum_{r=1}^{n}(\tan 3^r\theta - \tan 3^{r-1}\theta)$$

$$= \frac{1}{2}[\tan 3^n\theta - \tan\theta]$$

9. $2f(xy) = (f(x))^y + (f(y))^x \ \forall\ x, y \in R$

Put $y = 1$

$\Rightarrow \quad 2f(x) = f(x) + (f(1))^x$

$\Rightarrow \quad f(x) = p^x \ (\because f(1) = p)$

$$\therefore \sum_{r=1}^{n} f(r) = \sum_{r=1}^{n} p^r = \frac{p^{n+1} - p}{p - 1}$$

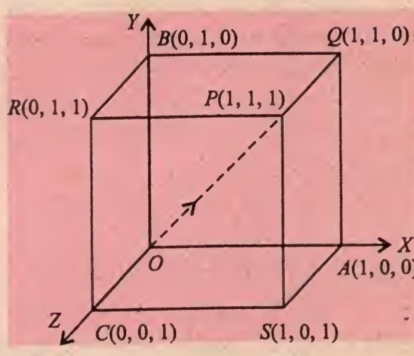
$$(p-1)\sum_{r=1}^{n} f(r) = (p^{n+1} - p)$$

10. D.C's of $OP = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

D.C's of $AR = \left(-\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

D.C's of $BS = \left(\frac{1}{\sqrt{3}}, \frac{-1}{\sqrt{3}}, \frac{1}{\sqrt{3}}\right)$

D.C's of $CQ = \left(\frac{1}{\sqrt{3}}, \frac{1}{\sqrt{3}}, -\frac{1}{\sqrt{3}}\right)$



Let l, m, n be the D.C's of required line

$$\cos\alpha = ll_1 + mm_1 + nn_1$$

$$= l\cdot\frac{1}{\sqrt{3}} + m\cdot\frac{1}{\sqrt{3}} + n\cdot\frac{1}{\sqrt{3}} = \frac{1}{\sqrt{3}}(l + m + n)$$

Similarly

$$\cos\beta = \frac{1}{\sqrt{3}}(l - m + n), \cos\gamma = \frac{1}{\sqrt{3}}(l + m - n)$$

$$\cos\delta = \frac{1}{\sqrt{3}}(-l + m + n)$$

$\therefore \quad \cos^2\alpha + \cos^2\beta + \cos^2\gamma + \cos^2\delta$

$$= \frac{1}{3}((l+m+n)^2 + (l-m+n)^2 + (l+m-n)^2 + (-l+m+n)^2)$$

$$= \frac{4}{3}$$

SOLVED PAPER

AMU - 2009 (Engg.)

Aligarh Muslim University

1. In an ellipse, if the lines joining focus to the extremities of the minor axis form an equilateral triangle with the minor axis, then the eccentricity of the ellipse is

(a) $\frac{\sqrt{3}}{2}$ (b) $\frac{\sqrt{3}}{4}$ (c) $\frac{1}{\sqrt{2}}$ (d) $\sqrt{\frac{2}{3}}$

2. If the planes $x = cy + bz$,
$y = az + cx$,
$z = bx + ay$.
pass through a line then $a^2 + b^2 + c^2 + 2abc$ is

(a) 0 (b) 1 (c) 2 (d) 3

3. If $\cos^{-1}x + \cos^{-1}y + \cos^{-1}z + \cos^{-1}t = 4\pi$, then the value of $x^2 + y^2 + z^2 + t^2$ is

(a) $xy + zy + zt$ (b) $1 - 2xyzt$
(c) 4 (d) 6

4. Four dice are rolled. The number of possible outcomes in which at least one dice shows 2 is

(a) 625 (b) 671 (c) 1023 (d) 1296

5. If $f(x + y) = f(x)\, f(y)$ for all x and y and if $f(5) = 2$ and $f'(0) = 3$, then $f'(5)$ is

(a) 0 (b) 2 (c) 5 (d) 6

6. The equation of the curve satisfying the differential equation $y_2(x^2 + 1) = 2xy_1$ passing through the point (0, 1) and having slope of tangent at $x = 0$ as 3 is

(a) $y = x^3 + 3x + 1$ (b) $y = x^3 - 3x + 1$
(c) $y = x^2 + 3x + 1$ (d) $y = x^2 - 3x + 1$

7. For the function $f(x) = \lim_{n\to\infty} \frac{\log(2+x) - x^{2n}\sin x}{1 + x^{2n}}$, which of the following is true?

(a) $\lim_{x\to 1^-} f(x)$ does not exist
(b) $\lim_{x\to 1^+} f(x)$ does not exist
(c) Both limits exist and $\lim_{x\to 1^-} f(x) = \lim_{x\to 1^+} f(x)$
(d) Both limits exist and $\lim_{x\to 1^-} f(x) \neq \lim_{x\to 1^+} f(x)$

8. The curve $y - e^{xy} + x = 0$ has a vertical tangent at the point

(a) (1, 1) (b) (1, 0)
(c) (0, 1) (d) none of these

9. If a hyperbola passes through the foci of the ellipse $\frac{x^2}{25} + \frac{y^2}{16} = 1$ and its transverse and conjugate axes coincide with the major and minor axes of the ellipse and product of their eccentricities be 1, then the equation of hyperbola is

(a) $\frac{x^2}{9} - \frac{y^2}{25} = 1$ (b) $\frac{x^2}{9} - \frac{y^2}{16} = 1$
(c) $\frac{x^2}{16} - \frac{y^2}{25} = 1$ (d) none of these

10. If p, q, r are positive and are in A.P., then roots of the quadratic equation $px^2 + qx + r = 0$ are complex for

(a) $\left|\frac{r}{p} - 7\right| \geq 4\sqrt{3}$ (b) $\left|\frac{p}{r} - 7\right| < 4\sqrt{3}$
(c) all p and r (d) no p and r

11. For any two sets A and B if $A \cap X = B \cap X = \phi$ and $A \cup X = B \cup X$ for some set X, then

(a) $A - B = A \cap B$ (b) $A = B$
(c) $B - A = A \cap B$ (d) none of these

12. If the coefficient of variation of a distribution is 45% and the mean is 12, then its standard deviation is

(a) 5.2 (b) 5.3
(c) 5.4 (d) none of these

13. The largest term in the expansion of $(4 + 2x)^{49}$ where $x = 1/3$ is

(a) 3rd (b) 5th
(c) 8th (d) none of these

14. The curve described parametrically by $x = t^2 + t$ and $y = t^2 - t$ represents

(a) a pair of straight lines (b) an ellipse
(c) a parabola (d) a hyperbola

15. Let r be a relation from R (set of real numbers) to R defined by $r = \{(a, b) \mid a, b \in R$ and $a - b + \sqrt{3}$ is an irrational number$\}$. The relation r is

(a) an equivalence relation (b) reflexive only
(c) symmetric only (d) transitive only

16. The set $C = \{z \mid z\bar{z} + a\bar{z} + \bar{a}z + b = 0, b \in R$ and $b < |a|^2\}$ is

$\Rightarrow 5y < -8 \Rightarrow y < \frac{-8}{5}$

$\therefore x > \frac{8}{15}, y < -\frac{8}{5}$

20. (a) : For the first 11 terms in A.P. , $d = 2$

Middle term of the A.P. is 6^{th} term.

$a_6 = a + 5d = a + 10,$

$a_{11} = a + 10d = a + 20$

For next 11 terms in G.P. $r = 2$

Middle term of the G.P. is 6^{th} terms = $b(2)^5$ where b = last term of A.P.

$\Rightarrow (a + 20)32 = a + 10$

$\Rightarrow 32a + 20 \times 32 = a + 10 \Rightarrow 31a = 10 - 20 \times 32$

$\Rightarrow a = \frac{-630}{31}$

$\therefore$ Middle term of entire sequence is 11^{th} term

$= \frac{-630}{31} + 10 \times d = \frac{-630}{31} + 10 \times 2 = -\frac{10}{31}$

$\therefore$ required term is $\frac{-10}{31}$

21. (b) : $\begin{vmatrix} x^n & x^{n+2} & x^{n+3} \\ y^n & y^{n+2} & y^{n+3} \\ z^n & z^{n+2} & z^{n+3} \end{vmatrix} = x^n y^n z^n \begin{vmatrix} 1 & x^2 & x^3 \\ 1 & y^2 & y^3 \\ 1 & z^2 & z^3 \end{vmatrix}$

$= x^n y^z z^n \begin{vmatrix} 1 & x^2 & x^3 \\ 0 & y^2 - x^2 & y^3 - x^3 \\ 0 & z^2 - x^2 & z^3 - x^3 \end{vmatrix} \begin{matrix} (R_3 \to R_3 - R_1 \\ R_2 \to R_2 - R_1) \end{matrix}$

$= x^n y^n z^n (y - x)(z - x) \begin{vmatrix} 1 & x^2 & x^3 \\ 0 & x + y & x^2 + y^2 + xy \\ 0 & x + z & x^2 + z^2 + xz \end{vmatrix}$

$= x^n y^n z^n (x - y)(x - z)(x^3 + xz^2 + x^2 z + yx^2 + yz^2 + xyz - x^3 - xy^2 - x^2 y - zx^2 - zy^2 - xyz)$

$= x^n y^n z^n (x - y)(x - z)[xz^2 + yz^2 - xy^2 - zy^2]$

$= x^n y^n z^n (x - y)(x - z)[x (z^2 - y^2) + yz (z - y)]$

$= (x^n y^n z^n)(x - y)(x - z)(z - y)(xy + yz + zx)$

$= x^n y^n z^n (x - y)(x - z) (z - y)(xy + yz + zx)$

$= x^{n+1} y^{n+1} z^{n+1} (x - y)(y - z)(z - x)\left(\frac{1}{x} + \frac{1}{y} + \frac{1}{z}\right)$

Comparing with given value of determinant

$n + 1 = 0 \Rightarrow n = -1$

22. (d) : Total number of cards = 52

Chance of getting spade = $\frac{13}{52} = \frac{1}{4}$

$\therefore$ Chance of not getting spade $= 1 - \frac{1}{4} = \frac{3}{4}$

$\therefore$ Chance that the person will not get spade in first two draws $= \frac{3}{4} \cdot \frac{3}{4} = \frac{9}{16}$

23. (c) : Force $(\bar{f}) = \frac{6 \times (9\bar{i} + 6\bar{j} + 2\bar{k})}{\sqrt{81 + 36 + 4}}$

$= \frac{6}{11}(9\bar{i} + 6\bar{j} + 2\bar{k})$

Displacement vector $(\bar{d}) = (7 - 3)\bar{i} + (-6 - 4)\bar{i} + (8 + 15)\bar{k}$

$= 4\bar{i} - 10\bar{j} + 23\bar{k}$

$\therefore$ work done $= \bar{f}.\bar{d} = \frac{6}{11}(9 \times 4 - 6 \times 10 + 23 \times 2)$

$= \frac{6}{11}(36 - 60 + 46) = \frac{6}{11} \times 22 = 12$

24. (b) : Equation to line in intercept form

$\frac{x}{a} + \frac{y}{b} = 1$. It passes through $\left(\frac{1}{5}, \frac{1}{5}\right)$

$\Rightarrow \quad a + b = 5ab \qquad .....(i)$

point $P(x,y)$ divides AB joining $A(a, 0)$ and $B(0, b)$ internally in ratio 3 : 1

$\Rightarrow x = \frac{a}{4}, y = \frac{3b}{4} \Rightarrow a = 4x, b = \frac{4y}{3}$

Keeping values of a and b in eq. (i) we get

$4x + \frac{4y}{3} = 5(4x)\left(\frac{4y}{3}\right)$

$\Rightarrow 3x + y = 20xy$

25. (d) : Since

$-\sqrt{3^2 + 4^2} \le 3\cos x + 4\sin x \le \sqrt{3^2 + 4^2} \quad \forall x \in R$

$\Rightarrow -5 + 5 \le 3\cos x + 4\sin x + 5 \le 5 + 5$

$\Rightarrow 0 \le 3\cos x + 4\sin x + 5 \le 10 \quad \forall x \in R$

$\therefore$ Max. value = 1

26. (c)	27. (d)	28. (c)	29. (a)	30. (d)	31. (a)
32. (b)	33. (d)	34. (c)	35. (b)	36. (c)	37. (a)
38. (b)	39. (b,d)	40. (c)	41. (c)	42. (c)	43. (b)
44. (c)	45. (a)	46. (d)	47. (d)	48. (d)	49. (b)
50. (d)	51. (c)	52. (a)	53. (b)	54. (d)	55. (b)
56. (c)	57. (d)	58. (c)	59. (a)	60. (b)	61. (b)
62. (b)	63. (b)	64. (b)	65. (d)	66. (d)	67. (c)
68. (d)	69. (a)	70. (d)			

For more detailed solutions Read MTG's New updated AMU Explorer.

CONCEPTUAL PROBLEMS

Functions

1. Range of $f(x) = \sin^{-1}(\log_2(\sin x - \cos x))$ is :

(a) $\left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$ (b) $\left[-\frac{\pi}{2}, \frac{\pi}{6}\right]$ (c) $\left[0, \frac{\pi}{6}\right]$ (d) $\left[-\frac{\pi}{6}, \frac{\pi}{6}\right]$

2. Let $f(x) = \frac{\sin x}{\sqrt{1+\tan^2 x}} - \frac{\cos x}{\sqrt{1+\cot^2 x}}$, the range of $f(x)$ is :

(a) $\{0\}$ (b) $[-1, 0]$ (c) $[-1, 1]$ (d) $[0, 1]$

3. Domain of $f(x) = \sqrt{p.\log_3(x^2-1).\sin(-\theta)}$, $p = \cos\theta + \sin\theta, -\pi < \theta < \frac{-3\pi}{4}$

(a) $(-\infty, -\sqrt{2})$ (b) $[\sqrt{2}, \infty)$

(c) $[-\sqrt{2}, \sqrt{2}]$ (d) $(1, \sqrt{2}] \cup [-\sqrt{2}, -1)$

4. A function $f: R \to R$ is defined by $f(x) = \frac{x^2-2}{x - 2\sin\alpha}$ is onto function then range of α is

(a) $\left[\frac{\pi}{4}, \frac{3\pi}{4}\right]$ (b) $\left[\frac{-\pi}{4}, \frac{\pi}{4}\right]$

(c) $\left[\frac{5\pi}{4}, \frac{7\pi}{4}\right]$ (d) R

5. A quadratic equation $(\cos 2)x^2 + bx + c = 0$, has positive roots, then domain of $f(x) = \sqrt{c\log_{0.4}(x-x^2)}$

(a) $(0, 1)$ (b) $(-\infty, 0) \cup (1, \infty)$

(c) $(1, \infty)$ (d) ϕ

6. If $x^3 f(x) = \sqrt{1+\cos 2x} + |f(x)|, -\frac{3\pi}{4} < x < -\frac{\pi}{2}$ and $f(x) = \frac{\lambda \cos x}{x^3+1}$. Then λ is :

(a) $\sqrt{2}$ (b) $-\sqrt{2}$ (c) 1 (d) 2

7. If the graph of the function $f(x) = \frac{a^{\sin x}-1}{x^n(a^{\sin x}+1)}$ is symmetric about y-axis, then n =

(a) 1/4 (b) −1/3 (c) 2/3 (d) 2

8. If $f(x) + 2f(1-x) = x^2 + 1$, $\forall x \in R$, then range of $f(x)$ is :

(a) $(-\infty, 1)$ (b) $\left[-\frac{1}{3}, \infty\right)$ (c) $(-1, 1)$ (d) $[0, \infty)$

9. Range of the function f defined by $f(x) = \left[\frac{1}{\cos\{x\}}\right]$, where [.] and {.} denote the greatest integer and fractional part function respectively, is :

(a) set of integers

(b) set of all natural numbers

(c) $\{2, 3, 4, \ldots\ldots\ldots\}$ (d) $\{1\}$

10. If $y = f(x)$ is one-one function and (5, 1) is a point on its graph, which one of the following statements is correct ?

(a) (1, 5) is a point on the graph of the inverse function $y = f^{-1}(x)$

(b) $f(5) = f(1)$

(c) The graph of the inverse function $y = f^{-1}(x)$ will be symmetric about the y-axis

(d) $f(f^{-1}(5)) = 1$

SOLUTIONS

1. **(b)** : $f(x)$ is real valued function when $\sin x - \cos x > 0$ and $-1 \le \log_2(\sin x - \cos x) \le 1$

($\because$ $\log_2 x$ is defined when $x > 0$ and $\sin^{-1}x$ is defined when $-1 \le x \le 1$)

$\sin x - \cos x > 0$ and $\frac{1}{2} \le \sin x - \cos x \le 2$

But $-\sqrt{2} \le \sin x - \cos x \le \sqrt{2}$

$\therefore \frac{1}{2} \le \sin x - \cos x \le \sqrt{2}$

$\Rightarrow \log_2(1/2) \le \log_2(\sin x - \cos x) \le \log_2\sqrt{2}$

$\Rightarrow -1 \le \log_2(\sin x - \cos x) \le \frac{1}{2}$

$\Rightarrow \sin^{-1}(-1) \le \sin^{-1}(\log_2(\sin x - \cos x)) \le \sin^{-1}\frac{1}{2}$

$\Rightarrow -\frac{\pi}{2} \le f(x) \le \frac{\pi}{6}$ So, Range is $\left[-\frac{\pi}{2}, \frac{\pi}{6}\right]$

2. **(c)** : $f(x) = \frac{\sin x}{|\sec x|} - \frac{\cos x}{|\operatorname{cosec} x|}$

$= \sin x\,|\cos x| - \cos x\,|\sin x|$

clearly, domain of $f(x)$ is $R - \left\{n\pi, (2n+1)\frac{\pi}{2}\right\}$

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$n \in I$, and period of $f(x)$ is 2π

$$f(x)=\begin{cases} 0 & , x\in(0,\pi/2) \\ -\sin 2x & , x\in(\pi/2,\pi) \\ 0 & , x\in(\pi,3\pi/2) \\ \sin 2x & , x\in(3\pi/2,2\pi) \end{cases} \Rightarrow \text{range of } f(x) \text{ is } [-1,1]$$

3. (d) : $p=\cos\theta+\sin\theta=\sqrt{2}\sin\left(\theta+\frac{\pi}{4}\right)$

and $\sin(-\theta)>0$ when $\theta\in\left(-\pi,-\frac{3\pi}{4}\right)$

$\because\quad -\pi<\theta<-\frac{3\pi}{4}\Rightarrow\frac{-3\pi}{4}<\theta+\frac{\pi}{4}<-\frac{\pi}{2}$

$\Rightarrow\quad -1<\sin\left(\theta+\frac{\pi}{4}\right)<-\frac{1}{\sqrt{2}}$, so $p<0$

$\because\quad p\log_3(x^2-1)\sin(-\theta)\geq 0$

$\therefore\quad \log_3(x^2-1)\leq 0\Rightarrow x^2-1>0$ and $x^2-1\leq 3^0$

$\Rightarrow\quad x^2>1$ and $x^2\leq 2$

$\Rightarrow\quad x>1$ or $x<-1$ and $-\sqrt{2}\leq x\leq\sqrt{2}$

$\Rightarrow\quad x\in(1,\sqrt{2}]\cup[-\sqrt{2},-1)$

4. (b) : Let $\sin\alpha=\lambda$, Let $\frac{x^2-2}{x-2\lambda}=y$

$\therefore\quad yx-2\lambda y=x^2-2\Rightarrow x^2-yx+2\lambda y-2=0$

As x is real, $D\geq 0$

$\therefore\quad y^2-4(2\lambda y-2)\geq 0$ or $y^2-8\lambda y+8\geq 0$

This has to be true for all real y

Hence its $D\leq 0$, *i.e.*, $(-8\lambda)^2-4.8\leq 0$

$\Rightarrow 64\lambda^2-32\leq 0$ $\Rightarrow 32(2\lambda^2-1)\leq 0$

$\Rightarrow -\frac{1}{\sqrt{2}}\leq\lambda\leq\frac{1}{\sqrt{2}}$ or $-\frac{1}{\sqrt{2}}\leq\sin\alpha\leq\frac{1}{\sqrt{2}}$

So Range of α is $\left[-\frac{\pi}{4},\frac{\pi}{4}\right]$

5. (d) : Both roots are positive and real

Let α and β be roots then

$\alpha.\beta=\frac{c}{\cos 2}$ $\left(\cos 2=\cos\left(\frac{2\times 180°}{\pi}\right)=\cos 115°<0\right)$

$\therefore\quad c<0\quad (\because \alpha\beta>0)$

$\therefore\quad$ For domain $c\log_{0.4}(x-x^2)\geq 0$

$\therefore\quad \log_{0.4}(x-x^2)\leq 0\quad (\because c<0)$

$\Rightarrow\quad x-x^2>0$ and $x-x^2\geq 1$

(Since base 0.4 lies in (0, 1), so sign of inequality changes)

$\Rightarrow\quad x(x-1)<0$ and $x^2-x+1\leq 0$

$\Rightarrow\quad 0<x<1$ and ϕ $\left(\because x^2-x+1=\left(x-\frac{1}{2}\right)^2+\frac{3}{4}>0\right)$

$\therefore\quad x\in\phi$

6. (b) : $x^3f(x)=\sqrt{1+\cos 2x}+|f(x)|=\sqrt{2\cos^2 x}+|f(x)|$

$=\sqrt{2}|\cos x|+|f(x)|$

Since R.H.S. is positive, so L.H.S. must be positive

So $x^3f(x)>0\Rightarrow f(x)<0$,

$\left(\because x^3<0 \text{ when } -\frac{3\pi}{4}<x<-\frac{\pi}{2}\right)$

$\left(|f(x)|=-f(x),\ f(x)<0 \text{ and } |\cos x|=-\cos x. \text{ when } x\in\left(-\frac{3\pi}{4},-\frac{\pi}{2}\right)\right)$

$\therefore\quad x^3f(x)=-\sqrt{2}\cos x-f(x)$

$\therefore\quad f(x)=\frac{-\sqrt{2}\cos x}{x^3+1}$, Since $f(x)=\frac{\lambda\cos x}{x^3+1}$

$\therefore\quad \lambda=-\sqrt{2}$.

7. (b) : $f(x)=\frac{1}{x^n}\times\frac{a^{\sin x}-1}{a^{\sin x}+1}$ is even function

($\therefore$ graph of f is symmetric about y-axis)

Let $h(x)=\frac{a^{\sin x}-1}{a^{\sin x}+1}$ and $g(x)=\frac{1}{x^n}$

$h(-x)=\frac{a^{-\sin x}-1}{a^{-\sin x}+1}=\frac{1-a^{\sin x}}{1+a^{\sin x}}=-h(x)$

i.e., h is an odd function

Hence $g(x)\times$ odd function = even function

$\therefore\quad g(x)$ is an odd function i.e. $g(-x)=-g(x)$

$\Rightarrow\quad \frac{1}{(-x)^n}=-\frac{1}{x^n}\Rightarrow\frac{1}{(-1)^nx^n}=-\frac{1}{x^n}\quad\Rightarrow(-1)^{n+1}=1$

This is possible only when $n=-\frac{1}{3}$

8. (b) : $f(x)+2f(1-x)=x^2+1$ ..... (i)

replacing $x\to(1-x)$, we get

$f(1-x)+2f(x)=(1-x)^2+1$

$\Rightarrow\quad f(1-x)+2f(x)=2+x^2-2x$ ..... (ii)

From (i) and (ii), we get $f(x)=\frac{1}{3}(x^2-4x+3)$

$\therefore\quad f(x)=\frac{(x-2)^2}{3}-\frac{1}{3}\geq-\frac{1}{3}$

So range of $f(x)$ will be $\left[-\frac{1}{3},\infty\right)$

9. (d) : $0\leq\{x\}<1\quad\Rightarrow\cos 1<\cos\{x\}\leq\cos 0$

$\therefore\ 1\leq\frac{1}{\cos\{x\}}<\frac{1}{\cos 1}<2$

$\therefore\quad\left[\frac{1}{\cos\{x\}}\right]=1$, i.e. range of the function is $\{1\}$

10. (a) : Graph of $f^{-1}(x)$ is symmetric about line $y=x$, so if a point (a, b) lie on $f(x)$ then the point (b, a) lie on $f^{-1}(x)$, so option (A) is correct.

MATHS MUSING

SOLUTION SET - 79

1. **(a), (b), (c)** : The result follows from the fact $a^n + b^n$ is divisible by $a + b$ for n odd and $a^n - b^n$ is divisible by $a - b$ for all n

2. **(d)** : $P(i) = ki$, $i = 1$ to 6

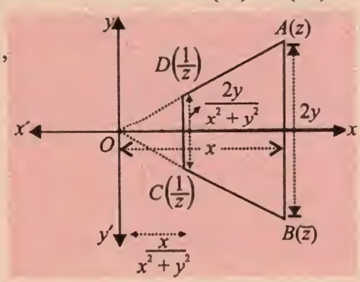
The desired probability $= \dfrac{P(5)}{P(1)+P(3)+P(5)} = \dfrac{5}{1+3+5}$
$= \dfrac{5}{9}$

3. **(a), (b), (c), (d)** : $f(x) = 2x^3 - 3(\lambda + 2)x^2 + 2\lambda x + 5$
$f'(x) = 6\,(x^2 - (\lambda + 2)x + (\lambda/3)) = 0$ has two distinct roots

if $(\lambda+2)^2 - \dfrac{4\lambda}{3} = \lambda^2 + \dfrac{8\lambda}{3} + 4 = \left(\lambda + \dfrac{4}{3}\right)^2 + \dfrac{20}{9} > 0$ for all λ

4. **(c)** : Let the points be $A(z), B(\bar{z}), C\left(\dfrac{1}{z}\right), D\left(\dfrac{1}{\bar{z}}\right)$

where $z = x + iy$
Area of $\Delta OAB = |xy|$,



Area of $\Delta OCD = \dfrac{|xy|}{(x^2+y^2)^2}$

$\therefore$ The desired area = |area of ΔOCD – area of ΔOAB|

$$= |xy|\left|1 - \frac{1}{(x^2+y^2)^2}\right|$$

$$= \frac{1}{4}|z^2 - \bar{z}^2|\left|1 - \frac{1}{|z|^4}\right| \quad \left(\because\ x = \frac{z+\bar{z}}{2}, y = \frac{z-\bar{z}}{2i}\right)$$

5. **(d)** : $\begin{array}{c|cc} & C & D \\ \hline A & 4 & 5 \\ B & 5 & 4 \end{array}$

The possible selections are

$$\left|\begin{matrix}4 & 0\\0 & 4\end{matrix}\right. \left|\begin{matrix}3 & 1\\1 & 3\end{matrix}\right. \left|\begin{matrix}2 & 2\\2 & 2\end{matrix}\right. \left|\begin{matrix}1 & 3\\3 & 1\end{matrix}\right. \left|\begin{matrix}0 & 4\\4 & 0\end{matrix}\right.$$

The number of ways is

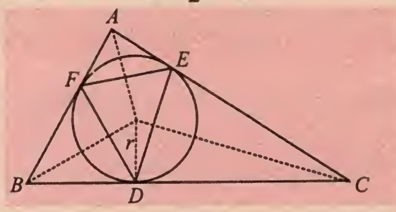
$$\binom{4}{4}^2\binom{5}{0}^2 + \binom{4}{3}^2\binom{5}{1}^2 + \binom{4}{2}^2\binom{5}{2}^2 + \binom{4}{1}^2\binom{5}{3}^2$$

$$+\binom{4}{0}^2\binom{5}{4}^2 = 1 + 20^2 + 60^2 + 40^2 + 5^2 = 5626$$

6. **(c)** : r, R and Δ be the inradius, circumradius and area of ΔABC, I is the incentre of ΔABC

In ΔBFD, $\angle BDF = \angle BFD = 90° - \dfrac{\angle B}{2}$ $(BD = BF)$

Similarly $\angle CDE = 90° - \dfrac{\angle C}{2}$



$\therefore\ \angle EDF = 180° - (\angle BDF + \angle CDF)$

$= \dfrac{\angle B + \angle C}{2} = 90° - \dfrac{\angle A}{2}$

$\therefore\ \angle D = \dfrac{\pi}{2} - \dfrac{\angle A}{2}, \angle E = \dfrac{\pi}{2} - \dfrac{\angle B}{2}, \angle F = \dfrac{\pi}{2} - \dfrac{\angle C}{2}$

$\Delta' = 2r^2 \sin D \sin E \sin F$ (r is circumradius of ΔDEF)

$= 2r^2 \cos\dfrac{A}{2}\cos\dfrac{B}{2}\cos\dfrac{C}{2} = \dfrac{r^2 s}{2R}$

7. **(a)** : $\dfrac{r'}{r} = 4\sin\dfrac{D}{2}\sin\dfrac{E}{2}\sin\dfrac{F}{2}$

$= 4\sin\dfrac{B+C}{4}\sin\dfrac{C+A}{4}\sin\dfrac{A+B}{4} = -1 + \sum\sin\dfrac{A}{2}$

8. **(b)** : $\dfrac{r_1'}{r} = 4\sin\dfrac{D}{2}\cos\dfrac{E}{2}\cos\dfrac{F}{2}$

$= 4\sin\dfrac{B+C}{4}\cos\dfrac{C+A}{4}\cos\dfrac{A+B}{4}$

$= 1 - \sin\dfrac{A}{2} + \sin\dfrac{B}{2} + \sin\dfrac{C}{2}$

9. **(a) – (t), (b) – (p), (c) – (r), (d) – (t)**

Continuity at $x = 1 \Rightarrow b = 0$
Differentiability at $x = 1 \Rightarrow a = 1$
Continuity at $x = 3 \Rightarrow d = 3c$
Differentiability at $x = 3 \Rightarrow c = 1/3, d = 1$.

10. If $P(x_1, y_1)$ is the midpoint of BC, its equation is $S_1 = S_{11}$, $yy_1 - 3(x + x_1) = y_1^2 - 6x_1$
It passes through (9, 5) $\quad \therefore 5y_1 - 3(9 + x_1) = y_1^2 - 6x_1$
$\therefore$ The locus of P is $y^2 - 5y = 3x - 27$

$$\left(y - \frac{5}{2}\right)^2 = 3\left(x - \frac{83}{12}\right)$$

Comparing with $y^2 = 4AX$, so $4A = 3$
So, latus-rectum is $4A$, *i.e.* 3.

Concept Boosters

Class XII

Limits, Continuity & Differentiability

– MTG Editorial Board

This column is especially aimed at Class XII so that they can prepare for competitive exams such as IIT, AIEEE, DCE, etc and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

PART - A

Evaluating Limits

1. In all the problems given below find the value of the limits.

(i) $\lim_{x\to 4} \dfrac{(\cos\alpha)^x-(\sin\alpha)^x-\cos 2\alpha}{x-4}$, $\alpha\in(0,\ \pi/2)$

(ii) $\lim_{x\to 0} \dfrac{(e^x-1)-(e^{x\cos x}-1)}{x+\sin x}$

(iii) $\lim_{x\to 0} \dfrac{\tan([-\pi^2]x^2)-\tan([-\pi]^2)x^2}{\sin^2 x}$

(v) $\lim_{x\to 0}\left(\dfrac{1^x+2^x+3^x+...+n^x}{n}\right)$

(vi) $\lim_{n\to\infty} n^{-n^2}\left\{(n+1)\left(n+\dfrac{1}{2}\right)\left(n+\dfrac{1}{4}\right)....\left(n+\dfrac{1}{2^{n-1}}\right)\right\}$

2.(i) A square is inscribed in a circle of radius R, a circle is then inscribed in this square, then a square in this circle and so on for n times. Evaluate the limit of the sum of areas of all squares as $n\to\infty$.

(ii) Find $\lim_{x\to 2}\dfrac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$

(iii) Calculate the values of α and β in order that

$$\lim_{x\to 0}\frac{x(1+\alpha\cos x)-\beta\sin x}{x^3}=1.$$

(iv) $\lim_{\theta\to\pi/4}\dfrac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$

(v) Without using L'Hospital rule evaluate

$$\lim_{x\to 1}\frac{x^{k+1}-(k+1)x+k}{(x-1)^2}$$

Testing for continuity

3. (i) Let $f(x)=\dfrac{\cos^{-1}(1-\{x\}^2)\cdot\sin^{-1}(1-\{x\})}{\sqrt{2}(\{x\}-\{x\}^3)}$ for $x\neq 0$

$= \dfrac{\pi}{2}$ for $x=0$

Consider another function $g(x)$ defined by

$$g(x)=\begin{cases} f(x), & x\ge 0\\ 2\sqrt{2}\,f(x), & x<0\end{cases}$$

(ii) Let $f(x)=x^3-3x^2+6\ \forall\ x\in R$

$$g(x)=\begin{cases}\max\{f(t): x+1\le t\le x+2,\ -3\le x<0\}\\ 1-x,\ \text{for } x\ge 0\end{cases}$$

Find the points where the function $g(x)$ is discontinuous in the interval $x\in[-3,1]$.

(iii) Let $[x]$ denote the greatest integer function and $f(x)$ be defined in a neighbourhood of 2 by $f(x)$ given by

$$f(x)=\begin{cases}\dfrac{\left(\exp\{(x+2)\ln 4\}^{\left(\frac{[x+1]}{9}\right)}\right)-16}{4^x-16}, & x<2\\[2ex] A\dfrac{1-\cos(x-2)}{(x-2)\tan(x-2)}, & x>2\end{cases}$$

Find the values of Δ and $f(2)$ in order that $f(x)$ may be continuous at $x=2$.

Mixed Problems on Continuity and Differentiability

4.(i) Let $f(x)=\begin{cases}x+2, & 0\le x<2\\ 6-x, & x\ge 2\end{cases}$,

$$g(x)=\begin{cases}1+\tan x, & 0\le x<\pi/4\\ 3-\cot x, & \pi/4\le x<\pi\end{cases}$$

Find the composite function $fog(x)$ and test its continuity and derivability.

(ii) Given that f and g are differentiable functions, evaluate (using L'Hospitals rule)

$$\lim_{x\to 2}\frac{f(x)g(4-x)-f(4-x)g(x)}{x-2}.$$ Assume $f(2)=2$, $f'(2)=-3$, $g(2)=4$ and $g'(2)=1$.

5.(i) Let $g(x)$ be a polynomial of degree 1 and $f(x)$ be defined by

$$f(x)=\begin{cases} g(x), & x\le 0 \\ \left(\dfrac{1+x}{2+x}\right)^{1/x}, & x>0 \end{cases}$$

Find the continuous function satisfying $f'(1)=f(-1)$.

(ii) Test continuity for the function

$$f(x)=\lim_{n\to\infty}\frac{(1+\sin x)^n+\ln x}{2+(1+\sin x)^n}$$

(iii) If a function $f(x)$ satisfies

$$f\left(\frac{x+2y}{3}\right)=\frac{f(x)+2f(y)}{3}\ \forall\ x,y\in R$$

and $f'(0)=1$, then prove that $f(x)$ is continuous for all $x\in R$.

6.(i) Let $f:[0,\infty)\to[1,\infty)$ be a one–one function satisfying $f(x)f(y)+2=f(x)+f(y)+f(xy)\forall\ x.\ y\ge 0$ and $f(1)=2\neq f(0)$ Evaluate $\int_0^1 f(x)dx$

(ii) Define $f(x)=x^2-2x$, $x\in R$. Let $g(x)$ be defined by $g(x)=f(f(x)-1)+f(5-f(x))$. Show that $g(x)\ge 0\ \forall\ x\in R$. Also find the critical points of $g(x)$.

O Application to function-based questions

7.(i) A function $f(x)$ is defined for all $x\in R$ and satisfies $f(x+y)=f(x)+2y^2+kxy\ \forall\ x,y\in R^*$ where k is a given constant. If $f(1)=2$, $f(2)=8$, find $f(x)$ and show that $f(x+y)\cdot f\left(\dfrac{1}{x+y}\right)=k,\ x+y\neq 0.$

(ii) Let f and g be real function such that $f(x+y)+f(x-y)=2f(x)g(y)\ \forall\ x,y\in R.$

If $f(x)$ is not identically zero and $|f(x)|\le 1\ \forall\ x\in R$, then prove that $|g(y)|\le 1\ \forall\ y\in R$.

(iii) If $e^{-xy}f(xy)=e^{-x}f(x)+e^{-y}f(y),\ \forall\ x,y\in R^+$, and $f'(1)=e$, determine $f(x)$.

O Miscellaneous

8.(i) Find the values of a, b, c if the function $f(x)=a|\sin x|+be^{|x|}+c|x|^3$ is differentiable at $x=0$.

(ii) Test for differentiability of the function

$$f(x)=(x^2-1)|x^2-3x+2|+\cos|x|$$

9.(i) Find $\displaystyle\lim_{x\to 0}\frac{\cot x\tan^{-1}(m\tan x)-m\cos^2(x/2)}{\sin^2(x/2)}$

(ii) Find $\displaystyle\lim_{x\to\infty}\left(\frac{a^{1/x}+b^{1/x}+c^{1/x}}{3}\right)^{3x}$.

10.

(i) Let $f:R\to R$ be a function satisfying

$$f(x+2y)=f(x)e^{2y}+f(2y)e^x+x^2(1-e^{2y})+4y^2(1-e^x)+4xy\ \forall\ x,\ y\in R.$$

Also $f'(0)=1$. Find $f(x)$.

(ii) Let $f(x)$ be a continuous function in $[-1, 1]$ and satisfies $f(2x^2-1)=2xf(x)\ \forall\ x\in[-1,1]$. Show that $f(x)$ is identically zero $\forall\ x\in[-1,1]$.

PART - B

Multiple Choice Questions

Single Option Correct

1. The value of $\displaystyle\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3}$ is

(a) $\sqrt{2}$ (b) $1/\sqrt{2}$ (c) $-1/2$ (d) $1/2$

2. A function $f(x)$ satisfies $3f(x)+4f\left(\dfrac{1}{x}\right)=\dfrac{1}{x}-5$. Then the value of $f(2)$ is

(a) $-3/14$ (b) $3/7$ (c) $3/14$ (d) $-3/7$.

3. Let $F(x)=f(x)g(x)h(x)$ when f, g, h are differentiable function. Assume $F'(x_0)=13\,F(x_0)$, $f'(x_0)-9f(x_0)$, $g'(x_0)=4\,g(x_0)$, then $\dfrac{h'(x_0)}{h(x_0)}$, $(h(x_0)\neq 0)$ is

(a) 0 (b) 26 (c) -26 (d) 17.

4. About the function represented parametrically as $x=2t-|t|, y=t^3+t^2|t|$, which of the following statement is correct?

(a) f is differentiable everywhere, except $x=0$ and $x=2$

(b) f is differentiable everywhere except $x=0$

(c) f is differentiable at $x=0$ and $f'(0)=0$

(d) f is differentiable at $x=0$ but $f'(0)\neq 0$.

5. $f(x)$ is a real valued function not identically equal to zero such that $f(x+y^n)=f(x)+(f(y))^n$, $y\in R$ and n is natural number >1 and $f'(0)\ge 0$, then which of the following statements is correct?

(a) $f(15)=-15,\ f'(20)=-1$

(b) $f(15) = 15,\ f'(20) = -1$
(c) $f(15) = -15,\ f'(20) = 1$
(d) $f(15) = 15,\ f'(20) = 1$

6. Let D be the domain and R the range of

$$f(x) = \left[\ln(\sin^{-1}\sqrt{x^2+3x+2}\right]$$

where $[\cdot]$ denotes the greatest integer function, then which of the following statements is correct?

(a) $D = \left[\frac{-3-\sqrt{5}}{2}, -2\right) \cup \left(-1, \frac{-3+\sqrt{5}}{2}\right]$,
R = Set of non-positive integers

(b) $D = \left[-\frac{3-\sqrt{5}}{2}, -2\right) \cup \left[-1, \frac{-3+\sqrt{5}}{2}\right)$,
R = Set of non-positive integers

(c) $D = \left(\frac{-3-\sqrt{5}}{2}, -2\right) \cup \left[-1, \frac{-3+\sqrt{5}}{2}\right)$,
R = Set of non-negative integer

(d) $D = \left[-\frac{3-\sqrt{5}}{2}, -2\right) \cup \left(-1, \frac{-3+\sqrt{5}}{2}\right)$,
R = Set of non-negative integers.

7. If $x + y = e^{x-y}$, then $\frac{d^2y}{dx^2}$ at $x = x_0$ is given by

(a) $\frac{4e^{y_0}}{e^{x_0}+e^{y_0}}$ (b) $\frac{4e^{x_0}}{e^{x_0}+e^{y_0}}$

(c) $\frac{4e^{(x_0-y_0)}}{(e^{x_0-y_0}+1)^3}$ (d) $\frac{4(x_0+y_0)}{(x_0+y_0+1)^2}$

8. The value of $\lim\limits_{x\to 0}\frac{27^x-9^x-3^x+1}{\sqrt{5}-\sqrt{4+\cos x}}$ is

(a) $4\sqrt{5}(\ln 3)^2$ (b) $8\sqrt{5}(\ln 3)^2$
(c) $4\sqrt{5}(\ln 3)$ (d) $8\sqrt{5}(\ln 3)$

9. The value of a such that f is continuous at $x = 0$, where $f(x) = \frac{\sin 2x + a\sin x}{x^3}$, $x \neq 0$ is

(a) -2 and then $f(0) = -1$ (b) 2 and then $f(0) = -1$
(c) -2 and then $f(0) = 1$ (d) 2 and then $f(0) = 1$

10. The value of $\lim\limits_{n\to\infty}\cos\left(\frac{x}{2}\right)\cos\frac{x}{4}\cdot\cos\frac{x}{8}\ldots.\cos\frac{x}{2^n}$ is

(a) $\frac{\sin x}{2x}$ (b) $\frac{x}{\sin x}$ (c) $\frac{\sin x}{x}$ (d) $\frac{2x}{\sin x}$

11. The values of a and b if the function given by

$$f(x) = \begin{cases} x^2+ax+1, & x \text{ rational} \\ ax^2+bx+1, & x \text{ irrational} \end{cases}$$

are respectively

(a) 1 and 2 (b) 1 and 1 (c) 2 and 1 (d) 1 and -1.

12. The value of $\lim\limits_{n\to\infty}(4^n+5^n)^{1/n}$ is given by

(a) $\sqrt{20}$ (b) 4 (c) 5 (d) 4.5.

13. Let $f(x) = \cos 2x \cot\left(\frac{\pi}{4}-x\right)$. Given that f is continuous at $x = \pi/4$, the value of $f(\pi/4)$ is

(a) -2 (b) 2 (c) 1 (d) -1.

14. Let f and g be two continuous and differentiable functions satisfying $f(x+y) = f(x)+f(y)\ \forall\ x, y \in R$. Also $f(x) = x^2g(x)$. Then $|f(15)-f(-15)|$ is

(a) -30 (b) 30
(c) 0 (d) cannot be determined.

15. The value of $\lim\limits_{x\to 1}\frac{x^{1/3}-x^{1/4}-2}{x^3-1}$ is

(a) 1/36 (b) $-1/36$ (c) $-1/12$ (d) 1/12.

SOLUTIONS

PART - A

1. $\lim\limits_{x\to 4}\frac{(\cos\alpha)^x-(\sin\alpha)^x-\cos 2\alpha}{x-4}$

$= \lim\limits_{x\to 4}\frac{(\cos\alpha)^x-(\sin\alpha)^x-(\cos^2\alpha-\sin^2\alpha)(\cos^2\alpha+\sin^2\alpha)}{x-4}$

$= \lim\limits_{x\to 4}\frac{(\cos\alpha)^x-(\sin\alpha)^x-\cos^4\alpha+\sin^4\alpha}{x-4}$

$= \lim\limits_{x\to 4}\frac{(\cos\alpha)^x-\cos^4\alpha-(\sin\alpha)^x+\sin^4\alpha}{x-4}$

$= \lim\limits_{x\to 4}\frac{(\cos\alpha)^4\{(\cos\alpha)^{x-4}-1\}-\sin^4\alpha\{(\sin\alpha)^{x-4}-1\}}{x-4}$

$= \cos^4\alpha\cdot\lim\limits_{x\to 4}\frac{(\cos\alpha)^{x-4}-1}{x-4}-\sin^4\alpha\cdot\lim\limits_{x\to 4}\frac{(\sin\alpha)^{x-4}-1}{x-4}$

$= \cos^4\alpha\cdot\lim\limits_{h\to 0}\frac{a^h-1}{h}-\sin^4\alpha\cdot\lim\limits_{h\to 0}\frac{b^h-1}{h}$ ...(1)

where $a = \cos\alpha$ and $b = \sin\alpha$

Now $\lim\limits_{h\to 0}\frac{a^h-1}{h} = \lim\limits_{h\to 0}\frac{e^{h\log_e a}-1}{h} = \ln a$

Using this we have from (1) the desired limit $= \cos^4\alpha\cdot\ln(\cos\alpha)-\sin^4\alpha\cdot\ln(\sin\alpha)$.

(ii) $\lim\limits_{x\to 0}\frac{(e^x-1)-(e^{x\cos x}-1)}{x+\sin x}$

$= \lim\limits_{x\to 0}\frac{e^x-1}{x+\sin x}-\lim\limits_{x\to 0}\frac{e^{x\cos x}-1}{x+\sin x}$

$= \lim\limits_{x\to 0}\frac{e^x-1}{x\left(1+\frac{\sin x}{x}\right)}-\lim\limits_{x\to 0}\frac{e^{x\cos x}-1}{x\cos x\left(\sec x+\frac{\sin x}{x\cos x}\right)}$

$$= \lim_{x\to0}\frac{e^x-1}{x}\cdot\frac{1}{\left(1+\frac{\sin x}{x}\right)} - \lim_{x\to0}\frac{e^{x\cos x}-1}{x\cos x}\cdot\frac{1}{\sec x+\frac{\sin x}{x\cos x}}$$

$$= \left(\lim_{x\to0}\frac{e^x-1}{x}\right)\left(\lim_{x\to0}\frac{1}{1+\frac{\sin x}{x}}\right) -$$

$$\left(\lim_{x\to0}\frac{e^{x\cos x}-1}{x\cos x}\right)\cdot\left(\lim_{x\to0}\frac{1}{\sec x+\frac{\tan x}{x}}\right)$$

$$= 1\cdot\frac{1}{1+1} - 1\cdot\frac{1}{1+1} = \frac{1}{2}-\frac{1}{2} = 0.$$

Remark : *Now we have pulled x and $x\cos x$ from the numerator while evaluating limits $e^x - 1$ and $e^{x\cos x} - 1$, to make use of the fact that $\lim_{x\to0}\frac{e^x-1}{x} = 1$. Such obervations help greatly when you want to simplify your caluclations and avoid the traps laid by L'Hospital rule.*

(iii) $\lim_{x\to0}\frac{\tan([-\pi^2]x^2)-\tan([-\pi^2])x^2}{\sin^2 x}$

(Observe that $\pi = 3.14 \Rightarrow \pi^2 = 9.86$

$\Rightarrow -\pi^2 = -9.86 \Rightarrow [-\pi^2] = -10$).

$$= \lim_{x\to0}\frac{\tan(-10x^2)-(\tan(-10))x^2}{\sin^2 x}$$

$$= \lim_{x\to0}\frac{-\tan(10x^2)+(\tan10)x^2}{\sin^2 x}$$

$$= \lim_{x\to0}\frac{\frac{[(-\tan(10x^2)+(\tan10)x^2)]}{x^2}}{\left(\frac{\sin^2 x}{x^2}\right)}$$

$$= \lim_{x\to0}\frac{-10\cdot\frac{(\tan(10x^2))}{10x^2}+\tan10}{\left(\frac{\sin x}{x}\right)^2}$$

$$= \frac{-10\cdot1+\tan10}{1^2} = \tan10-10.$$

(iv) $\lim_{x\to0}\left(\frac{1^x+2^x+3^x+\ldots+n^x}{n}\right)^{\frac{a}{x}}$

$$= e^{\lim_{x\to0}\left(\frac{1^x+2^x+3^x+\ldots+n^x}{n}-1\right)\cdot\frac{a}{x}}$$

$$= e^{\lim_{x\to0}\left(\frac{1^x+2^x+\ldots+n^x-n}{n}\right)\cdot\frac{a}{x}}$$

$$= e^{\lim_{x\to0}\left\{\frac{(1^x-1)+(2^x-1)+\ldots+(n^x-1)}{x}\right\}\cdot\frac{a}{n}}$$

$$= e^{\left\{\lim_{x\to0}\frac{1^x-1}{x}+\lim_{x\to0}\frac{2^x-1}{x}+\ldots+\lim_{x\to0}\frac{n^x-1}{x}\right\}\cdot\frac{a}{n}}$$

$$= e^{(\ln1+\ln2+\ldots+\ln n)\cdot\frac{a}{n}} = e^{\{\ln(n!)\}\frac{a}{n}}$$

$$= e^{\frac{a}{n}\ln(n!)} = e^{\ln(n!)^{a/n}} = (n!)^{a/n}$$

(v) $\lim_{n\to\infty} n^{-n^2}\left\{(n+1)\left(n+\frac{1}{2}\right)\ldots\left(n+\frac{1}{2^{n-1}}\right)\right\}^n$

$$= \lim_{n\to\infty}\left[\frac{(n+1)\left(n+\frac{1}{2}\right)\ldots\left(n+\frac{1}{2^{n-1}}\right)}{n^n}\right]^n$$

$$= \lim_{n\to\infty}\left\{\frac{n+1}{n}\cdot\frac{n+\frac{1}{2}}{n}\ldots\frac{n+\frac{1}{2^{n-1}}}{n}\right\}^n$$

$$= \lim_{n\to\infty}\left[\left(\frac{n+1}{n}\right)^n\cdot\left(\frac{n+\frac{1}{2}}{n}\right)^n\ldots\left(\frac{n+\frac{1}{2^{n-1}}}{n}\right)^n\right]$$

$$= \lim_{n\to\infty}\left[\left(1+\frac{1}{n}\right)^n\cdot\left(1+\frac{1}{2n}\right)^{\frac{2n}{2}}\ldots\left(1+\frac{1}{2^{n-1}\cdot n}\right)^{\frac{2^{n-1}\cdot n}{2^{n-1}}}\right]$$

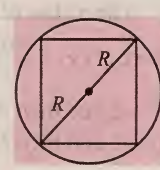
$$= \left\{\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^n\right\}\left\{\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{\frac{2n}{2}}\right\}\ldots$$

$$\left\{\lim_{n\to\infty}\left(1+\frac{1}{2^{n-1}\cdot n}\right)^{\frac{2^{n-1}\cdot n}{2^{n-1}}}\right\}$$

$$= e\cdot e^{1/2}\cdot e^{1/4}\ldots e^{1/2^{n-1}} = e^{1+\frac{1}{2}+\frac{1}{4}+\ldots+} = e^{\frac{1}{1-(1/2)}}$$

$= e^2$.

2.(i) If in a circle of radius R, we inscribed a square, then its side is given by



$$= \frac{\text{diagonal}}{\sqrt{2}} = \frac{2R}{\sqrt{2}} = \sqrt{2}R = a \text{ (say)}$$

Let a_1 be the side of another square,

then $a_1\sqrt{2} = a \Rightarrow a_1 = \frac{a}{\sqrt{2}}$

Again $a_2\sqrt{2} = a_1 \Rightarrow a_2 = \frac{a_1}{\sqrt{2}} = \frac{a}{2}$

S_n = Sum of areas of all squares

$= a^2 + a_1^2 + a_2^2 + \ldots + a_n^2$

$= a^2 + \frac{a^2}{2} + \frac{a^2}{4} + \ldots +$ to n terms

$$= a^2\left(\frac{1-\frac{1}{2^n}}{1-\frac{1}{2}}\right) = 2a^2\left(1-\frac{1}{2^n}\right)$$

$$\lim_{n\to\infty} S_n = 2a^2 = 4R^2.$$

(ii) $\displaystyle\lim_{x\to 2}\frac{\sqrt{x+7}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-2\sqrt[3]{3x-5}}$

$$= \lim_{x\to 2}\frac{\sqrt{x+7}-\sqrt{9}+\sqrt{9}-3\sqrt{2x-3}}{\sqrt[3]{x+6}-\sqrt[3]{8}+\sqrt[3]{8}-2\sqrt[3]{3x-5}}$$

$$= \lim_{x\to 2}\frac{\{(x+7)^{1/2}-9^{1/2}\}-3\{(2x-3)^{1/2}-1\}}{\{(x+6)^{1/3}-8^{1/3}\}-2\{(3x-5)^{1/3}-1\}}$$

$$= \lim_{x\to 2}\frac{\dfrac{(x+7)^{1/2}-9^{1/2}}{x+7-9}-3\left\{\dfrac{(2x-3)^{1/2}-1}{(2x-3)-1}\right\}\cdot 2}{\dfrac{(x+6)^{1/3}-8^{1/3}}{x+6-8}-2\left\{\dfrac{(3x-5)^{1/3}-1^{1/3}}{(3x-5)-1}\right\}\cdot 3}$$

$$= \frac{\frac{1}{2}\cdot(9)^{\frac{1}{2}-1}-6\cdot\frac{1}{2}\cdot(1)^{\frac{1}{2}-1}}{\frac{1}{3}(8)^{\frac{1}{3}-1}-6\cdot\frac{1}{3}\cdot(1)^{\frac{1}{3}-1}} = \frac{\frac{1}{2}\cdot\frac{1}{3}-3}{\frac{1}{3}\cdot\frac{1}{4}-2}$$

$$= -\frac{17}{6}\times\frac{12}{-23} = \frac{34}{23}.$$

(iii) $\displaystyle\lim_{x\to 0}\frac{x(1+\alpha\cos x)-\beta\sin x}{x^3} = 1$

i.e.

$$\lim_{x\to 0}\frac{x\left[1+\alpha\left(1-\frac{x^2}{2}+\frac{x^4}{4!}-\dots\right)\right]-\beta\left[x-\frac{x^3}{3!}+\frac{x^5}{5!}-\dots\right]}{x^3} = 1$$

i.e. $$\lim_{x\to 0}\frac{x(1+\alpha-\beta)+x^3\left(-\frac{\alpha}{2!}+\frac{\beta}{3!}\right)+x^5\left(\frac{\alpha}{4!}-\frac{\beta}{5!}\right)\dots}{x^3} = 1$$

$$\Rightarrow \lim_{x\to 0}\frac{1+\alpha-\beta}{x^2}+\lim_{x\to 0}\left(-\frac{\alpha}{2!}+\frac{\beta}{3!}\right)+\lim_{x\to 0}\left(\frac{\alpha}{4!}-\frac{\beta}{5!}\right)x^2\dots = 1$$

As the limit tends to a finite value, we have

$1+\alpha-\beta = 0$ and $-\frac{\alpha}{2!}+\frac{\beta}{3!} = 1$ *i.e.* $-3\alpha+\beta = 6$

Solving we get $\alpha = -5/2$, $\beta = -3/2$.

(iv) $\displaystyle\lim_{\theta\to\frac{\pi}{4}}\frac{\sqrt{2}-\cos\theta-\sin\theta}{(4\theta-\pi)^2}$

$$= \sqrt{2}\lim_{\theta\to(\pi/4)}\frac{1-\left(\frac{1}{\sqrt{2}}\cos\theta+\frac{1}{\sqrt{2}}\sin\theta\right)}{(4\theta-\pi)^2}$$

$$= \sqrt{2}\lim_{\theta\to(\pi/4)}\frac{1-\cos(\theta-(\pi/4))}{16(\theta-(\pi/4))^2}$$

Set $\theta = \frac{\pi}{4}+h$, so that when $\theta\to\pi/4$, $h\to 0$

$$\therefore \text{Above limit} = \sqrt{2}\lim_{h\to 0}\frac{1-\cos h}{16h^2} = \frac{\sqrt{2}}{16}\cdot\lim_{h\to 0}\frac{1-\cos h}{h^2}$$

$$= \frac{1}{8\sqrt{2}}\cdot\lim_{h\to 0}\left(\frac{2\sin^2(h/2)}{h^2}\right) = \frac{1}{8\sqrt{2}}\cdot 2\cdot\lim_{h\to 0}\left(\frac{\sin(h/2)}{h/2}\right)^2\cdot\frac{1}{4}$$

$$= \frac{1}{16\sqrt{2}}\cdot 1 = \frac{1}{16\sqrt{2}}.$$

(v) $\displaystyle\lim_{x\to 1}\frac{x^{k+1}-(k+1)x+k}{(x-1)^2} = \lim_{x\to 1}\frac{(x^{k+1}-x)-k(x-1)}{(x-1)^2}$

$$= \lim_{x\to\infty}\frac{x(x^k-1)-k(x-1)}{(x-1)^2}$$

$$= \lim_{x\to 1}\frac{x(x-1)(x^{k-1}+x^{k-2}+\dots+x+1)-k(x-1)}{(x-1)^2}$$

$$= \lim_{x\to 1}\frac{(x-1)\{x(x^{k-1}+x^{k-2}+\dots+x+1)-k\}}{(x-1)^2}$$

$$= \lim_{x\to 1}\frac{x^k+x^{k-1}+\dots+x^2+x-k}{x-1}$$

$$= \lim_{x\to 1}\frac{x^k+x^{k-1}+\dots+x^2+x-(1+1+\dots\text{ to } k \text{ times})}{x-1}$$

$$= \lim_{x\to 1}\frac{(x^k-1)+(x^{k-1}-1)+\dots+(x^2-1)+(x-1)}{x-1}$$

$$= \lim_{x\to 1}\frac{x^k-1}{x-1}+\lim_{x\to 1}\frac{x^{k-1}-1}{x-1}+\dots+\lim_{x\to 1}\frac{x-1}{x-1}$$

$$= k+(k-1)+(k-2)+\dots+2+1 = \frac{k(k+1)}{2}.$$

3.(i) For $x > 0$

Right hand limit $= \displaystyle\lim_{x\to 0^+} f(0) = \lim_{h\to 0} f(0+h)$

$$= \lim_{h\to 0}\frac{\cos^{-1}(1-\{h\}^2)\cdot\sin^{-1}(1-\{h\})}{\sqrt{2}\left(\{h\}-\{h\}^3\right)}$$

$$= \lim_{h\to 0}\frac{\cos^{-1}(1-h^2)\cdot\sin^{-1}(1-h)}{\sqrt{2}(h-h^3)}$$

$$= \lim_{h\to 0}\frac{\cos^{-1}(1-h^2)\cdot\sin^{-1}(1-h)}{\sqrt{2}\cdot h(1-h^2)}$$

$$= \lim_{h\to 0}\frac{\cos^{-1}(1-h^2)\cdot\sin^{-1}(1-h)}{\sqrt{2}\cdot h(1-h)(1+h)}$$

$$= \lim_{h\to 0}\frac{\sin^{-1}(1-h)}{\sqrt{2}(1-h)(1+h)}\cdot\lim_{h\to 0}\frac{\cos^{-1}(1-h^2)}{h}$$

$$= \frac{\sin^{-1}1}{\sqrt{2}} \cdot \lim_{\theta\to 0} \frac{2\theta}{\sqrt{2}\sin\theta}$$

$$= \frac{\sin^{-1}1}{\sqrt{2}} \cdot \frac{2}{\sqrt{2}} \lim_{\theta\to 0} \frac{\theta}{\sin\theta} = (\sin^{-1}1)1 = \frac{\pi}{2}$$

For $x < 0$

Left hand limit $= \lim_{x\to 0^-} f(x) = \lim_{h\to 0} f(0-h)$

$$= \lim_{h\to 0} \frac{\cos^{-1}(1-\{0-h\}^2)\cdot\sin^{-1}(1-\{0-h\})}{\sqrt{2}(\{0-h\}-\{0-h\}^3)}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}(1-\{1-h\}^2)\cdot\sin^{-1}(1-\{1-h\})}{\sqrt{2}(\{1-h\}-\{1-h\}^3)}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}(h(2-h))\cdot\sin^{-1}h}{\sqrt{2}(1-h)\cdot(2-h)\cdot h}$$

$$= \lim_{h\to 0} \frac{\cos^{-1}h(2-h)}{\sqrt{2}\cdot(1-h)(2-h)} \cdot \lim_{h\to 0} \frac{\sin^{-1}h}{h}$$

$$= \frac{\cos^{-1}0}{2\sqrt{2}} \cdot 1 = \frac{\pi}{4\sqrt{2}}$$

Also $f(0) = \frac{\pi}{2}$

Thus $f(x)$ is discontinuous at $x = 0$.
Checking the continuity for $g(x)$ at $x = 0$
$g(0) = f(0) = \pi/2$

Right hand limit $= \lim_{x\to 0^+} g(x) = \lim_{x\to 0^+} f(x) = \frac{\pi}{2}$

Left hand limit $= \lim_{x\to 0^-} g(x) = \lim_{x\to 0^-} 2\sqrt{2}\, f(x)$

$$= 2\sqrt{2} \lim_{x\to 0-} f(x) = 2\sqrt{2}\cdot\frac{\pi}{4\sqrt{2}} = \frac{\pi}{2}$$

Thus $g(x)$ is continuous at $x = 0$.

(ii) $f(x) = x^3 - 3x^2 + 6$
$f'(x) = 3x^2 - 6x \Rightarrow x = 0, 2$
These are the critical points of the function $f(x)$
$f''(x) = 6x, f'''(x) = 6,\ f''(2) = 12 > 0$
$\therefore\ x = 2$ is a point of local minima and $x = 0$ is a point of local maxima.
Clearly $f(x)$ is increasing in $(-\infty, 0)$ and $(2, \infty)$ and decreasing in $(0, 2)$.
$x + 2 \le 0 \Rightarrow x \le -2 \Rightarrow g(x) = f(x + 2), -3 \le x \le -2$
If $x + 1 < 0$ and $0 < x + 2 < 2$ *i.e.* $x < -1$ and $-2 < x < 0$
Thus $-2 < x < -1,\ g(x) = f(0)$
Now, for $0 \le x + 1, x + 2 \le 2 \Rightarrow -1 \le x \le 0$,
$g(x) = f(x + 1)$

$$\Rightarrow g(x) = \begin{cases} f(x+2) & -3\le x<-2 \\ f(0) & -2\le x<-1 \\ f(x+1) & -1\le x<0 \\ 1-x & x\ge 0 \end{cases}$$

Thus $g(x)$ is continuous in the interval $[-3, 1]$.

(iii) For finding left hand limit, observe that $1 < x < 2$, so that $[x + 1] = 2$

$$\text{LHL} = \lim_{x\to 2^-} \frac{(4^{x+2})^{1/2}-16}{4^x-16} = \lim_{x\to 2^-} \frac{4(2^x-4)}{(2^x-4)(2^x+4)}$$

$$= \lim_{x\to 2^-}\left(\frac{4}{2^x+4}\right) = \frac{4}{8} = \frac{1}{2}$$

$$\text{RHL} = A \lim_{x\to 2^+} \frac{1-\cos(x-2)}{(x-2)\tan(x-2)} = A \lim_{h\to 0} \frac{1-\cos h}{h\tan h}$$

where $x = h + 2$, so that when $x \to 2^+, h \to 0$

$$= A \lim_{h\to 0} \frac{2\sin^2(h/2)}{h\tan h} = A \lim_{h\to 0} \frac{2\sin^2(h/2)}{h^2\cdot(\tan h/h)}$$

$$= A \lim_{h\to 0} \frac{\left(\frac{\sin(h/2)}{h/2}\right)^2 \cdot \frac{1}{4}\cdot 2}{\frac{\tan h}{h}} = A\cdot\frac{2}{4} = \frac{A}{2}$$

For continuity at $x = 2$, LHL = RHL $= f(2)$
$\Rightarrow\quad A = 1$ and $f(2) = 1/2$.

4.(i) $f(x) = \begin{cases} x+2, & 0\le x<2 \\ 6-x, & 0\ge 2 \end{cases}$

$$g(x) = \begin{cases} 1+\tan x, & 0\le x<\frac{\pi}{4} \\ 3-\cot x, & \frac{\pi}{4}\le x<\pi \end{cases}$$

For $0 \le x < \pi/4$, $g(x) = 1 + \tan x$
So $f(g(x)) = f(1+\tan x) = 1+\tan x+2$
Now $x \in [0, \pi/4) \Rightarrow 1 + \tan x \in [1, 2)$
and for $x \in \left[\frac{\pi}{4}, \pi\right)$, $g(x) = 3 - \cot x$

Also for $x \in \left[\frac{\pi}{4}, \pi\right)$, $3 - \cot x \in [2, \infty)$

So $f(g(x)) = f(3 - \cot x) = 6 - (3 - \cot x)$
Again denote by $h(x)$ the composition of f w.r.t. g

$$h(x) = f(g(x)) = \begin{cases} 3+\tan x, & 0\le x<\pi/4 \\ 3+\cot x, & \pi/4\le x<\pi \end{cases}$$

$f(x)$ is obviously continuous in $[0, \pi]$

$$h'+\left(\frac{\pi}{4}\right) = \lim_{x\to\frac{\pi}{4}+} (-\mathrm{cosec}^2 x) = -2h'\left(\frac{\pi^-}{4}\right)$$

$$= \lim_{x\to\pi/4^-} (\sec^2 x) = 2$$

Thus $f(g(x))$ is differentiable everywhere in $[0, \pi)$ except at $x = \pi/4$.

(ii) $\displaystyle\lim_{x\to 2} \frac{f(x)g(4-x)-f(4-x)g(x)}{x-2}$

Let $h = x-2$, so that when $x \to 2$, $h \to 0$

$$= \lim_{h\to 0} \frac{f(h+2)g(2-h)-f(2-h)\cdot g(2+h)}{h}$$

$$= \lim_{h\to 0} \frac{\begin{array}{c} f(2+h)g(2-h)-f(2)g(2-h)+f(2)g(2-h) \\ -f(2-h)g(2+h)+f(2-h)g(2)-f(2-h)g(2)\end{array}}{h}$$

$$= \lim_{h\to 0} \frac{\begin{array}{c} g(2-h)[f(2+h)-f(2)]-f(2-h)(g(2+h)-g(2)) \\ +g(2-h)f(2)-f(2-h)g(2)\end{array}}{h}$$

$$= g(2)\cdot f'(2) - f(2)\cdot g'(2) + \lim_{h\to 0}\frac{g(2-h)f(2)-f(2-h)\cdot g(2)}{h}$$

$$= 4\cdot(-3) - 2\cdot 1 + \lim_{h\to 0}\frac{f(2)g(2-h)-f(2)g(2)+f(2)g(2)-f(2-h)g(2)}{h}$$

$$= -12-2+f(2)\left\{\lim_{h\to 0}\frac{g(2-h)-g(2)}{h}\right\} - g(2)\left\{\lim_{h\to 0}\frac{f(2-h)-f(2)}{h}\right\}$$

$$= -14 + f(2)\cdot(-g'(2)) + g(2)f'(2)$$

$$= -14 - 2\cdot(+1) + 4\cdot(-3) = -14-2-12 = -28.$$

5.(i) Consider $g(x) = ax + b$, where a and b are constants to be determined.

$$f(x) = \begin{cases} ax+b, & x \le 0 \\ \left(\dfrac{1+x}{2+x}\right)^{1/x}, & x > 0 \end{cases}$$

$$\lim_{x\to 0^+} f(x) = \lim_{x\to 0^-} f(x) \Rightarrow \left(\frac{1}{2}\right)^{\infty} = b \Rightarrow b = 0$$

$$f(1) = \frac{2}{3}$$

Now $f(x) = \left(\dfrac{1+x}{2+x}\right)^{1/x}$ on taking logarithm both sides

$$\ln f(x) = \frac{1}{x}\{\ln|x+1| - \ln|2+x|\}$$

$$\Rightarrow \frac{f'(x)}{f(x)} = -\frac{1}{x^2}\ln\frac{1+x}{2+x} + \frac{1}{x(x+1)(x+2)}$$

$$\Rightarrow \frac{f'(1)}{f(1)} = -1\ln\frac{2}{3} + \frac{1}{1\cdot 2\cdot 3} = \ln\frac{3}{2} + \frac{1}{6}$$

$$\Rightarrow f'(1) = \frac{2}{3}\ln\frac{3}{2} + \frac{2}{3}\times\frac{1}{6} = \frac{2}{3}\ln\frac{3}{2} + \frac{1}{9}$$

Now $f'(1) = f(-1) = b - a$

$$\Rightarrow b - a = \frac{2}{3}\ln\frac{3}{2} + \frac{1}{9} \Rightarrow 0 - a = \frac{2}{3}\ln\frac{3}{2} + \frac{1}{9}$$

$$\therefore a = -\frac{2}{3}\ln\frac{3}{2} - \frac{1}{9}$$

Thus $f(x) = -\left(\dfrac{2}{3}\ln\dfrac{3}{2} + \dfrac{1}{9}\right)x.$

(ii) $f(x)$ is defined for $\forall\ x \in (0, \infty)$

Now, for $x = n+1$, $1 + \sin x = 1$

$x \in ((2n-1)\pi,\ 2n\pi),\ 0 \le 1+\sin x < 1$

For $x \in (2n\pi,\ (2n+1)\pi),\ 2 \ge 1+\sin x > 1$

Thus $f(x)$ can be described as

$$f(x) = \begin{cases} \dfrac{1+\ln x}{3}, & x = n\pi \\ \dfrac{\ln x}{2}, & x \in ((2n-1)\pi,\ 2n\pi) \\ 1, & x \in (2n\pi,\ (2n+1)\pi) \end{cases}$$

It is seen from the above that $f(x)$ is discontinuous at integral multiple of π.

(iii) Given relation is

$$f\left(\frac{x+2y}{3}\right) = \frac{f(x)+2f(y)}{3} \quad \forall\ x, y \in R$$

Replacing x by $3x$ and y by 0, we get

$$f(x) = \frac{f(3x)+2f(0)}{3} \Rightarrow f(3x)+2f(0) = 3f(x)$$

$$\Rightarrow f(3x) = 3f(x) - 2f(0)$$

Now $f'(x) = \displaystyle\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$$= \lim_{h\to 0}\frac{f\left(\dfrac{3x+2\cdot(3h/2)}{3}\right)-f(x)}{h}$$

$$= \lim_{h\to 0}\frac{f(3x)+2f(3h/2)-3f(x)}{3h}$$

$$= \lim_{h\to 0}\frac{3f(x)-2f(0)+2f(3h/2)-3f(x)}{3h}$$

$$= \lim_{h\to 0}\frac{f(3h/2)-f(0)}{(3h/2)} = f'(0) = 1$$

Thus $f'(x) = 1 \Rightarrow f(x) = x + c$, which being a linear function in x, is always continuous.

6.(i) We have $f(x)f(y) + 2 = f(x) + f(y) + f(xy)$

Let $x = y = 1$, we get $f^2(1) + 2 = 3f(1) \Rightarrow f(1) = 1, 2$

Setting $x = y = 0$, we obtain

$f^2(0) + 2 = 3f(0) \Rightarrow f(0) = 1, 2$

As f is $1-1$ function and $f(0) \neq 2 \Rightarrow f(0) = 1$ and $f(1) = 2$

$$f'(x)=\lim_{h\to 0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to 0}\frac{f\left(x\left(1+\frac{h}{x}\right)\right)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{f(x)\cdot f\left(1+\frac{h}{x}\right)+2-f(x)-f\left(1+\frac{h}{x}\right)-f(x)}{h}$$

$$=\lim_{h\to 0}\frac{(f(x)-1)f(1+(h/x))-2(f(x)-1)}{h}$$

$$=\lim_{h\to 0}\frac{(f(x)-1)(f(1+(h/x))-2)}{h}$$

$$=\lim_{h\to 0}\frac{(f(x)-1)(f(1+(h/x))-f(1))}{h}$$

$$=\lim_{h\to 0}\frac{(f(x)-1)(f(1+(h/x))-f(1))}{x\times(h/x)}=\frac{f(x)-1}{x}\cdot f'(1)$$

$\Rightarrow xf'(x) = 2(f(x) - 1)$

Integrating both sides w.r.t x in between 0 and 1, we obtain

$$\int_0^1 x f'(x)dx = 2\int_0^1 (f(x)-1)dx$$

$$\Rightarrow xf(x)\Big|_0^1 - \int_0^1 f(x)dx = 2\int_0^1 f(x)dx - 2$$

$$\Rightarrow \int_0^1 f(x)dx = \frac{4}{3}$$

(iii) $g(x) = f(f(x)-1) + f(5-f(x))$

$g'(x) = f'(f(x)-1)\,f'(x) - f'(5-f(x))\,f'(x)$

Since $f(x)$ is differentiable everywhere $\Rightarrow g(x)$ exist for all $x \in R$ and then there is no point on which function is not differentiable.

Thus $g(x)$ is continuous as well.

For critical points $g'(x) = 0$

$\Rightarrow f'(x)\ \{f'(f(x)-1) - f'(5-f(x))\} = 0$

Either $f'(x) = 0$ or $f(x) - 1 = 5 - f(x)$

$\Rightarrow x = 1$ or x is given by $f(x) = 3$

$f(x) = 3 \Rightarrow x^2 - 2x - 3 = 0 \Rightarrow x = -1, 3$

Thus $x = -1, 1, 3$ are the critical points.

$g(x) = f(f(x)-1) + f(5-f(x))$

$= (f(x)-1)^2 - 2(f(x)-1) + f(5-f(x))^2 - 2(5-f(x))$

$= f^2 - 2f + 1 - 2f + 2 + 25 - 10f + f^2 - 10 + 2f$

$= 2f^2 - 12f + 18 = 2(f^2 - 6f + 9) = 2\ \{f(x) - 3\}^2 \geq 0$

7.(i) $f(x+y) = f(x) + 2y^2 + kxy$

Setting $x = 1, y = 1$, we get $f(2) = f(1) + 2 + k$

$\Rightarrow 8 = 2 + 2 + k \Rightarrow k = 4$

Now $\dfrac{f(x+y)-f(x)}{y} = 2y + kx$

Let $y = h \Rightarrow \dfrac{f(x+h)-f(x)}{h} = 2h + kx$

Taking limits $\lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h} = \lim\limits_{h\to 0}(2h+kx)$

$$\Rightarrow f'(x) = kx = 4x \Rightarrow f(x) = 4.\frac{x^2}{2} + C = 2x^2 + C$$

$f(1) = 2 \Rightarrow C = 0$

Here $f(x) = 2x^2$ and

$$f(x+y)\cdot f\left(\frac{1}{x+y}\right) = 2(x+y)^2\cdot\frac{2}{(x+y)^2} = 4 = k$$

(ii) Let $\max f(x) = M$, where $0 < M \leq 1$ (Becauses $f(x)$ is not zero identically and $|f(x)| \leq 1\ \forall\ x \in R$)

Now $f(x+y) + f(x-y) = 2f(x)\cdot g(y)$

$\Rightarrow\ 2f(x)\,g(y) = f(x+y) + f(x-y)$

$\Rightarrow\ |2f(x)\,g(y)| = |f(x+y) + f(x-y)|$

$\Rightarrow\ 2|f(x)|\ |g(y)| \leq |f(x+y)| + |f(x-y)| \leq M + M$

$\Rightarrow\ 2|f(x)|\ |g(y)| \leq 2M \Rightarrow |g(y)| \leq \dfrac{M}{|f(x)|}$

$\Rightarrow\ |g(y)| \leq \dfrac{M}{M}$ *i.e.* $|g(y)| \leq 1$

(iii) $e^{-xy}f(xy) = e^{-x}f(x) + e^{-y}f(y)$ .... (1)

Let $x = y = 1$ in (1) to obtain

$e^{-1}f(1) = e^{-1}f(1) + e^{-1}f(1)$

$\Rightarrow f(1) = 2f(1)\ \therefore\ f(1) = 0$

By definition $f'(x) = \lim\limits_{h\to 0}\dfrac{f(x+h)-f(x)}{h}$

$$=\lim_{h\to 0}\frac{f(x(1+(h/x)))-f(x\cdot 1)}{h}$$

$$=\lim_{h\to 0}\frac{e^{x+h}\{e^{-x}f(x)+e^{-1-\frac{h}{x}}f(1+(h/x))\}-e^{x}(e^{-x}f(x)+e^{-1}f(1))}{h}$$

$$=\lim_{h\to 0}\frac{e^{h}f(x)+e^{x+h-1-\frac{h}{x}}f(1+(h/x))-f(x)-e^{x-1}f(1)}{h}$$

$$=f(x)\cdot\lim_{h\to 0}\frac{e^{h}-1}{h}+e^{x-1}\lim_{h\to 0}\frac{e^{h-\frac{h}{x}}f(1+(h/x))}{x\cdot(h/x)}$$

$$=f(x)\cdot 1+e^{x-1}\cdot\frac{f'(1)}{x}=f(x)+\frac{e^{x-1}\cdot e}{x}=f(x)+\frac{e^{x}}{x}$$

$$\Rightarrow f'(x)=f(x)+\frac{e^{x}}{x}\Rightarrow e^{-x}f'(x)-e^{-x}f(x)=\frac{1}{x}$$

$$\Rightarrow \frac{d}{dx}\left(e^{-x}f(x)\right)=\frac{1}{x}$$

On integrating, we have $e^{-x}f(x) = \ln x + k$

As $f(1) = 0 \Rightarrow k = 0, \therefore f(x) = lnx \cdot e^x$

8.(i) $f(x) = a|\sin x| + be^{|x|} + c|x|^3$

$$f'(0^+) = \lim_{h\to 0}\frac{f(0+h)-f(0)}{h}$$

$$= \lim_{h\to 0}\frac{a\sin h + be^h + ch^3 - b}{h}$$

$$= \lim_{h\to 0}\left[a\left(\frac{\sin h}{h}\right)+b\left(\frac{e^h-1}{h}\right)+ch^2\right] = a+b$$

$$f'(0^-) = \lim_{h\to 0}\frac{f(0)-f(0-h)}{h}$$

$$= \lim_{h\to 0}\frac{(b-a\sin h+be^h+ch^3)}{h}$$

$$= \lim_{h\to 0} -a\left(\frac{\sin h}{h}\right)-b\left(\frac{e^h-1}{h}\right)+ch^2 = -(a+b)$$

For f to be differentiable at $x = 0$,
we have $a + b = -(a + b)$

$\Rightarrow a + b = 0$. Thus the values of a, b, c for f to be differentiable at $a, b \in R$: $a + b = 0$ and $c \in R$.

(ii) $f(x) = (x^2-1)\,|x^2 - 3x + 2| + \cos|x|$
$= (x^2 - 1)\,|(x - 1)(x - 2)| + \cos x$

Thus $f(x) = -(x^2 - 1)(x - 1)(x - 2) + \cos x, \quad 1 \le x \le 2$
$= (x^2 - 1)(x - 1)(x - 2) + \cos x, \ x \in (-\infty, 1) \cup (2, \infty)$

The only points where the function may fail to be differentiable is $x = 1, 2$.

For x = 1

$$f'(1^+) = \lim_{h\to 0}\frac{f(1+h)-f(1)}{h}$$

$$= \lim_{h\to 0}\frac{\left[\{1-(1+h)^2\}(1+h-1)(1+h-2)+\cos(1+h)\right]-\cos 1}{h}$$

$$= \lim_{h\to 0}\frac{(h^2+2h)\cdot h(1-h)}{h} + \lim_{h\to 0}\frac{\cos(1+h)-\cos 1}{h}$$

$$= 0 + \lim_{h\to 0}\frac{-\sin(1+(h/2))}{1} = -\sin$$

$$f'(1^-) = \lim_{h\to 0}\frac{f(1)-f(1-h)}{h}$$

$$= \lim_{h\to 0}\frac{\cos 1-\left[((1-h)^2-1)(1-h-1)(1-h-2)+\cos(1-h)\right]}{h}$$

$$= \lim_{h\to 0}\frac{-(h^2-2h)h(1+h)}{h} + \lim_{h\to 0}\frac{\cos 1-\cos(1-h)}{h}$$

$$= 0 + \lim_{h\to 0}\frac{-\sin(1-(h/2))}{1} = 0 - \sin 1 = -\sin 1$$

Hence f is derivable at $x = 1$

For x = 2

$$f'(2^+) = \lim_{h\to 0}\frac{f(2+h)-f(2)}{h}$$

$$= \lim_{h\to 0}\frac{[\{(2+h)^2-1\}(2+h-1)(2+h-2) + \cos(2+h)]-\cos 2}{h}$$

$$= \lim_{h\to 0}\frac{(h^2+4h+3)(1+h)h}{h} + \lim_{h\to 0}\frac{\cos(2+h)-\cos 2}{h}$$

$$= 3 + \lim_{h\to 0}\frac{-\sin(2+(h/2))}{1} = 3 - \sin 2$$

$$f'(2^-) = \lim_{h\to 0}\frac{f(2)-f(2-h)}{h}$$

$$= \lim_{h\to 0}\frac{\cos 2-[(1-\{(2-h)^2\})(2-h-1)(2-h-2)+\cos(2-h)]}{h}$$

$$= \lim_{h\to 0}\frac{-(h^2-4h+3)h(1-h)}{h} + \lim_{h\to 0}\frac{\cos 2-\cos(2-h)}{h}$$

$$= -3 + \lim_{h\to 0}\frac{-\sin(2-(h/2))}{h} = -3 - \sin 2$$

As $f'(2^+) \ne f'(2^-)$. Thus $f(x)$ is not differentiable at $x = 2$. Then the only point where the function is not differentiable is $x = 2$.

9. (i) : $$\lim_{x\to 0}\frac{\cot x\tan^{-1}(m\tan x)-m\cos^2(x/2)}{\sin^2(x/2)}$$

$$= \lim_{x\to 0}\frac{\cot x\tan^{-1}(m\tan x)-m\left(1-\sin^2(x/2)\right)}{\sin^2(x/2)}$$

$$= \lim_{x\to 0}\frac{\cot x\tan^{-1}(m\tan x)-m}{\sin^2(x/2)}+m$$

$$= \lim_{x\to 0}\frac{\tan^{-1}(m\tan x)-m\tan x}{\tan x\sin^2(x/2)}+m$$

$$= \lim_{x\to 0}\frac{\tan^{-1}(m\tan x)-m\tan x}{x\cdot(x/2)^2}+m$$

$$= \lim_{x\to 0}\frac{\tan^{-1}(m\tan x)-m\tan x}{x^3/4}+m$$

$$= \lim_{t\to 0}\frac{t-\tan t}{(t^3/4m^3)}+m \quad (\text{set } m\tan x = \tan t)$$

$$= m+4m^3\lim_{t\to 0}\frac{t-\tan t}{t^3} = m+4m^3\lim_{t\to 0}\frac{1-\sec^2 t}{3t^2}$$

(applying L'Hospital rule)

$$= m+4m^3\lim_{t\to 0}\frac{-2\sec^2 t\tan t}{6t} = m+4m^3\left(-\frac{1}{3}\right) = m-\frac{4}{3}m^3$$

(ii) $$\lim_{x\to\infty}\left(\frac{a^{1/x}+b^{1/x}+c^{1/x}}{3}\right)^{3x}$$

$$= \lim_{t\to 0}\left(\frac{a^t+b^t+c^t}{3}\right)^{3/t} \quad \text{Set } t=\frac{1}{x}$$

$$= \lim_{t\to 0} e^{\frac{3}{t}\ln\left(\frac{a^t+b^t+c^t}{3}\right)}$$

Now,

$$\lim_{t\to 0}\frac{3}{t}\ln\left(\frac{a^t+b^t+c^t}{3}\right) = \lim_{t\to 0}\frac{3}{t}\ln\left(1+\frac{a^t+b^t+c^t-3}{3}\right)$$

$$= \lim_{t\to 0}\frac{a^t+b^t+c^t-3}{t}\cdot\frac{\ln\left(1+\frac{a^t+b^t+c^t-3}{3}\right)}{\frac{a^t+b^t+c^t-3}{3}}$$

$$= \lim_{t\to 0}\frac{(a^t-1)+(b^t-1)+(c^t-1)}{t}\cdot\lim_{t\to 0}\frac{\ln(1+u)}{u}$$

$$= \lim_{t\to 0}\left[\frac{(a^t-1)}{t}+\frac{(b^t-1)}{t}+\frac{(c^t-1)}{t}\right]\cdot 1$$

$$= \left(\lim_{t\to 0}\frac{a^t-1}{t}\right)+\left(\lim_{t\to 0}\frac{b^t-1}{t}\right)+\left(\lim_{t\to 0}\frac{c^t-1}{t}\right)$$

$$= \ln a+\ln b+\ln c = \ln(abc)$$

Hence the limit is $e^{\ln(abc)} = abc$.

10. (i) We have $f(x+2y) = f(x)\,e^{2y} + f(2y)\,.\,e^x + x^2(1-e^{2y}) + 4y^2(1-e^x) + 4xy$

Set $x = y = 0$, we obtain $f(0) = 2f(0) \Rightarrow f(0) = 0$

By definition $f'(x) = \lim_{h\to 0}\frac{f(x+2h)-f(x)}{2h}$

$$= \lim_{h\to 0}\frac{f(x)e^{2h}+e^x f(2h)+x^2(1-e^{2h})+4h^2(1-e^x)+4xh-f(x)}{2h}$$

$$= \lim_{h\to 0}\frac{f(x)(e^{2h}-1)+e^x(f(2h)-f(0)-(e^{2h}-1)x^2+4xh-4h^2(1-e^x)}{2h}$$

$$= f(x)\lim_{h\to 0}\frac{e^{2h}-1}{2h}+e^x\lim_{h\to 0}\frac{f(2h)-f(0)}{2h} - x^2\lim_{h\to 0}\frac{e^{2h}-1}{2h}+2x$$

$= f(x) + e^x \cdot f'(0) - x^2 + 2x$

$\Rightarrow f'(x) = f(x) + e^x - x^2 + 2x$

$\Rightarrow e^{-x}f'(x) - e^{-x}f(x) = 1 + 2xe^{-x} - x^2e^{-x}$

$$\Rightarrow \frac{d}{dx}\left(e^{-x}f(x)\right) = \frac{d}{dx}\left(x^2e^{-x}+x\right)$$

$\Rightarrow e^{-x}f(x) = x + x^2e^{-x} + k$

As $f(0) = 0 \Rightarrow k = 0 \quad \therefore f(x) = xe^x + x^2$

(ii) $f(2x^2-1) = 2xf(x)\ \forall\ x \in [-1, 1]$

Replacing x by $-x$, we obtain

$f(2x^2-1) = -2x\,f(-x) \Rightarrow 2x\,f(-x) + f(2x^2-1) = 0$

$\Rightarrow 2x\,f(-x) + 2x\,f(x) = 0$

$\Rightarrow 2x\,(f(x) + f(-x)) = 0\ \forall\ x \in [-1, 1]$

$\therefore f(x) + f(-x) = 0.$ Thus $f(x)$ is odd.

As $f(x)$ is odd and continuous in $[-1, 1] \Rightarrow f(0) = 0$

$$\Rightarrow f(x) = \frac{f(2x^2-1)}{2x} \Rightarrow \lim_{x\to 0} f(x) = \lim_{x\to 0}\frac{f(2x^2-1)}{2x} = 0$$

Replacing $x = \cos\theta$ in the given functional equation, we obtain

$$f(\cos 2\theta) = 2\cos\theta\, f(\cos\theta) = 2\cos\theta\left(2\cos\frac{\theta}{2}f\left(\cos\frac{\theta}{2}\right)\right)$$

$$= 2^2\cos\theta\cos\frac{\theta}{2}f\left(\cos\frac{\theta}{2}\right)$$

$$= 2^2\cos\theta\cos\frac{\theta}{2}\left(2\cos\frac{\theta}{4}f\left(\cos\frac{\theta}{4}\right)\right)$$

$= \ldots\ldots\ldots\ldots\ldots\ldots$

$$= 2^{n+1}\cdot\cos\theta\cdot\cos\frac{\theta}{2}\cdot\cos\frac{\theta}{2^{n-1}}\cdot\cos\frac{\theta}{2^n}f\left(\cos\frac{\theta}{2^n}\right)$$

$$= \sin 2\theta\frac{f\left(\cos\frac{\theta}{2^n}\right)}{\sin\frac{\theta}{2^n}}$$

Taking limit on both sides as $n\to\infty$, we get

$$\lim_{n\to\infty} f(\cos 2\theta) = \sin 2\theta\lim_{n\to\infty}\frac{f\left(\cos\frac{\theta}{2^n}\right)}{\sin\frac{\theta}{2^n}}$$

$$= \sin 2\theta\lim_{n\to\infty}\frac{f\left(2\cos^2\frac{\theta}{2^{n+1}}-1\right)}{2\sin\frac{\theta}{2^{n+1}}\cdot\cos\frac{\theta}{2^{n+1}}}$$

$$= \sin 2\theta\cdot\left[\lim_{n\to\infty}\frac{f\left(2\sin^2\frac{\theta}{2^{n+1}}-1\right)}{2\sin\frac{\theta}{2^{n+1}}\cdot\cos\frac{\theta}{2^{n+1}}}\right] = 0$$

Thus $f(\cos 2\theta) = 0\ \forall\ \theta \in R$

This gives $f(x) = 0\ \ \forall x \in [-1, 1]$

PART - B

1. (d) : $$\lim_{x\to 0}\frac{\sin^{-1}x-\tan^{-1}x}{x^3} = \lim_{x\to 0}\frac{\frac{1}{\sqrt{1-x^2}}-\frac{1}{1+x^2}}{3x^2}$$

$$= \lim_{x\to 0} \frac{(1+x^2)-\sqrt{1-x^2}}{3x^2} \frac{1}{(1+x^2)\sqrt{1-x^2}}$$

$$= \lim_{x\to 0} \frac{(1+x^2)^2-(1-x^2)}{3x^2\left[(1+x^2)+\sqrt{1-x^2}\right]}\cdot 1$$

$$= \lim_{x\to 0} \frac{x^4+3x^2}{3x^2}\cdot\frac{1}{(1+x^2)+\sqrt{1-x^2}} = \lim_{x\to 0}\frac{x^2+3}{3}\cdot\frac{1}{2}=\frac{1}{2}$$

2. **(c) :** The functional relation is

$$3f(x) + 4f\left(\frac{1}{x}\right) = \frac{1}{x} - 5 \quad ....(1)$$

Changing x to $1/x$ in (1) we have

$$3f\left(\frac{1}{x}\right) + 4f(x) = x-5 \quad ....(2)$$

Multiplying (1) by 3 and (2) by 4 and then by subtracting

$$9f(x)-16f(x)=3\left(\frac{1}{x}-5\right)-4(x-5)$$

$$\Rightarrow -7f(x) = -15+20+\frac{3}{x}-4x$$

$$\Rightarrow f(x)=\frac{5+(3/x)-4x}{-7} \quad \Rightarrow f(x)=\frac{4x-(3/x)-5}{7}$$

$$f(2) = \frac{8-(3/2)-5}{7} = \frac{16-3-10}{2\times 7} = \frac{3}{14}$$

3. **(a) :** $F(x) = f(x)\,g(x)\,h(x)$

Taking logarithm, we have

$\ln F(x) = \ln f(x) + \ln g(x) + \ln h(x)$

Differentiate w.r.t x we obtain

$$\frac{F'(x)}{F(x)}=\frac{f'(x)}{f(x)}+\frac{g'(x)}{g(x)}+\frac{h'(x)}{h(x)}$$

$$\Rightarrow \frac{F'(x_0)}{F(x_0)}=\frac{f'(x_0)}{f(x_0)}+\frac{g'(x_0)}{g(x_0)}+\frac{h'(x_0)}{h(x_0)}$$

$$\Rightarrow 13=9+4+\frac{h'(x_0)}{h(x_0)} \quad \Rightarrow 0=\frac{h'(x_0)}{h(x_0)} \quad \therefore h'(x_0)=0$$

4. **(c) :** $x = 2t - |t|, y = t^3 + t^2|t|$

Function can be written in the form

$x = 3t, y = 0$, when $t < 0$

$x = t, y = 2t^3$, when $t > 0$

Eliminating the parameter, we obtain

$y = 0 \quad x < 0$

$\;\; = 2x^3 \quad x \geq 0$

Differentiate w.r.t x gives

$$\frac{dy}{dx} = 0, \quad x < 0$$

$$= 6x^2,\ x \geq 0$$

Hence the function is differentiable at $x = 0$ and $f'(0) = 0$

5. **(d) :** $f(x+y^n) = f(x) + (f(y))^n$

Set $x = y = 0 \Rightarrow f(0) = f(0) + f(0) = f(0) = 0$

$$\text{Now } \lim_{y\to 0}\frac{f(x+y^n)-f(x)}{y^n} = \lim_{y\to 0}\frac{(f(y))^n}{y^n}$$

$$\Rightarrow f'(x) = \left(\lim_{y\to 0}\frac{f(y)}{y}\right)^n = \left(\lim_{y\to 0}\frac{f(y+(0))-f(0)}{y}\right)^n$$

$$=(f''(0))^n \text{ constant}$$

$\therefore f(x) = ax + b, f'(0) = 1, f(0) = 0 \quad \therefore a = 1, b = 0$

Thus $f(x) = x \quad \therefore f(15) = 15, f'(20) = 1$

6. **(a) :** For domain $0 < x^2 + 3x + 2 \leq 1$

$x^2 + 3x + 2 > 0 \Rightarrow (x + 1)(x + 2) > 0$

$\Rightarrow x \in (-\infty, -2) \cup (-1, \infty)$

$x^2 + 3x + 2 \leq 1 \Rightarrow x^2 + 3x + 1 \leq 0$

$$\Rightarrow x \in \left[\frac{-3-\sqrt{5}}{2}, \frac{-3+\sqrt{5}}{2}\right]$$

$$\text{Thus domain } D = \left[\frac{-3-\sqrt{5}}{2}, -2\right) \cup \left(-1, \frac{-3+\sqrt{5}}{2}\right]$$

$$\text{For range, } 0 < \sin^{-1}\sqrt{x^2+3x+2} \leq \frac{\pi}{2}$$

$$\Rightarrow -\infty < \ln\left(\sin^{-1}\sqrt{x^2+3x+2}\right) \leq \ln\frac{\pi}{2}$$

Thus $\ln\left(\sin^{-1}\sqrt{x^2+3x+2}\right)$ can take all non-positive integral values.

7. **(c) :** $x + y = e^{x-y}$

Differentiating w.r.t x, we get

$$1+\frac{dy}{dx} = e^{x-y}\left(1-\frac{dy}{dx}\right) \Rightarrow y'+1 = e^{x-y}(1-y')$$

$$\Rightarrow y'+1 = (x+y)(1-y') \Rightarrow y'(x+y+1) = x+y-1$$

Differentiating again

$$y''(x+y+1)+y'(1+y') = 1+y'$$

$$\Rightarrow y''(x+y+1)+y'^2-1 = 0$$

$$\Rightarrow y''(x+y+1)+\left(\frac{x+y-1}{x+y+1}\right)^2-1 = 0$$

$$y''(x+y+1)+\frac{(x+y-1)^2-(x+y+1)^2}{(x+y+1)^2} = 0$$

$$\Rightarrow y''(x+y+1)+\frac{(-4(x+y))}{(x+y+1)^2} = 0 \Rightarrow y'' = \frac{4(x+y)}{(x+y+1)^3}$$

$$\left.\frac{d^2y}{dx^2}\right|_{x=x_0} = \frac{4(x_0+y_0)}{(x_0+y_0+1)^3} = \frac{4e^{x_0-y_0}}{\left(1+e^{x_0-y_0}\right)^3}$$

8. (b) : $\lim_{x\to0}\dfrac{27^x-9^x-3^x+1}{\sqrt5-\sqrt{4+\cos x}}=\lim_{x\to0}\dfrac{(9^x-1)(3^x-1)}{\sqrt5-\sqrt{4+\cos x}}$

$=\lim_{x\to0}\dfrac{(9^x-1)(3^x-1)(\sqrt5+\sqrt{4+\cos x})}{5-(4+\cos x)}$

$=\lim_{x\to0}\dfrac{(9^x-1)(3^x-1)(\sqrt5+\sqrt{4+\cos x})}{1-\cos x}$

$=\left(\lim_{x\to0}\dfrac{9^x-1}{x}\right)\cdot\left(\lim_{x\to0}\dfrac{3^x-1}{x}\right)\lim_{x\to0}\left(\dfrac{x^2}{2\sin^2\frac{x}{2}}\right)\cdot\left(\sqrt5+\sqrt5\right)$

$=\ln9\cdot\ln3\cdot2\cdot2\sqrt5=2\ln3(\ln3)\cdot4\sqrt5$

$=8\sqrt5(\ln3)^2$

9. (a) : $\lim_{x\to0}f(x)=\lim_{x\to0}\dfrac{\sin2x+a\sin x}{x^3}$

$=\lim_{x\to0}\dfrac{2\cos2x+a\cos x}{3x^2}$

For the limit to exist *i.e.* to be finite, the numerator must tend to zero when denominator tends to zero.

$2+a=0\Rightarrow a=-2$

Putting the value of a, we obtain

$\lim_{x\to0}f(x)=\lim_{x\to0}\dfrac{2(\cos2x-\cos x)}{3x^2}$

$=\lim_{x\to0}\dfrac{2(-2\sin2x+\sin x)}{6x}$

$=\lim_{x\to0}\left(-\dfrac{4}{3}\right)\cdot\left(\dfrac{\sin2x}{2x}\right)+\lim_{x\to0}\dfrac{1}{3}\left(\dfrac{\sin x}{x}\right)=-\dfrac{4}{3}+\dfrac{1}{3}=-1.$

10. (c) : $\lim_{n\to\infty}\cos\dfrac{x}{2}\cos\dfrac{x}{4}\cos\dfrac{x}{8}\,.....\cos\dfrac{x}{2^n}$

$=\lim_{n\to\infty}\dfrac{\cos\frac{x}{2}\cos\frac{x}{4}\,.....\cos\frac{x}{2^n}\cdot\sin\frac{x}{2^n}}{\sin\frac{x}{2^n}}$

$=\lim_{n\to\infty}\dfrac{\cos\frac{x}{2}\cos\frac{x}{4}\,.....\frac{1}{2}\cdot\sin\frac{x}{2^{n-1}}}{\sin\frac{x}{2^n}}$

$=\lim_{n\to\infty}\dfrac{\frac{1}{4}\cos\frac{x}{2}\cos\frac{x}{4}.....\sin\frac{x}{2^{n-2}}}{\sin\frac{x}{2^n}}$

$=\lim_{n\to\infty}\dfrac{\frac{\sin x}{2^n}}{\sin\left(\frac{x}{2^n}\right)}=\lim_{n\to\infty}\dfrac{\frac{x}{2^n}}{\sin\left(\frac{x}{2^n}\right)}\cdot\dfrac{\sin x}{x}$

$=\dfrac{\sin x}{x}\lim_{t\to0}\dfrac{t}{\sin t}=\dfrac{\sin x}{x}\cdot1=\dfrac{\sin x}{x}$

(We have repeatedly made use of the identity $\sin\theta=2\sin\dfrac{\theta}{2}\cos\dfrac{\theta}{2}$)

11. (b) : f is continuous at $x=1$

$\Rightarrow a+2=a+b+1\Rightarrow b=1$

f is continuous at $x=2\Rightarrow 2a+5=4a+2b+1$

$\Rightarrow 2a+5=4a+3\Rightarrow 2a=2\therefore a=1$

12. (c) : $\lim_{n\to\infty}\left(4^n+5^n\right)^{\frac{1}{n}}=\lim_{n\to\infty}\left\{5^n\left(\dfrac{4^n}{5^n}+1\right)\right\}^{\frac{1}{n}}$

$=\lim_{n\to\infty}\left\{5^n\left\{1+\left(\dfrac{4}{5}\right)^n\right\}\right\}^{\frac{1}{n}}=5\lim_{n\to\infty}\left(1+\left(\dfrac{4}{5}\right)^n\right)^{\frac{1}{n}}$

$=5(1+0)=5$

13. (b) : Since t is continuous at $x=\pi/4$

$\therefore f(\pi/4)=\lim_{x\to\pi/4}\cos2x\cot\left(\dfrac{\pi}{4}-x\right)$

$=\lim_{x\to\frac{\pi}{4}}\left(\dfrac{-2\sin2x}{-\sec^2\left(\frac{\pi}{4}-x\right)}\right)=\dfrac{-2\cdot1}{-1}=2$

14. (c) : We have $f(x+y)=f(x)+f(y)$ ..... (A)

and $f(x)=x^2g(x)$

As $f(x)$ is differentiable, by definition,

$f'(x)=\lim_{h\to0}\dfrac{f(x+h)-f(x)}{h}=\lim_{h\to0}\dfrac{f(h)}{h}$ (using A)

$=\lim_{h\to0}\dfrac{h^2g(h)}{h}=\lim_{h\to0}hg(h)=0$

$\Rightarrow f'(x)=0\therefore f(x)=\text{constant}=c$ (says)

Thus $|f(15)-f(-15)|=|c-c|=0$

15. (a) : $\lim_{x\to1}\dfrac{x^{1/3}-x^{1/4}-2}{x^3-1}=\lim_{x\to1}\dfrac{(x^{1/3}-1)-(x^{1/4}-1)}{x^3-1}$

$=\lim_{x\to1}\dfrac{(x^{1/3}-1)-(x^{1/4}-1)}{(x-1)(x^2+x+1)}$

$\left(\lim_{x\to1}\dfrac{x^{1/3}-1}{x-1}-\lim_{x\to1}\dfrac{x^{1/4}-1}{x-1}\right)\dfrac{1}{(1^2+1+1)}$

$=\left[\left(\dfrac{1}{3}\right)(1)^{-2/3}-\dfrac{1}{4}(1)^{-3/4}\right]\dfrac{1}{3}$

$=\left(\dfrac{1}{3}-\dfrac{1}{4}\right)\dfrac{1}{3}=\dfrac{1}{12}\times\dfrac{1}{3}=\dfrac{1}{36}$

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers comments and suggestions regarding the problems and solutions offered are always welcome.

1. If $x\sqrt{1+y}+y\sqrt{1+x}=0$ then $\frac{dy}{dx}=$

(a) $\frac{1}{1+x^2}$ (b) $\frac{-1}{1+x^2}$

(c) $\frac{1}{1+x}$ (d) none of these

2. If $\sin y = x\sin(a+y)$, then $\frac{dy}{dx}=$

(a) $\frac{\sin^2(a+y)}{\sin a}$ (b) $\sin(a+y)$

(c) $\sin^2(a+y)$ (d) $\frac{\sin(a+y)}{\sin a}$

3. $\lim_{x\to\infty}\sqrt{\left(\frac{x+\sin x}{x-\cos x}\right)}=$

(a) 0 (b) 1

(c) −1 (d) none of these

4. If $f(x)=\left(\frac{x^2+5x+3}{x^2+x+2}\right)^x$, then $\lim_{x\to\infty} f(x)$ is

(a) e^4 (b) e^3

(c) e^2 (d) 2^4

5. Let α and β be the roots of $ax^2+bx+c=0$ then $\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}$ is equal to

(a) 0 (b) $\frac{1}{2}(\alpha-\beta)^2$

(c) $\frac{a^2}{2}(\alpha-\beta)^2$ (d) $-\frac{a^2}{2}(\alpha-\beta)^2$

6. Find the domain of the function $f(x)=\frac{2}{[x/3]}-5^{\sin^{-1}x^2}+\frac{3x+1!}{\sqrt{x+1}}$, where [.] denotes the greatest integer function.

7. If $\alpha_1, \alpha_2, \alpha_3, ..., \alpha_n$ are the roots of equation $x^n - nax - b = 0$ and $(\alpha_1-\alpha_2).(\alpha_1-\alpha_3) .... (\alpha_1-\alpha_n) = A$, then find the value of $A - n\alpha_1^{n-1}$.

8. If left hand derivative and right hand derivative of a function f at 'a' are finite, then show that f is continuous at 'a'.

9. Find the value(s) of 'a' for which $\lim_{x\to 0}\frac{\sin 3x + a\sin 2x}{x^3}$ exists finitely. Find the value of the limit also.

10. If α, β are distinct real roots of the quadratic equation $ax^2+bx+c=0$, then show that

$$\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2}=\frac{b^2-4ac}{2}.$$

SOLUTIONS

1. **(d)** : Squaring the given equation we will get

$$y=\frac{-x}{1+x}$$

Hence,

$$\frac{dy}{dx}=\frac{-1}{(1+x)^2}$$

2. **(a)** : Differntiating both sides w.r.t. x we get

$$\cos y\frac{dy}{dx}=\sin(a+y)+x\cos(a+y)\frac{dy}{dx}$$

$$\cos y\frac{dy}{dx}=\sin(a+y)+x\cos(a+y)\frac{dy}{dx}$$

$$\frac{dy}{dx}[\sin(a+y).\cos y-\sin y.\cos(a+y)]=\sin^2(a+y)$$

$$\frac{dy}{dx}=\frac{\sin^2(a+y)}{\sin a}$$

3. **(b)** :

$$\lim_{x\to\infty}\sqrt{\frac{1+\frac{\sin x}{x}}{1-\frac{\cos x}{x}}} \text{ and } \lim_{x\to\infty}\frac{\sin x}{x}=\lim_{x\to\infty}\frac{\cos x}{x}=0$$

4. **(a)** : Use the fact $\lim_{x\to\infty}(f(x))^{g(x)}$

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$e^{\lim_{x\to\infty}(f(x)-1)g(x)}$

Where $f(x) \to 1$ as $x \to \infty \Rightarrow g(x) \to \infty$

5. (c) : Use the fact

$$\lim_{x\to 0}\frac{1-\cos ax}{x^2} = \frac{a^2}{2}$$

6. As $\left[\frac{x}{3}\right] = 0 \Rightarrow 0 \le \frac{x}{3} < 1 \Rightarrow 0 \le x < 3$

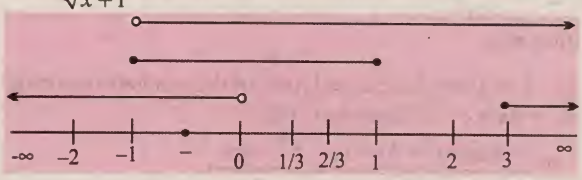
i.e., $\left[\frac{x}{3}\right]$ is defined for $x \in R - [0,3)$.

$\sin^{-1}x^2$ is defined for $0 \le x^2 \le 1 \Rightarrow x \in [-1, 1]$

$3x + 1!$ is defined for $3x + 1 \ge 0$ and $(3x+1) \in \{0, 1, 2, 3, ...\}$

$$\Rightarrow x \in \left\{\frac{-1}{3}, 0, \frac{1}{3}, \frac{2}{3}, ...\right\}$$

$\frac{1}{\sqrt{x+1}}$ is defined for $x \in (-1, \infty)$



$\therefore f(x)$ is defined for $x \in \left\{\frac{-1}{3}\right\}$.

7. Given $x^n - nax - b = 0$

Roots are $\alpha_1, \alpha_2, \alpha_3, \ldots, \alpha_n$

$\therefore x^n - nax - b = (x-\alpha_1)(x-\alpha_2)(x-\alpha_3)\ldots(x-\alpha_n)$

or $\frac{x^n - nax - b}{(x-\alpha_1)} = (x-\alpha_2)(x-\alpha_3)....(x-\alpha_n)$

or $\lim_{x\to\alpha_1}\frac{x^n - nax - b}{(x-\alpha_1)} = \lim_{x\to\alpha_1}(x-\alpha_2)(x-\alpha_3)....(x-\alpha_n)$

or $\lim_{x\to\alpha_1}\frac{nx^{n-1} - na}{1} = \lim_{x\to\alpha_1}(x-\alpha_2)(x-\alpha_3)...(x-\alpha_n)$

or $n\alpha_1^{n-1} - na = (\alpha_1-\alpha_2)(\alpha_1-\alpha_3)(\alpha_1-\alpha_4)...(\alpha_1-\alpha_n)$

$\Rightarrow n\alpha_1^{n-1} - na = A$

$\therefore A - n\alpha_1^{n-1} = -na$

8. $\lim_{h\to 0^+}(f(a)-f(a-h)) = \lim_{h\to 0^+}\frac{f(a)-f(a-h)}{h}\cdot h$

$\lim_{h\to 0} f'(a^-)\cdot h$ (as $f'(a^-)$ is finite)

Similarly

$\lim_{h\to 0^+}(f(a+h)-f(a)) = \lim_{h\to 0^+} f'(a^+).h = 0$

Hence

$\lim_{h\to 0^+} f(a-h) = \lim_{h\to 0^+} f(a+h) = f(a)$

Hence f is continuous at a.

9. Let $l = \lim_{x\to 0}\frac{\sin 3x + a\sin 2x}{x^3}\left(\frac{0}{0}\text{form}\right)$

$$= \lim_{x\to 0}\frac{3\cos 3x + 2a\cos 2x}{3x^2}$$

We should have $g(0) = 0$, where $g(x) = 3\cos 3x + 2a\cos 2x$, as the given limit exists finitely $\Rightarrow a = -3/2$.

Further $l = \lim_{x\to 0}\frac{3\cos 3x - 3\cos 2x}{3x^2}$

$$= \lim_{x\to 0}\frac{\cos 3x - \cos 2x}{x^2}\left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0}\frac{2\sin 2x - 3\sin 3x}{2x}\left(\frac{0}{0}\text{form}\right)$$

$$= \lim_{x\to 0}\frac{\cos 2x - 9\cos 3x}{2} = -\frac{5}{2}$$

Thus $a = -3/2$ and limit value $= -5/2$.

10.

$$\lim_{x\to\alpha}\frac{1-\cos(ax^2+bx+c)}{(x-\alpha)^2} = \lim_{x\to\alpha}\frac{2\sin^2\left(\frac{ax^2+bx+c}{2}\right)}{(x-\alpha)^2}$$

$$= \lim_{x\to\alpha}\frac{2\sin^2\left(\frac{a(x-\alpha)(x-\beta)}{2}\right)}{\left(\frac{a(x-\alpha)(x-\beta)}{2}\right)^2}\cdot\frac{a^2}{4}(x-\beta)^2$$

$$= \frac{a^2(\alpha-\beta)^2}{2} = \frac{a^2}{2}[(\alpha+\beta)^2 - 4\alpha\beta]$$

$$= \frac{a^2}{2}\left[\frac{b^2}{a^2} - \frac{4c}{a}\right] = \frac{b^2-4ac}{2}.$$

12. Let n be a positive integer and k be a whole number, $k \le 2n$.

Statement-1 : The maximum value of ${}^{2n}C_k$ is ${}^{2n}C_n$.

Statement-2 : ${}^nC_r = {}^nC_{n-r}$, for $n = 0, 1, 2, ..., n$.

13. Let $y = x + 3, y = 2x + 3, y = 3x + 2$ and $y + x = 3$ are four straight lines

Statement-1 : The number of triangles formed is ${}^4C_3$.

Statement-2 : Number of distinct point of intersection between various lines will determine the number of possible triangle.

14. Let A and B are two events such that $P(A) = \frac{3}{5}$ and $P(B) = \frac{2}{3}$, then

Statement-1 : $\frac{4}{15} \le P(A \cap B) \le 1$.

Statement-2 : $\frac{2}{5} \le P\left(\frac{A}{B}\right) \le \frac{9}{10}$.

15. Statement-1 : Every square matrix can be uniquely expressed as the sum of a symmetric matrix and a skew symmetric matrix.

Statement-2 : The elements on the main diagonal of a skew symmetric matrix are all different.

16. Consider the following system of equation $ax + y + z = 0,\ x + by + z = 0,\ x + y + cz = 0$ where a, b and c are satisfying $\frac{a}{1-a} + \frac{b}{1-b} + \frac{c}{1-c} = 2$.

Statement-1 : Above system of equation have infinitely many solutions.

Statement-2 : $\begin{vmatrix} a & 1 & 1 \\ 1 & b & 1 \\ 1 & 1 & c \end{vmatrix} = 0.$

17. From the point $P(\sqrt{2}, \sqrt{6})$, tangents PA and PB are drawn to the circle $x^2 + y^2 = 4$.

Statement-1 : Area of the quadrilateral $OAPB$ is 4.

Statement-2 : Tangents PA and PB are perpendicular to each other.

18. Statement-1 : Tangents drawn from the point (3, 4) on to the ellipse $\frac{x^2}{16} + \frac{y^2}{9} = 1$ will be mutually perpendicular.

Statement-2 : The points (3, 4) lies on the circle $x^2 + y^2 = 25$.

19. Statement-1 : The shortest distance between the skew lines $\vec{r} = \vec{a} + \alpha\vec{b}$ and $\vec{r} = \vec{c} + \beta\vec{d}$ is $\frac{|[\vec{a} - \vec{c}\ \vec{b}\ \vec{d}]|}{|\vec{b} \times d|}$.

Statement-2 : Two lines are skew lines if there exist no plane passing through them.

20. The equation of two straight lines are $\frac{x-1}{2} = \frac{y+3}{1} = \frac{z-2}{-3}$ and $\frac{x-2}{1} = \frac{y-1}{-3} = \frac{z+3}{2}$.

Statement-1 : The given lines are coplanar

Statement-2 : The equation $2x_1 - y_1 = 1$, $x_1 + 3y_1 = 4$, $3x_1 + 2y_1 = 5$ are consistent.

ANSWERS

1. (d)	2. (c)	3. (a)	4. (d)	5. (b)
6. (a)	7. (a)	8. (d)	9. (a)	10. (d)
11. (a)	12. (b)	13. (a)	14. (d)	15. (c)
16. (a)	17. (a)	18. (a)	19. (b)	20. (a)

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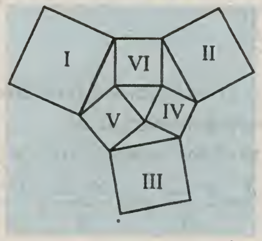
OLYMPIAD CORNER

International Olympiad Problems

Challenging problems for Olympiads, IIT-JEE and other contests.

1. The vertices of six squares coincide in such a way that they enclose triangles; see the picture. Prove that the sum of the areas of the three outer squares (I, II and III) equals three times the sum of the areas of the three inner squares (IV, V and VI).



2. $ABCD$ is a square with incircle T. A tangent l to T meets the sides AB and AD and the diagonal AC at P, Q and R respectively. Prove that

$$\frac{AP}{PB}+\frac{AR}{RC}+\frac{AQ}{QD}=1.$$

3. Let ΔABC be an equilateral. On side AB produced, we choose a point P such that A lies between P and B. We now denote a as the length of sides of ΔABC; r_1 as the radius of incircle of ΔPAC; and r_2 as the exradius of ΔPBC with respect to side BC. Determine the sum r_1+r_2 as a function of a alone.

4. If the tangents at A, B, C to the circumcircle of triangle ΔABC meet the opposite sides at D, E, F, respectively, prove that D, E, F, are collinear.

5. For triangle ABC such that $R\,(a+b)= c\sqrt{ab}$, prove that

$$r<\frac{3}{10}a.$$

Here, a, b, c, R, and r are the three sides, the circumradius and the inradius of ΔABC.

6. Consider the infinite sum

$$S=\frac{a_0}{10^0}+\frac{a_1}{10^2}+\frac{a_2}{10^4}+\frac{a_3}{10^6}+\ldots,$$

where the sequence $\{a_n\}$ is defined by $a_0=a_1=1$, and the recurrence relation $a_n=20a_{n-1}+12a_{n-2}$ for all positive integers $n\ge 2$. If $\sqrt{S}$ can be expressed in the form $\frac{a}{\sqrt{b}}$, where a and b are relatively prime positive integers, determine the ordered pair (a, b).

7. Show that $r^2+r_a^2+r_b^2+r_c^2\ge 4K$, where r, r_a, r_b, r_c, and K are respectively the inradius, exradii and area of a triangle ABC.

8. The surface area of a closed cylinder is twice the volume. Determine the radius and height of the cylinder given that the radius and height are both integers.

9. It is given that $\cos x=\cos y$ and $\sin x=-\sin y$. Prove that $\sin 1994\,x+\sin 1994\,y=0$.

10. Let a, b, c, d, e, f be the lengths of edges of a given tetrahedron and S be its surface area. Prove that

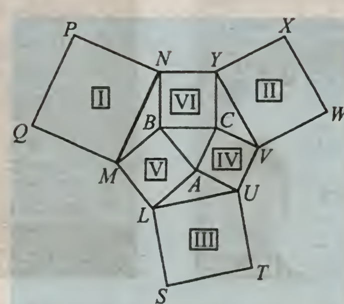
$$S\le\frac{\sqrt{3}}{6}(a^2+b^2+c^2+d^2+e^2+f^2).$$

SOLUTIONS

1. Let $MN=x_1, LU=x_3, VY=x_2, AC=x_4, AB=x_5, BC=x_6$, $\angle MBN=\alpha, \angle LAU=\beta, \angle VCY=\gamma, \angle BAC=A, \angle ACB=C$ and $\angle ABC=B$.

Then we have $\alpha+B=\pi$, $\beta+A=\pi$, $\gamma+C=\pi$

$$\left.\begin{aligned} x_1^2&=x_6^2+x_5^2-2x_5x_6\cos\alpha\\ x_2^2&=x_4^2+x_6^2-2x_4x_6\cos\gamma\\ x_3^2&=x_4^2+x_5^2-2x_4x_5\cos\beta\end{aligned}\right\}\ldots\ldots \quad \text{(i)}$$



$$\left.\begin{aligned} x_4^2 &= x_5^2 + x_6^2 - 2x_5x_6 \cos B \\ x_5^2 &= x_4^2 + x_6^2 - 2x_4x_6 \cos C \\ x_6^2 &= x_4^2 + x_5^2 - 2x_4x_5 \cos A \end{aligned}\right\} \ldots\ldots \qquad \text{(ii)}$$

From (ii), we have

$$x_4^2 + x_5^2 + x_6^2 = 2x_4x_5 \cos A + 2x_5x_6 \cos B + 2x_4x_6 \cos C$$
$$= -2x_4x_5 \cos \beta - 2x_5x_6 \cos \alpha - 2x_4x_6 \cos \gamma \quad \ldots \text{(iii)}$$

From (i), we have

$$x_1^2 + x_2^2 + x_3^2 = 2(x_4^2 + x_5^2 + x_6^2) - 2x_5x_6 \cos \alpha$$
$$- 2x_4x_5 \cos \beta - 2x_4x_6 \cos \gamma$$
$$= 2(x_4^2 + x_5^2 + x_6^2) + x_4^2 + x_5^2 + x_6^2 \qquad \text{using (iii)}$$
$$= 3(x_4^2 + x_5^2 + x_6^2)$$

That is, Area of (I + II + III) = 3 Area of (IV + V + VI).

2. We suppose that the equation of T is $x^2 + y^2 = 1$, and the coordinates of A and C are (1, 1) and (–1, –1) respectively. Let L ($\cos t$, $\sin t$) be the coordinates of the point of tangency of line l, with $0 < t < \frac{\pi}{2}$.

Then the equation of PQ is

$$y \cdot \sin t + x \cdot \cos t = 1,$$

and the coordinates of the involved points are

$$R\left(\frac{1}{\sin t + \cos t}, \frac{1}{\sin t + \cos t}\right), P\left(1, \frac{1 - \cos t}{\sin t}\right), Q\left(\frac{1 - \sin t}{\cos t}, 1\right),$$

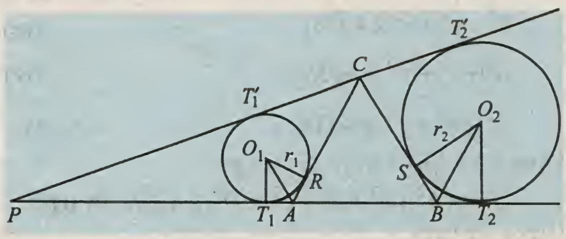
and so

$$\frac{AP}{PB} = \frac{\sin t + \cos t - 1}{\sin t - \cot + 1}, \quad \frac{AR}{RC} = \frac{\sin t + \cos t - 1}{\sin t + \cos t + 1},$$
$$\frac{AQ}{QD} = \frac{\cos t + \sin t - 1}{\cos t - \sin t + 1},$$

from which an easy calculation shows that

$$\frac{AP}{PB} + \frac{AQ}{QD} + \frac{AR}{RC} = 1.$$

3. Looking at the figure, we see that $\angle T_1 O_1 R = 60°$ since it is the supplement of $\angle T_1 AR = 120°$ (as an exterior angle for ΔABC). Hence, $\angle AO_1R = 30°$. Similarly, we obtain $\angle BO_2S = 30°$.



Since tangents drawn to a circle from an external point are equal, we have

$$T_1T_2 = T_1A + AB + BT_2 = RA + AB + SB$$
$$= r_1 \tan 30° + a + r_2 \tan 30° = \frac{r_1 + r_2}{\sqrt{3}} + a.$$

and

$$T_1'T_2' = T_1'C + CT_2' = CR + CS = (a - RA) + (a - SB)$$
$$= 2a - \frac{r_1 + r_2}{\sqrt{3}}.$$

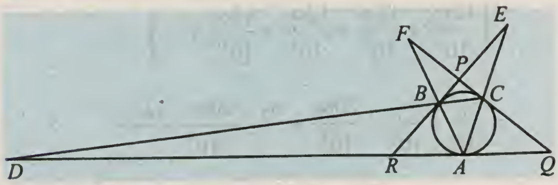
Since common external tangents to two circles are equal, $T_1T_2 = T_1'T_2'$.

Hence,

$$\frac{r_1 + r_2}{\sqrt{3}} + a = 2a - \frac{r_1 + r_2}{\sqrt{3}},$$

Hence we find that $r_1 + r_2 = \dfrac{a\sqrt{3}}{2}$.

4.



For notational ease, we write $PB = PC = \alpha$, $QC = QA = \beta$ and $RB = RA = \gamma$.

Apply Menelaus' Theorem for triangle ΔPQR with transversals BDC, EAC and BAF

$$-1 = \frac{PB}{BR}\frac{RD}{DQ}\frac{QC}{CP} = \frac{QC}{BR}\frac{RD}{DQ},$$
$$-1 = \frac{PE}{ER}\frac{RA}{AQ}\frac{QC}{CP} = \frac{PE}{ER}\frac{RA}{CP},$$
$$-1 = \frac{PB}{BR}\frac{RA}{AQ}\frac{QF}{FP} = \frac{PB}{AQ}\frac{QF}{FP}.$$

These lead to

$$\frac{\gamma + \beta + QD}{QD} = -\frac{BR}{QC} = -\frac{\gamma}{\beta}, \qquad \ldots\ldots \text{(i)}$$
$$\frac{\alpha + \beta + ER}{ER} = -\frac{CP}{RA} = -\frac{\alpha}{\gamma}, \qquad \ldots\ldots \text{(ii)}$$
$$\frac{QF}{FQ + \alpha + \beta} = -\frac{AQ}{PB} = -\frac{\beta}{\alpha}. \qquad \ldots\ldots \text{(iii)}$$

Now consider the product of ratios to prove D, E and F collinear by the converse of Menelaus :

$$\frac{PE}{ER}\frac{RD}{DQ}\frac{QF}{FP}.$$

This is equal to

$$\frac{\alpha + \beta + ER}{ER}\frac{\gamma + \beta + QD}{QD}\frac{QF}{FQ + \alpha + \beta}.$$

Using (i), (ii) and (iii) above, we get a value of –1, and the result is proved.

5. Since $c = 2R \sin C$, $R(a + b) = 2R \sin C\sqrt{ab}$.

Using the AM-GM inequality

$$\sin C = \frac{a + b}{2\sqrt{ab}} \geq 1.$$

Hence, $\sin C = 1$, $C = 90°$, $a = b$ and $c = \sqrt{2}a$. Thus

$$r = \frac{\text{Area of } \Delta ABC}{\text{Semi-perimeter of } \Delta ABC} = \frac{a^2}{2a+\sqrt{2}a} = \frac{a}{2+\sqrt{2}} < \frac{3}{10}a,$$

as required.

6. We have

$$S - \frac{20S}{10^2} - \frac{12S}{10^4} = \left(\frac{a_0}{10^0} + \frac{a_1}{10^2} + \frac{a_2}{10^4} + \frac{a_3}{10^6} + \ldots\right)$$
$$-\left(\frac{20a_0}{10^2} + \frac{20a_1}{10^4} + \frac{20a_2}{10^6} + \frac{20a_3}{10^8} + \ldots\right)$$
$$-\left(\frac{12a_0}{10^4} + \frac{12a_1}{10^6} + \frac{12a_2}{10^8} + \frac{12a_3}{10^{10}} + \ldots\right)$$
$$= \frac{a_0}{10^0} + \frac{a_1}{10^2} - \frac{20a_0}{10^2} + \frac{a_2 - 20a_1 - 12a_0}{10^4}$$
$$+ \frac{a_3 - 20a_2 - 12a_1}{10^6} + \frac{a_4 - 20a_3 - 12a_2}{10^8} + \ldots.$$

Since $a_n - 20a_{n-1} - 12a_{n-2} = 0$ for all positive integers $n \geq 2$, we have

$$S - \frac{20S}{10^2} - \frac{12S}{10^4} = \frac{a_0}{10^0} + \frac{a_1}{10^2} - \frac{20a_0}{10^2},$$

and substituting in $a_0 = a_1 = 1$, we have

$\frac{7988S}{10000} = \frac{81}{100}$, so $S = \frac{2025}{1997}$. Hence, $\sqrt{S} = \frac{45}{\sqrt{1997}}$, and

So the desired ordered pair is $(a, b) = (45, 1997)$

7. We have

$$r = \frac{K}{s},\ r_a = \frac{K}{s-a},\ r_b = \frac{K}{s-b}, \text{ and } r_c = \frac{K}{s-c}.$$

Thus, by the AM-GM inequality,

$$r^2 + r_a^2 + r_b^2 + r_c^2 = \frac{K^2}{s^2} + \frac{K^2}{(s-a)^2} + \frac{K^2}{(s-b)^2} + \frac{K^2}{(s-c)^2}$$
$$\geq 4\sqrt[4]{\frac{K^8}{s^2(s-a)^2(s-b)^2(s-c)^2}} = 4\frac{K^2}{K} = 4K.$$

8. Let r be the radius and h be the height of the closed cylinder, where r and h are both positive integers.
If the surface area of the closed cylinder is twice the volume, then

$$2\pi rh + 2\pi r^2 = 2\pi r^2 h.$$

It follows that $h + r = rh$. Thus

$$(r-1)(h-1) = rh - r - h + 1 = 0 + 1 = 1.$$

Hence $r - 1 = 1$ and $h - 1 = 1$ (since r and h are both positive integers), so that $r = 2$ and $h = 2$.
It is easily checked that if $r = 2$ and $h = 2$, then the surface area of the closed cylinder is twice the volume.

9. More generally, we prove that $\sin mx + \sin my = 0$ where m is an integer.
Since $\sin mx + \sin my =$

$2\sin\left(m\frac{x+y}{2}\right)\cos\left(m\left(\frac{x-y}{2}\right)\right)$, it is sufficient to show

that $\sin\left(\frac{x+y}{2}\right) = 0$ since $\sin\left(m\frac{x+y}{2}\right) = 0$ follows easily by induction from

$$\sin\left((m+1)\frac{(x+y)}{2}\right) = \sin\left(m\frac{(x+y)}{2}\right)\cos\left(\frac{x+y}{2}\right)$$
$$+ \cos\left(m\frac{(x+y)}{2}\right)\sin\left(\frac{x+y}{2}\right).$$

Now,

$$\cos x = \cos y \Leftrightarrow \cos x - \cos y = 0$$
$$\Leftrightarrow -2\sin\frac{x+y}{2}\sin\frac{x-y}{2} = 0 \quad \text{..... (i)}$$

and

$$\sin x = -\sin y \Leftrightarrow \sin x + \sin y = 0$$
$$\Leftrightarrow 2\sin\frac{x+y}{2}\cos\frac{x-y}{2} = 0. \quad \text{.... (ii)}$$

Squaring each of (i) and (ii) and adding, we find

$$4\sin^2\left(\frac{x+y}{2}\right)\underbrace{\left(\sin^2\frac{(x-y)}{2} + \cos^2\frac{(x-y)}{2}\right)}_{=1} = 0.$$

Hence $\sin\frac{x+y}{2} = 0$.

10. In tetrahedron $ABCD$ we put $AB = a$, $AC = b$, $AD = c$, $BC = d$, $CD = e$ and $BD = f$, and we denote the areas of ΔABC, ΔACD, ΔABD, and ΔBCD by S_1, S_2, S_3 and S_4 respectively.
Then the surface area S of the tetrahedron is equal to the sum of S_1, S_2, S_3 and S_4, *i.e.*

$$S = S_1 + S_2 + S_3 + S_4 \quad \text{...... (i)}$$

Using well known Geometric Inequalities we get

$$a^2 + b^2 + d^2 \geq 4\sqrt{3}S_1 \quad \text{...... (ii)}$$
$$b^2 + c^2 + e^2 \geq 4\sqrt{3}S_2 \quad \text{...... (iii)}$$
$$a^2 + c^2 + f^2 \geq 4\sqrt{3}S_3 \quad \text{...... (iv)}$$
$$d^2 + e^2 + f^2 \geq 4\sqrt{3}S_4. \quad \text{...... (v)}$$

From (ii) + (iii) + (iv) + (v) we get

$$2(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) \geq 4\sqrt{3}S, \text{ by (i)}$$

Hence we have

$$S \leq \frac{\sqrt{3}}{6}(a^2 + b^2 + c^2 + d^2 + e^2 + f^2) \text{ as required.}$$

CONCEPTUAL PROBLEMS

Limit, Continuity, Differentiability

1. If p and q are positive roots $(p > q)$ of the quadratic equation $ax^2 + bx + c = 0$, $(a > 0)$ then

$$\lim_{x\to\frac{1}{p}^+}\sqrt{\frac{1-\cos 2(cx^2+bx+a)}{2(1-px)^2}}$$ is

(a) $\frac{p-q}{pq}c$ (b) $\frac{q-p}{pq}c$

(c) 0 (d) $\frac{q-p}{c}$

2. If $f(x) = x^2 + x^3 + \log_e x$ and g is inverse of f then $g'(2)$ is equal to

(a) 6 (b) 1/6 (c) 2 (d) 3

3. The graph of function $y = f(x)$ has a unique tangent at $(e^a, 0)$ through which the graph passes, then

$$\lim_{x\to e^a}\frac{\log(1+7f(x))-\sin(f(x))}{3f(x)}$$ equals to :

(a) 1 (b) 2

(c) 7 (d) none of these

4. Number of non-differentiable points of

$$f(x) = ||x|-1| + |\cos\pi x| + \left|\sin\frac{\pi}{2}x\right| \text{ in } -2 < x < 2$$

is

(a) 7 (b) 8

(c) 5 (d) 2

5. Let $f(x) = \begin{cases} \frac{x}{2}-1, & 0 \le x < 1 \\ \frac{1}{2}, & 1 \le x \le 2 \end{cases}$

$\phi(x) = (2x + 1)(x - \lambda) + 3$, $0 \le x < \infty$

Then $\phi(f(x))$ is continuous at $x = 1$, if λ equals

(a) 1/2 (b) 11/6

(c) 1/6 (d) 13/6

6. If $f(x) = \begin{cases} x^{[a]}\sin\left(\frac{1}{x}\right), & x \ne 0 \\ 0, & x = 0 \end{cases}$, where [.] denotes the greatest integer function is continuous but non-differentiable at $x = 0$, then

(a) $a \in (0, 2)$ (b) $a \in [2, \infty)$

(c) $a \in (0, 1)$ (d) $a \in [1, 2)$

7. If the tangent to the curve $y = f(x)$ at $x = 0$ be given by the equation $x + 3y + 3 = 0$, then find the value of $\lim_{x\to 0}\frac{x^2}{\{f(x^2)-5f(4x^2)+4f(7x^2)\}}$.

(a) – 1/3 (b) 1/3

(c) 2/3 (d) – 2/3

8. Number of points where the function $f(x) = \max\{|\tan\pi x|, \cos\pi x\}$ is non-differentiable in the interval $(-1, 1)$ is

(a) 4 (b) 6

(c) 3 (d) 1

9. Set of all values of x such that

$$\lim_{n\to\infty}\frac{1}{1+\left(\frac{4\tan^{-1}(2\pi x)}{\pi}\right)^{4n}}$$ is non-zero and finite number, when $n \in N$ is

(a) $\left[0, \frac{1}{2\pi}\right]$ (b) $\left(-\frac{1}{\pi}, \frac{1}{\pi}\right)$

(c) $\left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]$ (d) $\left[-\frac{1}{2\pi}, 0\right]$

10. Number of non-differentiable points of

$$f(x) = \min\left([x], \{x\}, \left|x-\frac{3}{2}\right|\right) \text{ in } (0, 2)$$ where [.] and {.} denotes the greatest integer function and fractional part respectively, is

(a) 3 (b) 4

(c) 2 (d) 1

SOLUTIONS

1. **(a) :** $\frac{1}{p}, \frac{1}{q}$ are roots of $cx^2 + bx + a = 0$.

so $cx^2+bx+a = c\left(x-\frac{1}{p}\right)\left(x-\frac{1}{q}\right)$

so, $$\lim_{x\to 1/p^+}\sqrt{\frac{2\sin^2(cx^2+bx+a)}{2(1-px)^2}}$$

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$$= \lim_{n\to 1/p^+} \frac{|\sin(cx^2+bx+a)|}{|1-px|}$$

$$= \lim_{n\to 1/p^+} \frac{\left|\sin c\left\{\left(x-\frac{1}{p}\right)\left(x-\frac{1}{q}\right)\right\}\right|}{|p|\left|x-\frac{1}{p}\right|}$$

$[\because p > q$

$\therefore \frac{1}{p} < \frac{1}{q}$ (p and q are positive)

$\because \frac{1}{p} - \frac{1}{q} < 0 \quad (\because c > 0)]$

$$= \lim_{n\to 1/p^+} \frac{-\sin c\left\{\left(x-\frac{1}{p}\right)\left(x-\frac{1}{q}\right)\right\}.c\left(x-\frac{1}{q}\right)}{cp\left(x-\frac{1}{p}\right)\left(x-\frac{1}{q}\right)}$$

$$\left(\lim_{x\to 0}\frac{\sin x}{x} = 1\right)$$

$$= -c\left(\frac{1}{p}-\frac{1}{q}\right) = \frac{p-q}{pq}c$$

2. **(b)** : $g(x) = f^{-1}(x)$ (given)

$g(f(x)) = f^{-1}(f(x)) = x$

Differentiate both sides

$g'(f(x))\cdot f'(x) = 1$

$\therefore g'(f(x)) = \frac{1}{f'(x)}$ ...(i)

$f(x) = x^2 + x^3 + \log_e x,\ f'(x) = 2x + 3x^2 + \frac{1}{x}$

Put $x = 1$, then

$f(1) = 2, f'(1) = 6$

put $x = 1$ in equation (i)

so, $g'(f(1)) = \frac{1}{f'(1)}$

$\Rightarrow g'(2) = \frac{1}{f'(1)}$ [by putting $f(1) = 2$]

$\therefore g'(2) = \frac{1}{6}$

3. **(b)**: Here, $\lim_{x\to e^a} \frac{\log(1+7f(x)) - \sin(f(x))}{3f(x)} \left(\frac{0}{0} \text{ form}\right)$

Using L-Hospital's Rule :

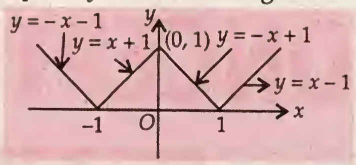
$$= \lim_{x\to e^a} \frac{7f'(x) - \{\cos(f(x))\cdot f'(x)\}\{1+7f(x)\}}{3f'(x)\cdot\{1+7f(x)\}}$$

$$= \lim_{x\to e^a} \frac{7-\cos(f(x))\{1+7f(x)\}}{3(1+7f(x))}$$

$$= \frac{7-1}{3} = 2$$

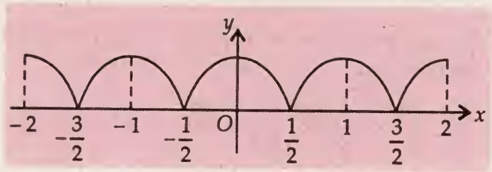
4. **(a)** : We know that, if $f(x)$ is differentiable and $g(x)$ is also differentiable then $f(x) \pm g(x)$ is also differentiable but if $f(x)$ is differentiable and $g(x)$ is not differentiable then $f(x) \pm g(x)$ is not differentiable.

(i) graph of $y = ||x| - 1|$ is given



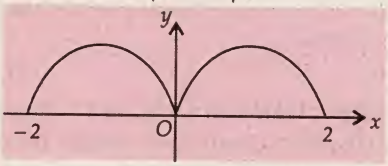
so $||x| - 1|$ is not differentiable at $x = -1$, $x = 0$ and $x = 1$ in $(-2, 2)$

(ii) Graph of $y = |\cos\pi x|$ is



So; $-\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2}$ are non differentiable points.

(iii) Graph of $y = \left|\sin\frac{\pi}{2}x\right|$ is



so; $\left|\sin\frac{\pi}{2}x\right|$ is not differentiable at $x = 0$ in $(-2, 2)$

so $f(x)$ is not defferentiable at $x = 0$

So number of non-differentiable points is 7

5. **(a)** : $\phi(f(x)) = \begin{cases} \phi\left(\frac{x}{2}-1\right) & ,0 \le x < 1 \\ \phi\left(\frac{1}{2}\right) & ,1 \le x \le 2 \end{cases}$

$$= \begin{cases} \frac{(x-1)(x-2-2\lambda)}{2} + 3 & ,0 \le x < 1 \\ 4 - 2\lambda & ,1 \le x \le 2 \end{cases}$$

$\lim_{x\to 1^-} \phi(f(x)) = 3,\ \phi(f(1)) = 4 - 2\lambda$

$\lim_{x\to 1^+} \phi(f(1)) = 4 - 2\lambda$

For $\phi(f(x))$ to be continuous at $x = 1$,

$4 - 2\lambda = 3$

$\Rightarrow \lambda = \frac{1}{2}$

6. **(b)** : $f'(0) = \lim_{h\to 0} \frac{f(h)-f(0)}{h} = \lim_{h\to 0} \frac{h^{[a]}\sin\left(\frac{1}{h}\right)}{h}$

$$= \lim_{h\to 0} h^{([a]-1)}.\sin\left(\frac{1}{h}\right)$$

This limit does not exist if $[a] - 1 > 0$

i.e., $a > 1 \Rightarrow a \ge 2$

Now, $\lim_{x\to0} f(x) = \lim_{x\to0} x^{[a]-1}\sin\left(\frac{1}{x}\right) = 0$

If $[a] - 1 > 0 \Rightarrow a \ge 2$

7. **(a)** : The slope of tangent $x + 3y + 3 = 0$ be given by $f'(0) = -1/3$

Now required limit is

$$l = \lim_{x\to0}\frac{x^2}{f(x^2)-5f(4x^2)+4f(7x^2)}\quad\left(\frac{0}{0}\text{ form}\right)$$

$$= \lim_{x\to0}\frac{2x}{f'(x^2)\cdot 2x - 5f'(4x^2)\cdot 8x + 4f'(7x^2)\cdot 14x}$$

(applying L'Hospital's rule)

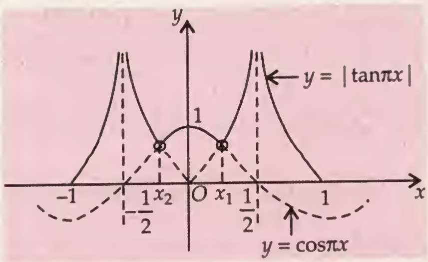
$$= \lim_{x\to0}\frac{2}{2f'(x^2)-40f'(4x^2)+56f'(7x^2)}$$

$$= \frac{2}{2f'(0)-40f'(0)+56f'(0)}$$

$$= \frac{2}{2\left(-\frac{1}{3}\right)-40\left(-\frac{1}{3}\right)+56\left(-\frac{1}{3}\right)}$$

$$\Rightarrow l = -\frac{1}{3}$$

8. **(a)** : The function is not differentiable and continuous at two points $x = -\frac{1}{2}$ and $x = \frac{1}{2}$ also function is continous but not differentiable at x_1 and x_2. Hence at four points, function is not differentiable.



9. **(c)** : $-1 \le \frac{4\tan^{-1}(2\pi x)}{\pi} \le 1$

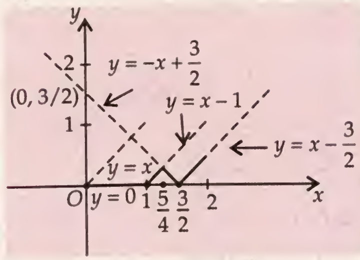
$\Rightarrow -\frac{\pi}{4} < \tan^{-1}(2\pi x) \le \frac{\pi}{4}$

$\Rightarrow -1 \le 2\pi x \le 1$

$\Rightarrow x \in \left[-\frac{1}{2\pi}, \frac{1}{2\pi}\right]$

10. **(a)** : Function is defined as below

$$f(x) = \begin{cases} 0, & 0 < x < 1 \\ x-1, & 1 \le x < \frac{5}{4} \\ -x+\frac{3}{2}, & \frac{5}{4} \le x < \frac{3}{2} \\ x-\frac{3}{2}, & \frac{3}{2} \le x < 2 \end{cases}$$



Function is continuous for (0, 2)

$$f'(x) = \begin{cases} 0, & 0 < x < 1 \\ 1, & 1 < x < \frac{5}{4} \\ -1, & \frac{5}{4} < x < \frac{3}{2} \\ 1, & \frac{3}{2} < x < 2 \end{cases}$$

so, function will not be differentiable at $x = 1$, $x = \frac{5}{4}$ and $x = \frac{3}{2}$

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers comments and suggestions regarding the problems and solutions offered are always welcome.

1. If a, b, c, d are positive real numbers then $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$

(a) $e^{d/b}$ (b) $e^{c/a}$

(c) $e^{(c+d)/(a+b)}$ (d) e

2. $\lim_{x\to 1}\frac{\sqrt{1+\cos 2(x-1)}}{x-1}$

(a) exists and it equals $\sqrt{2}$

(b) exists and it equals $-\sqrt{2}$

(c) does not exist because $(x-1)\to 0$

(d) does not exist because L.H.L. is not equal to R.H.L.

3. The function $\frac{\log(1+ax)-\log(1-bx)}{x}$ is not defined at $x = 0$. The value which should be assigned to f at $x = 0$, so that it is continuous at $x = 0$ is

(a) $a - b$ (b) $1 + b$

(c) $\log a + \log b$ (d) none of these

4. If a line OP through the origin O makes an angle α, 45° and 60° with x, y and z axis respectively then the direction cosines of OP are

(a) $\frac{1}{\sqrt{2}}, \frac{1}{2}, \frac{1}{2}$ (b) $\frac{1}{2}, \frac{1}{2}, \frac{1}{\sqrt{2}}$

(c) $\frac{1}{2}, \frac{1}{\sqrt{2}}, \frac{1}{2}$ (d) none of these

5. If O is the origin and the line OP of length r makes an angle α with x-axis and lies in the xz-plane, then the co-ordinates of P are

(a) $(r\cos\alpha, 0, r\sin\alpha)$ (b) $(0, 0, r\sin\alpha)$

(c) $(0, 0, r\cos\alpha)$ (d) $(r\cos\alpha, 0, 0)$

6. Evaluate $\lim_{x\to\pi/2}\frac{\cos x}{(1-\sin x)^{2/3}}$.

7. Evaluate $\lim_{x\to\infty}\left(\sin\frac{1}{x}+\cos\frac{1}{x}\right)^{x}$.

8. If f and g are differentiable at $a \in R$ such that $f(a) = g(a) = 0$ and $g'(a) \neq 0$, then show that $\lim_{x\to a}\frac{f(x)}{g(x)} = \frac{f'(a)}{g'(a)}$.

9. Show that the function g, defined by $g(x) = \sin\alpha + \cos\alpha - 1$, $\alpha = \sin^{-1}\sqrt{\{x\}}$, { . } denotes fractional part function, is an even function.

10. Evaluate $\lim_{x\to-\infty}\frac{\sqrt{2x^2+1}}{4x+1}$.

SOLUTIONS

1. (a) : $\lim_{x\to\infty}\left(1+\frac{1}{a+bx}\right)^{c+dx}$ (1 form)

$$= \lim_{x\to\infty}\left[\left(1+\frac{1}{a+bx}\right)^{a+bx}\right]^{\frac{c+dx}{a+bx}} = e^{\lim_{x\to\infty}\left(\frac{\frac{c}{x}+d}{\frac{a}{x}+b}\right)} = e^{d/b}$$

(If $\lim_{x\to\infty} f(x) = 0$ then $\lim_{x\to\infty}(1+f(x))^{\frac{1}{f(x)}} = e$)

2. (c) : Put $x - 1 = t$

$$\lim_{x\to 1}\frac{\sqrt{1+\cos 2(x-1)}}{x-1} = \lim_{t\to 0}\frac{\sqrt{2}|\cos t|}{t}$$

$= \lim_{t\to 0}\frac{\sqrt{2}\cos t}{t}$ does not exist because $t\to 0, (x-1)\to 0$

3. (d) : For continuity at $x = 0$

$\lim_{x\to 0} f(x) = f(0)$

$$\Rightarrow \lim_{x\to 0}\frac{a.\log(1+ax)}{ax}+\frac{b.\log(1-bx)}{-bx} = f(0)$$

$$\left(\because \lim_{x\to 0}\frac{\log(1+x)}{x} = 1\right)$$

$\Rightarrow a + b = f(0)$

4. (c): Direction cosines of OP are $l = \cos\alpha$, $m = \cos 45°$,

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$n = \cos 60°$ and we know that $l^2 + m^2 + n^2 = 1$

$\Rightarrow \cos^2\alpha + \cos^2 45° + \cos^2 60° = 1$

$$\Rightarrow \cos^2\alpha + \left(\frac{1}{2}\right) + \left(\frac{1}{4}\right) = 1 \Rightarrow \cos^2\alpha = \frac{1}{4} \quad \Rightarrow \cos\alpha = \frac{1}{2}$$

5. **(a)** : Let the co-ordinates of P be (x, y, z)

Since OP lies in xz-plane and makes an angle α with the x-axis.

$\therefore$ It makes angle $\frac{\pi}{2} - \alpha$ with z-axis and $\frac{\pi}{2}$ with y-axis.

So, $x = r\cos\alpha,\ y = r\cos\frac{\pi}{2},\ z = r\cos\left(\frac{\pi}{2} - \alpha\right)$ are the required co-ordinates and therefore are $(r \cos \alpha, 0, r \sin \alpha)$

6. $$\lim_{x \to \pi/2} \frac{\cos x}{(1 - \sin x)^{2/3}} = \lim_{t \to 0} \frac{\sin t}{(1 - \cos t)^{2/3}}$$

$$= \lim_{t \to 0} \frac{2\sin(t/2)\cos(t/2)}{(2\sin^2(t/2))^{2/3}} = \lim_{t \to 0} \frac{1}{2^{1/3}} \cdot \frac{\cos(t/2)}{(\sin(t/2))^{1/3}}.$$

Limit value is ∞ as $t \to 0^+$ and limit value is $-\infty$ as $t \to 0^-$. Thus limit does not exist.

7. $$\lim_{x \to \infty} \left(\sin\frac{1}{x} + \cos\frac{1}{x}\right)^x = e^{\lim\limits_{x \to \infty} x\left(\sin\frac{1}{x} + \cos\frac{1}{x} - 1\right)}$$

$$= e^{\lim\limits_{x \to \infty} \frac{\sin(1/2x)}{1/2x}\left(\cos\frac{1}{2x} - \sin\frac{1}{2x}\right)}$$

$$= e \quad \left(\begin{array}{l} \sin\frac{1}{x} = 2\sin\frac{1}{2x}\cos\frac{1}{2x}, \\ \cos\frac{1}{x} = 1 - 2\sin^2\frac{1}{2x} \end{array}\right)$$

8. $$\lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{\frac{f(x) - f(a)}{(x - a)}}{\frac{g(x) - g(a)}{(x - a)}} = \frac{f'(a)}{g'(a)},$$ as $f(a) = g(a) = 0$ and $g'(a) \neq 0$.

9. $$g(x) = \sin(\sin^{-1}\sqrt{\{x\}}) + \cos(\sin^{-1}\sqrt{\{x\}}) - 1$$

$$= \sqrt{\{x\}} + \cos(\cos^{-1}\sqrt{1 - \{x\}}) - 1 = \sqrt{\{x\}} + \sqrt{1 - \{x\}} - 1.$$

If $x \in I$ then $\{x\} = 0$

$\Rightarrow g(x) = 0 \Rightarrow g(x) = g(-x)$

If $x \notin I$, then $\{-x\} = 1 - \{x\}$

$\Rightarrow g(-x) = \sqrt{1 - \{x\}} + \sqrt{\{x\}} - 1 = g(x)$.

Hence, g is an even function.

10. $$\lim_{x \to -\infty} \frac{\sqrt{2x^2 + 1}}{4x + 1} = \lim_{x \to -\infty} -\frac{\sqrt{2 + \frac{1}{x^2}}}{4 + \frac{1}{x}}$$

$$= -\frac{1}{2\sqrt{2}}. \qquad \text{as } x < 0$$

NEW PATTERN

Sample Paper IIT-JEE 2010

PAPER - I

SECTION - I

Straight Objective Type

[3 marks for correct answer and –1 for wrong answer]
This section contains 8 multiple choice questions numbered 1 to 8. Each question has 4 choices (a), (b), (c) and (d), out of which ONLY ONE is correct.

1. On a railway there are 10 stations. The number of types of tickets required in order that it may be possible to book a passenger from every station to every other is

(a) $\frac{10!}{8!}$ (b) $10!\ 2!$
(c) $\frac{10!}{2!}$ (d) $\frac{10!}{8!2!}$

2. If the roots of $ax^2 + bx + c = 0$ are of the form $\frac{m}{m-1}$ and $\frac{m+1}{m}$ then the value of $(a + b + c)^2$ is

(a) $b^2 - 2ac$ (b) $2b^2 - ac$
(c) $b^2 - 4ac$ (d) $2(b^2 - 2ac)$

3. If A and B are two square matrices of order 3×3 which satisfy $AB = A$ and $BA = B$, then $(A + B)^7$ is
(a) $7(A + B)$ (b) $7\ I_{3\times 3}$
(c) $64(A + B)$ (d) $128\ I_{3\times 3}$

4. If a_1, a_2, a_3, a_4, a_5 are in H.P., then $a_1a_2 + a_2a_3 + a_3a_4 + a_4a_5$ is equal to
(a) $2a_1a_5$ (b) $3a_1a_5$
(c) $8a_1a_5$ (d) $4a_1a_5$

5. The maximum value of $(\sin\alpha_1)(\sin\alpha_2)....(\sin\alpha_n)$ under the restrictions $0 \le \alpha_1, \alpha_2, ...\alpha_n < \frac{\pi}{2}$ and $(\tan\alpha_1)(\tan\alpha_2)....(\tan\alpha_n) = 1$ is

(a) $\frac{1}{2^n}$ (b) $\frac{1}{2n}$
(c) $\frac{1}{2^{n/2}}$ (d) 1

6. In a conference 10 speakers are to give their speaches one after another. Find the probability of the event if S_1 speaks before S_2 and S_2 speaks before S_3 and the remaining 7 speakers have no objection to speak at any number?

(a) $\frac{1}{6}$ (b) $\frac{1}{3}$
(c) $\frac{1}{45}$ (d) $\frac{3}{10}$

7. i^i is a
(a) complex number
(b) purely imaginary number
(c) real number (d) none of these

8. If k and K are minimum and maximum values of $\sin^{-1}x + \cos^{-1}x + \tan^{-1}x$ respectively, then

(a) $k = \frac{\pi}{4}, K = \frac{3\pi}{4}$ (b) $k = 0, K = \pi$
(c) $k = \pi/2, K = \pi$ (d) not defined

SECTION - II

Multiple Correct Choice Type

[4 marks for correct answer and –1 for wrong answer]
This section contains 4 multiple choice questions numbered 9 to 12. Each question has 4 choices (a), (b), (c) and (d), out of which ONE OR MORE is/are correct.

9. If $f(x)$ and $g(x)$ are functions such that $f(x + y) = f(x) \cdot g(y) + g(x) \cdot f(y)$ then

$$\begin{vmatrix} f(\alpha) & g(\alpha) & f(\alpha+\theta) \\ f(\beta) & g(\beta) & f(\beta+\theta) \\ f(\gamma) & g(\gamma) & f(\gamma+\theta) \end{vmatrix}$$ is independent of

(a) α (b) β
(c) γ (d) θ

10. Sum of the roots of the equation $(x + 1) = 2\log_2(2^x + 3) - 2\log_4(1980 - 2^{-x})$ is greater than
(a) 2 (b) 3
(c) 4 (d) 5

11. Let $0 < P(A) < 1$, $0 < P(B) < 1$ and $P(A \cup B) = P(A) + P(B) - P(A) \cdot P(B)$; then

(a) $P\left(\frac{B}{A}\right) = \frac{P(B)}{P(A)}$
(b) $P(A^c \cup B^c) = P(A^c) + P(B^c)$
(c) $P((A \cup B)^c) = P(A^c)P(B^c)$

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40. $f(x)=\dfrac{\sin^4 x+\cos^4 x}{x+x^2\tan x}$ is

(a) even (b) odd

(c) periodic with period π

(d) periodic with period 2π

ASSERTION AND REASON

Directions: In each of the following questions, a statement of assertion is given and a corresponding statement of reason is given just below it. Out of the statements, mark the correct answer as

(a) If both Assertion and Reason are true and Reason is the correct explanation of Assertion

(b) If both Assertion and Reason are true but Reason is not the correct explanation of Assertion

(c) If Assertion is true but Reason is false

(d) If Assertion is false and Reason is true.

41. Let α, β, γ be the roots of the equation $x^3+ax+b=0$; $a, b \in R$.

Assertion : $\begin{vmatrix}\alpha & \beta & \gamma\\ \beta & \gamma & \alpha\\ \gamma & \alpha & \beta\end{vmatrix}=0.$

Reason : Any cubic equation over reals has atleast one real root.

42. **Assertion :** Let $L_1 : a_1x+b_1y+c_1=0$, $L_2 : a_2x+b_2y+c_2=0$ and $L_3 : a_3x+b_3y+c_3=0$ are three concurrent lines, then $\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=0.$

Reason : If $\begin{vmatrix}a_1 & b_1 & c_1\\ a_2 & b_2 & c_2\\ a_3 & b_3 & c_3\end{vmatrix}=0$, then the lines L_1, L_2 and L_3 must be concurrent.

43. **Assertion :** The numbers sin18° and –sin54° are equations of such quadratic equations with integer coefficients.

Reason : If x = 18°, then $5x$ = 90° if y = –54°, then $5y$ = –270°.

44. **Assertion :** If $a > 0$ and $b^2 - 4ac < 0$ then the value of the integral $\int \dfrac{dx}{ax^2+bx+c}$ will be of the type $\mu \tan^{-1}\left(\dfrac{x+A}{B}\right)+C$, where A, B, C, μ are constants.

Reason : If $a > 0$, $b^2 - 4ac < 0$, then ax^2+bx+c can be written as sum of two squares.

45. **Assertion :** The sequence $a_n=\left(1+\dfrac{1}{n}\right)^n$ is monotonically increasing.

Reason : The function $f(x)=\left(x+\dfrac{1}{x}\right)^x$ is monotonically increasing over $(0, \infty)$.

46. **Assertion :** The area bounded by $y = x(x-1)^2$, the y-axis and the line $y = 2$ is $\int_0^2 (x(x-2)^2-2)dx$ is equal to $\dfrac{10}{3}$.

Reason : The curve $y = x(x-1)^2$ is intersected to $y = 2$ at $x = 2$ only and for $0 < x < 2$, the curve $y = x(x-1)^2$ lies below the line $y = 2$.

47. **Assertion :** If a normal drawn at a point P on the parabola $y^2 = 4ax$ meets the curve again at Q, then the least distance of the point Q from the axis of the parabola is $4\sqrt{2}a$.

Reason : If normal drawn at point $P(at^2, 2at)$ on parabola $y^2 = 4ax$ meets the curve again at $Q(at_1^2, 2at_1)$, then $t_1 = t+\dfrac{2}{t} \Rightarrow$ minimum value of $t_1 = 2\sqrt{2}$.

48. **Assertion :** $\lim\limits_{x\to 0}\dfrac{x}{a}\left(\dfrac{1}{x}\right)$ doesnot exist, (where [·] denotes the greatest integer function)

Reason : $\lim\limits_{x\to 0}\left(\dfrac{1}{x}\right)$ does not exist.

49. **Assertion :** $\vec{b}$ and $\vec{c}$ are two non-collinear vectors, such that $\vec{a}\cdot(\vec{b}+\vec{c})=4$ and $\vec{a}\times(\vec{b}\times\vec{c})=(x^2-2x+6)\vec{b}+(\sin y)\vec{c}$, where x and y are real numbers. The point (x, y) lies on $x = 1$.

Reason : The vector $\vec{a}$ lies in the plane of $\vec{b}$ and $\vec{c}$.

50. Let f: [1, 13] $\to R$ be an integrable function with $f''(x) > 0 \ \forall \ x \in R$.

Assertion : $\int_1^3 f(x)dx+\int_{11}^{13} f(x)dx \geq \int_5^9 f(x)dx$

Reason : If $a < b < c$ and $f''(x) > 0$, then $f(a-b+c) \leq f(a)+f(c)-f(b)$

ANSWER KEY

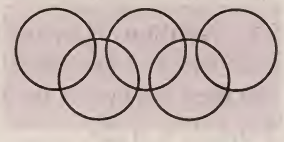
1. (c)	2. (b)	3. (d)	4. (a)	5. (a)	6. (b)
7. (c)	8. (c)	9. (c)	10. (b)	11. (b)	12. (d)
13. (a)	14. (b)	15. (c)	16. (b)	17. (a)	18. (b)
19. (b)	20. (a)	21. (a)	22. (d)	23. (a)	24. (a)
25. (c)	26 (b)	27. (c)	28. (d)	29. (a)	30. (b)
31. (a)	32. (d)	33. (d)	34. (c)	35. (b)	36. (b)
37. (b)	38. (c)	39. (d)	40. (b)	41. (b)	42. (c)
43. (a)	44. (a)	45. (a)	46. (a)	47. (c)	48. (d)
49. (c)	50. (a)				

OLYMPIAD CORNER

International Olympiad Problems

Challenging problems for Olympiads, IIT-JEE and other contests.

1. There are 9 regions inside the 5 rings of the Olympics. Put a different whole number from 1 to 9 in each, so that the sum of the numbers in each ring is the same. What are the largest and the smallest values of this common sum?



2. Let R^+ be the set of all non-negative real numbers. Two positive real numbers a and b are given. Suppose that a mapping $f : R^+ \to R^+$ satisfies the functional equation
$f(f(x)) + af(x) = b(a + b)x$.
Prove that there exists a unique solution of this equation.

3. Let $f(x)$ be a polynomial with rational coefficients and α be a real number such that
$\alpha^3 - \alpha = (f(\alpha))^3 - f(\alpha) = 33^{1992}$.
Prove that for each $n \geq 1$. $(f^{(n)}(\alpha))^3 - f^{(n)}(\alpha) = 33^{1992}$, where $f^{(n)}(x) = f(f(...f(x)))$, and n is a positive integer.

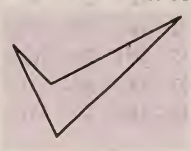
4. In a triangle ABC, let D and E be the intersections of the bisectors of $\angle ABC$ and $\angle ACB$ with the sides AC, AB respectively. Determine the angles $\angle A$, $\angle B$, $\angle C$, if $\angle BDE = 24°$, $\angle CED = 18°$

5. Prove that $n = \dfrac{5^{125}-1}{5^{25}-1}$ is a composite number.

6. Triangle ABC is inscribed in a circle. The chord AD bisects $\angle BAC$. Assume that $AB = \sqrt{2}BC = \sqrt{2}AD$. Determine the angles of ΔABC.

7. Show that any triangle can be dissected into 9 or fewer convex pentagons of equal area.

8. Define a boomerang as a quadrilateral whose opposite sides do not intersect and one of whose internal angles is greater than 180 degrees. (See figure displayed). Let C be a convex polygon having s sides. Suppose that the interior region of C is the union of q quadrilaterals, none of whose interiors intersect one another. Also, suppose that b of these quadrilaterals are boomerangs. Show that $q \geq b + (s - 2)/2$.

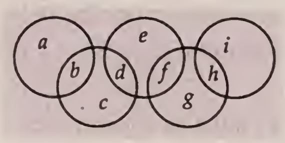


9. Let $A_1A_2 .... A_n$ be a convex n-gon.
(a) Prove that
$A_1A_2 + A_2A_3 + ... + A_nA_1 \leq A_1A_3 + A_2A_4 + ... + A_nA_2$.
(b) Prove or disprove that
$$2\cos\left(\frac{\pi}{n}\right)(A_1A_2 + A_2A_3 + ... + A_nA_1) \geq A_1A_3 + A_2A_4 + ... + A_nA_2.$$

10. In an equilateral triangle ABC of side 2, consider the incircle Γ. Show that for all points P of Γ, $\overline{PA}^2 + \overline{PB}^2 + \overline{PC}^2 = 5$.

SOLUTION

1. For the five rings, we have :
$a + b = b + c + d = d + e + f = f + g + h = h + i = N$ ...(i)
Since, we are dealing with the nine non-zero decimal digits so we have $\sum_{j=1}^{9} j = \frac{9(10)}{2} = 45$. The five regions sum to a common N for 45/5 = 9, but then one pair must be 9 + 0 or one triplet 9 + 0 + 0, which is not allowed. So $N > 9$. Since $a + b = h + i$, there must be atleast two pairs of decimal digits that sum to N. For $10 \leq N \leq 15$, we have



$N = 9 + a = 8 + (1 + a) = ...$, for $1 \leq a \leq 6$
while $N = 16 = 9 + 7$ and $N = 17 = 9 + 8$ only.
So $N \leq 15$.
From (i) : $a + b = b + c + d$ or $a = c + d$ ...(ii)
and $h + i = f + g + h$ or $i = f + g$ ...(iii)
The five central digits must equal $45 - 2N$:
$(c + d) + e + (f + g) = a + e + i = 45 - 2N$.
So we have

N	$2N$	$45 - 2N$	a, e, i
10	20	25	9, 8, -no digit available
11	22	23	9, 8, 6; ....
12	24	21	9, 8, 4; ....
13	26	19	9, 8, 2; ....

14	28	17	9, 7, 1; ....
15	30	15	9, 5, 1; ....

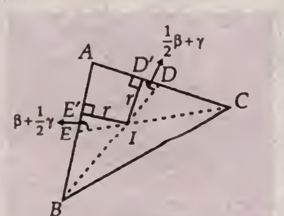
So, $11 \le N \le 15$.

2. Let $x_0 \ge 0$ and let $x_n = f^{(n)}(x_0) = f(f^{n-1}(x_0))$, $f^{(0)}(x_0) = x_0$. Since $f: R^+ \to R^+$, $x_n \ge 0$ for all $n \in N$. Putting x_n into the given equation we obtain
$x_{n+2} + ax_{n+1} = b(a+b)x_n$
for starting values x_0 and $x_1 = f(x_0)$. Solving this recurrence, we get
$x_n = Ab^n + B(-1)^n(a+b)^n$
for some real constants A and B. If $B \ne 0$, then taking n large enough we would get $|Ab^n| < |B(-1)^n(a+b)^n|$ (because $|b| < |a+b|$). Therefore, for some large n, x_n would be negative. Hence, $B = 0$, which gives $x_n = Ab^n$ and in particular, $x_0 = A, f(x_0) = x_1 = Ab = bx_0$, for any $x_0 \ge 0$. This gives a unique solution $f(x) = bx$.

3. Let $g(x) = x^3 - x$. It is easy to see that if $c > \frac{2\sqrt{3}}{9}$, then the equation $g(x) = c$ has a real solution which is unique. We have, $g(\alpha) = g(f(\alpha)) = 33^{1992}$.

But, since $33^{1992} > \frac{2\sqrt{3}}{9}$, $f(\alpha) = \alpha$.
Thus, $f^{(n)}(\alpha) = \alpha$ and we get
$(f^{(n)}(\alpha))^3 - f^{(n)}(\alpha) = \alpha^3 - \alpha = 33^{1992}$.

4. Let I be the incentre of triangle ABC. Let D' and E' be the projection onto AC and AB such that $ID' = IE' = r$ (inradius) of I, and let $\angle A = \alpha$, $\angle B = \beta$ and $\angle C = \gamma$, as usual.



Now $\angle ADB = \frac{1}{2}\beta + \gamma$, so

$$ID = \frac{r}{\sin\left(\frac{1}{2}\beta+\gamma\right)} \text{ and } IE = \frac{r}{\sin(\beta+\frac{1}{2}\gamma)} \quad \text{...(i)}$$

Applying the law of sines to triangle IDE we have

$$\frac{\sin(\beta+\frac{1}{2}\gamma)}{\sin(\frac{1}{2}\beta+\gamma)} = \frac{\sin 18^\circ}{\sin 24^\circ} \quad \text{...(ii)}$$

As $\angle BDE = 24^\circ$ and $\angle CED = 18^\circ$ we have

$\angle DIE = 138^\circ = 90^\circ + \frac{1}{2}\alpha \Rightarrow \alpha = 96^\circ$.
Thus,

$$\beta+\gamma = 84^\circ \text{ so } \gamma = 84^\circ - \beta \text{ and } \frac{1}{2}\gamma = 42^\circ - \frac{1}{2}\beta \quad \text{...(iii)}$$

Now using (iii) in (ii), we get

$$\frac{\sin(42^\circ+\frac{1}{2}\beta)}{\sin(84^\circ-\frac{1}{2}\beta)} = \frac{\sin 18^\circ}{\sin 24^\circ}.$$

Expanding and doing some calculations gives

$$\tan\frac{1}{2}\beta = \frac{\sin 18^\circ \sin 84^\circ - \sin 24^\circ \sin 42^\circ}{\sin 24^\circ \cos 42^\circ + \sin 18^\circ \cos 84^\circ},$$ and this is equal to $\tan 6^\circ$

[To see this we have

$$\frac{\sin 18^\circ \sin 84^\circ - \sin 24^\circ \sin 42^\circ}{\sin 24^\circ \cos 42^\circ + \sin 18^\circ \cos 84^\circ} = \frac{\sin 6^\circ}{\cos 6^\circ},$$

by cross multiplication

$\sin 18^\circ \sin^2 84^\circ - \sin 24^\circ \sin 42^\circ \cos 6^\circ$
$= \sin 6^\circ \sin 24^\circ \cos 42^\circ + \sin 18^\circ \sin^2 6^\circ$
$\Rightarrow \sin 18^\circ[\cos^2 6^\circ - \sin^2 6^\circ]$
$= \sin 24^\circ[\sin 6^\circ \cos 42^\circ + \cos 6^\circ \sin 42^\circ]$
$\Rightarrow \sin 18^\circ \cos 12^\circ = \sin 24^\circ \sin 48^\circ$
$\Rightarrow \sin 18^\circ = 2\sin 12^\circ \sin 48^\circ$
$\Rightarrow \sin 18^\circ = \cos 36^\circ - \cos 60^\circ$

$$\sin 18^\circ = -2\sin^2 18^\circ + \frac{1}{2}$$

giving the equivalent condition

$$2\sin^2 18^\circ + \sin 18^\circ - \frac{1}{2} = 0.$$

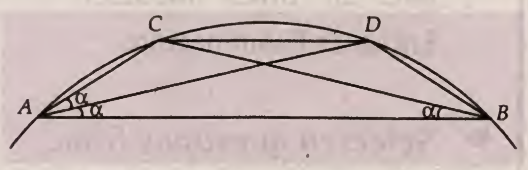
This is the same as $\sin 18^\circ = -\frac{1}{4}(-1+\sqrt{5})$, which is true.]

So, $\frac{1}{2}\beta = 6^\circ \Rightarrow \beta = 12^\circ$ and $\alpha = 96^\circ, \beta = 12^\circ, \gamma = 72^\circ$.

5. Let $x = 5^{25}$, then
$5^{125} - 1 = x^5 - 1 = (x-1)(x^4 + x^3 + x^2 + x + 1)$
$= (x^4 + 9x^2 + 1 + 6x^3 + 6x + 2x^2 - 5x^3 - 10x^2 - 5x)(x-1)$
$= ((x^2 + 3x + 1)^2 - 5x(x+1)^2)(x-1)$
$= ((x^2 + 3x + 1)^2 - 5^{13}(x+1))^2)(x-1)$
$= (x^2 + 3x + 1 + 5^{13}(x+1))$
$(x^2 + 3x + 1 - 5^{13}(x+1))(x-1)$

which implies $\frac{5^{125}-1}{5^{25}-1}$ is a composite number.

6.



Since $BC = AD$, the cyclic quadrilateral $ABDC$ must be an isosceles trapezoid. So either $AC = BD$ or $AB = CD$. The diagonal of an isosceles trapezoid is greater than its arm, while here $AB = \sqrt{2}BC$, so the latter case is not possible. Therefore $AC = BD$ (see the figure). D is the mid point of the arc BC, so that $BD = CD$. Thus, $AC = BD = CD = x$. Let $a = BC$ (= AD). Then $AB = a\sqrt{2}$. Using the Ptolemy theorem on $ABDC$ we get
$AC \cdot BD + AB \cdot CD = AD \cdot BC$,
or $x^2 + xa\sqrt{2} - a^2 = 0$, which gives

$$\frac{x}{a}=\frac{\sqrt{6}-\sqrt{2}}{2}.$$

Let $\alpha = \angle ABC$. Clearly, $\angle CAB = 2\angle ABC = 2\alpha$.
Using the law of sines on ΔABC, we obtain

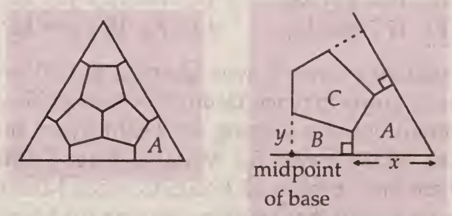
$$2\cos\alpha=\frac{\sin 2\alpha}{\sin\alpha}=\frac{a}{x}=\frac{2}{\sqrt{6}-\sqrt{2}},$$

which gives $\cos\alpha=\frac{\sqrt{6}+\sqrt{2}}{4}$. Hence,

$$\cos 2\alpha = 2\cos^2\alpha - 1 = \frac{8+2\sqrt{12}}{8}-1=\frac{\sqrt{3}}{2},$$

or equivalently $2\alpha = 30°$. Therefore, the angles of the triangle ABC are 15°, 30° and 135°.

7. We claim that any triangle can be dissected into 9 convex pentagons of equal area. Take an equilateral triangle and dissect it as follows :

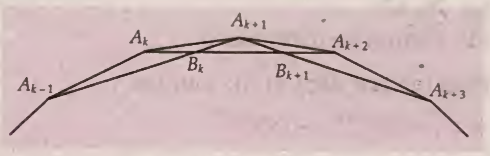


This can be done if we consider the corner A. We can find some x such that the area of A is 1/9 the area of the triangle. Once we have this we can find y so that the area of B is1/18 the area of the triangle. Then the area of C will be

$\frac{1}{3}-2\cdot\frac{1}{18}-\frac{1}{9}=\frac{1}{9}$ (the area of the triangle). By symmetry all nine pentagons will have equal areas. Further all other triangles are shears of the equilateral triangle which preserves the ratio of areas.

8. For convenience, the interior angle in a boomerang which is greater than 180° will be called a "reflex angle".
Clearly, there are b reflex angles, each occurring in a different boomerang and each with the corresponding vertex in the interior of C. Angles around these vertices add up to $2b\pi$. On the other hand, the sum of all the interior angles of C is $(s-2)\pi$ and the sum of the interior angles of all the q quadrilaterals is $2\pi q$.
Therefore, $2\pi q \geq 2b\pi + (s-2)\pi$
from which $q \geq b + (s-2)/2$ follows.

9.



(a) Let $n > 3$ and take all indices modulo n. For each k, define point B_k as the intersection of A_kA_{k+2} and $A_{k-1}A_{k+1}$. Then A_k, B_k, B_{k+1}, A_{k+2} are on the line segment A_kA_{k+2} and exactly in that order because of convexity of the n-gon. Thus

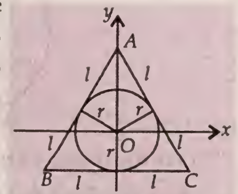
$$\begin{aligned}\sum A_kA_{k+2} &\geq \sum (A_kB_k + B_{k+1}A_{k+2})\\ &= \sum (A_kB_k + B_kA_{k+1}) \text{ by shifting of indices}\\ &\geq \sum A_kA_{k+1} \text{ by the triangle inequality}\end{aligned}$$

(b) This inequality is false.
Again let $n > 3$ and take all indices modulo n. Trivially,

$$\begin{aligned}\sum A_kA_{k+2} &\leq \sum (A_kA_{k+1} + A_{k+1}A_{k+2})\\ &\qquad\text{by the triangle inequality}\\ &= \sum (A_kA_{k+1} + A_kA_{k+1})\\ &\qquad\text{by shifting of indices}\\ &= 2\sum A_kA_{k+1}.\end{aligned}$$

However, the constant "2" cannot be lowered. Fix two points X and Y with distance 1. Consider the (degenerate) n-gon $A = A_1A_2 \ldots A_n$, where
$A_1 \equiv A_2 \equiv X$, $A_3 \equiv \ldots \equiv A_n \equiv Y$.
(A is degenerate, but it could easily be approximated by a sequence of proper convex n-gons).
Then $\sum A_kA_{k+2} = 4$ and $\sum A_kA_{k+1} = 2$.

10. Take O as origin, with the x-axis parallel to BC, and y-axis along OA. Then the incircle has radius



$r=\frac{1}{\sqrt{3}}$ and A, B, C have co-ordinates

$$A\left(0,\frac{2}{\sqrt{3}}\right),\ B\left(-1,\frac{-1}{\sqrt{3}}\right),\ C\left(1,\frac{-1}{\sqrt{3}}\right).$$

A point on the incircle has co-ordinates parameterized by θ, $0 \leq \theta < 2\pi$ given by $P(\frac{1}{\sqrt{3}}\cos\theta, \frac{1}{\sqrt{3}}\sin\theta)$.

Let $c = \cos\theta$, $s = \sin\theta$. Then
$AP^2 + BP^2 + CP^2$

$$\begin{aligned}&=\left[\frac{1}{3}c^2+\frac{1}{3}(2-s)^2\right]+\left[\frac{1}{3}(c+\sqrt{3})^2+\frac{1}{3}(s+1)^2\right]\\ &\qquad\left[\frac{1}{3}(c-\sqrt{3})^2+\frac{1}{3}(s+1)^2\right]\\ &=\frac{1}{3}[c^2+4-4s^2+s^2+c^2+2\sqrt{3}c+3+\\ &\qquad s^2+2s+1+c^2-2\sqrt{3}c+3+s^2+2s+1]\\ &=\frac{1}{3}[3(c^2+s^2)+12]=5\end{aligned}$$

Since, $c^2 + s^2 = 1$.

Mock Test Paper

COMMON ADMISSION TEST

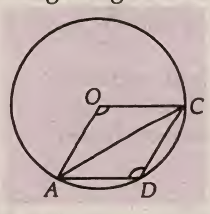
CAT 2009

Exam on
28th November 2009
to
10th December 2009

There are 25 multiple choice questions numbered 1 to 25. Each question has 4 choices (a), (b), (c) & (d) out of which ONLY ONE is correct.

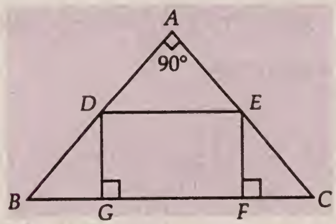
1. A man marks up his goods by $x\%$ and then gives a discount of $y\%$ and thus makes a profit of 53%. If he had marked his goods up by $y\%$ and then given a discount of $x\%$, then he would have suffered a loss of 77%. What is the value of x, if the cost price of the goods is Rs.100?
   (a) 45
   (b) 15
   (c) 80
   (d) cannot be determined

2. Consider the diagram given below.



   O is the centre of the circle with radius 4 cm. AC is $4\sqrt{3}$ cm. Find the length of side AD, if $AD = DC$.
   (a) 4 cm (b) $4\sqrt{3}$ cm
   (c) 5 cm (d) none of these

3. Consider the figure given below.



   $AB = 3$ cm, $AC = 4$ cm and $BC = 5$ cm. Points D and E divide sides AB and AC in such a manner that $\frac{AD}{AB} = k$ and $\frac{AE}{AC} = k$. Find the area of $\square DEFG$ if the ratio of area of ΔADE to that of ΔABC is 2 : 5.
   (a) 2.38 cm^2 (b) 2.78 cm^2
   (c) 2.98 cm^2 (d) 4.8 cm^2

4. A grocer mixes two varieties of pulses A and B in the ratio of 2 : 5 and 3 : 7 so as to make two mixtures M_1 and M_2 respectively. He now mixes M_1 and M_2 in the ratio of 1 : 4 and sells the compound mixture at Rs. 15 per kg making a 25% profit. Find the price of A (dearer variety) if price of B is Rs. 10 per kg?
   (a) Rs. 16.7 per kg (b) Rs. 12.2 per kg
   (c) Rs. 17.7 per kg (d) Rs. 18.3 per kg

5. A Rajdhani express leaves Mumbai at 7.00 am. It takes 12 hours to reach Delhi. There is exactly one Rajdhani express leaving for Delhi every hour. A Ranidhani leaves for Mumbai from Delhi at 7.30 am and it takes 12 hours to reach Mumbai. There is a Ranidhani express leaving for Mumbai every hour. How many Ranidhani expresses will the Rajdhani express (leaving Mumbai at 7.00 a.m.) come across?
   (a) 12 (b) 11
   (c) 1 (d) None of these

6. In a family, Sita is married to Anil and Geeta is married to Sunil. Geeta's age is the average of Anil's & Sunil's age. Sita's age is two-third of Geeta's age. Four years ago, Sunil's age was four times current age of his son Bunty. Keshav and Parvati are Bunty and Babli's grandparents. Difference between ages of Parvati and Bunty is equal to the difference in ages of Keshav and Babli. Babli is twice as old as Bunty. Anil's age is equal to sum of ages of Bunty and Babli. Sum of the ages of the family members except the grandparents is 134. Sum of ages of Keshav and Parvati is equal to the sum of ages of Sunil, Anil, Sita and Geeta. Find the ages (in years) of Keshav and Parvati.
   (a) 63, 55
   (b) 59, 51
   (c) 70, 62
   (d) Cannot be determined

7. Find the last digit of the number
   $8^{323^{129}} - 256^{643^{247}} - 795^{1403^{327}}$
   (a) 1 (b) 8
   (c) 7 (d) none of these

Radius of circle $S_2 = \dfrac{a}{(\sqrt{2})^2}$ and so on.

$$\therefore \text{ sum of radii} = a + \frac{a}{\sqrt{2}} + \frac{a}{2} + \frac{a}{2\sqrt{2}} + \ldots = \frac{a}{1-\frac{1}{\sqrt{2}}} = 2$$

$$\Rightarrow \quad a = 2\left(1 - \frac{1}{\sqrt{2}}\right) = 2 - \sqrt{2}$$

7. (d) : Shifting the origin to (2, 3)
So equation of circle will be $(x-2)^2 + (y-3)^2 = 8$

8. (c) : $AP = 2r\cos 30° = \dfrac{2r\sqrt{3}}{2} = r\sqrt{3}$

$AP = AC = CP = r\sqrt{3}$ Now $AC + CP = 2r\sqrt{3}$

$\Rightarrow \quad AR + RC + CS + SP = 2r\sqrt{3}$

$\Rightarrow \quad AR + SP + RC + CS = 2r\sqrt{3}$

$\Rightarrow \quad RT + TS + RS + RS = 2r\sqrt{3}$

$[\because \; RC = RS \text{ and } CS = RS]$

$\Rightarrow \quad 3RS = 2r\sqrt{3} \quad \Rightarrow \quad RS = \dfrac{2r}{\sqrt{3}}$

9. (a) : Let $P(\theta), Q\left(\theta + \dfrac{2\pi}{3}\right), R\left(\theta + \dfrac{4\pi}{3}\right)$

then $P' \equiv (a\cos\theta, b\sin\theta)$,

$$Q' \equiv \left(a\cos\left(\theta + \frac{2\pi}{3}\right), b\sin\left(\theta + \frac{2\pi}{3}\right)\right),$$

$$R'\left(a\cos\left(\theta + \frac{4\pi}{3}\right), b\sin\left(\theta + \frac{4\pi}{3}\right)\right)$$

Let centroid of $\Delta P'Q'R' \equiv (x', y')$

$$x' = a\left[\frac{\cos\theta + \cos\left(\theta + \frac{2\pi}{3}\right) + \cos\left(\theta + \frac{4\pi}{3}\right)}{3}\right]$$

$$= \frac{a}{3}\left[\cos\theta + 2\cos(\theta + \pi)\cos\frac{\pi}{3}\right] = 0$$

$$y' = \frac{a}{3}\left[\sin\theta + \sin\left(\theta + \frac{2\pi}{3}\right) + \sin\left(\theta + \frac{4\pi}{3}\right)\right]$$

$$= \frac{a}{3}\left[\sin\theta + 2\sin(\theta + \pi)\cos\frac{\pi}{3}\right] = 0$$

10. (c) : Equation of normal is
$ax\sec\theta - by\,\text{cosec}\theta = a^2 - b^2$

$$Q \equiv \left(\frac{a^2 - b^2}{a}\cos\theta, 0\right), R \equiv \left(0, \frac{b^2 - a^2}{b}\right)$$

$$\therefore \quad h = \frac{a^2 - b^2}{2a}\cos\theta, k = \frac{b^2 - a^2}{2b}\sin\theta$$

Eliminating θ we get the equation of locus as

$$\frac{x^2}{\frac{(a^2 - b^2)^2}{4a^2}} + \frac{y^2}{\frac{(a^2 - b^2)^2}{4b^2}} = 1$$

which is an ellipse with $\dfrac{(a^2 - b^2)^2}{4b^2} > \dfrac{(a^2 - b^2)^2}{4a^2}$

$$\text{So, } \frac{(a^2 - b^2)^2}{4a^2} = \frac{(a^2 - b^2)^2}{4b^2}\left(1 - e'^2\right)$$

$$\Rightarrow \; e^4 = \frac{e^4}{(1 - e^2)}(1 - e'^2) \; \Rightarrow \; e = e'$$

11. (c, d) : Let the equation of the circle be
$x^2 + y^2 + 2gx + 2fy + c = 0$ ...(i)
It passes through the points (0,0) and (1,0)

$\therefore \quad c = 0$ and $1 + 2g + c = 0$; $\therefore \; g = -1/2$

Radius of circle (i) is $r_1 = \sqrt{(g^2 + f^2 - c)} = \sqrt{\left(\frac{1}{4} + f^2\right)}$

The centre of the circle $x^2 + y^2 = 9$ is (0, 0) and radius $r_2 = 3$

Since the circle (i) passes through the centre (0, 0) of given circle and it also touches the circle internally.

$\therefore$ Diameter of circle = radius of given circle

i.e. $2r_1 = r_2$

or $2\sqrt{\left(\frac{1}{4} + f^2\right)} = 3 \Rightarrow f^2 = 2 \Rightarrow f = \pm\sqrt{2}$

Hence the centre of circle (1) is $(-g, -f)$

i.e., $\left(\dfrac{1}{2}, -\sqrt{2}\right)$ or $\left(\dfrac{1}{2}, \sqrt{2}\right)$.

12. (a) 13. (d) 14. (a)

Maths Musing Solution Senders

Solution Set 80

1. G. Sarath Chandra, Hyderabad
2. Atdanu Roy, Medinipur (W.B.)
3. R. Vishnoo Prashanth, Mehsana (N.G.)
4. Prasanjeet Mohanty, Cuttack
5. Satya Dev, Bangalore
6. J. Suruli Vel, Tirupati
7. Debabrota Basu, Bandel, Hooghly
8. Ranu Vikram, Ranchi
9. Neelabja Chatterjee, Tamluk (W.B.)
10. Shourya Dipta Ghosh, Howrah (W.B.)

Solution Set 81

1. Shourya Dipta Ghosh, Howrah (W.B.)
2. Sourav Koner, Burdwan (W.B.)
3. J. Suruli Vel, Tirupati
4. Mohsin Shiraz, Kushinagar (U.P.)
5. Saptarshee Chatterjee, Burdwan (W.B.)
6. Anish Chakraborty, Howrah (W.B.)
7. Amipara Nikunj, Rajkot (Gujarat)
8. S. Vijay Kumar, Warangal (A.P.)
9. B. Karthikeya, Warangal (A.P.)

CHALLENGING PROBLEMS

for various Engineering Exams

1. If $z = i\log_e(3+\sqrt{8})$ then $\cos z$ = ?

(a) $\sqrt{8}$ (b) 3
(c) $-2\sqrt{8}$ (d) 6

2. If $a_1, a_2, a_3, \ldots, a_n$ be in H.P. then terms $\dfrac{a_r}{\sum_{i=1}^{n} a_i - a_r}$ (where r = 1, 2, 3, ..., n) are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

3. If $\overline{OA} = 2\hat{i} - 3\hat{j} + 4\hat{k}$, $\overline{OB} = 4\hat{i} + 3\hat{j} - 2\hat{k}$ then bisector of $\angle AOB$ is position vector

(a) $3\hat{i} + \hat{k}$ (b) $-2\hat{i} - 6\hat{j} + 6\hat{k}$
(c) $\hat{i} + 3\hat{j} - 3\hat{k}$ (d) none of these

4. If $\alpha = 1 + (\sqrt{2} - 1) + (3 - 2\sqrt{2}) + (5\sqrt{2} - 7) + \ldots$ to ∞ and $\alpha\beta = 1$ then α, β are roots of

(a) $(2-\sqrt{2})(x^2+1) + (4\sqrt{2}-7)x = 0$
(b) $x^2 - 4x + 1 = 0$
(c) $x^2 - 2\sqrt{2}x + 1 = 0$
(d) none of these

5. $\underbrace{27\ 27\ 27\ \ldots\ 27}_{2n \text{ times}} = A(10^B - 1)$ then

(a) $A = \dfrac{9}{11}, B = 2n$ (b) $A = \dfrac{3}{11}, B = n$
(c) $A = \dfrac{3}{11}, B = 2n$ (d) none of these

6. If $(a^2 - 1)x^2 + 2(a-1)x + 2 > 0 \ \forall x \in R$ then

(a) $a > 1$ or $a < -3$ (b) $a \le -3$ or $a \ge 1$
(c) $a \le -1$ or $a \ge 1$ (d) none of these

7. If $f(x+y) = f(x)h(y) + h(x)f(y)$ then $\begin{vmatrix} f(\alpha+\theta) & f(\alpha) & h(\alpha) \\ f(\beta+\theta) & f(\beta) & h(\beta) \\ f(\gamma+\theta) & f(\gamma) & h(\gamma) \end{vmatrix}$ is independent of

(a) θ, α (b) θ
(c) α, β, γ (d) $\alpha, \beta, \gamma, \theta$

8. $\left\{\dfrac{3^{2009}}{26}\right\}$ = ? (where $\{x\}$ is fractional part of x)

(a) 1/26 (b) 9/26
(c) 4/13 (d) none of these

9. If $A = \sum_{r=1}^{4} \sin(2r-1)\theta$, $B = \sum_{r=1}^{4} \cos(2r-1)\theta$ then $A : B$ = ?

(a) $\tan\theta$ (b) $\tan 2\theta$
(c) $\tan 4\theta$ (d) $\cot 4\theta$

10. If $12^{\tan^{-1}x} + 12^{\sin^{-1}x} + 12^{\cos^{-1}x} > 3.K^{\pi/K}$ then K = ?

(a) 12 (b) 4
(c) $2\sqrt{3}$ (d) none of these

11. The domain of $f(x) = \sqrt{\sin^{-1}\sqrt{1-x^2} + \cos^{-1}x}$ is

(a) $\left[-\dfrac{1}{2}, \dfrac{1}{2}\right]$ (b) [–1, 1]
(c) (–1, 1) (d) none of these

12. Find the greatest angle of triangle with sides $x, y, \sqrt{x^2+xy+y^2}$

(a) 90° (b) 150°
(c) 120° (d) none of these

13. If $I_1 = \displaystyle\int_0^1 \frac{x^2 dx}{(2-x^3)e^{x^3}}$, $I_2 = \displaystyle\int_0^1 \frac{e^x dx}{1+x}$ then $I_1 : I_2$ = ?

(a) 1 : 3 (b) 3 : 1
(c) 1 : 3e (d) 3e : 1

14. $\displaystyle\lim_{x \to -\pi} \frac{\sin x}{|x+\pi|}$ = ?

(a) 1 (b) –1
(c) $-\pi$ (d) does not exist

15. What is constant for hyperbola $(x-9)^2\cos^2\theta - y^2\sin^2\theta = \sin^2 2\theta$?

(a) ordinates of latera-recta
(b) abscissae of foci
(c) eccentricity
(d) none of these

16. $\dfrac{6}{1!} + \dfrac{16}{2!} + \dfrac{30}{3!} + \dfrac{48}{4!} + \ldots$ to ∞ = ?

By : Anil Kr. Gupta(AKG), ASANSOL (W.B.) Mob.: 09832230099

(a) $8e$ (b) $8e-1$
(c) $8e-2$ (d) none of these

17. Find the no. of rational terms in expansion of $(\sqrt[3]{3}+\sqrt{2})^{36}$ is
(a) 6 (b) 7
(c) 8 (d) none of these

18. Find solution set of $|\sin x| - \sin x < 1$ in $x \in (0, 2\pi)$
(a) $\left(0, \frac{7\pi}{6}\right)$ (b) $(0, \pi) \cup \left(\frac{11\pi}{6}, 2\pi\right)$
(c) $\left(0, \frac{7\pi}{6}\right) \cup \left(\frac{11\pi}{6}, 2\pi\right)$ (d) $(0, \pi) \cup \left(\frac{7\pi}{6}, 2\pi\right)$

19. If $ax + by = 1$ touches $x^2 + y^2 = p^2$ then locus of (a, b) is
(a) parabola of latus-rectum $4p$
(b) circle of radius $1/p^2$
(c) ellipse having centre as origin
(d) none of these

20. If $\int \{x(1-x^2)\}^{1/3} \cdot x^{-4} dx = A(x^B - 1)^C + K$ then
(a) $A = -\frac{3}{8}$ (b) $B = -2$
(c) $C = \frac{4}{3}$ (d) all of these

21. No. of ways of forming 10 couples from 10 men & 10 women is
(a) 576 (b) 490
(c) 385 (d) none of these

22. If $\int_{\pi/6}^{x} \sqrt{4-\sin^2 t} \cdot dt + \int_0^y \cos t \cdot dt = 0$ then at $\left(0, \frac{\pi}{4}\right)$, $\frac{dy}{dx} = ?$
(a) $-2\sqrt{2}$ (b) $2\sqrt{2}$
(c) $\sqrt{2}$ (d) $\frac{1}{\sqrt{2}}$

23. Number of real solutions of $\left(\frac{5}{7}\right)^x + x^2 + 3 = x$ is
(a) 0 (b) 2
(c) 4 (d) infinitely many

24. If α, β be roots of $x^2 + ax + b = 0$ then roots of $x^2 + (2b - a^2)x + b^2 = 0$ are
(a) $\alpha + \beta, \alpha - \beta$ (b) α^2, β^2
(c) $\frac{\alpha}{\beta}, \frac{\beta}{\alpha}$ (d) none of these

25. If one root of $x^3 + \sqrt{3}(3x^2 - 1) = 3x$ is $\tan\theta$ then $\theta = ?$
(a) 30° (b) 40°
(c) 60° (d) none of these

26. If $\frac{a^2+1-\cos^2 x}{\cos 2x} + \frac{a^2}{1-\cot^2 x} = 0$ has atleast one real solution of x then a lies in
(a) $(-1, 1)$ (b) $[-1, 1]$
(c) $\{0\}$ (d) none of these

27. A plane cutting the axes in P, Q, R passes through $(\alpha - \beta, \beta - \gamma, \gamma - \alpha)$. If O be the origin then locus of centre of sphere $OPQR$ is
(a) $\alpha x + \beta y + \gamma z = 4$
(b) $(\alpha - \beta)x + (\beta - \gamma)y + (\gamma - \alpha)z = 0$
(c) $(\alpha - \beta)yz + (\beta - \gamma)zx + (\gamma - \alpha)xy = 2xyz$
(d) none of these

28. Find $f(x)$ if $\{f'(x)\}^2 + 3f'(x)f(x) + \{f(x)\}^2 = 0$
(a) $c.e^x$ (b) $c.e^{-x}$
(c) $c.e^{(-3\pm\sqrt{5})x}$ (d) $c.e^{-(3\pm\sqrt{5})\frac{x}{2}}$

29. If $\sin A$, $\sin B$, $\sin C$ are in an increasing A.P. in ΔABC then
(a) $B \not> 60°$ (b) $A < 60°$
(c) $C > 60°$ (d) all correct

30. $f(x) = \tan^{-1}(\cos x + \sin x)^3$ is an increasing function in
(a) $\left(0, \frac{\pi}{2}\right)$ (b) $\left(\frac{\pi}{4}, \frac{\pi}{2}\right)$
(c) $\left(-\frac{\pi}{4}, \frac{\pi}{4}\right)$ (d) none of these

31. If $I_n = \int_0^{\pi/2} x^n \cos x \, dx$ and $2^k(I_k + 7kI_6) = \pi^k$ then $k = ?$
(a) 6 (b) 8
(c) 5 (d) none of these

32. The locus of the centre of the circle touching circles $|z - z_1| = r_1$ and $|z - z_2| = r_2$ externally is
(a) circle (b) ellipse
(c) hyperbola (d) none of these

33. If $(x^2 + y^2)^5 = (ax^5 - 10x^3y^2 + bxy^4)^2 + (bx^4y - 10x^2y^3 + ay^5)^2$ then
(a) $a = 1, b = 3$ (b) $a = 1, b = 5$
(c) $a = 5, b = 1$ (d) none of these

34. If $f: N \to I$ defined as
$$f(x) = \begin{cases} -\frac{x}{2}, & \text{if } x \text{ is even} \\ \frac{x-1}{2}, & \text{if } x \text{ is odd} \end{cases}, \text{ then } f \text{ is}$$
(a) one-one and onto
(b) one-one but not onto
(c) neither one-one nor onto
(d) none of these

35. If $\sin\theta = k\sin(\theta + \phi)$ implies $\tan(\theta + \phi) = \frac{\sin\phi}{\cos\phi + p}$ then $p = ?$

(a) k (b) $-k$
(c) $2k$ (d) $-2k$

36. If $a_i > 0$, $b_i > 0$ ($i = 1, 2, 3, ..., n$) so that $\sum_{i=1}^{n} a_i = \sum_{i=1}^{n} b_i$ then $\sum_{i=1}^{n} \frac{a_i^2}{a_i + b_i} \geq ?$

(a) $\sum_{i=1}^{n} b_i$ (b) $2\sum_{i=1}^{n} b_i$
(c) $\sum_{i=1}^{n} \frac{b_i}{2}$ (d) none of these

37. If $f(x) = \int_{1/2}^{x} \sqrt{4-t^2}\,dt$ then $f(x)$ increases strictly when $x \in$
(a) $(-2, 2)$ (b) $[-2, 2]$
(c) $(2, \infty)$ (d) none of these

38. If $k < 0$ and $f(x) = e^{kx} + e^{-kx}$ is monotonically decreasing then $x \in$
(a) $(-k, k)$ (b) $(0, \infty)$
(c) $(-\infty, 0)$ (d) none of these

39. Find the sum of those factors of 7! which are of the form $3n + 1$ and odd ($n \in N$)
(a) 48 (b) 42
(c) 12 (d) 8

40. The total number of selections of atmost n boys from a group of $(2n + 1)$ boys is 31, find n
(a) 2 (b) 5/2
(c) 3 (d) none of these

41. If $\int_{0}^{2a} (|x - a| + 5)\,dx$ does not exceed $\frac{21}{4}$ then $a \in$
(a) $\left(-\infty, -\frac{21}{2}\right) \cup \left(\frac{1}{2}, \infty\right)$ (b) $\left(-\infty, \frac{1}{2}\right)$
(c) $\left(-\frac{21}{2}, \frac{1}{2}\right)$ (d) none of these

42. $\int_{0}^{\pi/4} (\tan^n(x - [x]) + \tan^{n-2}(x - [x]))\,dx = ?$
(a) $\frac{1}{n-1}$ (b) $\frac{1}{1-n}$
(c) $\frac{2}{n-1}$ (d) none of these

43. For every odd integer n, if
$\sin n\theta = \sum_{r=0}^{n} a_r \sin^r \theta \quad \forall \theta \in R$ then
(a) $a_0 = 1, a_1 = 2$ (b) $a_0 = 0, a_1 = 2$
(c) $a_0 = 0, a_1 = 1$ (d) none of these

44. $\int_{0}^{2} x^3 \left[\cos\frac{\pi x}{2} + 1\right] dx = ?$ (where $[x]$ is Greatest Integer Function)
(a) 1/4 (b) 4
(c) 0 (d) none of these

45. If differential equation $\frac{dx}{3y + f} + \frac{dy}{px + g} = 0$ represents a circle then $p = ?$
(a) g (b) f
(c) 3 (d) 4

46. If $x^2 - 4x < \sin^{-1}(\sin 5)$ then
(a) $x \in (2 - \sqrt{2\pi - 1}, 2 + \sqrt{2\pi - 1})$
(b) $x \in (-1, 5)$
(c) $x \in (2 - \sqrt{9 - 2\pi}, 2 + \sqrt{9 - 2\pi})$
(d) none of these

47. If $\int_{0}^{\infty} f(x + \sqrt{1 + x^2})\,dx = a\int_{b}^{\infty} \left(c + \frac{1}{x^2}\right) f(x)\,dx$, then
(a) $a = \frac{1}{2}$ (b) $b = 1$
(c) $c = 1$ (d) all correct

48. $\lim_{x \to 1} \sec(\pi 2^{-x}) \log_e x = ?$
(a) $2\pi^{-1}\log_{10} 2$ (b) $2\pi^{-1}\log_2 e$
(c) $2\pi^{-1}\log_e 2$ (d) none of these

49. While travelling at a speed of x km/hr, a bus burns petrol $\left(\frac{3}{x} + \frac{x}{300}\right)$ litres/km. If cost of petrol be Rs.30/litre and other cost is also Rs.30/hr., then speed for a journey of 75 km. having minimum cost is
(a) $20\sqrt{3}$ km/hr (b) 20 km/hr
(c) $5\sqrt{3}$ km/hr (d) none of these

50. In ΔABC, A and B are 2 solutions of $k\sec^2 x = 2\tan x$ $(0 < k < 1, A \neq B)$ then $\sin C = ?$
(a) $\frac{1}{2}$ (b) 1
(c) $\frac{\sqrt{3}}{2}$ (d) none of these

SOLUTIONS

1. **(b) :** $\because \cos z = \frac{1}{2}(e^{iz} + e^{-iz})$

$$= \frac{1}{2}\{e^{-\log_e(3+\sqrt{8})} + e^{\log_e(3+\sqrt{8})}\}$$
$$= \frac{1}{2}\{(3+\sqrt{8})^{-1} + (3+\sqrt{8})\}$$
$$= \frac{1}{2}\left\{\frac{3-\sqrt{8}}{9-8} + 3 + \sqrt{8}\right\} = 3$$

2. **(c) :** $\frac{1}{a_1}, \frac{1}{a_2}, \frac{1}{a_3}, ..., \frac{1}{a_n}$ are in A.P.

$\Rightarrow \frac{a_1 + a_2 + ... + a_n}{a_1} - 1, \frac{a_1 + a_2 + ... + a_n}{a_2} - 1, ...$ are in A.P.

$\Rightarrow \frac{a_2 + a_3 + ... + a_n}{a_1}, \frac{a_1 + a_3 + ... + a_n}{a_2}, ...$ are in A.P.

$\Rightarrow \dfrac{a_1}{a_2+a_3+...+a_n}, \dfrac{a_2}{a_1+a_3+...+a_n}$, ... are in H.P.

$\Rightarrow \dfrac{a_r}{\sum_{i=1}^{n} a_i - a_r}$ $(r = 1, 2, 3, ..., n)$ are in H.P.

3. **(a)** : Here, $|\overline{OA}| = |\overline{OB}|$. If midpoint of AB be P then $\overline{OP}$ is position vector of bisector of $\angle AOB$

$\Rightarrow \overline{OP} = \dfrac{\overline{OA}+\overline{OB}}{2} = 3\hat{i}+\hat{k}$

4. **(a)** : $\alpha = 1+(\sqrt{2}-1)+(\sqrt{2}-1)^2+(\sqrt{2}-1)^3+...$ to ∞

$= \dfrac{1}{1-(\sqrt{2}-1)} = \dfrac{1}{2-\sqrt{2}}$

$\because$ $\alpha\beta = 1$ $\quad\therefore$ $\beta = 2-\sqrt{2}$

$\therefore$ $\alpha+\beta = \dfrac{1}{2-\sqrt{2}}+2-\sqrt{2} = \dfrac{7-4\sqrt{2}}{2-\sqrt{2}}$

$\therefore$ Equation is $x^2 - \dfrac{7-4\sqrt{2}}{2-\sqrt{2}}\cdot x+1 = 0$

or $(2-\sqrt{2})(x^2+1)+(4\sqrt{2}-7)x = 0$

5. **(c)** : $A(10^3-1) = \underbrace{27\ 27\ 27\ ...\ 27}_{2n \text{ times}}$

$= 27 + 27 \times 10^2 + 27 \times 10^4 + ...$ to n terms

$= \dfrac{27\{(10^2)^n-1\}}{10^2-1} = \dfrac{27(10^{2n}-1)}{99} = \dfrac{3}{11}(10^{2n}-1)$

$\therefore$ $A = \dfrac{3}{11}, B = 2n$

6. **(a)** : Let $f(x) = (a^2-1)x^2+2(a-1)x+2$

$\therefore$ $f(x) > 0$ if (i) $a^2-1 > 0$ (ii) $D < 0$

$\Rightarrow$ $a^2 > 1$ and $4(a-1)^2 - 4(a^2-1)\cdot 2 < 0$

$\Rightarrow$ $(a+1)(a-1) > 0$ and $(a+3)(a-1) > 0$

$\Rightarrow$ $a < -1$ or $a > 1$ and $a < -3$ or $a > 1$

$\Rightarrow$ $a < -3$ or $a > 1$

7. **(d)** : $\begin{vmatrix} f(\alpha)h(\theta)+h(\alpha)f(\theta) & f(\alpha) & h(\alpha) \\ f(\beta)h(\theta)+h(\beta)f(\theta) & f(\beta) & h(\beta) \\ f(\gamma)h(\theta)+h(\gamma)f(\theta) & f(\gamma) & h(\gamma) \end{vmatrix}$

$= \begin{vmatrix} 0 & f(\alpha) & h(\alpha) \\ 0 & f(\beta) & h(\beta) \\ 0 & f(\gamma) & h(\gamma) \end{vmatrix}$ $[C_1 \to C_1 - h(\theta).C_2 - f(\theta).C_3]$

$= 0$

8. **(b)** : $\because$ $3^{2009} = (3^3)^{669}.3^2 = 9[(26+1)^{669}]$

$= 9[26^{669} + {}^{669}C_1.26^{668} + ... + 1]$

$=$ (A multiple of 26) + 9

$\therefore$ $\dfrac{3^{2009}}{26} =$ (A multiple of 26) $+ \dfrac{9}{26}$

$\therefore$ $\left\{\dfrac{3^{2009}}{26}\right\} = \dfrac{9}{26}$

9. **(c)** : $A : B = \dfrac{(\sin 7\theta+\sin\theta)+(\sin 5\theta+\sin 3\theta)}{(\cos 7\theta+\cos\theta)+(\cos 5\theta+\cos 3\theta)}$

$= \dfrac{2\sin 4\theta\cos 3\theta+2\sin 4\theta\cos\theta}{2\cos 4\theta\cos 3\theta+2\cos 4\theta\cos\theta} = \dfrac{2\sin 4\theta}{2\cos 4\theta} = \tan 4\theta$

10. **(a)** : $\because$ $\sin^{-1}x$ & $\cos^{-1}x$ both are defined if $-1 \le x \le 1$

$\therefore$ $\tan^{-1}x \ge -\dfrac{\pi}{4}$

$\Rightarrow$ $\tan^{-1}x+\sin^{-1}x+\cos^{-1}x \ge \dfrac{\pi}{2}-\dfrac{\pi}{4} = \dfrac{\pi}{4}$ ...(i)

Further, $\sin^{-1}x$, $\cos^{-1}x$ & $\tan^{-1}x$ can't be equal simultaneously

$\therefore$ A.M. > G.M.

$\Rightarrow$ $12^{\tan^{-1}x}+12^{\sin^{-1}x}+12^{\cos^{-1}x} > 3.(12^{\tan^{-1}x}.12^{\sin^{-1}x}.12^{\cos^{-1}x})^{1/3}$

$= 3.(12^{\tan^{-1}x+\sin^{-1}x+\cos^{-1}x})^{1/3} \ge 3.(12^{\pi/4})^{1/3}$

$\therefore$ $12^{\tan^{-1}x}+12^{\sin^{-1}x}+12^{\cos^{-1}x} > 3.12^{\pi/12}$

$\Rightarrow$ $k = 12$

11. **(b)** : Here, $-1 \le x \le 1$, $1-x^2 \ge 0$ and $\sin^{-1}\sqrt{1-x^2}+\cos^{-1}x \ge 0$

All are satisfied when $x \in [-1, 1]$

12. **(c)** : $\because$ $\sqrt{x^2+xy+y^2}$ is greatest

$\therefore$ $\cos\theta = \dfrac{x^2+y^2-(x^2+xy+y^2)}{2xy} = -\dfrac{1}{2}$

$\therefore$ $\theta = 120°$

13. **(c)** : Put $x^3 = z$

$\therefore$ $3x^2dx = dz$

$\Rightarrow I_1 = \dfrac{1}{3}\int_0^1 \dfrac{dz}{(2-z)e^z} = \dfrac{1}{3}\int_0^1 \dfrac{dz}{\{2-(1-z)\}e^{1-z}}$

$= \dfrac{1}{3}\int_0^1 \dfrac{e^z dz}{(1+z)e} = \dfrac{1}{3e}\int_0^1 \dfrac{e^x dx}{1+x} = \dfrac{1}{3e}\cdot I_2$

$\therefore$ $I_1 : I_2 = 1 : 3e$

14. **(d)**: L.H.L. $= \lim\limits_{x\to-\pi^-} \dfrac{\sin x}{-(x+\pi)} = \lim\limits_{h\to 0} \dfrac{\sin(-\pi-h)}{-(-\pi-h)-\pi} = 1$

$|x+\pi| = -(x+\pi), x < -\pi$

R.H.L. $= \lim\limits_{x\to-\pi^+} \dfrac{\sin x}{x+\pi} = \lim\limits_{h\to 0} \dfrac{\sin(-\pi+h)}{-\pi+h+\pi} = -1$

$|x+\pi| = x+\pi, x > -\pi$

L.H.L. is 1 and R.H.L. is -1

$\therefore$ Limit does not exist.

15. **(b)** : $\dfrac{(x-9)^2}{4\sin^2\theta} - \dfrac{y^2}{4\cos^2\theta} = 1$

Here, $a^2 = 4\sin^2\theta$, $b^2 = 4\cos^2\theta$

$\Rightarrow e = \sqrt{1+\dfrac{b^2}{a^2}} = |\operatorname{cosec}\theta|$

$\Rightarrow$ $ae = \pm 2 \Rightarrow$ Abscissae of foci are constant

16. **(a)** : Series is $\dfrac{2\cdot 3}{1!}+\dfrac{4\cdot 4}{2!}+\dfrac{6.5}{3!}+\dfrac{8.6}{4!}+...$ to ∞

Here $t_n = \frac{2n(n+2)}{n!} = \frac{2\cdot(n+2)}{(n-1)!} = \frac{2\{(n-1)+3\}}{(n-1)!}$

$= \frac{2}{(n-2)!} + \frac{6}{(n-1)!}$

$\therefore$ $S_\infty = 2e + 6e = 8e$

17. **(a) :** $t_{r+1} = {}^{36}C_r(\sqrt[3]{3})^{36-r}\cdot(\sqrt{2})^r$, which is rational if $36 - r = 3m$ and $r = 2n$

$\Rightarrow m = \frac{36-2n}{3}$

Possible n are 3, 6, 9, 12, 15, 18

$\Rightarrow$ No. of rational terms = 6

18. **(c) :** $\because$ $|\sin x| = \begin{cases}\sin x & , x \in (0,\pi] \\ -\sin x, & x\in(\pi, 2\pi)\end{cases}$

$\therefore$ If $x \in (0, \pi]$, $\sin x - \sin x < 1$ is always true

If $x \in (\pi, 2\pi)$, $-\sin x - \sin x < 1$

$\Rightarrow \sin x > -\frac{1}{2} \quad \Rightarrow x \in \left(\frac{11\pi}{6}, 2\pi\right) \cup \left(\pi, \frac{7\pi}{6}\right)$

Hence, $x \in \left(0, \frac{7\pi}{6}\right) \cup \left(\frac{11\pi}{6}, 2\pi\right)$

19. **(d) :** $ax + by = 1$ touching $x^2 + y^2 = p^2$

$\Rightarrow \left|\frac{0+0-1}{\sqrt{a^2+b^2}}\right| = p \;\Rightarrow a^2 + b^2 = \frac{1}{p^2}$

$\Rightarrow$ Locus of (a, b) is $x^2 + y^2 = \frac{1}{p^2}$, which is a circle of radius $1/p$

20. **(d) :** $I = \int \frac{(x-x^3)^{1/3}}{x^4}dx = \int\left(\frac{1}{x^2}-1\right)^{1/3}\cdot\frac{1}{x^3}dx$

Put $\frac{1}{x^2} - 1 = t \Rightarrow \frac{1}{x^3}dx = -\frac{dt}{2}$

$= -\frac{1}{2}\int t^{1/3}\cdot dt = -\frac{1}{2}\cdot\frac{t^{4/3}}{(4/3)} + k$

$= -\frac{3}{8}\left(\frac{1}{x^2}-1\right)^{4/3} \Rightarrow A = -\frac{3}{8}, B = -2, C = \frac{4}{3}$

21. **(c) :** First couple can be formed in ${}^{10}C_1 \times {}^{10}C_1 = 10^2$; 2nd couple in ${}^{9}C_1 \times {}^{9}C_1 = 9^2$ and so on.

$\therefore$ Reqd. no. of ways $= 10^2 + 9^2 + ... + 2^2 + 1^2$

$= \frac{10(10+1)(2\times10+1)}{6} = 385$

22. **(a) :** Diff. w.r. to x, we get

$\sqrt{4-\sin^2 x} + \cos y \cdot \frac{dy}{dx} = 0$

At $\left(0, \frac{\pi}{4}\right)$, $\sqrt{4-0} + \frac{1}{\sqrt{2}}\cdot\frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = -2\sqrt{2}$

23. **(a) :** We have, $-x^2 + x - 3 = \left(\frac{5}{7}\right)^x$ ...(i)

In L.H.S., coefficient of $x^2 = -1 < 0$ and

$D = 1^2 - 4(-1)(-3) < 0$

$\therefore$ L.H.S. < 0. But, R.H.S. > 0

$\Rightarrow$ No solution.

24. **(b) :** $\alpha + \beta = -a$, $\alpha\beta = b$

$\because$ $2b - a^2 = 2\alpha\beta - \{-(\alpha+\beta)\}^2 = -(\alpha^2+\beta^2)$ &

$b^2 = \alpha^2.\beta^2$

$\Rightarrow$ α^2, β^2 are 2 roots

25. **(b) :** Given equation is

$3x - x^3 = -\sqrt{3}(1-3x^2)$

$\Rightarrow \frac{3x-x^3}{1-3x^2} = -\sqrt{3}$

If $\tan\theta$ be its one root then

$\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta} = -\tan 60°$

$\Rightarrow$ $\tan 3\theta = \tan 120° \Rightarrow \theta = 40°$

26. **(c) :** $\frac{a^2+\sin^2 x}{\cos 2x} = \frac{a^2}{\cot^2 x - 1}$

$\Rightarrow \frac{a^2+\sin^2 x}{\cos 2x} = \frac{a^2\cdot\sin^2 x}{\cos 2x}$

$\Rightarrow \sin^2 x = \frac{a^2}{a^2-1} \quad \because 0 \le \sin^2 x \le 1$

$\therefore 0 \le \frac{a^2}{a^2-1} \le 1$

(i) $a^2 - 1 > 0$ or $a = 0$ and (ii) $\frac{a^2}{a^2-1} - 1 \le 0$

i.e., $a \notin [-1, 1]$ or $a = 0$ and $\frac{1}{a^2-1} \le 0$

i.e., $a \notin [-1, 1]$ or $a = 0$ and $a^2 - 1 < 0$

i.e., $a \notin [-1, 1]$ or $a = 0$ and $a \in (-1, 1)$

$\Rightarrow$ $a = 0$ is only possibility

27. **(c) :** Let $\frac{x}{a} + \frac{y}{b} + \frac{z}{c} = 1$ ...(1)

be the plane passing through $(\alpha-\beta, \beta-\gamma, \gamma-\alpha)$

$\Rightarrow \frac{\alpha-\beta}{a} + \frac{\beta-\gamma}{b} + \frac{\gamma-\alpha}{c} = 1$ ...(2)

Let sphere be

$x^2 + y^2 + z^2 + 2ux + 2vy + 2wz + d = 0$ ...(3)

$\therefore$ $d = 0$ for passing through origin

$\because$ (3) passes through $P(a, 0, 0)$, $Q(0, b, 0)$ & $R(0, 0, c)$

$\therefore$ $a = -2u$, $b = -2v$, $c = -2w$

From (2) : $\frac{\alpha-\beta}{-2u} + \frac{\beta-\gamma}{-2v} + \frac{\gamma-\alpha}{-2w} = 1$

$\Rightarrow$ Locus of centre $(-u, -v, -w)$ is

$\frac{\alpha-\beta}{2x} + \frac{\beta-\gamma}{2y} + \frac{\gamma-\alpha}{2z} = 1$

$\Rightarrow$ $(\alpha-\beta)yz + (\beta-\gamma)zx + (\gamma-\alpha)xy = 2xyz$

28. **(d) :** $f'(x) = \frac{-3f(x) \pm \sqrt{9\{f(x)\}^2 - 4\{f(x)\}^2}}{2}$

$= \frac{-3f(x) \pm \sqrt{5}f(x)}{2} = \frac{-(3\pm\sqrt{5})}{2}f(x)$

$\Rightarrow \dfrac{f'(x)}{f(x)} = \dfrac{-(3 \pm \sqrt{5})}{2}$

On integrating, $\log\{f(x)\} = \dfrac{-(3 \pm \sqrt{5})x}{2} + \log c$

$\Rightarrow f(x) = c.e^{-(3\pm\sqrt{5})\frac{x}{2}}$

29. (d) : $\because\ 2\sin B = \sin A + \sin C$

$= 2\sin\dfrac{A+C}{2}\cos\dfrac{A-C}{2}$

$\therefore\ 2\cdot 2\sin\dfrac{B}{2}\cos\dfrac{B}{2} = 2\cos\dfrac{B}{2}\cos\dfrac{A-C}{2}$

$\Rightarrow 2\sin\dfrac{B}{2} = \cos\dfrac{A-C}{2}$ $\left[\because\ \cos\dfrac{B}{2} \neq 0 \text{ as } \dfrac{B}{2} \neq \dfrac{\pi}{2} \text{ i.e. } B \neq \pi\right]$

$\Rightarrow 2\sin\dfrac{B}{2} \leq 1$ $\left[\because \cos\dfrac{A-C}{2} \leq 1\right]$

$\Rightarrow \sin\dfrac{B}{2} \leq \sin 30°$

$\Rightarrow$ $B \leq 60°$ $\therefore B \ngtr 60°$ $\Rightarrow$ $A + C \geq 120°$

$\therefore$ $A < 60°$ and $C > 60°$ $[\because \sin A < \sin B < \sin C]$

30. (c) : $\because$ $\tan^{-1}x$ and x^3 both are increasing function

$\therefore$ If $\cos x + \sin x$ is increasing then $f(x)$ will be increasing function

$\therefore$ We must have $\dfrac{d}{dx}(\cos x + \sin x) > 0$

i.e. $-\sin x + \cos x > 0$

i.e. $\cos x > \sin x$, which is satisfied for (c)

31. (b) : $I_n = [x^n \sin x]_0^{\pi/2} - \displaystyle\int_0^{\pi/2} nx^{n-1} \cdot \sin x\, dx$

$= \left(\dfrac{\pi}{2}\right)^n - n[x^{n-1}(-\cos x)]_0^{\pi/2} + n\displaystyle\int_0^{\pi/2}(n-1)x^{n-2}(-\cos x)dx$

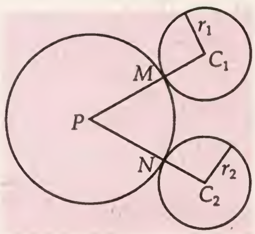
$= \left(\dfrac{\pi}{2}\right)^n - n(n-1)I_{n-2}$

$\therefore$ When $n = 8$,

$I_8 = \left(\dfrac{\pi}{2}\right)^8 - 56 \cdot I_6 \Rightarrow 2^8(I_8 + 56 \cdot I_6) = \pi^8$

clearly, $k = 8$

32. (c) :



Let $PM = PN = r$

$\therefore$ $PC_1 - PC_2 = (r_1 + r) - (r_2 + r)$

$\therefore$ $|PC_1 - PC_2| = |r_1 - r_2|$ = constant

$\Rightarrow$ Locus of P is a hyperbola.

33. (b) : $\because (x + iy)^5 = x^5 + {}^5C_1x^4iy + {}^5C_2x^3i^2y^2 + {}^5C_3x^2i^3y^3 + {}^5C_4xi^4y^4 + i^5y^5$

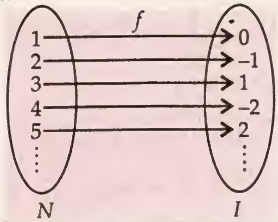
$= (x^5 - 10x^3y^2 + 5xy^4) + i(5x^4y - 10x^2y^3 + y^5)$

On taking modulus to both sides and squaring,

$(x^2 + y^2)^5 = (x^5 - 10x^3y^2 + 5xy^4)^2 + (5x^4y - 10x^2y^3 + y^5)^2$

Hence, $a = 1$, $b = 5$ (on comparing)

34. (a) : $\because f(1) = 0, f(2) = -1, f(3) = 1, f(4) = -2, f(5) = 2$, ... etc.



$\because$ Every element of N has unique image in I and no element left in I

$\Rightarrow$ f is one-one & onto

35. (b) : $\sin\{(\theta + \phi) - \phi\} = k\sin(\theta + \phi)$

$\Rightarrow$ $\sin(\theta + \phi)\cos\phi - \cos(\theta + \phi)\sin\phi = k\sin(\theta + \phi)$

$\Rightarrow$ $\sin(\theta + \phi)(\cos\phi - k) = \cos(\theta + \phi)\sin\phi$

$\therefore\ \tan(\theta + \phi) = \dfrac{\sin\phi}{\cos\phi - k} \Rightarrow p = -k$

36. (c) : $\dfrac{a_i^2}{a_i + b_i} = \dfrac{a_i(a_i + b_i) - a_ib_i}{a_i + b_i} = a_i - \dfrac{a_ib_i}{a_i + b_i}$

$= a_i - \dfrac{1}{2}\cdot\dfrac{2a_ib_i}{a_i + b_i} \geq a_i - \dfrac{1}{2}\cdot\dfrac{a_i + b_i}{2}$ $[\because H.M. \leq A.M.]$

$\therefore\ \dfrac{a_i^2}{a_i + b_i} \geq \dfrac{3a_i - b_i}{4}$

$\therefore\ \displaystyle\sum_{i=1}^{n}\dfrac{a_i^2}{a_i + b_i} \geq \dfrac{1}{4}\sum_{i=1}^{n}(3a_i - b_i) = \dfrac{1}{4}\left\{3\sum_{i=1}^{n}a_i - \sum_{i=1}^{n}b_i\right\}$

$= \dfrac{1}{4}\displaystyle\sum_{i=1}^{n}2b_i$ $\left[\because \displaystyle\sum_{i=1}^{n}a_i = \sum_{i=1}^{n}b_i\right]$

37. (a) : $f(x) = \displaystyle\int_{1/2}^{x}\sqrt{4 - t^2}\,dt$ and for strictly increasing function, $f'(x) > 0$

$\Rightarrow \sqrt{4 - x^2} > 0 \quad \Rightarrow 4 - x^2 > 0$

$\Rightarrow$ $(x + 2)(x - 2) < 0 \Rightarrow x \in (-2, 2)$

38. (c) : $\because f'(x) = k(e^{kx} - e^{-kx})$ ...(1)

$\because$ $k < 0 \therefore e^{kx} - e^{-kx} > 0$ for $f(x)$ to be decreasing

i.e. $e^{-kx}(e^{2kx} - 1) > 0$

i.e. $e^{2kx} - 1 > 0$ $[\because e^{-kx} > 0]$

i.e. $e^{2kx} > 1$ for which $x < 0$ $(\because k < 0)$

$\therefore$ $x \in (-\infty, 0)$

39. (d) : $\because$ $7! = 2^4.3^2.5.7$ and factor should be of $(3n + 1)$ form and odd, only

$\therefore$ Their sum = 8

40. (d) : Here, $31 = {}^{2n+1}C_1 + {}^{2n+1}C_2 + ... + {}^{2n+1}C_n$ ...(1)

$\because$ ${}^{2n+1}C_0 + {}^{2n+1}C_1 + ... + {}^{2n+1}C_{2n+1}) = 2^{2n+1}$

$\therefore\quad 2({}^{2n+1}C_0 + {}^{2n+1}C_1 + \ldots + {}^{2n+1}C_n) = 2^{2n+1}$

$[\because {}^nC_r = {}^nC_{n-r}]$

$\Rightarrow\quad 2(1 + 31) = 2^{2n+1}$

$\Rightarrow\quad 2^6 = 2^{2n+1} \Rightarrow n = 5/2$, not possible (being fraction)

41. (d) : $I = \int_0^a \{(a-x)+5\}\,dx + \int_a^{2a} \{(x-a)+5\}\,dx$

$$= \left[(a+5)x - \frac{x^2}{2}\right]_0^a + \left[\frac{x^2}{2} + (5-a)x\right]_a^{2a} = a^2 + 10a$$

$\because\ I \not> \frac{21}{4} \quad \therefore\ I \le \frac{21}{4}$

$\Rightarrow\quad 4a^2 + 40a - 21 \le 0 \Rightarrow (2a+21)(2a-1) \le 0$

$\Rightarrow \left(a + \frac{21}{2}\right)\left(a - \frac{1}{2}\right) \le 0 \Rightarrow a \in \left[-\frac{21}{2}, \frac{1}{2}\right]$

42. (a) : In $0 < x < \frac{\pi}{4}, [x] = 0$

$\therefore\ I = \int_0^{\pi/4} (\tan^n x + \tan^{n-2} x)\,dx$

$= \int_0^{\pi/4} \tan^{n-2} x(\tan^2 x + 1)\,dx = \int_0^{\pi/4} \tan^{n-2} x \sec^2 x\,dx$

$= \int_0^1 z^{n-2} \cdot dz$, where $z = \tan x$ and

x	0	$\pi/4$
z	0	1

$= \left[\frac{z^{n-1}}{n-1}\right]_0^1 = \frac{1}{n-1}$

43. (d) : $\sin n\theta = a_0 + a_1\sin\theta + a_2\sin^2\theta + \ldots \quad \ldots(1)$

If $\theta = 0$, $0 = a_0$

On diff. (1) w.r. to θ,

$n\cos n\theta = a_1\cos\theta + a_2 2\sin\theta\cos\theta + \ldots$

Putting $\theta = 0$, $n = a_1$.

44. (a) : If $0 < x < 1,\ 0 < \frac{\pi x}{2} < \frac{\pi}{2}$

$\Rightarrow \cos 0 > \cos\frac{\pi x}{2} > \cos\frac{\pi}{2}$

$\Rightarrow 2 > \cos\frac{\pi x}{2} + 1 > 1 \Rightarrow \left[\cos\frac{\pi x}{2} + 1\right] = 1$

If $1 < x < 2 \Rightarrow \frac{\pi}{2} < \frac{\pi x}{2} < \pi \quad \Rightarrow \cos\frac{\pi}{2} > \cos\frac{\pi x}{2} > \cos\pi$

$\Rightarrow 0 > \cos\frac{\pi x}{2} > -1 \quad \Rightarrow 1 > \cos\frac{\pi x}{2} + 1 > 0$

$\Rightarrow \left[\cos\frac{\pi x}{2} + 1\right] = 0$

$\therefore\ I = \int_0^1 x^3 \cdot 1 \cdot dx + \int_1^2 x^3 \cdot 0 \cdot dx = \frac{1}{4}$

45. (c) : $(px + g)dx + (3y + f)dy = 0$

On integrating, $\frac{px^2}{2} + gx + \frac{3y^2}{2} + fy + c = 0$

$\Rightarrow \frac{p}{2} = \frac{3}{2}$ (for circle) $\Rightarrow\ p = 3$

46. (c) : $\because\ \frac{-\pi}{2} \le \sin^{-1} x \le \frac{\pi}{2}$

$\therefore\quad \sin^{-1}(\sin 5) = 5 - 2\pi \in \left[-\frac{\pi}{2}, \frac{\pi}{2}\right]$

$\therefore\quad x^2 - 4x < 5 - 2\pi \quad \Rightarrow (x-2)^2 < 9 - 2\pi$

$\Rightarrow |x - 2| < \sqrt{9 - 2\pi}$

$2 - \sqrt{9 - 2\pi} < x < 2 + \sqrt{9 - 2\pi}$

47. (d) : Put $x + \sqrt{1 + x^2} = t$

$\Rightarrow\quad 1 + x^2 = t^2 + x^2 - 2tx$

$\Rightarrow x = \frac{1}{2}\left(t - \frac{1}{t}\right)$

x	0	∞
t	1	∞

$\Rightarrow dx = \frac{1}{2}\left(1 + \frac{1}{t^2}\right)dt$

$\therefore\ I = \int_1^\infty f(t) \cdot \frac{1}{2}\left(1 + \frac{1}{t^2}\right)dt = \frac{1}{2}\int_1^\infty \left(1 + \frac{1}{x^2}\right) f(x)dx$

$\Rightarrow a = \frac{1}{2}, b = 1, c = 1$

48. (b) : Limit $= \lim_{x\to 1} \frac{\log_e x}{\cos(\pi 2^{-x})} \quad \left(\frac{0}{0} \text{ form}\right)$

$= \lim_{x\to 1} \frac{1/x}{-\sin(\pi 2^{-x}) \cdot \pi \cdot 2^{-x} \cdot (\log_e 2)(-1)}$

$= \frac{2}{\pi \log_e 2} = 2\pi^{-1} \log_2 e$

49. (a) : Let speed of bus is x km/hr

Time taken $= \frac{75}{x}$ hr.

$\therefore\quad$ Other cost $= 30 \times \frac{75}{x}$

Petrol cost $= 30\left(\frac{3}{x} + \frac{x}{300}\right)75$

$\therefore\quad$ Total cost, $C = 30 \times \frac{75}{x} + 30\left(\frac{3}{x} + \frac{x}{300}\right)75$

$= 75\left(\frac{120}{x} + \frac{x}{10}\right)$

$\frac{dC}{dx} = 75\left(\frac{-120}{x^2} + \frac{1}{10}\right)$

For min. cost, $\frac{dC}{dx} = 0$

$\Rightarrow\quad x^2 = 1200 \Rightarrow\ x = 20\sqrt{3}$

$\frac{d^2C}{dx^2} = 75 \times \frac{240}{x^3} \quad \therefore\ \frac{d^2C}{dx^2} > 0$ if $x = 20\sqrt{3}$

50. (b) : $k = \frac{2\tan x}{1 + \tan^2 x} = \sin 2x \quad \because\ 0 < k < 1$

$\therefore\quad 0 < \sin 2x < 1$

$\because\quad A$ and B are 2 solutions

$\therefore\quad k = \sin 2A$ and $k = \sin 2B$

$\Rightarrow\quad \sin 2A = \sin 2B$

$\Rightarrow\quad 2A = 180° - 2B$

$\Rightarrow\quad A + B = 90° \quad \Rightarrow C = 90° \ \therefore\ \sin C = 1$

OLYMPIAD CORNER

Challenging problems for Olympiads, IIT-JEE and other contests.

1. Let $ABCD$ be a quadrilateral with $AD = BC$ and let $\angle A + \angle B = 120°$. Three equilateral triangles ΔACP, ΔDCQ and ΔDBR are drawn on AC, DC and DB away from AB. Prove that the three new vertices P, Q and R are collinear.

2. There are n black marbles and two red marbles in a jar. One by one, marbles are drawn at random out of the jar. Amit wins as soon as two black marbles are drawn, and Prashant wins as soon as two red marbles are drawn. The game continues until one of the two wins. Let $J(n)$ and $F(n)$ be the two probabilities that Amit and Prashant win, respectively.

1. Determine the value of $F(1) + F(2) + ... + F(3992)$.
2. As n approaches infinity, what does $J(2) \times J(3) \times J(4) \times ... \times J(n)$ approach?

3. Consider a square $ABCD$ with side length 1. Select a point M exterior to the square so that $\angle AMB$ is 90°. Let $a = AM$ and $b = BM$. Now, determine the point N exterior to the square so that $CN = a$ and $DN = b$. Find, as a function of a and b, the length of the line segment MN.

4. Given positive real numbers a, b, and c such that $a + b + c = 1$, show that $a^a b^b c^c + a^b b^c c^a + a^c b^a c^b \le 1$.

5. Assume that a and b are integers. Prove that the equation $a^2 + b^2 + x^2 = y^2$ has an integer solution x, y if and only if the product ab is even.

6. Find the remainder when the polynomial $x^{135} + x^{125} - x^{115} + x^5 + 1$ is divided by the polynomial $x^3 - x$.

7. Let ABC be an equilateral triangle and Γ its incircle. If D and E are points on AB and AC, respectively, such that DE is tangent to Γ, show that

$$\frac{AD}{DB} + \frac{AE}{EC} = 1.$$

8. Given a quadrilateral $ABCD$, with $AD = \sqrt{3}$, $AB + CD = 2AD$, $\angle A = 60°$ and $\angle D = 120°$, find the length of the line segment from D to the mid-point of BC.

9. Let x, y be positive integers with $y > 3$ and

$$x^2 + y^4 = 2[(x - 6)^2 + (y + 1)^2].$$

Prove that $x^2 + y^4 = 1994$.

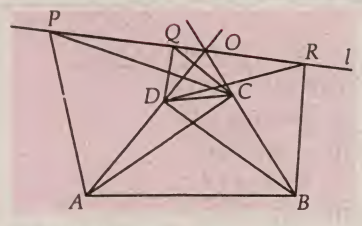
10. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{ab}{a^5 + b^5 + ab} + \frac{bc}{b^5 + c^5 + bc} + \frac{ca}{c^5 + a^5 + ca} \le 1.$$

When does the equality hold?

SOLUTIONS

1.



Let O be the intersection of AD and BC.

Since, $\angle A + \angle B = 120°$, we get $\angle AOB = 60°$

Let l be the exterior bisector of $\angle AOB$.

Since, $\angle APC = 60° = \angle AOC$, we have that O, P, A, C are concyclic.

Hence, $\angle POA = \angle PCA = 60°$.

The exterior angle of $\angle AOB$ is 120°, showing that PO bisects the exterior angle of $\angle AOB$. Thus, P lies on l.

Similarly, Q and R lie on l.

Hence, P, Q and R are collinear.

2. If Prashant wins, the balls must be drawn in one of the following three ways : red, red; red, black, red; or black, red, red. This must be the case, as otherwise two black balls will be drawn and Amit will win. Hence, the probability that Prashant wins is the sum of the probabilities of each of three cases above.

Thus,

$$F(n) = \frac{2}{n+2} \cdot \frac{1}{n+1} + \frac{2}{n+2} \cdot \frac{n}{n+1} \cdot \frac{1}{n} + \frac{n}{n+2} \cdot \frac{2}{n+1} \cdot \frac{1}{n}$$

$$= \frac{2}{(n+1)(n+2)} + \frac{2}{(n+1)(n+2)} + \frac{2}{(n+1)(n+2)}$$

$$= \frac{6}{(n+1)(n+2)}.$$

Note that $F(n) = \frac{6}{n+1} - \frac{6}{n+2}$, so we can use a telescoping series to calculate the desired sum.

We have

1. $F(1) + F(2) + ... + F(3992)$

$$=\left(\frac{6}{2}-\frac{6}{3}\right)+\left(\frac{6}{3}-\frac{6}{4}\right)+\left(\frac{6}{4}-\frac{6}{5}\right)+\ldots+\left(\frac{6}{3993}-\frac{6}{3994}\right)$$

$$=\frac{6}{2}-\frac{6}{3994}=3-\frac{3}{1997}=\frac{5988}{1997}.$$

2. Now $J(n) = 1 - F(n)$

$$=1-\frac{6}{(n+1)(n+2)}=\frac{n^2+3n-4}{(n+1)(n+2)}$$

$$=\frac{(n-1)(n+4)}{(n+1)(n+2)}$$

Thus, $J(2) \times J(3) \times J(4) \times \ldots \times J(n)$

$$=\frac{1\cdot 6}{3\cdot 4}\cdot\frac{2\cdot 7}{4\cdot 5}\cdot\frac{3\cdot 8}{5\cdot 6}\cdot\frac{4\cdot 9}{6\cdot 7}\cdots\frac{(n-1)\cdot(n+4)}{(n+1)\cdot(n+2)}$$

$$=\frac{1\cdot 2\cdot 3\cdot 4\cdot\ldots\cdot(n-2)\cdot(n-1)}{3\cdot 4\cdot 5\cdot 6\cdot\ldots\cdot n\cdot(n+1)}\cdot\frac{6\cdot 7\cdot 8\cdot 9\cdot\ldots\cdot(n+3)\cdot(n+4)}{4\cdot 5\cdot 6\cdot 7\cdot\ldots\cdot(n+1)\cdot(n+2)}$$

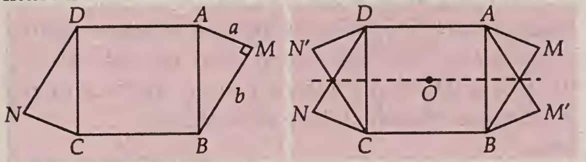
$$=\frac{1\cdot 2}{n\cdot(n+1)}\cdot\frac{(n+3)\cdot(n+4)}{4\cdot 5}$$

$$=\frac{1}{10}\cdot\frac{(n+3)(n+4)}{n(n+1)}$$

$$=\frac{1}{10}\cdot\frac{n+3}{n}\cdot\frac{n+4}{n+1}$$

Thus, as n approaches infinity, $\frac{n+3}{n}=1+\frac{3}{n}$ and $\frac{n+4}{n+1}=1+\frac{3}{n+1}$ both approach 1, and so $J(2) \times J(3) \times J(4) \times \ldots \times J(n)$ approaches 1/10.

3. **Method - I :** Let O be the centre of the square. Consider a horizontal reflection through the line parallel to DA and passing through O. Let the image of M and N about this line be M' and N', respectively. There exists a point where MN intersects $M'N'$. Let this point be K. Since K lies on both lines, this point must lie on this horizontal line of reflection.



Now, the above diagram is also symmetrical about the line parallel to DC passing through O. Then K must lie on this line as well. This leaves us with K coinciding with O, since the two lines intersect at O.

Now, because of the symmetry, $NO = OM$. That is, $OM=\frac{MN}{2}$. Now, consider quadrilateral $OAMB$. Since, $\angle AOB$ is a right angle and so is $\angle AMB$, then the quadrilateral is cyclic.

By Ptolemy's Theorem on this quadrilateral, we have $OM\cdot AB = AM\cdot OB + MB\cdot OA$

$$=a\cdot\frac{1}{\sqrt{2}}+b\cdot\frac{1}{\sqrt{2}},$$

since

$$OA=OB=\frac{1}{\sqrt{2}}.$$

Since $OB = 1$, we have $OM=\frac{\sqrt{2}}{2}(a+b)$, and so $MN=\sqrt{2}(a+b)$, since $MN = 2OM$. Thus, the length of line segment MN is $\sqrt{2}(a+b)$.

Method - II : Let $\angle BAM = x$. Then $a = \cos x$ and $b = \sin x$. Let O be the centre of $ABCD$ and R be the midpoint of AB. Then $\angle BAM = \angle RAM = \angle RMA = x$, and so, $\angle ORM = \angle ARO + \angle ARM$

$$=\frac{\pi}{2}+(\pi-2x)=\frac{3\pi}{2}-2x$$

Now, $OR = RA = RM = \frac{1}{2}$. By the Cosine Law on triangle ROM, we have :

$$OM^2 = OR^2 + RM^2 - 2OR\cdot OM\cos\angle ORM$$

$$=\frac{1}{4}+\frac{1}{4}-2\cdot\frac{1}{2}\cdot\frac{1}{2}\cdot\cos\left(\frac{3\pi}{2}-2x\right)$$

$$=\frac{1}{2}-\frac{1}{2}\cdot(-\sin 2x)$$

$$=\frac{1}{2}-\frac{1}{2}\cdot 2\sin x\cos x=\frac{1}{2}+ab$$

Thus, using the fact that $a^2 + b^2 = 1$, we have

$$4OM^2 = 2(1 + 2ab) = 2(a^2 + b^2 + 2ab) = 2(a + b)^2.$$

Hence, $2OM=\sqrt{2}(a+b)$, by taking the square root of both sides (Note : a, b and OM are all positive).

Since, $2OM = MN$, we have $MN=\sqrt{2}(a+b)$

4. Using the weighted $A.M. - G.M.$ inequality three times, we have the following :

$$\frac{c\cdot a+a\cdot b+b\cdot c}{c+a+b}\geq\left(a^cb^ac^b\right)^{\frac{1}{a+b+c}},$$

$$\frac{b\cdot a+c\cdot b+a\cdot c}{b+c+a}\geq\left(a^bb^cc^a\right)^{\frac{1}{a+b+c}},$$

$$\frac{a\cdot a+b\cdot b+c\cdot c}{a+b+c}\geq\left(a^ab^bc^c\right)^{\frac{1}{a+b+c}}.$$

Adding these inequalities together, we get

$$1=a+b+c=\frac{(a+b+c)^2}{a+b+c}$$

$$=\frac{a^2+b^2+c^2+2ab+2ac+2bc}{a+b+c}\geq a^ab^bc^c+a^bb^cc^a+a^cb^ac^b.$$

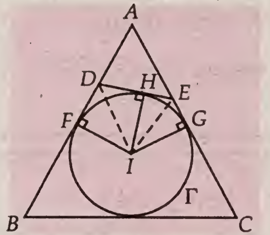
5. First, we prove that this condition is necessary. Suppose ab is odd. Then a, b are odd and $a^2 \equiv b^2 \equiv 1 \bmod 4$. Now, $x^2 \equiv 0$ or $1 \bmod 4$, and $y^2 \equiv 0$ or $1 \bmod 4$. Therefore, $a^2 + b^2 + x^2 = y^2$ is not possible, since if we consider this modulo 4, $2 + x^2 \equiv y^2 \bmod 4$, which is impossible since $2 + x^2 \equiv 2$ or $3 \bmod 4$. Therefore, ab must be even.

If ab is even then, without loss of generality, $a = 2k$.

Consider $4k^2 + b^2 + x^2 = y^2$.
If $4k^2 + b^2 = 2t + 1$, t is an integer, then set $y - x = 1$ and $y + x = 2t + 1$, $2y = 2(t + 1)$, $y = t + 1$ and $x = t$.
Then $2t + 1 + t^2 = (t + 1)^2$.
If $4k^2 + b^2$ is even, then $b = 2s$ and
$4k^2 + b^2 = 4(k^2 + s^2) = 4m$.
Again if $y^2 - x^2 = 4m$, then set $y - x = 2$ and $y + x = 2m$, we get
$y = m + 1$ and $x = y - 2 = m - 1$.
Now $4m + (m - 1)^2 = (m + 1)^2$.
Hence, $a^2 + b^2 + x^2 = y^2$ always has a solution when ab is even.

6. $$x^{135} + x^{125} - x^{115} + x^5 + 1 = (x^3 - x)Q(x) + ax^2 + bx + c$$
$$= x(x - 1)(x + 1)Q(x) + ax^2 + bx + c$$
This must be valid for all values of x. Substituting in $x = 0$, $x = 1$, and $x = -1$ we get :
$$x = 0 : 1 = 0 + c \quad \Rightarrow \quad c = 1$$
$$x = 1 : 3 = 0 + a + b + c \quad \Rightarrow \quad a + b = 2$$
$$x = -1 : -1 = 0 + a - b + c \quad \Rightarrow \quad a - b = -2$$
Solving the system
$$a + b = 2$$
$$a - b = -2$$
gives $a = 0$, $b = 2$. So, the remainder is $2x + 1$.

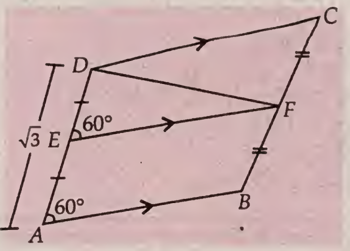
7. **Method - I :** Let I be the incentre and r the inradius. Let F, G and H be the points where Γ is tangent to AB, AC and DE. Then $DF = DH$ and $EG = EH$ (this is a well-known property of the two tangents from a point to a circle). This implies that $\angle FID = \angle DIH$ and $\angle GIE = \angle EIH$. Since, $\angle FIG = 120°$, $\angle FID + \angle GIE = 60°$. We can write $\angle FID = 30° + \phi$, and $\angle GIE = 30° - \phi$. Now, $FD = r\tan\angle FID$ and $AF = r\tan 60°$, so that



$$\frac{AD}{DB} = \frac{AF - FD}{AF + FD} = \frac{\tan 60° - \tan\angle FID}{\tan 60° + \tan\angle FID}$$
$$= \frac{\sqrt{3} - \dfrac{\tan 30° + \tan\phi}{1 - \tan 30°\tan\phi}}{\sqrt{3} + \dfrac{\tan 30° + \tan\phi}{1 - \tan 30°\tan\phi}}$$
$$= \frac{\sqrt{3} - \dfrac{1 + \sqrt{3}\tan\phi}{\sqrt{3} - \tan\phi}}{\sqrt{3} + \dfrac{1 + \sqrt{3}\tan\phi}{\sqrt{3} - \tan\phi}} = \frac{1 - \sqrt{3}\tan\phi}{2},$$
and similarly
$$\frac{AE}{EC} = \frac{1 + \sqrt{3}\tan\phi}{2}.$$
We then have
$$\frac{AD}{DB} + \frac{AE}{EC} = \frac{1 - \sqrt{3}\tan\phi}{2} + \frac{1 + \sqrt{3}\tan\phi}{2} = 1$$
as required.

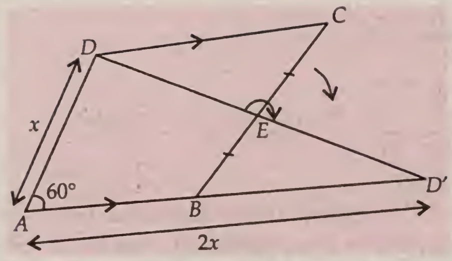
Method - II : Using the same diagram as shown above, assume, without loss of generality, that the sides of the equilateral triangle ABC have length 1. Let $x = AD$ and $y = AE$, where $0 \le x, y \le 1/2$. Now using the Cosine Law in triangle ADE, we have
$$(1 - x - y)^2 = x^2 + y^2 - 2xy\cos 60°$$
This is equivalent to each of the following :
$$1 + x^2 + y^2 - 2x - 2y + 2xy = x^2 + y^2 - xy,$$
$$\Rightarrow \quad 2x + 2y - 1 = 3xy,$$
$$\Rightarrow \quad x - xy + y - xy = 1 - x - y + xy$$
and finally,
$$\frac{x}{1 - x} + \frac{y}{1 - y} = 1$$
The last equation is valid since x and y cannot be equal to 1. This is our desired result.

8. **Method - I :** Given that AB and CD are parallel. Then by Thales' Theorem, a line through the mid-points E and F of AD and BC respectively will also be parallel to AB, and the length of EF will be $\dfrac{AB + CD}{2} = AD$. Now, applying the Cosine Law to triangle DEF,



we have
$$DF^2 = DE^2 + EF^2 - 2 \cdot DE \cdot EF\cos\angle DEF$$
$$= \left(\frac{AD}{2}\right)^2 + AD^2 - AD^2\cos 60°$$
$$= \frac{3AD^2}{4},$$
so $DF = \dfrac{3}{2}$

Method - II : Let the mid-point of BC be E. Note that lines AB and CD are parallel.
Rotate triangle DEC about point E so that C coincides with B, and D coincides with D' as shown in the figure. This is possible since E was chosen to be the mid-point of line segment BC.



Now A, B, and D' are collinear since lines CD and AB are parallel,
so that $\angle D'BE + \angle ABE = \angle DCE + \angle ABE = 180°$.
Let $x = AD$, then $AD' = AB + BD' = AB + CD = 2x$.
Thus, triangle DAD' is similar to the ubiquitous $1:2:\sqrt{3}$ triangle.

Hence, $DE = \dfrac{DD'}{2} = \dfrac{\sqrt{3}AD}{2} = \dfrac{3}{2}$

9. Rewriting we get

$$x^2 - 24x - y^4 + 2y^2 + 4y + 74 = 0 \quad \text{...(i)}$$

Now (i) has integer solutions only if the discriminant $4(y^4 - 2y^2 - 4y + 70)$ is a perfect square. It is easy to prove that for $y \geq 4$,

$$(y^2 - 2)^2 < y^4 - 2y^2 - 4y + 70 < (y^2 + 1)^2 \quad \text{...(*)}$$

(Indeed (*) $\Leftrightarrow y^2 - 2y + 33 > 0$ and $4y(y + 1) > 69$. The first inequality is true.
Since $y \geq 4$, $4y(y + 1) \geq 4\cdot4\cdot5 = 80 > 69$).
The only perfect squares between $(y^2 - 2)^2$ and $(y^2 + 1)^2$ are $(y^2 - 1)^2$ and $(y^2)^2$.
Now, $(y^2 - 1)^2 = y^4 - 2y^2 - 4y + 70$

$$\Leftrightarrow \quad y = \frac{69}{4} \notin \mathbb{Z},$$

and $y^4 - 2y^2 - 4y + 70 = y^4 \Leftrightarrow y^2 + 2y - 35 = 0$

$$\Leftrightarrow \quad y = 5 \text{ or } y = -7.$$

Thus, $y = 5$
Now (i) gives $x = 37$ and

$$x^2 + y^4 = 37^2 + 5^4 = 1369 + 625 = 1994$$

The result also works for $y = 1$ and $y = 2$ as well, but fails for $y = 3$ with $x = 1$.

10. We first note that

$$\frac{a^5+b^5}{2} \geq \left(\frac{a^3+b^3}{2}\right)\left(\frac{a^2+b^2}{2}\right), \text{ since}$$

$a^5 - a^3b^2 - a^2b^3 + b^5 = (a - b)^2(a + b)(a^2 + ab + b^2) \geq 0$ with equality if and only if $a = b$.
Similarly,

$$\frac{a^3+b^3}{2} \geq \left(\frac{a+b}{2}\right)\left(\frac{a^2+b^2}{2}\right)$$

because $a^3 - a^2b - ab^2 + b^3 = (a - b)^2(a + b) \geq 0$, with equality if and only if $a = b$. Thus,

$$\frac{a^5+b^5}{2} \geq \left(\frac{a^3+b^3}{2}\right)\left(\frac{a^2+b^2}{2}\right) \geq ab\left(\frac{a^3+b^3}{2}\right)$$

$$\geq ab\left(\frac{a+b}{2}\right)\left(\frac{a^2+b^2}{2}\right) \geq \frac{a^2b^2(a+b)}{2}.$$

It is enough, therefore, to prove

$$\frac{ab}{ab(a+b)ab+ab} + \frac{bc}{bc(b+c)bc+bc} + \frac{ca}{ca(c+a)ca+ca} \leq 1,$$

$$\text{or } \frac{1}{ab(a+b)+abc} + \frac{1}{bc(b+c)+abc} + \frac{1}{ca(c+a)+abc} \leq 1,$$

Equivalently,

$$\frac{1}{ab(a+b+c)} + \frac{1}{bc(a+b+c)} + \frac{1}{ca(a+b+c)} \leq 1,$$

$$\text{or } \frac{c}{abc(a+b+c)} + \frac{a}{abc(a+b+c)} + \frac{b}{abc(a+b+c)} \leq 1.$$

Again, because $abc = 1$, we get

$$\frac{a+b+c}{a+b+c} \leq 1, \text{ which is true.}$$

The equality requires $a = b = c = 1$.

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers comments and suggestions regarding the problems and solutions offered are always welcome.

1. Discuss the continuity of the function

$$f(x)=\lim_{n\to\infty}\left\{\frac{\cos\pi x-x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}\right\} \text{ at } x=1.$$

2. Show that $\lim_{n\to\infty}\sum_{K=0}^{n}\frac{{}^nC_K}{n^K(K+3)}=e-2$.

3. Suppose $\phi(.)$ is a differentiable function. If $\phi(x+y)=\phi(x)\cdot\phi(y)$ and $\phi(5)=2$, $\phi'(0)=3$, then find the value of $\phi'(1)$.

4. Find the domain and range of the function f, defined by $f(x)=\dfrac{x^2+1}{\ln(x^2+1)}$.

5. Show that $\sin[x]$, where [.] denotes the greatest integer function, is non-periodic. Also show that there is no x for which $\sin[x]=\cos[x]$ but there are infinitely many x for which $\sin[x]=\tan[x]$.

6. Let $f(x+y)=f(x)-f(y)+2xy-1$, $\forall\ x, y\in R$. If f is differentiable and $f'(0)=b$, then find the set of values of b, if $f(x)>0$, $\forall x$.

7. A function f is defined by

$$f(x)=\begin{cases}\dfrac{(b^2-a^2)}{2}, & 0\le x\le a\\[2mm] \dfrac{b^2}{2}-\dfrac{x^2}{6}-\dfrac{a^3}{3x}, & a<x\le b\\[2mm] \left(\dfrac{b^3-a^3}{3x}\right), & x>b\end{cases}$$

Discuss the continuity of f, f' and f'' in $[0,\infty)$.

8. A function $f(x)$ is defined for $x\in[0,1]$ and $f(x)+f(y)=f(xy-\sqrt{1-x^2}.\sqrt{1-y^2})$ and $f(0)=\dfrac{\pi}{2}, f\left(\dfrac{1}{\sqrt2}\right)=\dfrac{\pi}{4}$. Find the function $f(x)$.

9. Discuss the continuity and differentiability of the function f defined by $f(x)=\left[x^2\left[\dfrac{1}{x^2}\right]\right]$, $x\ne0$, where [.] denotes the G.I.F.

10. Find the set of values of 'a' for which the function $f:[-3,3]\to R$ defined by $f(x)=\left[\dfrac{x^2}{a}\right]\tan ax+\sec ax$ is an (i) even function (ii) odd function. ([.] denotes the G.I.F.)

SOLUTIONS

1. For, $0<x<1$, $\lim_{m\to\infty}x^m=0$

$$f(x)=\lim_{n\to\infty}\left\{\frac{\cos\pi x-x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}\right\}=\cos\pi x$$

For $x=1$

$$f(x)=\lim_{n\to\infty}\left\{\frac{\cos\pi-\sin(1-1)}{1+1-1}\right\}=\lim_{n\to\infty}\cos\pi=-1$$

For $x>1$

$$f(x)=\lim_{n\to\infty}\left\{\frac{\cos\pi x-x^{2n}\sin(x-1)}{1+x^{2n+1}-x^{2n}}\right\}$$

$$=\lim_{n\to\infty}\left\{\frac{\dfrac{\cos\pi x}{x^{2n+1}}-\dfrac{1}{x}\sin(x-1)}{\dfrac{1}{x^{2n+1}}+1-\dfrac{1}{x}}\right\}=\frac{\sin(x-1)}{(1-x)}$$

$\lim_{m\to\infty}\dfrac{1}{x^m}=0$ if $x>1$

Thus, we have

$$f(x)=\begin{cases}\cos\pi x, & 0<x<1\\ -1, & x=1\\ \dfrac{\sin(x-1)}{(1-x)}, & x>1\end{cases}$$

L.H.L.

$=\lim_{h\to0}f(1-h)=\lim_{h\to0}.\cos\pi(1-h)=\lim_{h\to0}\cos(\pi-\pi h)$

$=-1$

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R.H.L.

$= \lim_{h\to 0} f(1+h) = \lim_{h\to 0} \frac{\sin[(1+h)-1]}{[1-(1+h)]} = \lim_{h\to 0}\frac{\sin h}{-h} = -1$

and $f(1) = -1$

Thus $\lim_{h\to 0} f(1-h) = \lim_{h\to 0} f(1+h) = f(1)$

Hence $f(x)$ is continuous at $x = 1$.

2. $\lim_{n\to\infty}\sum_{K=0}^{n}\frac{{}^nC_K}{n^K(K+3)} = \lim_{n\to\infty}\sum_{K=0}^{n}\frac{1}{(K+3)}\,{}^nC_K\frac{1}{n^K}$

$= \lim_{n\to\infty}\sum_{K=0}^{n}{}^nC_K\frac{1}{n^K}\cdot\int_0^1 x^{K+2}dx$

$= \int_0^1\left(x^2\lim_{n\to\infty}\sum_{K=0}^{n}{}^nC_K\left(\frac{x}{n}\right)^K\right)dx$

$= \int_0^1 x^2\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^n dx$

$= \int_0^1 x^2\lim_{n\to\infty}\left(1+\frac{x}{n}\right)^{\frac{n}{x}\cdot x} dx \qquad \left(\lim_{x\to\infty}\left(1+\frac{1}{x}\right)^x = e\right)$

$= \int_0^1 x^2e^x dx$

$= \left[x^2\int e^x dx\right]_0^1 - \int_0^1\left\{\frac{d}{dx}(x^2)(\int e^x dx)\,dx\right\}$

$= [x^2e^x]_0^1 - 2\int_0^1 xe^x dx = e - 2\int_0^1 xe^x dx$

$= e - 2\left[x\int e^x dx - \int\frac{dx}{dx}(\int e^x dx)\,dx\right]_0^1$

$= e - 2[xe^x - e^x]_0^1 = e - 2$ Proved

3. Given $\phi(x+y) = \phi(x)\cdot\phi(y)$ ...(1)

or $\phi(x+0) = \phi(x)\cdot\phi(0)$ or $\phi(0) = 1$

since

$\lim_{h\to 0}\frac{\phi(x+h)-\phi(x)}{h} = \phi'(x)$

$\therefore \lim_{h\to 0}\frac{\phi(x).\phi(h)-\phi(x)}{h} = \phi'(x)$

or $\lim_{h\to 0}\frac{\phi(x).[\phi(h)-1]}{h} = \phi'(x)$

or $\lim_{h\to 0}\frac{\phi(x).[\phi(h)-\phi(0)]}{h} = \phi'(x)$

or $\phi(x)\,.\,\phi'(0) = \phi'(x)$

or $3\phi(x) = \phi'(x)$ ...(2)

or $3\phi(x) = \frac{d[\phi(x)]}{dx}$ or $3dx = \frac{d[\phi(x)]}{\phi(x)}$

Integrating both sides

$\therefore$ $3x = \ln\phi(x) + C$ ...(3)

Put $x = 5$

$15 = \ln\phi(5) + C \Rightarrow 15 = \ln 2 + C \quad \therefore C = 15 - \ln 2$

Putting the value of 'C' in (3)

$\therefore$ $3x = \ln\phi(x) + 15 - \ln 2$ or $\ln\phi(x) = 3x + (\ln 2) - 15$

$\therefore \phi(x) = e^{3x+(\ln 2)-15} = e^{3x-15}.e^{\ln 2} \quad \therefore \phi(x) = 2e^{3x-15}$

then $\phi(1) = 2e^{-12}$

From (2)

$3\phi(x) = \phi'(x) \quad \therefore \phi'(1) = 3\phi(1) = 6e^{-12}$

4. $f(x)$ is defined if $\ln(x^2+1) \neq 0$

$\Rightarrow x^2 + 1 \neq 1 \Rightarrow x \neq 0$.

Thus domain of $f = R - \{0\}$

Now let $t = x^2 + 1$, then $t > 1$.

Let $g(t) = \frac{t}{\ln t}$

$\Rightarrow g'(t) = \frac{\ln t - t\,.\,\frac{1}{t}}{\ln^2 t} = \frac{\ln t - 1}{\ln^2 t}$

Thus $g(t)$ decreases for $t \in (1, e]$ and increases for $t \in [e, \infty)$

$g(e) = \frac{e}{\ln e} = e$

We observe that $\lim_{t\to 1^+} g(t) = \infty$ and

$\lim_{t\to\infty} g(t) = \lim_{t\to\infty}\frac{t}{\ln t} = \lim_{t\to\infty}\frac{1}{1/t} = \infty$

Thus range of g is $[e, \infty)$

Hence range of f is $[e, \infty)$

5. Let $f(x) = \sin[x]$. If f is periodic and $T > 0$ is one of the periods, then $f(x+T) = f(x)$, $\forall\, x$

$\Rightarrow \sin[x+T] = \sin[x]$, $\forall\, x$

If $x = 0$, then $\sin[T] = 0$

$\Rightarrow [T] = n\pi$, for some integer n.

Since $[T]$ is an integer and $n\pi$ is an integer only when $n = 0$, $[T] = 0$

If $x = T$, then $\sin[2T] = \sin[T] = 0$

$\Rightarrow [2T] = 0$ proceeding this way.

$[T] = [2T] = [3T] = \ldots. = 0$,

which is not possible for any $T > 0$

Hence $\sin[x]$ is not a periodic function.

If $\sin[x] = \cos[x]$

$\Rightarrow \tan[x] = 1 = \tan\frac{\pi}{4}$, then $[x] = n\pi + \pi/4$, $n \in I$, which is not possible for any x as L.H.S. is an integer and R.H.S. is never an integer.

However if $\sin[x] = \tan[x]$

$\Rightarrow \cos[x] = 1 = \cos 0$, then $[x] = n\pi$, $n \in I$, which is possible only where

$[x] = 0 \Rightarrow 0 \le x < 1$

6. Putting $x = 0 = y$ in the given functional equation, we get $f(0) = -1$.

Now

$f'(x) = \lim_{h\to 0}\frac{f(x+h)-f(x)}{h}$

$= \lim_{h\to 0}\frac{f(x)-f(h)+2xh-1-f(x)}{h}$

$= 2x - \lim_{h\to 0}\frac{f(h)-f(0)}{h} = 2x - f'(0) = 2x - b$

$\Rightarrow\ f(x) = x^2 - bx + c$

Putting $x = 0$ we have $c = -1$. Hence $f(x) = x^2 - bx - 1$

The discriminant $D = b^2 + 4$.

For $f(x) > 0$, $\forall x$, $D < 0$, which is not possible for any b. Hence there is no such b *i.e.*, $b \in \{\}$

7. $f(0) = f(0+) = \dfrac{b^2 - a^2}{2}$

$$f(a) = f(a-) = \frac{b^2 - a^2}{2},\ f(a+) = \frac{b^2}{2} - \frac{a^2}{6} - \frac{a^3}{3a} = \frac{b^2 - a^2}{2}$$

$$f(b) = f(b-) = \frac{b^2}{2} - \frac{b^2}{6} - \frac{a^3}{3b} = \frac{b^2}{3} - \frac{a^3}{3b}$$

and $f(b+) = \dfrac{b^3 - a^3}{3b} = \dfrac{b^3 - a^3}{3b} = \dfrac{b^2}{3} - \dfrac{a^3}{3b}$.

Hence f is continuous everywhere in the domain *i.e.*, $[0, \infty)$

$$\text{Now } f'(x) = \begin{cases} 0, & 0 < x < a \\ -\dfrac{x}{3} + \dfrac{a^3}{3x^2}, & a < x < b \\ \dfrac{(a^3 - b^3)}{3x^2}, & x > b \end{cases}$$

$$f'(a-) = 0, f'(a+) = 0, f'(b-) = -\frac{b}{3} + \frac{a^3}{3b^2}, f'(b+) = \frac{a^3 - b^3}{3b^2}$$

Thus f is differentiable at 'a' and 'b'. Hence

$$f'(x) = \begin{cases} 0, & 0 < x \le a \\ -\dfrac{x}{3} + \dfrac{a^3}{3x^2}, & a < x \le b \\ \dfrac{a^3 - b^3}{3x^2}, & x > b \end{cases}$$

Thus f' is continuous every where in $(0, \infty)$.

$$\text{Again } f''(x) = \begin{cases} 0, & 0 < x < a \\ -\dfrac{1}{3} - \dfrac{2a^3}{3x^3}, & a < x < b \\ \dfrac{2(b^3 - a^3)}{3x^3}, & x > b \end{cases}$$

$$f''(a-) = 0,\ f''(a+) = \frac{-1}{3} - \frac{2a^3}{3a^3} \ne f''(a-)$$

$$f''(b-) = -\frac{1}{3} - \frac{2a^3}{3b^3},$$

$$f''(b+) = \frac{2(b^3 - a^3)}{3b^3} = \frac{2}{3} - \frac{2a^3}{3b^3} \ne f''(b-)$$

Hence $f''(a)$ and $f''(b)$ does not exist

Hence $f''(x)$ is continuous everywhere in $(0, \infty)$ except at 'a' and 'b'.

8. $f'(x) = f'(xy - \sqrt{1-x^2}\sqrt{1-y^2}) \left[y - \dfrac{\sqrt{1-y^2}(-2x)}{2\sqrt{1-x^2}} \right]$

Put $x = 0$

$$f'(0) = f'(-\sqrt{1-y^2})(y) \Rightarrow f'(-\sqrt{1-y^2}) = \frac{f'(0)}{y}$$

Put $-\sqrt{1-y^2} = t \Rightarrow y = \sqrt{1-t^2}$

$$\Rightarrow f'(t) = \frac{f'(0)}{\sqrt{1-t^2}}$$

$$\Rightarrow f(t) = -f'(0)\cos^{-1}t + c \quad \text{...(1)}$$

Put $t = 0$ in (1), we get

$$\frac{\pi}{2} = -f'(0)\frac{\pi}{2} + c \quad \text{...(2)}$$

Put $t = \dfrac{1}{\sqrt{2}}$ in (1), we get

$$\frac{\pi}{4} = -f'(0)\frac{\pi}{4} + c \quad \text{...(3)}$$

from (2) & (3) we get $f'(0) = -1$ and $c = 0$

$\therefore\ f(x) = \cos^{-1}x$.

9. $f(x) = [x^2 [1/x^2]]$. Obviously f is an even function. Hence it suffices to discuss the continuity and differentiability of f for $x > 0$ (f is not defined for $x = 0$). If $x^2 > 1$, then $[1/x^2] = 0$ and hence $f(x) = 0$, which is everywhere continuous and differentiable.

Further for $0 < x \le 1,\ \dfrac{1}{x^2} \ge 1$

If $n < \dfrac{1}{x^2} < n+1$, for some $n \in N$, then $[1/x^2] = n$ and hence for such x, $f(x) = [x^2 \cdot n]$

But

$$\frac{1}{n+1} < x^2 < \frac{1}{n} \Rightarrow \frac{n}{n+1} < x^2 n < 1$$

$\Rightarrow [x^2 n] = 0 \Rightarrow f(x) = 0$, which is everywhere continuous and differentiable.

If $\dfrac{1}{x^2} = n$, for some $n \in N$, then $f(x) = \left[\dfrac{1}{n}n\right] = 1$

Hence for $x > 0$ and

$$n \in N,\ f(x) = \begin{cases} 1, & \text{if } x^2 = \dfrac{1}{n} \\ 0, & \text{otherwise} \end{cases}$$

Thus f is not continuous at $x = \pm\dfrac{1}{\sqrt{n}}$, $n \in N$.

Hence f is not differentiable at $x = \pm\dfrac{1}{\sqrt{n}}$. Further for $x \ne 0$, $x \ne \pm\dfrac{1}{\sqrt{n}}$, f is a constant function, therefore f is continuous and differentiable.

10. (i) If f is an even function, then $f(-x) = f(x)$

$$\Rightarrow 2\left[\frac{x^2}{a}\right]\tan ax = 0 \Rightarrow \left[\frac{x^2}{a}\right] = 0,\ \forall x \in [-3, 3]$$

$$\Rightarrow 0 \le \frac{x^2}{a} < 1,\ \forall x \in [-3, 3]$$

$\Rightarrow 0 \le x^2 < a$ (as $a > 0$) $\forall x \in [-3, 3] \Rightarrow a > 3^2$

(ii) If f is an odd function, then

$f(-x) = -f(x)$

$\Rightarrow 2\sec ax = 0$, $\forall x \in [-3, 3]$,

which is not possible for any a.

MATHS MUSING

SOLUTION SET - 82

1. (a) : The number of all 5-element subsets is $\binom{14}{5}$

The number of 5-element subsets so that no two numbers are consecutive is $\binom{14-5+1}{5}=\binom{10}{5}$

$\therefore\quad m=\binom{14}{5}-\binom{10}{5}=2002-252=1750$

2. (c) : Area of $\Delta=\frac{1}{2}\left|\begin{vmatrix} x_1 & y_1 & 1\\ x_2 & y_2 & 1\\ x_3 & y_3 & 1\end{vmatrix}\right|$

$=\frac{1}{2}\left|\begin{vmatrix} m & n & 1\\ 12 & 19 & 1\\ 23 & 20 & 1\end{vmatrix}\right|=70\Rightarrow -m+11n=197\pm 140 \quad \ldots(1)$

Median through A has slope -5

$\Rightarrow 5m+n=107 \quad \ldots(2)$

(1), (2) $\Rightarrow m=20,\ n=7,\ m+n=27$

3. (b) : $N=\sum\limits_{n=1}^{25}((4n)^2+(4n-1)^2-(4n-2)^2-(4n-3)^2)$

$=\sum\limits_{n=1}^{25}((4n)^2-(4n-2)^2+(4n-1)^2-(4n-3)^2)$

$=\sum\limits_{n=1}^{25}(2(8n-2)+2(8n-4))=4\sum\limits_{n=1}^{25}(8n-3)$

$=10400-300=10100$

4. (a) : $\frac{1}{x}=13+(-1)^{1/10}=13+\text{cis}\,(2r+1)\frac{\pi}{10}$

$\frac{1}{z_1}=13+\text{cis}\frac{\pi}{10},\ \frac{1}{\bar{z}_1}=13+\text{cis}\left(-\frac{\pi}{10}\right)$

$\therefore\quad \frac{1}{|z_1|^2}=170+26\cos\frac{\pi}{10}$

$N=\sum\limits_{i=1}^{5}\frac{1}{|z_i|^2}$

$=\sum\limits_{i=1}^{5}(170+26\cos(2r+1)\frac{\pi}{10})$

$=850+26\left(\cos\frac{\pi}{10}+\cos\frac{3\pi}{10}+\cos\frac{5\pi}{10}+\cos\frac{7\pi}{10}+\cos\frac{9\pi}{10}\right)$

$=850$

5. (a, d) :

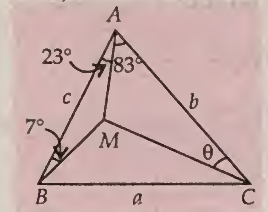
$\frac{MA}{c}=\frac{\sin 7^\circ}{\sin 150^\circ}=\frac{\sin\theta}{\sin(\theta+83^\circ)}$

($\because b=c$ and apply sine rule in ΔABM & ΔACM)

$\therefore\quad 2\sin 7^\circ\sin(\theta+83^\circ)=\sin\theta$

$\cos(\theta+76^\circ)-\cos(\theta+90^\circ)=\sin\theta$

$\therefore\quad \cos(\theta+76^\circ)=0\Rightarrow\theta=90^\circ-76^\circ=14^\circ$



6. (c) : $P(\sqrt{2}\cos\theta,\sin\theta),\ S(1,0),\ S_1(-1,0)$

$\Rightarrow\ G=\left(\frac{\sqrt{2}}{3}\cos\theta,\frac{\sin\theta}{3}\right)$

Locus of G is $\frac{9x^2}{2}+9y^2=1$

Latus-rectum $=\frac{2}{9}\times\frac{3}{\sqrt{2}}=\frac{\sqrt{2}}{3}$

7. (c) : $I=(\cos\theta,(\sqrt{2}-1)\sin\theta)$

Locus of I is $x^2+\frac{y^2}{(\sqrt{2}-1)^2}=1$

Latus-rectum $=2(\sqrt{2}-1)^2=6-4\sqrt{2}$

8. (a) : $S=\left(0,\frac{\cos^2\theta}{\sin\theta}\right)$ and $O(x,y)$ divides GS in the ratio $-2:3$.

$\therefore\quad x=\sqrt{2}\cos\theta,\ y=-\frac{\cos 2\theta}{\sin\theta}$

Locus of O is $y^2=\frac{2(1-x^2)^2}{2-x^2}$

9. (a) – (q); (b) – (p); (c) – (q); (d) – (t)

(a) $\int\limits_0^\infty\frac{e^x dx}{e^{2x}+1}=\tan^{-1}e^x\Big|_0^\infty=\frac{\pi}{2}-\frac{\pi}{4}=\frac{\pi}{4}$

(b) $e^x=1+t^2\Rightarrow I=2\int\limits_1^{\sqrt{3}}\frac{dt}{t^2+1}=2\tan^{-1}t\Big|_1^{\sqrt{3}}=\frac{\pi}{6}$

(c) $\tan x=t\ \Rightarrow I=\int\limits_0^\infty\frac{dt}{1+4t^2}=\frac{1}{2}\tan^{-1}2t\Big|_0^\infty=\frac{\pi}{4}$

(d) $\frac{\cos x}{x}=t\ \Rightarrow dt=-(\cos x+x\sin x)\frac{dx}{x^2}$

$I=\int\limits_0^\infty\frac{dt}{1+t^2}=\tan^{-1}t\Big|_0^\infty=\frac{\pi}{2}$

10. P traces the curve $x^3+(1+x)y^2=0$

Area $=2\int\limits_{-1}^{0}\sqrt{\frac{-x^3}{1+x}}dx=\frac{3\pi}{4}=\frac{m\pi}{n},m+n=7$

Math Archives

10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers comments and suggestions regarding the problems and solutions offered are always welcome.

1. Let $f(x)=\dfrac{3\sin x+2\cos x}{\sin x+\cos x}$. Show that f is increasing for all x in its domain. Discuss the reason of $f(\pi)$ being less than $f\left(\dfrac{\pi}{2}\right)$.

2. Find the maximum and minimum values of $\sqrt{3}\cos\dfrac{x}{2}+\sin\dfrac{x}{2}-\dfrac{x-3}{2}$, $x\in[-4\pi,0]$.

3. If $f(x) = (4\sin^2 x-1)^n (x^2-x+1)$, $n \in N$. Explain when $x=\dfrac{\pi}{6}$ can become a point of minima ?

4. Find the value of 'a' for which the function $f(x) = (4a-3)(x+\log 5)+2(a-7)\cot\dfrac{x}{2}\sin^2\dfrac{x}{2}$ does not possess critical points ?

5. Find the set of values of 'b' for which local extreme of the function $f(x)$ are positive where $f(x)=\dfrac{2}{3}a^2x^3-\dfrac{5a}{2}x^2+3x+b$ and maximum occurs at $x=\dfrac{1}{3}$.

6. Show that $\dfrac{\tan x}{x}>\dfrac{x}{\sin x}$ for $0<x<\dfrac{\pi}{2}$

7. If $\dfrac{v-u}{1+v^2}<\tan^{-1}v-\tan^{-1}u<\dfrac{v-u}{1+u^2}, 0<u<v$ then deduce that $\dfrac{\pi}{4}+\dfrac{3}{25}<\tan^{-1}\dfrac{4}{3}<\dfrac{\pi}{4}+\dfrac{1}{6}$.

8. Verify Rolle's theorem for $f(x) = x^2(x-1)^2$ in the interval $0 \le x < 1$.

9. Prove that the tangent to the curve $y=\dfrac{1+3x^2}{3+x^2}$ drawn at the point for which $y = 1$ intersect at the origin.

10. Find the interval in which $f(x)=e^{|x^2-6x+8|}$ decreases.

SOLUTIONS

1. $f(x)=\dfrac{3\sin x+2\cos x}{\sin x+\cos x}$

$f'(x)=\dfrac{1}{(\sin x+\cos x)^2}>0$ in its domain.

$\Rightarrow f(x)$ is increasing in its domain.

Now, $\lim\limits_{h\to 0} f\left(\dfrac{3\pi}{4}+h\right)=-\infty$,

whereas $\lim\limits_{h\to 0}\left(\dfrac{3\pi}{4}-h\right)=\infty$

It is obvious that $f\left(\dfrac{\pi}{2}\right)>f(\pi)$. The reason being function 'f' is discontinuous at $\dfrac{3\pi}{4}$.

2. Let $f(x)=\sqrt{3}\cos\dfrac{x}{2}+\sin\dfrac{x}{2}-\dfrac{(x-3)}{2}$

$f'(x)=\dfrac{-\sqrt{3}}{2}\sin\dfrac{x}{2}+\dfrac{1}{2}\cos\dfrac{x}{2}-\dfrac{1}{2}=0.$

For critical points,

$\Rightarrow \cos\left(\dfrac{x}{2}+\dfrac{\pi}{3}\right)=\dfrac{1}{2} \quad \Rightarrow \dfrac{x}{2}+\dfrac{\pi}{3}=2n\pi\pm\dfrac{\pi}{3}$

$\Rightarrow x=-\dfrac{4\pi}{3}, x\in\left(-\dfrac{4\pi}{3},0\right)$

Thus, the only points, where maxima/minima can occur are $-4\pi, -\dfrac{4\pi}{3}, 0$.

Since, f is continuous function, hence it obtain its maxima and minima.

We have $f(-4\pi)=\sqrt{3}+2\pi+\dfrac{3}{2}$.

$f\left(-\dfrac{4\pi}{3}\right)=-\sqrt{3}\left(\dfrac{1}{2}\right)-\dfrac{\sqrt{3}}{2}+\dfrac{2\pi}{3}+\dfrac{3}{2}=-\sqrt{3}+\dfrac{2\pi}{3}+\dfrac{3}{2}$

$f(0)=\sqrt{3}+\dfrac{3}{2}$

$\therefore$ Maximum of $f(x)=\sqrt{3}+\dfrac{3}{2}$

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Minimum $f(x) = -\sqrt{3} + \frac{2\pi}{3} + \frac{3}{2}$

3. $f(x) = (4\sin^2 x - 1)^n (x^2 - x + 1)$
$f'(x) = n(4\sin^2 x - 1)^{n-1} (8\sin x \cos x)(x^2 - x + 1)$
$+ (4\sin^2 x - 1)^n (2x - 1)$
$= (4\sin^2 x - 1)^{n-1} [n(x^2 - x + 1)\, 4(\sin 2x) + (2x - 1)$
$(4\sin^2 x - 1)]$
If f has local minimum at $x = \frac{\pi}{6}$, then $f'\left(\frac{\pi}{6}\right) = 0$
and $f'\left(\frac{\pi^-}{6}\right) < 0$ and $f'\left(\frac{\pi^+}{6}\right) > 0$, which is possible only if $(n - 1)$ is odd *i.e.* 'n' is even.

4. $f'(x) = (4a - 3) + (a - 7)\cos x$
For non-existence of critical points, $f'(x) \neq 0$ for any $x \in R$.
$\Rightarrow \quad \cos x \neq \frac{3-4a}{a-7}$ for any x.
$\Rightarrow \left|\frac{3-4a}{a-7}\right| > 1 \Rightarrow |4a-3| - |a-7| > 0$ ..... (i)
(i) If $a \geq 7 \Rightarrow$ from (i) we get $a > -\frac{4}{3}$
$\Rightarrow a \geq 7$
(ii) If $\frac{3}{4} \leq a < 7 \Rightarrow 4a - 3 + a - 7 > 0$
$\Rightarrow a > 2 \Rightarrow 2 < a < 7$
(iii) If $a < \frac{3}{4} \Rightarrow 3 - 4a + a - 7 > 0$
$\Rightarrow a < -\frac{4}{3}$ Hence, $a \in \left(-\infty, -\frac{4}{3}\right) \cup (2, \infty)$.

5. $f'(x) = 2a^2 x^2 - 5ax + 3 = (ax - 1)(2ax - 3) = 0$
$\Rightarrow \quad x = \frac{1}{a}, \frac{3}{2a}$. If $a > 0$ then local maxima occurs at $x = \frac{1}{a}$ and minima at $x = \frac{3}{2a}$.
$\because$ maxima occurred at $x = \frac{1}{a} = \frac{1}{3} \Rightarrow a = 3$
minima occurred at $x = \frac{3}{2a} = \frac{1}{2}$
If $a < 0$ then maximum shall occur at $x = \frac{3}{2a}$ and minima at $x = \frac{1}{a}$
$\Rightarrow \quad \frac{3}{2a} = \frac{1}{3} \Rightarrow a = \frac{9}{2} > 0$ not possible.
Hence, $b > -\frac{3}{8}$.

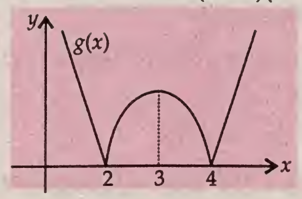
6. We have to show that $\frac{\tan x \sin x - x^2}{x \sin x} > 0$ for $0 < x < \frac{\pi}{2}$. Since, $x \sin x > 0$ for $0 < x < \frac{\pi}{2}$, it is enough to show that $\tan x \sin x - x^2 > 0$, $0 < x < \frac{\pi}{2}$
Let $f(x) = \tan x \sin x - x^2$, $0 < x < \frac{\pi}{2}$
$f'(x) = \sin x \sec^2 x + \sin x - 2x$
$f''(x) = \sec x + \cos x - 2 + 2\sin x \tan x \sec^2 x$
$= (\sqrt{\sec x} - \sqrt{\cos x})^2 + 2\sin x \cos x \sec^2 x > 0$ for $0 < x < \frac{\pi}{2}$
$\therefore f'(x)$ is increasing, also $f'(0) = 0$
$\Rightarrow f'(x) > 0$ for $0 < x < \frac{\pi}{2}$
$\Rightarrow f(x)$ is increasing in $0 < x < \frac{\pi}{2}$ and $f(0) = 0$
$\Rightarrow \tan x \sin x - x^2 > 0 \Rightarrow \frac{\tan x \sin x - x^2}{x \sin x} > 0$
$\Rightarrow \quad \frac{\tan x}{x} > \frac{x}{\sin x}, 0 < x < \frac{\pi}{2}$

7. Let $v = \frac{4}{3}, u = 1$ $\therefore \frac{3}{25} < \tan^{-1}\frac{4}{3} - \tan^{-1} 1 < \frac{1}{6}$
$\Rightarrow \quad \frac{3}{25} + \frac{\pi}{4} < \tan^{-1}\frac{4}{3} < \frac{1}{6} + \frac{\pi}{4}$.

8. Function $f(x) = x^2(x - 1)^2$ satisfy three conditions:
(I) It is continuous in $[0, 1]$
(II) It is differentiable in $(0, 1)$
(III) $f(0) = f(1)$
$\therefore$ By Rolle's theorem, $f'(0) = 0 \Rightarrow c^2(c - 1)(5c - 3) = 0$
$\Rightarrow c = 0, 1, \frac{3}{5}$ $\therefore \quad c = \frac{3}{5} \in (0, 1)$

9. Given $y = \frac{1+3x^2}{3+x^2} \quad \Rightarrow \quad \frac{dy}{dx} = \frac{16x}{(3+x^2)^2}$
when $y = 1$, $x = \pm 1$
$\frac{dy}{dx} = 1$ at $(1, 1)$ and $\frac{dy}{dx} = 1$ at $(-1, 1)$
Equation of tangent at $(1, 1)$ is $x - y = 0$
Equation of tangent at $(-1, 1)$ is $x + y = 0$
$\therefore$ Both intersect at $(0, 0)$.

10. Let $f(x) = e^{|x^2 - 6x + 8|}$
Let $g(x) = |x^2 - 6x + 8| = |(x - 2)(x - 4)|$



As e^x is an increasing function so if arguments of 'x' is increasing e^x will be increasing. So $f(x)$ will be increasing if $g(x)$ increasing and $f(x)$ will decrease when $g(x)$ decrease. From graph it is clear that $g(x)$ is decreasing in $(-\infty, 2) \cup (3, 4)$.

OLYMPIAD CORNER

Challenging problems for Olympiads, IIT-JEE and other contests.

1. How many six digit perfect squares are there each having the property that if each digit is increased by one, the resulting number is also a perfect square ?

2. Determine the radius r of a circle inscribed in a given quadrilateral if the lengths of successive tangents from the vertices of the quadrilateral to the circle are a, a, b, b, c, c, d, d respectively.

3. A hexagon is inscribed in a circle with radius r. Two of its sides have length 1, two have length 2 and the last two have length 3. Prove that r is a root of the equation $2r^3 - 7r - 3 = 0$.

4. Find the equation of the locus of a point P which moves so that the tangents from P to the circle $x^2 + y^2 = r^2$ cut off a line segment of length $2r$ on the line $x = r$.

5. For $x_1, x_2, \ldots, x_n > 0$, show that

$$\frac{x_1^2}{x_1+x_2}+\frac{x_2^2}{x_2+x_3}+\ldots+\frac{x_n^2}{x_n+x_1} \geq \frac{x_1+x_2+\ldots+x_n}{2}$$

6. Find the largest constant k such that

$$\frac{kabc}{a+b+c} \leq (a+b)^2+(a+b+4c)^2 \text{ for all } a, b, c > 0$$

7. On each of three cards was written a whole number from 1 to 10. These cards were shuffled and dealt to three people who recorded the numbers on their respective cards. The cards were collected, and the process was repeated again. After a few times, the three people computed the totals of their numbers. They turned out to be 13, 15 and 23. What were the numbers on the cards ?

8. Let a, b and c be positive real numbers. Prove that $a^a b^b c^c \geq (abc)^{(a+b+c)/3}$

9. Prove or disprove that

$$\sqrt{5+\sqrt{21}}+\sqrt{8+\sqrt{55}}=\sqrt{7+\sqrt{33}}+\sqrt{6+\sqrt{35}}.$$

10. Find all the whole numbers between 1 and 100 which can be written as a sum of integers constructed by using each of the digits 0 through 9 exactly once. (Example : $90 = 0 + 1 + 52 + 3 + 4 + 6 + 7 + 8 + 9$ is one such number.)

SOLUTIONS

1. If the six digit square is given by

$m^2 = a\cdot10^5 + b\cdot10^4 + c\cdot10^3 + d\cdot10^2 + e\cdot10 + f$, then

$n^2 = (a+1)\cdot10^5 + (b+1)\cdot10^4 + (c+1)\cdot10^3 + (d+1)\cdot10^2 + (e+1)\cdot10 + (f+1)$,

so that

$n^2 - m^2 = 111{,}111 = (111)(1{,}001) = (3\cdot37)(7\cdot11\cdot13)$

Hence,

$$n + m = d_i \text{ and } n - m = \frac{111{,}111}{d_i}$$

where d_i is one of the divisors of 111,111. Since 111,111 is a product of five primes it has 32 different divisors.

But since we must have $d_i > \frac{111{,}111}{d_i}$, there are atmost 16 solutions given by the form

$$lm = \frac{1}{2}\left(d_i - \frac{111{,}111}{d_i}\right)$$

Then since m^2 is a six digit number, we must have

$$632.46 \approx 200\sqrt{10} < 2m < 2{,}000$$

On checking the various divisors, there are four solutions. One of them corresponds to $d_i = 3\cdot13\cdot37 = 1,443$ so that

$$m = \frac{1}{2}(1,443 - 7\cdot11) = 683 \text{ and } m^2 = 466{,}489.$$

Then $466{,}489 + 111{,}111 = 577{,}600 = 760^2$.

The others are given by the table :

d_i	m	m^2	n^2	n
$3\cdot7\cdot37 = 777$	317	100,489	211,600	460
$3\cdot11\cdot37 = 1,221$	565	319,225	430,336	656
$7\cdot11\cdot13 = 1,001$	445	198,025	309,136	556

2. Let $2A$, $2B$, $2C$, $2D$ denote the angles between successive pairs of radii vectors to the points of tangency and let r be the inradius. Then

$$r = \frac{a}{\tan A} = \frac{b}{\tan B} = \frac{c}{\tan C} = \frac{d}{\tan D}$$

Also, since $A + B + C + D = \pi$,

we have $\tan(A + B) = \tan(C + D) = 0$

or, $$\frac{\tan A + \tan B}{1 - \tan A \tan B} + \frac{\tan C + \tan D}{1 - \tan C \tan D} = 0,$$

so that

$$\frac{r(a+b)}{r^2-ab}+\frac{r(c+d)}{r^2-cd}=0$$

Finally,

$$r^2=\frac{abc+bcd+cda+dab}{a+b+c+d}$$

3. Equal chords subtend equal angles at the centre of a circle; if each of sides of length i subtends an angle $\alpha_i (i = 1, 2, 3)$ at the centre of the given circle, then

$$2\alpha_1 + 2\alpha_2 + 2\alpha_3 = 360°,$$

whence

$$\frac{\alpha_1}{2}+\frac{\alpha_2}{2}=90°-\frac{\alpha_3}{2},$$

and $\cos\left(\frac{\alpha_1}{2}+\frac{\alpha_2}{2}\right)=\cos\left(90°-\frac{\alpha_3}{2}\right)=\sin\frac{\alpha_3}{2}$

Next we apply the addition formula for the cosine :

$$\cos\frac{\alpha_1}{2}\cos\frac{\alpha_2}{2}-\sin\frac{\alpha_1}{2}\sin\frac{\alpha_2}{2}=\sin\frac{\alpha_3}{2} \quad \text{...... (i)}$$

where, $\sin\left(\frac{\alpha_1}{2}\right)=\frac{(1/2)}{r}$, $\cos\left(\frac{\alpha_1}{2}\right)$

$$=\frac{\sqrt{4r^2-1}}{2r} \text{ From [Fig 1];}$$

$$\sin\left(\frac{\alpha_2}{2}\right)=\frac{1}{r},\ \cos\left(\frac{\alpha_2}{2}\right)=\frac{\sqrt{r^2-1}}{r} \text{ From [Fig. 2];}$$

$$\sin\left(\frac{\alpha_3}{2}\right)=\frac{3/2}{r}, \text{ From [Fig. 3]}$$

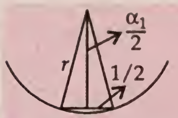


Fig. 1

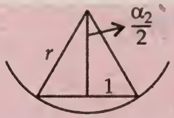


Fig. 2

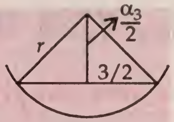


Fig. 3

We substitute these expressions into (i) and obtain, after multiplying both sides by $2r^2$,

$$\sqrt{4r^2-1}\cdot\sqrt{r^2-1}-1=3r$$

Now write it in the form

$$\sqrt{(4r^2-1)(r^2-1)}=3r+1,$$

Squaring both sides, we get

$$(4r^2-1)(r^2-1) = 9r^2 + 6r + 1,$$

which is equivalent to

$$r(2r^3 - 7r - 3) = 0$$

Since $r \neq 0$, we have $2r^3 - 7r - 3 = 0$.

4. From the circle, $x^2 + y^2 = a^2$, we find the points of intersection with the line $(y - q) = m(x - p)$.

Substituting $y = mx - mp + q$, we get $x^2 + (mx - mp + q)^2 = a^2$, leading to the two solutions :

$$x=\frac{m(mp-q)\pm\sqrt{a^2(m^2+1)-m^2p^2+q(2mp-q)}}{m^2+1}$$

For the line to be a tangent, the discriminant, $\Delta = a^2(m^2 + 1) - m^2p^2 + q(2mp - q)$, must be zero.

So, we solve $\Delta = 0$, to get

$$m=\frac{pq}{p^2-a^2}\pm\frac{a\sqrt{p^2+q^2-a^2}}{p^2-a^2}$$

So, we have that the equations of the tangent lines from (p, q) to the circle $x^2 + y^2 = a^2$ are

$$y=\left(-\frac{pq}{a^2-p^2}\pm\frac{a\sqrt{a^2-p^2-q^2}}{p^2-a^2}\right)x$$

$$-\left(-\frac{pq}{a^2-p^2}\pm\frac{\sqrt{a^2-p^2-q^2}}{p^2-a^2}\right)p+q$$

Setting $x = a$ in these, and subtracting the two values leads to $(a + p)^2 = p^2 + q^2 - a^2$

Thus the locus of P is given by $y^2 = 2a(a + x)$

This is a parabola opening to the right with central axis, the x-axis and nose at $x = -a$. There are, of course, three points that should be excluded :

$(-a, 0)$, $(a, 2a)$, and $(a, -2a)$.

5. By CSB

$$[(x_1 + x_2) + (x_2 + x_3) + ... + (x_n + x_1)]$$

$$\times\left[\frac{x_1^2}{x_1+x_2}+\frac{x_2^2}{x_2+x_3}+...+\frac{x_n^2}{x_n+x_1}\right]$$

$$\geq (x_1 + x_2 + ... + x_n)^2$$

The result then follows by dividing each side by $2(x_1 + x_2 + ... + x_n)$.

6. By the A.M. – G.M. inequality,

$$(a+b)^2+(a+b+4c)^2=(a+b)^2+(a+2c+b+2c)^2$$

$$\geq(2\sqrt{ab})^2+(2\sqrt{2ac}+2\sqrt{2bc})^2$$

$$=4ab+8ac+8bc+16c\sqrt{ab}$$

Therefore

$$\frac{(a+b)^2+(a+b+4c)^2}{abc}\cdot(a+b+c)$$

$$\geq\frac{4ab+8ac+8bc+16c\sqrt{ab}}{abc}\cdot(a+b+c)$$

$$=\left(\frac{4}{c}+\frac{8}{b}+\frac{8}{a}+\frac{16}{\sqrt{ab}}\right)(a+b+c)$$

$$=8\left(\frac{1}{2c}+\frac{1}{b}+\frac{1}{a}+\frac{1}{\sqrt{ab}}+\frac{1}{\sqrt{ab}}\right)\left(\frac{a}{2}+\frac{a}{2}+\frac{b}{2}+\frac{b}{2}+c\right)$$

$$\geq 8\left(5\sqrt[5]{\frac{1}{2a^2b^2c}}\right)\left(5\sqrt[5]{\frac{a^2b^2c}{2^4}}\right)=100,$$

again by the A.M. – G.M. inequality. Hence, the largest constant k is 100. For $k = 100$, equality holds if and only if $a = b = 2c > 0$.

7. Now $13 + 15 + 23 = 51 = 3\cdot 17$ so each person received three cards. Consider the triple $(x, y, z) x \leq y \leq z$ with $x + y + z = 17$. Now the total 23

shows $z \geq 8$. If $z = 10$, then the third 10 can be used for atmost one of 13, 15 whence $y \geq 5$ and $x = 3$. But then one player has atleast $10 + 3 + 3 = 16$, a contradiction. If $z = 8$ then $y = 7$ (for $8 + 8 + 7 = 23$), but the triple (2, 7, 8) cannot produce a sum of 13 or 15. This shows $z = 9$. If $x > 3$ someone must have a total of atleast $9 + 4 + 4 = 17$. This shows $x \leq 3$. Thus, $9 + 9 + 5 = 23$ makes $y = 5$ and $x = 3$, and the cards were distributed $(3 + 5 + 5 = 13)$, $(3 + 3 + 9 = 15)$ and $(5 + 9 + 9 = 23)$.

8. We prove equivalently that $a^{3a}b^{3b}c^{3c} \geq (abc)^{a+b+c}$. Due to complete symmetry in a, b, and c, we may assume, without loss of generality, that $a \geq b \geq c > 0$. Then $(a - b) \geq 0$, $(b - c) \geq 0$, $(a - c) \geq 0$ and

$\frac{a}{b} \geq 1, \frac{b}{c} \geq 1, \frac{a}{c} \geq 1.$

Therefore,

$$\frac{a^{3b}b^{3b}c^{3c}}{(abc)^{a+b+c}} = \left(\frac{a}{b}\right)^{1-b}\left(\frac{b}{c}\right)^{b-c}\left(\frac{a}{c}\right)^{1-c} \geq 1$$

9. In order to simplify the radicals, the radicands should be forced to equal square numbers (*e.g.*, $7 + \sqrt{33}$ should be a square of some number). Numbers whose squares have a rational and radical part are usually in the form $a + b$.

So let

$$\sqrt{7+\sqrt{33}} = a + b = \sqrt{(a+b)^2} = \sqrt{a^2 + b^2 + 2ab},$$

and set

$$a^2 + b^2 = 7 \text{ and } 2ab = \sqrt{33}, \text{ i.e. } b = \frac{\sqrt{33}}{2a}$$

Thus,

$$a^2 + \left(\frac{\sqrt{33}}{2a}\right)^2 = 7$$

which multiplying by $4a^2$ gives

$$(2a^2 - 3)(2a^2 - 11) = 4a^4 + 33 - 28a^2 = 0$$

So $2a^2 = 3$ or $2a^2 = 11$, *i.e.*

$$a = \sqrt{\frac{3}{2}} = \frac{\sqrt{6}}{2}, b = \frac{\sqrt{33}}{\sqrt{6}} = \frac{\sqrt{22}}{2}$$

or $$a = \sqrt{\frac{11}{2}} = \frac{\sqrt{22}}{2}, b = \frac{\sqrt{6}}{2},$$

and so $$\sqrt{7+\sqrt{33}} = a + b = \frac{\sqrt{6}+\sqrt{22}}{2}$$

Using the same process for the other radicals, we get

$$\sqrt{6+\sqrt{35}} = \frac{\sqrt{10}+\sqrt{14}}{2},$$

so $$\sqrt{7+\sqrt{33}} + \sqrt{6+\sqrt{35}} = \frac{\sqrt{6}+\sqrt{22}+\sqrt{10}+\sqrt{14}}{2}; \quad ..(i)$$

and $$\sqrt{8+\sqrt{55}} = \frac{\sqrt{10}+\sqrt{22}}{2}, \sqrt{5+\sqrt{21}} = \frac{\sqrt{6}+\sqrt{14}}{2},$$

so $$\sqrt{8+\sqrt{55}} + \sqrt{5+\sqrt{21}} = \frac{\sqrt{10}+\sqrt{22}+\sqrt{6}+\sqrt{14}}{2} \quad ..(ii)$$

From (1) and (2); we get

$$\sqrt{7+\sqrt{33}} + \sqrt{6+\sqrt{35}} = \sqrt{8+\sqrt{55}} + \sqrt{5+\sqrt{21}}.$$

10. Of course the smallest number that can be constructed this way is

$$0 + 1 + 2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 45$$

Suppose we put two of these digits together to form a two-digit number, say we use the digits a and b to form the two-digit number "ab". Since "ab" is really the number $10a + b$, in putting a and b together to form "ab" we are adding $9a$ to the sum. So, every time we stick two digits together we add a multiple of 9 to the sum. This says that the only possible sums that we can form are those which arise by adding a multiple of 9 to the sum 45, namely :

54, 63, 72, 81, 90, 99 as well as 45 of course. These sums are all possible : For example,

$54 = 0 + 12 + 3 + 4 + 5 + 6 + 7 + 8 + 9,$
$63 = 0 + 1 + 23 + 4 + 5 + 6 + 7 + 8 + 9,$
$72 = 0 + 1 + 2 + 34 + 5 + 6 + 7 + 8 + 9,$
$81 = 0 + 1 + 2 + 3 + 45 + 6 + 7 + 8 + 9,$
$90 = 0 + 1 + 2 + 3 + 4 + 56 + 7 + 8 + 9,$
$99 = 0 + 1 + 2 + 3 + 4 + 5 + 67 + 8 + 9.$

Solution Sender of Maths Musing

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4.	Joy Banerjee	–	Howrah (W.B.)

SET - 83

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3.	Joy Banerjee	–	Howrah, (W.B.)
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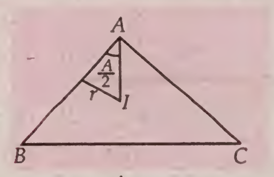
Regional Mathematical Olympiad 2009

* **ALOK KUMAR, B.Tech, IIT Kanpur**

1. Let ABC be a triangle in which $AB = AC$ and let I be its in-centre. Suppose $BC = AB + AI$. Find $\angle BAC$.
2. Show that there is no integer a such that $a^2 - 3a - 19$ is divisible by 289.
3. Show that $3^{2008} + 4^{2009}$ can be written as product of two positive integers each of which is larger than 2009^{182}.
4. Find the sum of all 3-digit natural numbers which contain atleast one odd digit and atleast one even digit.
5. A convex polygon Γ is such that the distance between any two vertices of Γ does not exceed 1.
   (i) Prove that the distance between any two points on the boundary of Γ does not exceed 1.
   (ii) If X and Y are two distinct points inside Γ, prove that there exists a point Z on the boundary of Γ such that $XZ + YZ \le 1$.
6. In a book with page numbers from 1 to 100, some pages are torn off. The sum of the numbers on the remaining pages is 4949. How many pages are torn off?

SOLUTIONS

1. **Ist solution :** (Trigonometric approach)



We have $AI = r\,\text{cosec}\dfrac{A}{2}$, where r is the inradius of the triangle ABC.

Given condition reads

$BC = AB + AI$

$\Rightarrow \quad a = c + r\ \text{cosec}\left(\dfrac{A}{2}\right)$

Using extended law of sines $a = 2R\sin A$, $c = 2R\sin C$; R being the circum-radius of the triangle and

$r = 4R\sin\left(\dfrac{A}{2}\right)\ \sin\left(\dfrac{B}{2}\right)\ \sin\left(\dfrac{C}{2}\right)$

We have

$$2R\sin A = 2R\sin C + 4R\left[\left(\sin\left(\frac{A}{2}\right)\ \sin\left(\frac{B}{2}\right)\ \sin\left(\frac{C}{2}\right)\right)\right]\text{cosec}\left(\frac{A}{2}\right)$$

$$\Rightarrow \sin A = \sin C + 2\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$\Rightarrow \sin A - \sin C = 2\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Rightarrow 2\sin\left(\frac{A-C}{2}\right)\cos\left(\frac{A+C}{2}\right) = 2\sin\frac{B}{2}\sin\frac{C}{2}$$

$$\Rightarrow 2\sin\left(\frac{A-C}{2}\right)\sin\left(\frac{B}{2}\right) = 2\sin\left(\frac{B}{2}\right)\sin\left(\frac{C}{2}\right)$$

$$\Rightarrow \sin\frac{B}{2}\left\{\sin\left(\frac{A-C}{2}\right) - \sin\left(\frac{C}{2}\right)\right\} = 0$$

As ABC is a triangle $0 < A, B, C < \pi$,

we have $\sin\left(\dfrac{B}{2}\right) \neq 0$, giving

$$\sin\left(\frac{A-C}{2}\right) = \sin\frac{C}{2}$$

Note that $0 < \dfrac{C}{2} < \dfrac{\pi}{2}$ and also $0 < \dfrac{|A-C|}{2} < \dfrac{\pi}{2}$

we have

$$\left(\frac{A-C}{2}\right) = \frac{C}{2} \quad \Rightarrow A = 2C$$

Triangle is isosceles $\Rightarrow B = C$

Thus $A + B + C = \pi$

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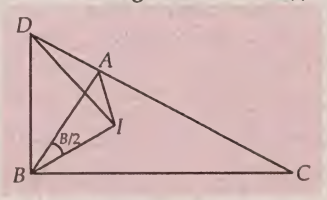
$$\Rightarrow \quad A + \frac{A}{2} + \frac{A}{2} = \pi$$

$$\Rightarrow \quad 2A = \pi \quad \therefore \quad A = \frac{\pi}{2}$$

Thus $\angle BAC = 90°$

2nd solution : (Geometric approach)

Here some ingenuity is needed to transform the hypothesis on length, *i.e.*, $BC = AB + AI$, to construct a segment whose length is $AB + AI$(!)



Note that $\angle AIB = 90° + \frac{C}{2}$

Produce CA to D such that $AD = AI$

Now, $BC = AB + AI$ reads

$BC = AC + AI \quad (AB = AC)$

$\quad = CA + AD = CD$

Now, $\angle BDC = \angle CBD = 90° - \frac{C}{2}$

We have $\angle AIB + \angle BDC = 180°$. Thus the quadrilateral $AIBD$ is cyclic.

Then $\angle API = \angle ABI = \frac{B}{2}$

But ADI is isosceles ($\because \; AD = AI$)

$\angle DAI = 180° - 2\,(\angle ADI) = 180° - B$

Thus $\angle CAI = B$.

But $\angle CAI = \frac{A}{2}$. We then have $A = 2B$

$A + B + C = 180°$

$\Rightarrow \quad 2B + B + 2B = 180° \;\Rightarrow\; B = 45°$, giving

$A = 2B = 90°$, as before.

2. **1st solution :**

We prove : If $a^2 - 3a - 19$ is divisible by 17, then it is not divisible by 289.

As $-17a + 119 \equiv 0 \pmod{17}$

We have

$a^2 - 3a - 19 \equiv a^2 - 20a + 100 \pmod{17}$

$\qquad\qquad \equiv (a - 10)^2 \pmod{17}$

Thus $17 \mid a^2 - 3a - 19$ if $a = 17k + 10$, for some $k \in Z$.

But then we have $a^2 - 3a - 19$

$= (17k + 10)^2 - 3(17k + 10) - 19$

$= 17^2k^2 + 340k + 100 - 51k - 30 - 19$

$= 17^2k^2 + 289k + 51$

$= 289k(k + 1) + 51$

Then $a^2 - 3a - 19$ is not divisible by 289, for 289 dividėd the first, but it doesnot divide the second.

2nd solution :

This is based on a clever way of writing the given expression as a sum of two linear factors in a and a number divisible by 17.

$a^2 - 3a - 19 = (a - 10)(a + 7) + 51$

Let us assume that 289 divides $a^2 - 3a - 19$ for some integer a. Then 17divedes it. Now 17 is a prime and 17 divides $(a - 10)(a + 7)$, so it must divide atleast one of them.

But $(a + 7) - (a - 10) = 17$, a multiple of 17.

Hence, wherever 17 divides one of $(a + 7)$ and $(a - 10)$, it has to divide other also, Then 289 divides $(a - 10)(a + 7)$. But then 289 would be forced to divide 51. Impossible contradiction.

Thus, there is no integer a for which 289 divides $a^2 - 3a - 19$.

3. This is based on Sophie-Germain identity

$x^4 + 4y^4 = (x^2 + 2xy + 2y^2)(x^2 - 2xy + 2y^2)$

Identifying 3^{502} with x and 4^{502} with y we have

$3^{2008} + 4^{2009} = (3^{1004} + 2.3^{502} \cdot 4^{502} + 2.4^{1004})$
$\qquad (3^{1004} - 2.3^{502} \cdot 4^{502} + 2.4^{1004})$

Now $x^2 - 2xy + 2y^2 = (x - y)^2 + y^2 > y^2$

$> 4^{1004} = 2^{2008} > 2^{2002} = (2^{11})^{182} = (2048)^{182}$

$> (2009)^{182}$

The smaller factor is larger than 2009^{182}, the larger factor must also be larger than 2009^{182}.

Hence $3^{2008} + 4^{2009}$ can be written as product of two positive integers each of which is larger than 2009^{182}.

4. Some set -theoretic notation would help to present the arguments better.

Let S denote the set of all 3-digit natural numbers, set A be those in S formed of odd digit only and B be those in S formed of even digits only.

Then $S - (A \cup B)$ is the set of all 3-digit natural numbers having atleast one odd digit and at least one even digit.

The sum then is

$$\sum_{s\in S} S - \sum_{a\in A} A - \sum_{b\in B} B$$

$$\sum_{s\in S} S = 100 + 101 + 999$$

$$= (1 + 2 + ... + 999) - (1 + 2 + ... + 99)$$

$$= \frac{999 \times 1000}{2} - \frac{99 \times 100}{2} = 9 \times 50\{1110 - 11\}$$

$$= 450 \times 1099 = 494550$$

Consider the set A, each number in A has its digits from the set $\{1, 3, 5, 7, 9\}$. Suppose the

digit in unit place is 1. We can fill the digit in ten's place in 5 ways and the digits in hundred's place in 5 ways. Thus there are 25 numbers in A that have 1 in its unit's place. The same holds for each number. Then the sum of the digits in unit's place is

$25(1 + 3 + 5 + 7 + 9) = 25 \times 25 = 625$

The sum of digit in ten's place and hundred's place also 625.

Then $\sum_{a \in A} A = 625(1+10+10^2) = 625 \times 111 = 69375$

Now come to set B, each number in B has its digit's from {0, 2, 4, 6, 8}, but the digit in hundred's place is never zero. The number of numbers having the digit 0 in unit's place is $4 \times 5 = 20$.

(Recall that the corresponding number in case of odd number was 25)

Similarly there are 20 numbers having unit digits 2, 4, 6 or 8.

Sum of digits in unit's place of all the numbers in $B = 20(0 + 2 + 4 + 6 + 8) = 20 \times 20 = 400$.

Sum of digits in ten's place is also same. But in the hundred's place 0 spoils this observation. The number of number having (2, 4, 6 or 8) in the hundred's place is $5 \times 5 = 25$.

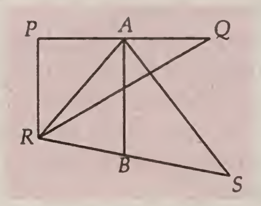
The sum of digits in hundredth place of all the numbers in B is $25(2 + 4 + 6 + 8) = 25 \times 20 = 500$

Thus $\sum_{b \in B} B = 400 + 400 \cdot 10 + 500 \cdot 10^2 = 54400$.

The desired sum is $494550 - 69375 - 54400$

$= 370775$.

5. (i) Consider two points A and B on the boundary of Γ, with A lying on side PQ and B lying on side RS of Γ.



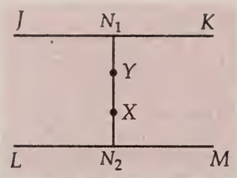
Now AB lies between AR and AS in triangle ARS. One of $\angle ABR$ and $\angle ABS$ is atleast 90°, says $\angle ABR \geq 90°$. Then $AR \geq AB$ ...(i)

But AR lies inside triangle PQR and one of $\angle RAP$ and $\angle RAQ$ is atleast 90°, say $\angle RAQ \geq 90°$. Then $RQ \geq RA$. On combining (i) and (ii)

$AB \leq RA \leq RQ \leq 1$

Thus, $AB \leq 1$.

(ii) Let X and Y be two points in the interior of Γ.



Join XY and extend them on either side to meet the sides JK and LM at N_1 and N_2 respectively.

$(XN_2 + YN_2) + (XN_1 + YN_1)$

$= (XN_1 + XN_2) + (YN_1 + YN_2)$

$= N_1N_1 + N_1N_2$

$= 2N_1N_2 \leq 2$.

by the result established in the first part.

Then one of the sums $XN_1 + YN_2$ and $XN_1 + YN_2$ is atmost one. We may identify Z accordingly as N_1 or N_2.

6. Let the number of pages torn off the book be k. The page number on both sides of a page are of form $2S - 1$ and $2S$ and their sum is

$4S - 1, S \geq 1$.

The sum of numbers on both sides of the pages form a arithmetic sequence

$\{3, 7, 11, ..., 4S - 1, 199\}$ ....(i)

The sum of numbers on all pages in the untorn box $= 1 + 2 + 3 + ... + 100 = 5050$

Hence sum of numbers on the torn pages

$= 5050 - 4949 = 101$.

Some k numbers from the set

$\{3, 7, 11, ... 4S - 1, ..., 199\}$ sum to 101.

We have to find k.

We have $(4S_1 - 1) + (4S_2 - 1) + ... + (4S_k - 1) = 101$

$\Rightarrow 4(S_1 + S_2 + ... + S_k) - k = 101$

Reducing modulo 4 we see $k \equiv 3 \pmod 4$

Put $S_1 + S_2 + ... + S_k = S$

$4S - k = 101$

Now k can be 3, 7, 11, ...

But if $k \geq 7$ we have

$(4S_1 - 1) + (4S_2 - 1) + ... + (4S_k - 1) =$ The sum of numbers on the torn off pages

$\Rightarrow 3 + 7 + 11 + 15 + 19 + 23 + 27$

$= \dfrac{7(3+27)}{2} = 105$

So the only possibility is $k = 3$

Thus $4(S_1 + S_2 + S_3) - 3 = 101$

$\Rightarrow S_1 + S_2 + S_3 = 26$

And we can choose S_1, S_2, S_3 desired positive integers in several ways.

Math Archives 10 Best Problems

Math Archives, as the title itself suggests, is a collection of various challenging problems related to the topics of IIT-JEE Syllabus. This section is basically aimed at providing an extra insight and knowledge to the candidates preparing for IIT-JEE. In every issue of MT, challenging problems are offered with detailed solution. The readers comments and suggestions regarding the problems and solutions offered are always welcome.

1. Discuss the monotonicity of the function g defined by $g(x) = f(x^2 - x - 10) + f(14 + x - x^2)$, $f''(x) > 0$ for all real numbers x except finite number of real numbers x, for which $f''(x) = 0$.

2. Suppose that f and g are non-constant differentiable, real valued functions on R. If for every $x, y \in R$,
$f(x + y) = f(x)f(y) - g(x)g(y)$,
$g(x + y) = g(x)f(y) + f(x)g(y)$ and $f'(0) = 0$, then prove that maximum and minimum value of the function $f^2(x) + g^2(x)$ are same for all $x \in R$.

3. Real valued function $f(x)$ satisfies the relation $f\left(\frac{x+y}{3}\right)=\frac{2f(x)+2f(y)-4}{6}$ $\forall\, x, y \in R$. If $f'(0) = 2$, prove that $f(x)$ is an increasing function for all x.

4. Let $f(x)=-\frac{1}{2}(2\theta^2-4x-2x^2)$, where '$\theta$' is a real parameter. Now let x_1, x_2 be the roots of $f(x)$ where $x_1 < x_2$. If $F(\theta)=\int_{x_1}^{x_2} f(x)\,dx$, find the minimum and the maximum value of $F(\theta)$ and the corresponding θ.

5. The function $f(x)=\sqrt{ax^3+bx^2+cx+d}$ has its non-zero local minimum and maximum values at -2 and 2 respectively. If a is a root of $x^2 - x - 6 = 0$. Find the possible values of a, b, c and d.

6. If $a = -1$, $b \geq 1$ and $f(x)=\frac{1}{|x|}$, show that the conditions of Lagrange's mean value theorem are not satisfied in the interval $[a, b]$, but the conclusion of the theorem is true if and only if $b>1+\sqrt{2}$.

7. If $f(x) = 2x^3 - 15x^2 + 24x$, and $g(x)=\int_0^a f(x)\,dx+\int_0^{5-a} f(x)\,dx, 0<a<5$. Find the interval in which $g(x)$ is increasing.

8. For what value of 'a' the point of local minima of $f(x) = x^3 - 3ax^2 + 3(a^2 - 1)x + 1$ is less than 4 and point of local maxima is greater than -2?

9. The equation $t^2 + 2xt + 4 = 0$ does not possess distinct real roots. Find the equation of the tangent of greatest slope to the curve $y = x^3 - 2x^2 + x$.

10. A point $P(x, y)$ moves on the curve $x^{2/3} + y^{2/3} = a^{2/3}$, $a > 0$. For each position (x, y) of P, perpendiculars are drawn from origin upon the tangent and normal at P, the length (absolute value) of them being $p_1(x)$ and $p_2(x)$ respectively. Prove that $\frac{dp_1}{dx}.\frac{dp_2}{dx}<0$.

SOLUTIONS

1. $f''(x) \geq 0 \Rightarrow f'(x)$ is an increasing function of x.
($f''(x) = 0$ at finitely many values of x does not affect the increasing ness of $f'(x)$)
Now $g'(x) = (2x - 1)[f'(x^2 - x - 10) - f'(14 + x - x^2)]$
Intervals of increase of g :
If $g(x)$ increases then $g'(x) \geq 0$.
$\Rightarrow (2x - 1)$ and $[f'(x^2 - x - 10) - f'(14 + x - x^2)]$ are of same sign.

Case I :
$2x - 1 \geq 0$ and $f'(x^2 - x - 10) - f'(14 + x - x^2) \geq 0$
$\Rightarrow x \geq \frac{1}{2}$ and $x^2 - x - 10 \geq 14 + x - x^2$, as f' is increasing
$\Rightarrow x \geq \frac{1}{2}$ and $x^2 - x - 12 \geq 0 \Rightarrow x \geq 4$

Case II :
$2x - 1 \leq 0$ and $f'(x^2 - x - 10) - f'(14 + x - x^2) \leq 0$
$\Rightarrow x \leq \frac{1}{2}$ and $-3 \leq x \leq 4$
$\Rightarrow -3 \leq x \leq \frac{1}{2}$
Hence, $g(x)$ increases for $x \in \left[-3, \frac{1}{2}\right] \cup [4, \infty)$

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Similarly, $g(x)$ decreases for $x\in(-\infty,-3]\cup\left[\frac{1}{2},4\right]$

2. We have $f(x+y)=f(x)f(y)-g(x)g(y)$
Differentiate both sides w.r.t. x keeping y constant, we get
$f'(x+y)=f'(x)(y)-g'(x)g(y)$
Putting $x=0$, we get
$f'(y)=-g'(0)\,g(y)$ ... (1) (as $f'(0)=0$)
We also have, $g(x+y)=g(x)f(y)+f(x)g(y)$
Differentiate both sides w.r.t. 'x' keeping 'y' constant, and put $x=0$, we get
$g'(y)=g'(0)f(y)$ ... (2) (as $f'(0)=0$)
we have
$f'(y)=-g'(0)g(y)$ & $g'(y)=g'(0)f(y)$

$$\Rightarrow\ f'(y)=\frac{-g'(y)}{f(y)}g(y)\Rightarrow\ f(y)f'(y)=-g'(y)g(y)$$

we get $f(y)f'(y)+g(y)g'(y)=0$

$$\Rightarrow\ \frac{d}{dy}(f^2(y)+g^2(y))=0$$

$\Rightarrow\ f^2(y)+g^2(y)=\lambda$ (const)
Now putting $x=y=0$ in both the given functional equations we get;

$$f(0)=f^2(0)-g^2(0),\ g(0)=2f(0)g(0)$$

$$\Rightarrow\ g(0)=1 \text{ or } f(0)=\frac{1}{2}$$

But if $f(0)=\frac{1}{2}$, first equation gives $g^2(0)=-\frac{1}{4}$, which is not possible.
Hence $g(0)=0$ and $f(0)=1\Rightarrow\lambda=1$
Hence, $f^2(x)+g^2(x)=1,\ x\in R$.
$\Rightarrow$ Maximum and minimum values of $f^2(x)+g^2(x)$ are same for all $x\in R$.

3. For $x=0$, $y=0$, the given equation gives

$$f(0)=\frac{4f(0)-4}{6}\Rightarrow f(0)=-2$$

Now,

$$f'(x)=\lim_{h\to0}\frac{f(x+h)-f(x)}{h}=\lim_{h\to0}\frac{f\left(\frac{3x+3h}{3}\right)-f(x)}{h}$$

$$=\lim_{h\to0}\frac{\frac{2f(3x)+2f(3h)-4}{6}-f(x)}{h}$$

$$=\lim_{h\to0}\frac{2f(3x)+2f(3h)-4-6f(x)}{6h}$$

For $y=0$, the given relation yields

$$f\left(\frac{x}{3}\right)=\frac{2f(x)+2f(0)-4}{6}$$

$$\Rightarrow\ f(x)=\frac{2f(3x)-4-4}{6}=\frac{f(3x)-4}{3}$$

[Replacing $x/3$ by x]

$\Rightarrow\ f(3x)=3f(x)+4$

Hence,

$$f'(x)=\lim_{h\to0}\frac{6f(x)+8+2f(3h)-4-6f(x)}{6h}$$

$$=\lim_{h\to0}\frac{f(3h)+2}{3h}=\lim_{h\to0}\frac{f(3h)-f(0)}{3h}$$

$\Rightarrow\ f(x)=2x+c$ at $x=0$, $c=-2$
$\Rightarrow\ f(x)=2x-2$
$\Rightarrow\ f'(x)=2>0\Rightarrow$ Always increasing.

4. Consider $g(x)=x^2+2x$
Clearly $-\theta^2$ will be a negative number. If θ increases then $-\theta^2$ will decrease or graph of $g(x)$ will come down by the quantity $-\theta^2$. Also $F(\theta)$ is algebraic area bounded by x-axis and the curve and will be negative. So if we have increasing θ, $F(\theta)$ will decrease. Hence maximum value of $F(\theta)$ will be corresponding to $\theta=0$ and this value is equal to

$$F(\theta)_{\max}=\int_{-2}^{0}(x^2+2x)\,dx=\left|\frac{x^3}{3}+x^2\right|_{-2}^{0}=\frac{-4}{3}$$

$\Rightarrow\ F(\theta)_{\max}=-\frac{4}{3}$ for $\theta=0$ and clearly $F(\theta)$ min does not exist.

5. Since minimum occurs before maximum, so $a<0$
Also 'a' is a root of $x^2-x-6=0\Rightarrow a=-2$
Let $g(x)=ax^3+bx^2+cx+d=-2x^3+bx^2+cx+d$
$\Rightarrow\ g'(x)=-6x^2+2bx+c$
roots of $g'(x)=0$ are -2 and 2 $\Rightarrow$ $b=0$, $c=24$
Since minimum value is non-zero $g(-2)>0$
$\Rightarrow\ d>32$ so $a=-2$, $b=0$, $c=24$, $d>32$.

6. Given, $f(x)=\frac{1}{|x|},\ x\neq0$
Let $f(0)=\lambda$, λ is definite real number.

$$\text{Now } Rf'(0)=\lim_{h\to0}\frac{f(0+h)-f(0)}{h}$$

$$=\lim_{h\to0}\frac{\frac{1}{|h|}-\lambda}{h}=\lim_{h\to0}\frac{1-\lambda h}{h^2}\Rightarrow\infty$$

Hence $f(x)$ is not differentiable at $x=0$. Thus, the conditions of Lagrange's Mean Value theorem are not satisfied in the interval which includes the origin.

Conclusion of the Lagrange's Mean Value theorem :

$$\frac{f(b)-f(a)}{b-a}=f'(c),\ a<c<b$$

$$\frac{\frac{1}{|b|}-\frac{1}{|a|}}{b-a}=\frac{1}{|c|^2}\quad\text{or}\quad\frac{1}{|b|}-\frac{1}{|a|}=(b-a)\left(-\frac{1}{|c|^2}\right)$$

$$\Rightarrow\ \frac{1}{|b|}-1=b+1\left(-\frac{1}{|c|^2}\right)=-\frac{b+1}{c^2}\ (\because a=-1)$$

$$\Rightarrow\ c^2=\frac{b^2+b}{b-1}\ \text{or}\ \frac{b^2+b}{b-1}<b^2\ (\text{as } b^2>c^2)$$

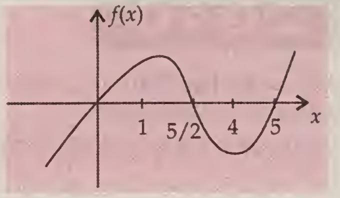
$$\therefore \quad \frac{b(b-1+\sqrt{2})(b-1-\sqrt{2})}{b-1} > 0$$

$\Rightarrow \; b > 1+\sqrt{2} \quad (\because \; b \geq 1)$

Hence the conclusion of the L.M.V. theorem is true if $b > 1+\sqrt{2}$

7. $f'(x) = 6x^2 - 30x + 24 = 6(x-4)(x-1)$
Graph of $f(x)$ will be as shown in fig.

$g'(a) = f(a) - f(5-a)$



if $a < 5 - a$

$\Rightarrow \; a < \frac{5}{2}$, then from the graph $f(a) > f(5-a)$

so $g'(0) > 0$

and if $a > \frac{5}{2}$, then $f(5-a) > f(a)$ so $g'(a) < 0$.

Hence $g(x)$ is increasing in $\left[0, \frac{5}{2}\right]$

8. $f'(x) = 3(x^2 - 2ax + a^2 - 1)$
Clearly roots of the equation $f'(x) = 0$ must be distinct and lie in the interval $(-2, 4)$

$\therefore \quad \Delta > 0 \;\; \Rightarrow a \in R$ ...(1)

$f'(-2) > 2 \;\; \Rightarrow \;\; a^2 + 4a + 3 > 0$

$\Rightarrow \; a < -3$ or $a > -1$ ...(2)

$f'(4) > 0 \Rightarrow a^2 - 8a + 15 > 0$

$\Rightarrow \; a > 5$ or $a < 3$ ...(3)

and $-2 < \frac{-B}{2A} < 4 \;\Rightarrow -2 < a < 4$ ...(4)

From (1), (2), (3) and (4) $-1 < a < 3$.

9. Since $t^2 + 2xt + 4 = 0$ does not possess distinct real roots, $4x^2 - 16 < 0 \Rightarrow -2 \leq x \leq 2$.

Slope of the tangent at any point (x, y) is

$$\frac{dy}{dx} = 3x^2 - 4x + 1$$

which has max. or min.

$$\frac{d^2y}{dy^2} = 0 \Rightarrow 6x - 4 = 0 \Rightarrow x = \frac{2}{3}$$

Hence $\left(\frac{dy}{dx}\right)_{at\,x=-2} = 21, \left(\frac{dy}{dx}\right)_{x=2} = 5, \left(\frac{dy}{dx}\right)_{at\,x=2/3} = -\frac{1}{3}$

At $x = -2$, $b = -8 - 8 - 2 = -18$

10. Any point $P(x, y)$ on the curve can be represented by using parameter θ, as $(a\cos^3\theta, a\sin^3\theta)$

$$\frac{dy}{dx} = \frac{\frac{dy}{d\theta}}{\frac{dx}{d\theta}} = \frac{3a\sin^2\theta\cos\theta}{-3a\cos^2\theta\sin\theta} = -\tan\theta$$

$\therefore$ equation of the tangent at P is

$y - a\sin^3\theta = -\tan\theta(x - a\cos^3\theta)$

$x\tan\theta + y = a\sin^3\theta + a\cos^3\theta\tan\theta$

$$p_1 = \left|\frac{a\sin^3\theta + a\cos^3\theta\tan\theta}{\sec\theta}\right| = \frac{1}{2}|a\sin 2\theta|$$

Also, equation of the normal at P is

$y - a\sin^3\theta = \cot\theta(x - a\cos^3\theta)$

or $\; x\cot\theta - y = a\cos^3\theta\cot\theta - a\sin^3\theta$

$$p_2 = \left|\frac{a\cos^3\theta.\cot\theta - a\sin^3\theta}{\mathrm{cosec}\,\theta}\right| = |a\cos 2\theta|$$

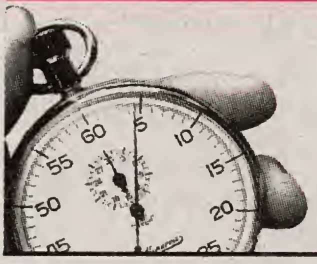
$\Rightarrow \; 4p_1^2 + p_2^2 = a^2 =$ constant point $P(x, y)$

$\Rightarrow$ If p_1 increases, p_2 decreases and commonly

$\Rightarrow \; \frac{dp_1}{dx}$ and $\frac{dp_2}{dx}$ are of opposite signs.

$$\frac{dp_1}{dx} . \frac{dp_2}{dx} < 0$$

CHALLENGING PROBLEMS



for various Engineering Exams

1. If $\left(\frac{y}{z}\right)^a\left(\frac{z}{x}\right)^b\left(\frac{x}{y}\right)^c = 1$ and $A = \left(\frac{y}{z}\right)^{\frac{1}{b-c}}$, $B = \left(\frac{z}{x}\right)^{\frac{1}{c-a}}$, $C = \left(\frac{x}{y}\right)^{\frac{1}{a-b}}$, then
(a) $A = B = C$ (b) $ABC = 1$
(c) $A + B + C = 0$ (d) none of these

2. The number of solutions of $\sqrt{\frac{49}{1+x}} - \sqrt{x+7} = \sqrt{\frac{x(x+8)}{x+1}}$ is
(a) 1 (b) 0
(c) 2 (d) none of these

3. If $a^3 = b^4$ such that $\left(\frac{a}{b}\right)^{4/3} + \left(\frac{b}{a}\right)^{3/4} = a^\alpha + b^\beta$, then find α and β
(a) $\alpha = \frac{1}{4}, \beta = -\frac{1}{3}$ (b) $\alpha = \frac{1}{3}, \beta = -\frac{1}{4}$
(c) $\alpha = \frac{-1}{3}, \beta = \frac{1}{4}$ (d) none of these

4. If $\frac{S_m}{m^2} = \frac{S_n}{n^2} = k$ in any A.P. [S_k means sum to k terms] then S_k =
(a) $3k$ (b) k^2
(c) k^3 (d) none of these

5. If $9(25a^2 + b^2) + 25(c^2 - 3ac) = 15b(3a + c)$, then
(a) b, c, a are in A.P. (b) a, b, c are in G.P.
(c) a, b, c are in A.P. (d) a, b, c are in H.P.

6. The sum of all terms in 11^{th} bracket of series $(1) + (2 + 2^2) + (2^3 + 2^4 + 2^5) + (2^6 + 2^7 + 2^8 + 2^9) + \ldots$ is $2^p - 2^q$ then $p : q$ =
(a) 5 : 6 (b) 5 : 1
(c) 55 : 1 (d) none of these

7. Three numbers (> 0) whose sum is 21 are in A.P. If 2, 2, 14 be added to them respectively then resulting numbers are in G.P. then which is not among the three numbers?
(a) 25 (b) 13
(c) 1 (d) 7

8. Find x, if $4^{\log_{10}x} + x^{\log_{10}4} = 32$.
(a) 10 (b) 100
(c) 1/100 (d) none of these

9. If $x = \log_{4a}3a$, $y = \log_{3a}2a$, $z = \log_{2a}a$, then $xyz(2 - z)$ =
(a) 1 (b) z
(c) 0 (d) $-x$

10. If x be real and $\frac{3}{x+1} - \frac{2}{x}$ does not lie between a and b then
(a) a and b both are of opposite sign
(b) both a and b are rational
(c) a and b are conjugate to each other
(d) none of these

11. If $A_r = \begin{pmatrix} x^r & y^r \\ y^r & x^r \end{pmatrix}$, $(|x|, |y| < 1)$, then $\sum_{r=1}^{\infty} \det.(A_r) =$
(a) $\frac{x^2 - y^2}{1 - x^2 - y^2}$ (b) $\frac{x^2 - y^2}{x^2 + y^2}$
(c) $\frac{x^2}{1-x^2} + \frac{y^2}{1-y^2}$ (d) $\frac{x^2 - y^2}{(x^2-1)(y^2-1)}$

12. If $\frac{1}{3.6} + \frac{1}{3.6.9} + \frac{1}{3.6.9.12} + \ldots$ to $\infty = e^k - (1+k)$, then k =
(a) 1/3 (b) −1
(c) 2/3 (d) none of these

13. If $2\omega + 1 = \sqrt{-3}$, then $\begin{vmatrix} 1 & 1 & 1 \\ 1 & -\omega^2 - 1 & \omega^2 \\ 1 & \omega^2 & \omega^7 \end{vmatrix} =$
(a) $3(\omega - \omega^2)$ (b) $3\omega^2$
(c) 3 (d) none of these

14. If a, b, c be in G.P., then $a + b$, $2b$, $b + c$ are in
(a) A.P. (b) G.P.
(c) H.P. (d) none of these

15. $\frac{1}{2} + \frac{1}{3} + \frac{1}{6} + \ldots$, 9^{th} term of the series =

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(a) $\frac{1}{4}$ (b) $-\frac{5}{6}$
(c) $-\frac{1}{6}$ (d) none of these

16. Total number of factors of 1800 (excluding 1) is
(a) 35 (b) 36
(c) 11 (d) none of these

17. Amplitude of $\left(\sin\frac{\pi}{12}+i\cos\frac{\pi}{12}\right)^8$ is
(a) $-\frac{\pi}{3}$ (b) $-\frac{2\pi}{3}$
(c) $-\frac{\pi}{6}$ (d) none of these

18. If $x:y:z=1:2:3$ and $\log_{xyz}(x+y+z)=\cos\left(\frac{\pi}{3}\right)$, then y =
(a) 6 (b) 9
(c) 12 (d) none of these

19. If x, y, z be in A.P. and x, $y\sqrt{2}$, z be in G.P. then
(a) x, $2y$, z are in H.P. (b) x, y, z are in H.P.
(c) x, $\frac{y}{2}$, z are in H.P. (d) none of these

20. If $P_n = 1 + 22 + 333 + 4444 + ...$ to n terms then
(a) $P_9 - P_8 = 10^9 + 1$ (b) $P_4 - P_3 = \frac{4}{9}P_9$
(c) $P_9 - P_8 = 10^9 - 1$ (d) none of these

21. Find the coefficient of x^9 in the expansion of $(1+x)^{2009} + (1+x)^{2010} + (1+x)^{2011} + ... + (1+x)^{2020}$
(a) ${}^{2021}C_{2011} - {}^{2021}C_{1999}$ (b) ${}^{2021}C_{10} - {}^{2021}C_{2009}$
(c) ${}^{2021}C_{2011} - {}^{2009}C_{1999}$ (d) none of these

22. $\{x + (x^3-1)^{1/2}\}^5 + \{x - (x^3-1)^{1/2}\}^5$ is a polynomial of degree
(a) 5 (b) 6
(c) 7 (d) not a polynomial

23. If $(C_0 + C_1)(C_1 + C_2)(C_2 + C_3)$ ... to n factors $= f(n).(C_1.C_2.C_3. ... C_{n-1})$, then
(a) $f(2)=\frac{32}{3}$ (b) $f(3)=\frac{9}{2}$
(c) $f(1) = 2$ (d) none of these

24. Form the quadratic equation whose roots are A.M. & H.M. of the roots of $x^2 + px + q = 0$
(a) $2px^2 + (p^2 + 4q)x + 2q = 0$
(b) $2px^2 + (p^2 - 4q)x + 2pq = 0$
(c) $2px^2 + (p^2 + 4q)x + 2p = 0$
(d) none of these

25. If $x = i(i+\sqrt{2})$, then $x^4 + 4x^3 + 6x^2 + 4x + 3$ =
(a) -6 (b) 6
(c) 9 (d) none of these

26. If in a G.P., $t_7 = 5$, $t_{17} = 20$, then t_{12} =
(a) 2 (b) 10
(c) $\frac{100}{9}$ (d) none of these

27. If $(1-p)$ is a root of $x^2 + px = p - 1$ then roots are
(a) 1, 0 (b) 0, -1
(c) -1, 2 (d) none of these

28. Find the sum of the solutions of $(2010-x)^2 = |x - 2010| + 6$
(a) 4020 (b) 0
(c) 2 (d) none of these

29. If $2 - i$ is rotated about origin by a right angle in complex plane in clockwise direction then find its new position
(a) $-2 - i$ (b) $-1 + 2i$
(c) $-1 - 2i$ (d) none of these

30. Number of arrangements of the word "ARTICLE" so that even places are filled by consonants is
(a) 72 (b) 96 (c) 62 (d) 576

31. If
$$\cos^{-1}\left(x-\frac{x^2}{2}+\frac{x^3}{4}-...\right)+\sin^{-1}\left(x^2-\frac{x^4}{2}+\frac{x^6}{4}-...\right)=\frac{\pi}{2}$$
$(0<|x|<\sqrt{2})$, then x =
(a) $\sqrt{2}-1$ (b) $\frac{1}{2}$
(c) $-\frac{1}{2}$ (d) none of these

32. Solve : $x^{\log_2(x/98)}.14^{\log_2 7} = 1$
(a) 7 (b) 14
(c) 7, 14 (d) none of these

33. If a set is finite and have n distinct elements then number of relations on the set is
(a) n^2 (b) 2^n (c) 2^{2n} (d) 2^{n^2}

34. If 3 non-zero numbers x, y, z be in A.P. and $\tan^{-1}x$, $\tan^{-1}y$, $\tan^{-1}z$ be also in A.P. then
(a) x, y, z are in G.P.
(b) $x^3 + y^3 + z^3 = 3xyz$
(c) $x^2 + y^2 + z^2 = xy + yz + zx$
(d) all of these

35. If $a^2 + b^2 = 1 = p^2 + q^2$, then
(a) $|ap + bq| \geq 1$ (b) $0 \leq |ap + bq| \leq 2$
(c) $|ap \pm bq| \leq 1$ (d) none of these

36. If $\log_{1/3}|z - i| < \log_{1/3}|z - 1|$, then locus of z is
(a) $x > y$ (b) $x < y$
(c) $x < y + 1$ (d) none of these

37. Find the largest integer n so that 33! is divisible by 2^{n-1}
(a) 32 (b) 33
(c) 31 (d) none of these

38. Solve : $1+\log_{10}x+(\log_{10}x)^2=\dfrac{26}{\log_{10}x-1}$

(a) 30 (b) 3^{10}
(c) 10^3 (d) none of these

39. If six line segments are of lengths 1, 2, 3, 4, 5, 6 units then number of triangles that can be formed is

(a) ${}^6C_3-6$ (b) ${}^6C_3-{}^4C_3$
(c) ${}^6C_3-13$ (d) ${}^6C_3-7$

40. A and B have equal number of sisters and 3 different gifts are to be distributed among them. It is given that probability that all gifts go to sisters of B is 1/20. Find the number of sisters that each have.

(a) 4 (b) 3
(c) 5 (d) none of these

41. If $x-\dfrac{1}{x}=2i\sin\theta$, then $x^4-\dfrac{1}{x^4}=$

(a) $2i\sin4\theta$ (b) $2\sin4\theta$
(c) $2\cos4\theta$ (d) none of these

42. If $A=\begin{pmatrix}0 & -\tan\left(\frac{\theta}{2}\right)\\ \tan\left(\frac{\theta}{2}\right) & 0\end{pmatrix}$ and I be unit matrix of order 2, then $(I-A)\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}=$

(a) I (b) $I+A$
(c) $A-I$ (d) 0

43. Solve : $5^2.5^4.5^6 \ldots 5^{2x}=(0.04)^{-28}$

(a) 5 (b) 6
(c) 4 (d) 7

44. If the ratio of total number of combinations of $2n$ different things to the total number of combinations of n different things be 513 : 1 then n =

(a) 9 (b) 10
(c) 11 (d) none of these

45. For what values of a, roots of $x^2+2(2+3a)x+a^2+2(a+1)=0$ are non-real?

(a) $-2<a<-4$ (b) $-1<a<-4$
(c) $-1<a<-\dfrac{1}{4}$ (d) none of these

46. If $t_r=\dfrac{1}{r!}$, then
$t_0t_n+2.t_1t_{n-1}+4.t_2t_{n-2}+\ldots+2^n.t_nt_0=$.

(a) $\dfrac{2^n}{n!}$ (b) $\dfrac{3^n}{n!}$
(c) 3^n (d) none of these

47. If both roots of $x^2-2009x+k=0$ be prime integers then number of possible values of k is

(a) 1005 (b) ${}^{2009}C_2$
(c) $\dfrac{\underline{|k}}{2}$ (d) none of these

48. If a, b, c be in A.P. and $\dfrac{w-x}{ax}=\dfrac{w-y}{by}=\dfrac{w-z}{cz}$, then x, y, z are in

(a) A.P. (b) G.P.
(c) H.P. (d) none of these

49. What is definite about the discriminant (D) of a quadratic equation with roots
$\alpha=\dfrac{\cot^2\theta-\cos^2\theta}{\cos^2.\cot^2\theta}$ & $\beta=\dfrac{\tan^2\theta-\sin^2\theta}{\sin^2\theta.\tan^2\theta}$ $\left(0<\theta<\dfrac{\pi}{2}\right)$

(a) $D\le0$ (b) $D\ge0$
(c) $D=0$ (d) none of these

50. In arrangements of the word 'MONDAY', if p = number of words that do not begin with M and q = number of words which begin with M but do not end with Y then $p:q$ =

(a) 25 : 4 (b) 4 : 25
(c) < 0 (d) none of these

SOLUTIONS

1. **(a)** : $\left(\dfrac{z}{x}\right)^b\cdot\left(\dfrac{x}{y}\right)^c=\left(\dfrac{z}{y}\right)^a=\left(\dfrac{z}{x}\cdot\dfrac{x}{y}\right)^a=\left(\dfrac{z}{x}\right)^a\cdot\left(\dfrac{x}{y}\right)^a$

$\Rightarrow \left(\dfrac{x}{y}\right)^{c-a}=\left(\dfrac{z}{x}\right)^{a-b}$

$\Rightarrow \left(\dfrac{x}{y}\right)^{\frac{1}{a-b}}=\left(\dfrac{z}{x}\right)^{\frac{1}{c-a}} \Rightarrow C=B$

Similarly, it can be proved $A=B$
$\therefore\ A=B=C.$

2. **(a)** : On simplification, we get

$\sqrt{x^2+8x+7}+\sqrt{x^2+8x}=7$ ...(1)

$\Rightarrow \sqrt{a+7}+\sqrt{a}=7$ [Put $x^2+8x=a$]

$\Rightarrow a+7=49+a-14\sqrt{a}$

$\Rightarrow \sqrt{a}=3 \Rightarrow x^2+8x-9=0$

$\Rightarrow x=-9, 1$

But $x=-9$ does not satisfy given equation
$\therefore\ x=1$ is the only solution.

3. **(b)** : L.H.S. $=\left(\dfrac{a^4}{b^4}\right)^{1/3}+\left(\dfrac{b^3}{a^3}\right)^{1/4}$

$=\left(\dfrac{a^4}{a^3}\right)^{1/3}+\left(\dfrac{b^3}{b^4}\right)^{1/4}$ $[\because a^3=b^4]$

$=a^{1/3}+b^{-1/4}=a^\alpha+b^\beta$

$\Rightarrow \alpha=\dfrac{1}{3},\beta=-\dfrac{1}{4}$

4. **(c)** : $S_m=m^2k$ gives $\dfrac{m}{2}[2a+(m-1)d]=m^2k$

$\Rightarrow 2a-d+md=2mk$ ...(1)

Similarly, $2a-d+nd=2nk$ ...(2)

On solving, $d = 2k$

$\Rightarrow\quad 2a - d + m.2k = 2mk \quad$ [from (1)]

$\therefore\quad 2a = d = 2k$

$\therefore\quad S_k = \frac{k}{2}[2a+(k-1)d] = \frac{k}{2}[d+kd-d]$

$= \frac{k}{2}(k.2k) = k^3$

5. **(a)** : Put $x = 15a$, $y = 3b$, $z = 5c$

Given equation becomes

$x^2 + y^2 + z^2 - xy - yz - zx = 0$

$\Rightarrow\quad \frac{1}{2}\{(x-y)^2+(y-z)^2+(z-x)^2\} = 0$

$\Rightarrow\quad x - y = 0 = y - z = z - x$

$\Rightarrow\quad 15a = 3b = 5c$

$\Rightarrow\quad \frac{a}{1} = \frac{b}{5} = \frac{c}{3} = k$ (say)

$\therefore\quad b = 5k,\ c = 3k,\ a = k$

$\Rightarrow\quad b, c, a$ are in A.P.

6. **(d)** : I^{st} term of successive brackets are

$2^0, 2^1, 2^{1+2}, 2^{1+2+3}, \ldots$

$\therefore\quad$ 1st term of 11^{th} bracket is

$2^{1+2+3+\ldots+10} = 2^{\frac{10(10+1)}{2}} = 2^{55}$

$\therefore\quad$ Sum of all terms in 11^{th} bracket is

$(2^{55} + 2^{56} + 2^{57} + \ldots$ to 11 terms)

$= \frac{2^{55}(2^{11}-1)}{2-1} = 2^{66} - 2^{55} = 2^p - 2^q$ (given)

$\Rightarrow\quad p : q = 66 : 55 = 6 : 5$

7. **(a)** : Let three numbers in A.P. be $a - d$, a, $a + d$

$\therefore\quad (a - d) + a + (a + d) = 21$

$\Rightarrow\quad a = 7$

Also, $2 + (a - d)$, $2 + a$, $14 + (a + d)$ are in G.P.

$\Rightarrow\quad (2 + a)^2 = (2 + a - d)(14 + a + d)$

$\Rightarrow\quad 81 = (9 - d)(21 + d) \quad [\because a = 7]$

$\Rightarrow\quad d^2 + 12d - 108 = 0$

$\Rightarrow\quad d = 6, -18$

$\therefore\quad d = 6$ [$\because d = -18$ makes 3rd no. negative)

$\therefore\quad$ Three numbers are 1, 7, 13.

8. **(b)** : $x^{\log_{10}4} = (x^{\log_x 4})^{\log_{10}x} = 4^{\log_{10}x}$

$\therefore\quad$ Given equation becomes $2(4^{\log_{10}x}) = 32 = 2(4^2)$

$\Rightarrow\quad \log_{10}x = 2$

$\therefore\quad x = 10^2 = 100$

9. **(b)** : $xyz(2 - z) = \log_{4a}3a.\log_{3a}2a.\log_{2a}a(2 - \log_{2a}a)$

$= \log_{4a}a.(2\log_{2a}2a - \log_{2a}a) = \log_{4a}a.\log_{2a}4a$

$= \log_{2a}a = z$

10. **(c)** : Let $y = \frac{3}{x+1} - \frac{2}{x}$

$\Rightarrow\quad yx^2 + (y - 1)x + 2 = 0 \quad \ldots(1)$

If x be real then discriminant ≥ 0

$\Rightarrow\quad (y - 1)^2 - 4.y.2 \geq 0$

$\Rightarrow\quad y^2 - 10y + 1 \geq 0$

$\Rightarrow\quad (y-5)^2 - (2\sqrt{6})^2 \geq 0$

$\Rightarrow\quad \{y-(5-2\sqrt{6})\}\{y-(5+2\sqrt{6})\} \geq 0$

$\Rightarrow\quad y \leq 5-2\sqrt{6}$ or $y \geq 5+2\sqrt{6}$

$\Rightarrow\quad y$ does not lie between $5-2\sqrt{6}$ & $5+2\sqrt{6}$

$\therefore\quad a = 5-2\sqrt{6}, b = 5+2\sqrt{6}$, which are conjugate to each other

11. **(d)** : $|A_r| = x^{2r} - y^{2r}$

$\therefore\quad \sum_{r=1}^{\infty}\det.(A_r) = (x^2-y^2)+(x^4-y^4)+\ldots$ to ∞

$= \frac{x^2}{1-x^2} - \frac{y^2}{1-y^2} \quad [\because |x|, |y| < 1]$

$= \frac{x^2(1-y^2)-y^2(1-x^2)}{(1-x^2)(1-y^2)} = \frac{x^2-y^2}{(x^2-1)(y^2-1)}$

12. **(a)** : Given series

$= \frac{1}{1\cdot 2}\cdot\frac{1}{3^2} + \frac{1}{1\cdot 2\cdot 3}\cdot\frac{1}{3^3} + \frac{1}{1\cdot 2\cdot 3\cdot 4}\cdot\frac{1}{3^4} + \ldots$ to ∞

$= \left\{1+\left(\frac{1}{3}\right)+\frac{(1/3)^2}{\underline{|2}}+\ldots \text{ to } \infty\right\} - \left(1+\frac{1}{3}\right)$

$= e^{1/3} - \left(1+\frac{1}{3}\right)$

$\therefore\quad k = \frac{1}{3}$

13. **(d)** :

$$\text{Det.} = \begin{vmatrix} 3 & 0 & 0 \\ 1 & -\omega^2-1 & \omega^2 \\ 1 & \omega^2 & \omega \end{vmatrix} \quad [R_1 \to R_1 + (R_2 + R_3)]$$

$[\because \omega^7 = \omega \ \&\ 1 + \omega^2 + \omega = 0]$

$= 3\{(-\omega^3 - \omega) - \omega^4\} = 3(-1 - \omega - \omega) = 3(\omega^2 - \omega)$

14. **(c)** : Let r be common ratio

$\therefore\quad b = ar,\ c = ar^2$

Now, $\frac{2(a+b)(b+c)}{(a+b)+(b+c)} = \frac{2(a+ar)(ar+ar^2)}{a+ar+ar+ar^2}$

$= \frac{2(a+ar).ar(1+r)}{(a+ar)(1+r)} = 2ar = 2b$

$\Rightarrow\quad (a + b), 2b, (b + c)$ are in H.P.

15. **(b)** : $\because\quad \frac{1}{3} - \frac{1}{2} = -\frac{1}{6} = \frac{1}{6} - \frac{1}{3}$

$\therefore\quad$ Series is an A.P.

$\therefore\quad t_9 = \frac{1}{2} + (9-1)\left(-\frac{1}{6}\right) = -\frac{5}{6}$

16. **(a)** : Since, $1800 = 2^3.3^2.5^2$

No. of factors of any +ve integer of the form $2^\alpha.3^\beta.5^\gamma\ldots$ is $(\alpha+1)(\beta+1)(\gamma+1)\ldots$

$\therefore\quad$ Total no. of factors $= (3 + 1)(2 + 1)(2 + 1) = 36$

$\therefore\quad$ Reqd. no. of factors $= 36 - 1 = 35$ (excluding 1)

17. **(b)** : $\left(\sin\frac{\pi}{12} + i\cos\frac{\pi}{12}\right)^8 = \left\{i\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right)\right\}^8$

$= i^8\left\{\cos\left(-\frac{\pi}{12}\right) + i\sin\left(-\frac{\pi}{12}\right)\right\}^8$

$$= 1.\left\{\cos\left(-\frac{8\pi}{12}\right) + i\sin\left(-\frac{8\pi}{12}\right)\right\}$$

$$= \cos\left(-\frac{2\pi}{3}\right) + i\sin\left(-\frac{2\pi}{3}\right)$$

$\Rightarrow$ Amplitude $= -\dfrac{2\pi}{3}$

18. (c) : $\dfrac{x}{1} = \dfrac{y}{2} = \dfrac{z}{3}$

$\Rightarrow x = \dfrac{y}{2}$ & $z = \dfrac{3y}{2}$

$\because \log_{xyz}(x+y+z) = \cos\left(\dfrac{\pi}{3}\right)$

$\therefore \log_{\frac{y}{2}.y.\frac{3y}{2}}\left(\dfrac{y}{2} + y + \dfrac{3y}{2}\right) = \dfrac{1}{2}$

$\Rightarrow 2\log_{\frac{3y^3}{4}} 3y = 1 \quad \Rightarrow \dfrac{3y^3}{4} = 9y^2$

$\Rightarrow y = 12$ [$\because y \neq 0$]

19. (a) : $y = \dfrac{x+z}{2}$ ...(1)

$2y^2 = xz$ ...(2)

$2y \cdot y = xz$

$\Rightarrow 2y = \dfrac{xz}{y} = \dfrac{2xz}{x+z}$ [from (1)]

$\Rightarrow$ x, $2y$, z are in H.P.

20. (c) : $P_9 - P_8 = 999999999 = 9 + 90 + 900 + \ldots$ to 9 terms

$= \dfrac{9(10^9 - 1)}{10 - 1}$ [Being G.P. having c.r. = 10]

$= 10^9 - 1$

21. (c) : Given exp. $= (1+x)^{2009}\{1 + (1+x) + \ldots$ to 12 terms$\}$

$$= (1+x)^{2009}\left\{\frac{(1+x)^{12} - 1}{(1+x) - 1}\right\}$$

$$= \frac{1}{x}\{(1+x)^{2021} - (1+x)^{2009}\}$$

$\therefore$ Reqd. coefficient of x^9 = coefficient of x^{10} in the expansion of $(1+x)^{2021} - (1+x)^{2009}\}$

$$= {}^{2021}C_{10} - {}^{2009}C_{10} = {}^{2021}C_{2011} - {}^{2009}C_{1999}$$

22. (c) : On expanding, given expn.

$= 2\{x^5 + {}^5C_2x^3.(x^3 - 1) + {}^5C_4x(x^3 - 1)^2\}$, which is a polynomial of degree 7

[$\because$ Highest power of $x = 7$]

23. (c) : $\because \dfrac{{}^nC_{r-1} + {}^nC_r}{{}^nC_{r-1}} = \dfrac{{}^{n+1}C_r}{{}^nC_{r-1}} = \dfrac{n+1}{r}$

(after simplification)

$$\therefore \frac{C_0 + C_1}{C_0} \cdot \frac{C_1 + C_2}{C_1} \cdot \frac{C_2 + C_3}{C_2} \cdots \frac{C_{n-1} + C_n}{C_{n-1}}$$

$$= \frac{n+1}{1} \cdot \frac{n+1}{2} \cdot \ldots \cdot \frac{n+1}{n}$$

$\Rightarrow (C_0 + C_1)(C_1 + C_2)(C_2 + C_3) \ldots (C_{n-1} + C_n)$

$$= \frac{(n+1)^n}{\underline{|n}}(C_0 \cdot C_1 \cdot C_2 \cdot \ldots \cdot C_{n-1})$$

$$= \frac{(n+1)^n}{\underline{|n}}(C_1 \cdot C_2 \cdot \ldots \cdot C_{n-1}) \qquad [\because C_0 = 1]$$

$\therefore f(n) = \dfrac{(n+1)^n}{\underline{|n}}$

$\therefore f(1) = 2, f(2) = \dfrac{9}{2}, f(3) = \dfrac{32}{3}$

24. (d) : $\alpha + \beta = -p$, $\alpha\beta = q$ (if α, β be the roots of given equation)

$\therefore$ A.M. $= -\dfrac{p}{2}$, H.M. $= \dfrac{2\alpha\beta}{\alpha+\beta} = \dfrac{-2q}{p}$

$\therefore$ Required equation is

$$x^2 - \left(-\frac{p}{2} - \frac{2q}{p}\right)x + \left(-\frac{p}{2}\right)\left(\frac{-2q}{p}\right) = 0$$

$\Rightarrow x^2 + \dfrac{(p^2 + 4q)x}{2p} + q = 0$

$\Rightarrow 2px^2 + (p^2 + 4q)x + 2pq = 0$

25. (b) : $x = i^2 + i\sqrt{2}$

$\Rightarrow (x+1)^2 = (i\sqrt{2})^2$

$\Rightarrow x^2 + 2x + 3 = 0$ ...(1)

Now, $x^4 + 4x^3 + 6x^2 + 4x + 3$

$= x^2(x^2 + 2x + 3) + 2x(x^2 + 2x + 3) - (x^2 + 2x + 3) + 6$

$= x^2.0 + 2x.0 - 0 + 6$ [using (1)]

$= 6$

26. (d) : If a, b, c are in A.P. then t_a, t_b, t_c of any G.P. are also in G.P.

$\because$ 7, 12, 17 are in A.P.

$\therefore$ t_7, t_{12}, t_{17} of any G.P will be in G.P.

$\therefore t_{12} = \pm\sqrt{t_7 \cdot t_{17}} = \pm\sqrt{5(20)} = \pm 10$

Alternative Solution :

In any G.P., if $t_p = m$, $t_q = n$, then $t_{\frac{p+q}{2}} = \pm\sqrt{mn}$ (if $p + q$ be even)

27. (b) : $(1 - p)$ is a root of $x^2 + px = p - 1$

$\Rightarrow (1-p)^2 + p(1-p) = p - 1$

$\Rightarrow p = 1$

$\therefore$ Given equation becomes $x^2 + x = 0$

$\Rightarrow x = 0, -1$

28. (a) : We can write $(x - 2010)^2 - |x - 2010| - 6 = 0$

or, $|x - 2010|^2 - |x - 2010| - 6 = 0$

or, $t^2 - t - 6 = 0$ [where $t = |x - 2010|$]

$\Rightarrow t = 3, -2$

$\Rightarrow t = 3$ [$\because t = |x - 2010| \neq -2$]

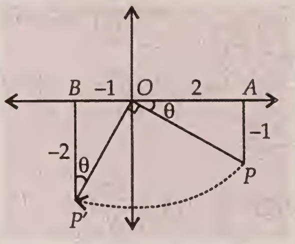
$\Rightarrow$ $|x-2010| = 3$ $\Rightarrow$ $x - 2010 = \pm 3$

$\Rightarrow$ $x = 2013, 2007$

$\therefore$ Their sum = 4020

29. **(c) :** $\tan\theta = \left|\frac{AP}{OA}\right| = \left|\frac{OB}{BP'}\right| = \frac{1}{2}$

$\therefore$ P' is $-1-2i$



30. **(d) :** 4 consonants, 3 vowels

3 even & 4 odd places

$\therefore$ 3 even places by 4 consonants $\rightarrow {}^4P_3$

Remaining 4 places $\rightarrow {}^4P_4$

$\therefore$ Reqd. no. of arrangements = ${}^4P_3 \times {}^4P_4 = 576$

31. **(d) :**

$$\cos^{-1}\left(x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots\right) = \frac{\pi}{2} - \sin^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\right)$$

$$= \cos^{-1}\left(x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots\right)$$

$$\Rightarrow x - \frac{x^2}{2} + \frac{x^3}{4} - \ldots = x^2 - \frac{x^4}{2} + \frac{x^6}{4} - \ldots$$

$$\Rightarrow \frac{x}{1-\left(-\frac{x}{2}\right)} = \frac{x^2}{1-\left(-\frac{x^2}{2}\right)} \quad \text{[Being infinite G.P.]}$$

$$\Rightarrow \frac{2x}{2+x} = \frac{2x^2}{2+x^2} \Rightarrow 2x + x^2 = 2 + x^2 \therefore x = 1$$

32. **(c) :** Taking logarithm with base 2

$(\log_2 x - \log_2 98)\log_2 x + \log_2 7 \log_2 14 = 0$

$\Rightarrow$ $(\log_2 x - 2\log_2 7 - 1)\log_2 x + \log_2 7(\log_2 7 + 1) = 0$

$\Rightarrow$ $(z - 2p - 1)z + p(p+1) = 0$

[where $z = \log_2 x$, $p = \log_2 7$]

$\Rightarrow$ $(z-p)(z-p-1) = 0$

$\Rightarrow$ $\log_2 x = \log_2 7$, $\log_2 7 + 1 = \log_2 x$

$\Rightarrow$ $x = 7, 14$

33. **(d) :** Let the set be A. $\therefore n(A \times A) = n \times n = n^2$

$\therefore$ No. of relations on A = No. of subsets of $A \times A$ $= 2^{n^2}$

34. **(d) :** $2y = x + z$ ...(1)

$2\tan^{-1}y = \tan^{-1}x + \tan^{-1}z$

$$\Rightarrow \frac{2y}{1-y^2} = \frac{x+z}{1-xz} = \frac{2y}{1-xz} \quad \text{[from (1)]}$$

$\Rightarrow$ $1 - y^2 = 1 - xz$ $[\because y \neq 0]$

$\Rightarrow$ $y^2 = xz$ $\Rightarrow$ $\left(\frac{x+z}{2}\right)^2 = xz$

$\Rightarrow$ $(x-z)^2 = 0$ $\Rightarrow$ $x = z$

$\therefore$ From (1), $2y = 2z$

$\Rightarrow$ $y = z$ $\therefore$ $x = y = z$

35. **(c) :** Let $a = \cos\alpha$, $p = \cos\beta$

$\because$ $a^2 + b^2 = 1 = p^2 + q^2$

$\therefore$ $b = \sin\alpha$, $q = \sin\beta$

$\therefore$ $|ap \pm bq| = |\cos\alpha\cos\beta \pm \sin\alpha\sin\beta|$

$= |\cos(\alpha \mp \beta)| \le 1$

36. **(a) :** Let $z = x + iy$

$\therefore$ Given equation gives $|z - i| > |z - 1|$

$[\because$ Base $< 1]$

$\Rightarrow$ $\sqrt{x^2 + (y-1)^2} > \sqrt{(x-1)^2 + y^2}$

$\Rightarrow$ $-2y > -2x$ $\Rightarrow$ $x > y$

37. **(a) :** $33! = (2\cdot4\cdot6\cdot \ldots 32)(1\cdot3\cdot5\cdot\ldots\cdot 33)$

$= 2^{16}(1\cdot2\cdot3\cdot \ldots \cdot16)(1\cdot3\cdot5\cdot\ldots\cdot33)$

$= 2^{16}(2\cdot4\cdot6 \ldots 16)(1\cdot3\cdot5\cdot \ldots 15)(1\cdot3\cdot5\cdot \ldots\cdot 33)$

$= 2^{16}\cdot2^8(1\cdot2\cdot3\cdot \ldots \cdot8)(1\cdot3\cdot5\cdot \ldots \cdot15)(1\cdot3\cdot5\cdot \ldots \cdot33)$

$= 2^{24}(2.4.6.8)(1.3.5.7)(1.3.5. \ldots 15)(1.3.5. \ldots 33)$

$= 2^{24}(2^{1+2+1+3}\cdot3)(1\cdot3\cdot5\cdot7)(1\cdot3\cdot5\cdot\ldots\cdot15)(1\cdot3\cdot5\cdot\ldots\cdot33)$

$= 2^{31}\cdot3(1\cdot3\cdot5\cdot7)(1\cdot3\cdot5\cdot \ldots\cdot 15)(1\cdot3\cdot5\cdot \ldots \cdot33)$

$\Rightarrow$ 33! is divisible by 2^{31}

$\Rightarrow$ largest $(n-1)$ is 31 $\therefore n = 32$ (largest)

38. **(c) :** $a^2 + a + 1 = \frac{26}{a-1}$ (taken $a = \log_{10}x$)

$\Rightarrow$ $a^3 - 1 = 26$ $\Rightarrow$ $a = 3$ $\Rightarrow$ $\log_{10}x = 3$ $\Rightarrow$ $x = 10^3$

39. **(c) :** Sum of any 2 sides > 3rd side in forming any triangle

$\therefore$ Possible combinations are (2, 3, 4), (2, 4, 5), (2, 5, 6), (3, 4, 5), (3, 4, 6), (3, 5, 6) & (4, 5, 6) only *i.e.* 7 ways

$\because$ ${}^6C_3 - 13 = 20 - 13 = 7$

40. **(b) :** Let x be the number of sisters that each have

$\therefore$ Total sisters = $2x$

Given, $\frac{{}^xC_3}{{}^{2x}C_3} = \frac{1}{20}$ $\Rightarrow$ $\frac{x(x-1)(x-2)}{2x(2x-1)(2x-2)} = \frac{1}{20}$

$\Rightarrow$ $\frac{x-2}{4(2x-1)} = \frac{1}{20}$ $\Rightarrow x = 3$

41. **(a) :** Let $x = \cos\theta + i\sin\theta$

$\therefore$ $\frac{1}{x} = \cos\theta - i\sin\theta$

$\Rightarrow$ $x^4 = \cos4\theta + i\sin4\theta$ & $\frac{1}{x^4} = \cos4\theta - i\sin4\theta$

$\therefore$ $x^4 - \frac{1}{x^4} = 2i\sin4\theta$

42. (b) : $I-A=\begin{pmatrix}1 & 0\\0 & 1\end{pmatrix}-\begin{pmatrix}0 & -x\\x & 0\end{pmatrix}$, where $x=-\tan\frac{\theta}{2}$

$$=\begin{pmatrix}1 & x\\-x & 1\end{pmatrix}$$

$$\therefore\quad (I-A)\begin{pmatrix}\cos\theta & -\sin\theta\\ \sin\theta & \cos\theta\end{pmatrix}=\begin{pmatrix}1 & x\\-x & 1\end{pmatrix}\begin{pmatrix}\frac{1-x^2}{1+x^2} & \frac{-2x}{1+x^2}\\ \frac{2x}{1+x^2} & \frac{1-x^2}{1+x^2}\end{pmatrix}$$

$$=\frac{1}{(1+x^2)^2}\cdot\begin{pmatrix}1 & x\\-x & 1\end{pmatrix}\begin{pmatrix}1-x^2 & -2x\\2x & 1-x^2\end{pmatrix}$$

$$=\frac{1}{(1+x^2)^2}\cdot\begin{pmatrix}1-x^2+2x^2 & -2x+x-x^3\\-x+x^3+2x & 2x^2+1-x^2\end{pmatrix}$$

$$=\begin{pmatrix}1 & -x\\x & 1\end{pmatrix}=I+A$$

43. (d) : $5^{2+4+6+\ldots+2x}=\left(\frac{4}{100}\right)^{-28}$

$\Rightarrow\ 5^{2(1+2+3+\ldots+x)}=\left(\frac{1}{25}\right)^{-28}\quad\Rightarrow\ 5^{2\cdot\frac{x(x+1)}{2}}=(5^{-2})^{-28}$

$\Rightarrow\ x(x+1)=56\ \Rightarrow\ x=7,-8$

$\because\ x\nless 0\quad\therefore\ x=7$

44. (a) : Total no. of combinations of n diff. things $=2^n-1$

and total no. of combinations of $2n$ diff. things $=2^{2n}-1$

Given that $(2^{2n}-1):(2^n-1)=513:1$

$\Rightarrow\ 2^n+1=513\qquad[\because 2^n-1\neq 0]$

$\Rightarrow\ 2^n=512=2^9\ \Rightarrow\ n=9$

45. (c) : Here, $D<0$

$\Rightarrow\ 4(2+3a)^2-4(a^2+2a+2)<0\ \Rightarrow\ 8a^2+10a+2<0$

$\Rightarrow\ (4a+1)(a+1)<0\ \Rightarrow\ \left(a+\frac{1}{4}\right)(a+1)<0$

$\Rightarrow\ -1<a<-\frac{1}{4}$

46. (b) : Given expansion

$$=\frac{1}{\lfloor n}\left[\frac{\lfloor n}{\lfloor 0\,\lfloor n}+\frac{2\lfloor n}{\lfloor 1\,\lfloor n-1}+\frac{2^2\lfloor n}{\lfloor 2\,\lfloor n-2}+\ldots+\frac{2^n\lfloor n}{\lfloor n\,\lfloor 0}\right]$$

$$=\frac{1}{\lfloor n}[{}^nC_0+{}^nC_1.2+{}^nC_2.2^2+\ldots+C_n2^n]$$

$$=\frac{1}{\lfloor n}(1+2)^n=\frac{3^n}{\lfloor n}.$$

47. (d) : $\because$ Sum of the roots = 2009, which is odd and both roots are prime

$\therefore$ One root must be even & other root must be odd

$\Rightarrow$ Their product must be even

$\Rightarrow$ k is even

But, product of 2 primes can never be even

$\therefore$ We get a contradiction

$\therefore$ No value of k possible here

48. (c) : Here, $2b=a+c$ ...(1)

Also, $\frac{\frac{w}{x}-1}{a}=\frac{\frac{w}{y}-1}{b}=\frac{\frac{w}{z}-1}{c}=\frac{1}{k}$ (say)

$\Rightarrow\ a=k\left(\frac{w}{x}-1\right), b=k\left(\frac{w}{y}-1\right), c=k\left(\frac{w}{z}-1\right)$

From (1), $2k\left(\frac{w}{y}-1\right)=k\left(\frac{w}{x}-1\right)+k\left(\frac{w}{z}-1\right)$

$\Rightarrow\ \frac{2}{y}=\frac{1}{x}+\frac{1}{z}\ \Rightarrow\ x, y, z$ are in H.P.

49. (c) : $\alpha=\operatorname{cosec}^2\theta-\cot^2\theta=1$,

$\beta=\sec^2\theta-\tan^2\theta=1$

$\therefore$ Roots are equal $\Rightarrow D=0$

50. (a) : $\because$ 6 different letters can be arranged in $\lfloor 6$ ways

$\therefore$ Number of words beginning with $M=\lfloor 5$

$\Rightarrow$ Number of words not beginning with $M=\lfloor 6-\lfloor 5=p$ ...(1)

Number of words beginning with M and ending with $y=\lfloor 4$

$\Rightarrow$ Numbrt of words beginning with M but not ending with $y=\lfloor 5-\lfloor 4=q$ ...(2)

$\therefore\ p:q=\frac{\lfloor 5(6-1)}{\lfloor 4(5-1)}=25:4$

CHALLENGING PROBLEMS

for various Engineering Entrance Exams

1. If $\dfrac{1}{(x-a)^2(x-b)} = \dfrac{\lambda}{(x-a)^2} + \dfrac{\mu}{x-a} + \dfrac{v}{x-b}$, then which is not true ?
(a) $\lambda^2 = v$ (b) $\lambda^2 + \mu = 0$
(c) $\mu + v = 0$ (d) none of these

2. If $xy > 0$ and $(x+y)\dfrac{y}{x} = -\dfrac{3}{2}, x + y + \dfrac{y}{x} = \dfrac{1}{2}$, then $\sum_{r=0}^{\infty}\left(\dfrac{x}{y}\right)^r = ?$
(a) 1 (b) $\dfrac{3}{2}$ (c) 3 (d) 0

3. If $3x^{\log_{45}2} = 16 - 2^{\log_{45}x}$, then $|14 - x| = ?$
(a) 2011 (b) 20 (c) 76 (d) 10

4. Which interval is included in the domain of
$$f(x) = \operatorname{cosec}^{-1}\left(\frac{x + \frac{22}{x} - 8}{x + \frac{22}{x} + 8}\right)?$$
(a) $(-\infty, -22)$ (b) $(-2, -1)$
(c) $(-\infty, 8)$ (d) all of these

5. If 13, 11, 27 be remainders when $\alpha x^2 + \beta x + \gamma$ is divided by $x + 1$, $x + 2$ and $x + 3$ respectively, then $\alpha^2 + \beta^2 + \gamma^2 = ?$
(a) 3861 (b) 2501
(c) 2011 (d) none of these

6. Sum of 4 nos. in A.P. is 6 so that the product of extremes is 10 times the product of the means, then sum of the numerical value of all the nos. is
(a) 36 (b) 6 (c) 45 (d) 7

7. If α, β be 2 distinct solutions of $3\cos\theta + 4\sin\theta = 6$ $(0° < \alpha, \beta < 360°)$ such that $\sin(\alpha + \beta) = \dfrac{\lambda\mu}{(\lambda - \mu)^2}$, then possible values of λ and μ are
(a) $\lambda = 8, \mu = 3$ (b) $\mu = -3, \lambda = -8$
(c) $\lambda = 3, \mu = 8$ (d) all of these

8. If 3 normals drawn from given point (h, k) to $y^2 = 4ax$ be real and different, then
(a) $9ak^2 + 4(h - 2a^2) < 0$ (b) $27ak^2 > 4(h - 2a)^3$
(c) $27ak^2 < 4(h - 2a)^3$ & $h > 2a$ (d) 0

9. If $y = f(x)$ be an invertible function and $h(x) = xf(x)$, then $\int_{f(a)}^{f(b)} f^{-1}(y)dy + \int_a^b f(x)dx =$
(a) $h(a) - h(b)$ (b) $h(b) - h(a)$
(c) $h(a) + h(b)$ (d) $bh(b) - ah(a)$

10. A metallic ellipse in $\dfrac{x^2}{100} + \dfrac{y^2}{91} = 1$ form is heated in such a way that it coincides with its auxiliary circle in K seconds, then K = ? (Given that e change at the rate of $\dfrac{10^{-2}}{67}$ per second)
(a) 2010 seconds (b) 670 seconds
(c) 33.5 seconds (d) none of these

For question nos. 11 to 14, write
(a) if answer is a +ve integer
(b) if answer is a proper fraction
(c) if answer is zero
(d) if answer is irrational number.

11. If z_1, z_2 represent 2 points P and Q respectively on complex plane and $|z_1| = |z_2|$, $\angle POQ = \pi/3$, then $z_1^3 + z_2^3 = ?$

12. If $x = 29.0292929....$to ∞, $y = 19.9101010.....$to ∞ and $\{x\} = 1 - \dfrac{\lambda}{990}, \{y\} = 1 - \dfrac{\mu}{990}$, then $2\lambda + \mu$ = ? (where $\{\cdot\}$ denotes fractional part function)

13. If α, β be the minimum values of $\dfrac{3 + \cos 4\theta}{1 - \cos 4\theta}$ and $\sin^6\theta + \cos^6\theta$, then $\alpha - \beta = ?$

14. If $f(x) = \sum_{r=1}^{10}\{x, r\}$, then $\left\{f(\{\sqrt{3}\})\right\} = ?$ (where $\{\cdot\}$ denotes the fractional part function

15. In ΔABC, if $\Sigma\cos^2 A = \alpha + \beta.\ \Pi(\cos A)$, then

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numerical value of the sum of α and β is
(a) -1 (b) 1 (c) 3 (d) 0

16. If $a_r = \dfrac{1}{r+1}$ and $a_1 + (1 - a_1)a_2 + (1 - a_1)(1 - a_2)a_3$ +........to 2010 terms = $1 - K$ then K = ?
(a) a_{2010} (b) a_{2011} (c) 2011 (d) 0

17. If a, p, m be the remainders when 32 is divided by 5, 32^{32} is divided by 3 and 32^{32} is divided by 7 respectively, then
(a) a, p, m are in A.P. (b) p, a, m are in G.P.
(c) $a = p = m$ (d) none of these

18. If $f(x) = \cos(2010\{x^3\}(2011^{[x^2]} + 2012x))$; $x \in R$, then $f_{max.}$ = ? [$\{\cdot\}$ denotes fractional part function and $[\cdot]$ denotes greatest integer function]
(a) 0 (b) $\dfrac{1}{\sqrt{2}}$ (c) 1 (d) -1

19. If $2\sin x(1005\cos x - 2011) = 2011\cos x - 4020\sin^2 x$ then how many values of $\tan\dfrac{x}{2}$ are possible ?
(a) 0 (b) 2 (c) 4 (d) 1

20. If between 2 positive numbers α and β ($2\alpha < 3\beta$), 3 A.M.s' a, b, c and 3 H.M.s' x, y, z be inserted such that $2(a + b + c) = 4(x^{-1} + y^{-1} + z^{-1}) = 9$, then $1 + 10(\beta + 100\alpha)$ = ?
(a) 1021 (b) 2001 (c) 2011 (d) 0

21. $\lim\limits_{x\to 0} \displaystyle\int_0^x \frac{2t\,dt}{(e^x - 1 - x)\sqrt{\frac{2a}{3} - \frac{t}{2} + 104}} = \frac{1}{19}$, then a = ?
(a) 2010 (b) 2011 (c) 2012 (d) 1

22. If A and B be 2 events and probability that exactly one of them occurs is $\lambda P(A) + \mu P(B) + \phi P(A \cap B)$, then
(a) $\lambda + \mu = \phi$ (b) $\lambda + \mu + \phi = 0$
(c) $\lambda + \mu = 2\phi$ (d) none of these

23. In ΔABC, $A = 60°$, $B = 45°$, $c = 5$ and $CM \perp AB$. A circle with CM as diameter cuts CA and CB at Q and S respectively so that $SQ = \dfrac{C\sqrt{\lambda}}{\mu\sqrt{\mu}}$, then $1 + 10(c^2\lambda - \mu^3)\lambda$ = ?
(a) 2001 (b) 2021 (c) 2013 (d) 2011

24. If $\displaystyle\int \frac{dx}{\sqrt{1-\tan^2 x}} = \frac{1}{\alpha}\sin^{-1}(\alpha \sin x) + c$, then $\alpha^2(1 + \alpha^2)(1 + \alpha^4)(1 + \alpha^2 + \alpha^{12})$ = ?
(a) 2008 (b) 2010 (c) 2012 (d) 2011

25. If the areas of n distinct squares be in A.P. such that the sum of areas of all n squares be n^2, then length of each side of which square is a prime number
(a) 13^{th} (b) 25^{th}
(c) 5^{th} (d) all of these

26. If S and S' be 2 foci and P is any point on $\dfrac{x^2}{48} + \dfrac{y^2}{36} = 1$ such that $\angle PSS' = \theta$, $\angle PS'S = \phi$, then the ratio of $\dfrac{1-\cos\theta}{1+\cos\phi}$ and $\dfrac{1+\cos\theta}{1-\cos\phi}$ is
(a) 1 : 3 (b) 3 : 1 (c) 1 : 9 (d) 9 : 1

27. If α, β be eccentric angles of 2 ends of a focal chord of ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and e be its eccentricity, then e =
(a) $\dfrac{\cos\frac{\alpha-\beta}{2}}{\cos\frac{\alpha+\beta}{2}}$ (b) $\dfrac{\sin\alpha - \sin\beta}{\sin(\alpha-\beta)}$
(c) $\dfrac{\sin(\alpha-\beta)}{\sin\beta - \sin\alpha}$ (d) none of these

28. If A be the area of the region bounded by $y = 2011^x \cdot \log 2011$, $y = 2011^{-x} \cdot \log_e 2011$ and $x = 1$, then $[A + 2]\{A\}$ = ? (where $[x]$ denotes integral part and $\{x\}$ denotes fractional part of x)
(a) 1 (b) 2011
(c) undefined (d) none of these

29. If in ΔABC, $2\cos A \sin C = \sin B$, then
$$\Delta = \begin{vmatrix} 1 & 1+\sin A & \sin A(\sin^2 A + 1) \\ 1 & 1+\sin B & \sin B(\sin^2 B + 1) \\ 1 & 1+\sin C & \sin C(\sin^2 C + 1) \end{vmatrix}$$
(a) 0 (b) -1 (c) 1 (d) 2

30. If m and M be the least and greatest value of $f(x) = \dfrac{2011 + x^{2012}}{x}$ $(x \neq 0)$, then
(a) $m = |M|$ (b) $|m| = M$
(c) $m < M$ (d) none of these

SOLUTIONS

1. (d) : $\dfrac{1}{(x-a)^2(x-b)} = \dfrac{1}{x-a}\left\{\dfrac{1}{x-a}\cdot\dfrac{1}{x-b}\right\}$

$= \dfrac{1}{x-a}\cdot\dfrac{1}{a-b}\left\{\dfrac{1}{x-a} - \dfrac{1}{x-b}\right\}$

$= \dfrac{1}{a-b}\cdot\left\{\dfrac{1}{(x-a)^2} - \dfrac{1}{(x-a)(x-b)}\right\}$

$= \dfrac{1}{a-b}\dfrac{1}{(x-a)^2} - \dfrac{1}{(a-b)^2}\cdot\dfrac{1}{x-a} + \dfrac{1}{(a-b)^2}\cdot\dfrac{1}{x-b}$

$\therefore \lambda = \dfrac{1}{a-b}$, $\mu = -\dfrac{1}{(a-b)^2}$, $v = \dfrac{1}{(a-b)^2}$

2. (c) : Let $y/x = a \quad \therefore \quad x + y = -3/2a$ ...(i)
and $x + y + a = 1/2$ ...(ii)

$\Rightarrow \quad -\dfrac{3}{2a} + a = \dfrac{1}{2} \quad \Rightarrow \quad 2a^2 - a - 3 = 0$

$\therefore \quad a = \dfrac{3}{2} \quad \left[\because\ a \neq -1 \text{ as } xy > 0 \ \Rightarrow\ \dfrac{y}{x} > 0 \ \Rightarrow\ a > 0\right]$

$\Rightarrow \quad \dfrac{y}{x} = \dfrac{3}{2} \quad \Rightarrow \quad \dfrac{x}{y} = \dfrac{2}{3}$

$\therefore \quad \displaystyle\sum_{r=0}^{\infty}\left(\frac{x}{y}\right)^r = 1 + \frac{x}{y} + \frac{x^2}{y^2} + \ldots\ldots + \text{to } \infty$

$= 1 + \dfrac{2}{3} + \left(\dfrac{2}{3}\right)^2 + \ldots\ldots + \infty = 3 \quad \left[\because\ r = \dfrac{2}{3} < 1\right]$

3. (a) : $\because \quad a^{\log_x b} = b^{\log_x a} \quad \therefore \quad x^{\log_{45} 2} = 2^{\log_{45} x}$

$\therefore$ Given equation becomes $3(2^{\log_{45} x}) + 2^{\log_{45} x} = 16$

$\Rightarrow \quad 4(2^{\log_{45} x}) = 16 \quad \Rightarrow \quad 2^{\log_{45} x} = 2^2$

$\Rightarrow \quad \log_{45} x = 2 \quad \therefore \quad x = 45^2 = 2025$

$\therefore \quad |14 - x| = |-2011| = 2011$

4. (a) : $\left|\dfrac{x + \dfrac{22}{x} - 8}{x + \dfrac{22}{x} + 8}\right| \geq 1 \Rightarrow \dfrac{|x^2 + 22 - 8x|}{|x^2 + 22 \pm 8x|} \geq 1$ ...(i)

$\Rightarrow \quad |x^2 + 22 - 8x| \Rightarrow |x^2 + 22 + 8x|$
[Here, $x^2 + 22 + 8x = (x + 4)^2 + 6 > 0$]
$\Rightarrow \quad (x^2 + 22 - 8x)^2 \ \geq\ (x^2 + 22 + 8x)^2$
$\Rightarrow \quad (2x^2 + 44)(-16x) \geq 0 \Rightarrow x \leq 0$ as $2x^2 + 44 > 0$ always.

But, $x \neq 0$ due to presence of the term $\dfrac{22}{x}$ in question.
Hence, $x < 0$ or $x \in (-\infty, 0)$
$\Rightarrow$ $(-\infty, -22)$ is included in the domain of $f(x)$.

5. (c) : Let $f(x) = \alpha x^2 + \beta x + \gamma$
$\therefore$ From Remainder theorem, $f(-1) = 13$, $f(-2) = 11$ and $f(-3) = 27$
$\Rightarrow \alpha - \beta + \gamma = 13$ ...(i), $4\alpha - 2\beta + \gamma = 11$ ..(ii)
$9\alpha - 3\beta + \gamma = 27$ ...(iii)
On solving we get $\alpha = 9$, $\beta = 29$, $\gamma = 33$
$\alpha^2 + \beta^2 + \gamma^2 = 81 + 841 + 1089 = 2011$.

6. (a) : Let nos. as $a - 3d$, $a - d$, $a + d$, $a + 3d$
$\therefore \quad (a - 3d) + (a - d) + (a + d) + (a + 3d) = 6 \quad \Rightarrow \quad a = \dfrac{3}{2}$
and, $\quad (a - 3d)(a + 3d) = 10\,(a - d)(a + d)$

$\Rightarrow a^2 - 9d^2 = 10\,(a^2 - d^2) \quad \Rightarrow \quad d^2 = 9a^2 \quad \Rightarrow \quad d = \pm\dfrac{9}{2}$

$\therefore$ Nos. are 15, 6, –3, –12 or –12, –3, 6, 15
$\therefore$ Sum of their numerical values
$= |15| + |6| + |-3| + |-12| = 36$

7. (d) : $3\cos\alpha + 4\sin\alpha = 6$ ...(i), $3\cos\beta + 4\sin\beta = 6$..(ii)
On subtracting, $3(\cos\beta - \cos\alpha) = 4(\sin\alpha - \sin\beta)$

$\Rightarrow \quad 3.2\sin\dfrac{\beta + \alpha}{2}\sin\dfrac{\alpha - \beta}{2} = 4.2\cos\dfrac{\alpha + \beta}{2}\sin\dfrac{\alpha - \beta}{2}$

$\therefore \quad \tan\dfrac{\alpha + \beta}{2} = \dfrac{4}{3} \quad \left[\because\ \alpha \neq \beta \ \therefore\ \sin\dfrac{\alpha - \beta}{2} \neq 0\right]$

$\Rightarrow \quad \sin(\alpha + \beta) = \dfrac{2\left(\dfrac{4}{3}\right)}{1 + \left(\dfrac{4}{3}\right)^2} = \dfrac{24}{25} = \dfrac{\lambda\mu}{(\lambda - \mu)^2}$ ...(iii) (given)

(iii) is satisfied for options (a), (b) and (c)

8. (c) : Equation of normal to $y^2 = 4ax$ ... (i)
at $P(am^2, -2am)$ is $am^3 + (2a - x)m + y = 0$ ... (ii)
If it passes through (h, k), then $am^3 + (2a - h)\,m + k = 0$
Let $f(m) = am^3 + (2a - h)m + k$

$f'(m) = 3am^2 + (2a - h) \Rightarrow f'(m) = 0$ has roots

$\pm\sqrt{\dfrac{h - 2a}{3a}} = \pm\,\alpha$ (say)

If α, $-\alpha$ be real then roots of (ii) will be all real and 3 normals to (i) drawn from (h, k) will be real and different.
$\therefore$ We must have $h - 2a > 0$ as $a > 0$ *i.e.* $h > 2a$
Further, $f(\alpha)$ and $f(-\alpha)$ must be of opp. sign
$\Rightarrow f(\alpha).f(-\alpha) < 0$
$\Rightarrow \{a\alpha^3 + (2a - h)\alpha + k\}\,\{a(-\alpha)^3 + (2a - h)(-\alpha) + k\} < 0$
On putting the values of α and simplification, we get
$27ak^2 < 4\,(h - 2a)^3$

9. (b) : $I = \underbrace{\int_a^b t.f'(t)dt}_{I_1} + \int_a^b f(x)dx$

$= \left[t \cdot f(t)\right]_a^b - \int_a^b 1 \cdot f(t)dt + \int_a^b f(t)dt$
$\left[\because\ \int_a^b f(x)dx = \int_a^b f(t)dt\right]$
$= bf(b) - af(a) = h(b) - h(a)$

Let $f^{-1}(y) = t$ in I_1
$\therefore y = f(t)$
$dy = f'(t)dt$

y	$f(a)$	$f(b)$
t	a	b

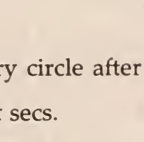
10. (a) : Before heating, $e = \sqrt{1 - \dfrac{91}{100}} = \dfrac{3}{10}$

$\because$ Eccentricity of any circle is zero
$\therefore$ When ellipse coincides with auxiliary circle after heating, e will change from $\dfrac{3}{10}$ to 0 in k secs.

$\therefore \quad \dfrac{de}{dt} = -\dfrac{10^{-2}}{67} \quad \Rightarrow \quad \displaystyle\int_{3/10}^{0} de = -\int_0^k \frac{10^{-2}}{67} \cdot dt$

$\Rightarrow \quad 0 - \dfrac{3}{10} = -\dfrac{10^{-2}}{67} \cdot (k - 0) \Rightarrow k = \dfrac{3}{10} \times \dfrac{67}{10^{-2}} = 2010 \text{ secs}$

11. (c) : Here, $z_2 - 0 = (z_1 - 0)\,e^{i \cdot \pi/3}$



$\Rightarrow \quad z_2^3 = z_1^3 \cdot e^{i\pi} = z_1^3(\cos\pi + i\sin\pi) = -z_1^3$

$\therefore \quad z_1^3 + z_2^3 = 0$

12. (a) : $\because \quad x = 29\cdot0\dot{2}\dot{9}$ and $y = 19\cdot9\dot{1}\dot{0}$

Contd. on page no. 88

$$= \frac{a(\varepsilon^{-k/2} - \varepsilon^{k/2}) + \varepsilon^{-k/2} + \varepsilon^{k/2}}{a(\varepsilon^{-k/2} + \varepsilon^{k/2}) + \varepsilon^{-k/2} - \varepsilon^{k/2}} = \frac{\cos\frac{\pi k}{5} - ia\sin\frac{\pi k}{5}}{a\cos\frac{\pi k}{5} - i\sin\frac{\pi k}{5}}$$

In particular, at $k = 0$ the solution is $x_0 = \frac{1}{a}$.

(ix) (a), (b)

The roots of the equation $x^7 = 1$ are

$\cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}$ $(k = 0, 1, 2, ..., 6)$.

Therefore, the roots of the equation

$x^6 + x^5 + x^4 + x^3 + x^2 + x + 1 = 0$ (*)

will be $x_k = \cos\frac{2k\pi}{7} + i\sin\frac{2k\pi}{7}$ $(k = 1, 2, 3, 4, 5, 6)$.

Put $x + \frac{1}{x} = y$,

then $x^2 + \frac{1}{x^2} = y^2 - 2$, $x^3 + \frac{1}{x^3} = y^3 - 3y$.

Equation (*) may be rewritten in the following way $\left(x^3 + \frac{1}{x^3}\right) + \left(x^2 + \frac{1}{x^2}\right) + \left(x + \frac{1}{x}\right) + 1 = 0$.

It is evident that $x_1 = \bar{x}_6, x_2 = \bar{x}_5, x_3 = \bar{x}_4$,

$x_k = \bar{x}_7 - k = x_k + \bar{x}_k = 2\cos\frac{2k\pi}{7}$.

Hence, we may conclude that the quantities $2\cos\frac{2\pi}{7}, 2\cos\frac{4\pi}{7}, 2\cos\frac{8\pi}{7}$ are the roots of the following equation $y^3 + y^2 - 2y - 1 = 0$ let the roots of a certain cubic equation be α, β, γ.

We then have $\alpha + \beta + \gamma = a$, $\alpha\beta + \alpha\gamma + \beta\gamma = b$, $\alpha\beta\gamma = c$.

Let the equation, whose roots are the quantities $\sqrt[3]{\alpha}, \sqrt[3]{\beta}, \sqrt[3]{\gamma}$, be $x^3 - Ax^2 + Bx - C = 0$.

Then $\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = A$,

$\sqrt[3]{\alpha}\sqrt[3]{\beta} + \sqrt[3]{\alpha}\sqrt[3]{\gamma} + \sqrt[3]{\beta}\sqrt[3]{\gamma} = B$, $\sqrt[3]{\alpha\beta\gamma} = C$.

Let us make use of the following identity

$$(m + p + q)^3 = m^3 + p^3 + q^3 + 3(m + p + q)(mp + mq + pq) - 3mpq.$$

Putting here instead of m, p and q first $\sqrt[3]{\alpha}, \sqrt[3]{\beta}, \sqrt[3]{\gamma}$, and then $\sqrt[3]{\alpha\beta}, \sqrt[3]{\alpha\gamma}, \sqrt[3]{\beta\gamma}$, we find

$A^3 = a + 3AB - 3C$, $B^3 = b + 3BCA - 3C^2$.

In our case we have $a = -1$, $b = -2$, $c = 1$, $C = 1$.

Hence $A^3 = 3AB - 4$, $B^3 = 3AB - 5$.

Multiplying these equations and putting $AB = z$, we find $z^3 - 9z^2 + 27z - 20 = 0$,

$(z - 3)^3 + 7 = 0$, $z = 3 - \sqrt[3]{7}$.

But $A^3 = 3z - 4 = 5 - 3\sqrt[3]{7}$, $A = \sqrt[3]{5 - 3\sqrt[3]{7}}$.

Therefore, indeed,

$$\sqrt[3]{\alpha} + \sqrt[3]{\beta} + \sqrt[3]{\gamma} = \sqrt[3]{2\cos\frac{2\pi}{7}} + \sqrt[3]{2\cos\frac{4\pi}{7}} + \sqrt[3]{2\cos\frac{8\pi}{7}} = \sqrt[3]{5 - 3\sqrt[3]{7}}.$$

The second identity is proved in the same way.

Contd. from page no. 26

$\therefore\ \{x\} = 0.0\dot{2}\dot{9} = \dfrac{29}{990} = 1 - \dfrac{961}{990} \Rightarrow \lambda = 961$

$\{y\} = 0.9\dot{1}\dot{0} = \dfrac{910-9}{990} = \dfrac{901}{990} = 1 - \dfrac{89}{990} \Rightarrow \mu = 89$

$\therefore\ 2\lambda + \mu = 2011$

13. (b) : $\because \dfrac{3+\cos 4\theta}{1-\cos 4\theta} = \dfrac{2+(1+\cos 4\theta)}{2\sin^2 2\theta} = \dfrac{2+2\cos^2 2\theta}{2(2\sin\theta\cos\theta)^2}$

$= \dfrac{1+\cos^2 2\theta}{4\sin^2\theta\cos^2\theta} = \dfrac{(\cos^2\theta+\sin^2\theta)^2 (\cos^2\theta - \sin^2\theta)^2}{4\sin^2\theta\cos^2\theta}$

$= \dfrac{2(\cos^4\theta + \sin^4\theta)}{4\sin^2\theta\cos^2\theta} = \dfrac{\cot^2\theta + \tan^2\theta}{2} \ge \sqrt{\cot^2\theta\cdot\tan^2\theta}$

$[\because\ AM \ge GM]$

$\therefore$ Minimum value of $\dfrac{3+\cos 4\theta}{1-\cos 4\theta} = 1 = \alpha$

Otherwise: $\dfrac{3+\cos 4\theta}{1-\cos 4\theta}$ will be min, if N^r is smallest and D^r is greatest, which is possible when $\cos 4\theta = -1$.

$\therefore\ \alpha = \dfrac{3+(-1)}{1-(-1)} = 1$

Also, $\sin^6\theta + \cos^6\theta = (\sin^2\theta + \cos^2\theta)^3 - 3\sin^2\theta.\cos^2\theta.(\sin^2\theta + \cos^2\theta)$

$= 1 - \dfrac{3}{4}\sin^2 2\theta \ge \dfrac{1}{4} \quad [\because\ 0 \le \sin^2 2\theta \le 1]$

$\therefore$ Minimum value of $\sin^6\theta + \cos^6\theta = \dfrac{1}{4} = \beta$

$\therefore\ \alpha - \beta = 1 - \dfrac{1}{4} = \dfrac{3}{4}$, which is a proper fraction.

14. (d) : $f(x) = \{x+1\} + \{x+2\} + \{x+3\} + \ldots + \{x+10\}$
$= \{x\} + \{x\} + \ldots$ to 10 terms $[\because\ \{x+n\} = \{x\}$, if $n \in I] = 10\{x\}$

$\therefore\ f(\{\sqrt{3}\}) = f(\sqrt{3}-1) = 10\{\sqrt{3}-1\} = 10(\sqrt{3}-1)$

$[\because\ 0 < \sqrt{3}-1 < 1]$

$= 10\ (0{\cdot}732)$ (app.) $= 7{\cdot}32$ (app.) $\therefore\ [f(\{\sqrt{3}\})] = 7$

$\therefore\ \{f(\{\sqrt{3}\})\} = f(\{\sqrt{3}\}) - [f(\{\sqrt{3}\})]$

$= 10(\sqrt{3}-1) - 7 = 10\sqrt{3} - 17$, which is irrational

15. (b) : $\because\ A + B + C = \pi$ ...(i)

$\therefore\ \Sigma\cos^2 A = \cos^2 A + (1 - \sin^2 B) + \cos^2 C$
$= 1 + (\cos^2 A - \sin^2 B) + \cos^2 C$
$= 1 + \cos(A+B)\cos(A-B) + \cos C\cdot\cos C$
$= 1 + \cos(\pi - C)\cos(A-B) + \cos C\cdot\cos(\pi - \overline{A+B})$

[from (i)]

$= 1 - \cos C\cdot\{\cos(A-B) + \cos(A+B)\}$
$= 1 - 2\cos A\cos B\cos C = 1 - 2\prod(\cos A)$

$\therefore\ \alpha = 1, \beta = -2$(on comparison) $\therefore |\alpha + \beta| = |1-2| = 1$

16. (a) : $1 - K = \dfrac{1}{2} + \left(1-\dfrac{1}{2}\right)\cdot\dfrac{1}{3} + \left(1-\dfrac{1}{2}\right)\left(1-\dfrac{1}{3}\right)\cdot\dfrac{1}{4}$

$+ \left(1-\dfrac{1}{2}\right)\left(1-\dfrac{1}{3}\right)\left(1-\dfrac{1}{4}\right)\cdot\dfrac{1}{5} + \ldots$ to 2010 terms

$= \dfrac{1}{2} + \dfrac{1}{2.3} + \dfrac{1}{3.4} + \dfrac{1}{4.5} + \ldots$ to 2010 terms

$= \left(1-\dfrac{1}{2}\right) + \left(\dfrac{1}{2}-\dfrac{1}{3}\right) + \left(\dfrac{1}{3}-\dfrac{1}{4}\right) + \left(\dfrac{1}{4}-\dfrac{1}{5}\right) + \ldots + \left(\dfrac{1}{2010}-\dfrac{1}{2011}\right)$

$= 1 - \dfrac{1}{2010} \Rightarrow K = \dfrac{1}{2011} = a_{2010}$

17. (b) : Clearly, $a = 2$ ...(i)

$\because\ 32^{32} = (2^5)^{32} = (3-1)^{160} = 3^{160} - {}^{160}C_1 3^{159} + {}^{160}C_2\cdot 3^{158} \ldots\ldots - {}^{160}C_{159}\cdot 3 + 1$

$=$ (a number multiple of 3) $+ 1 = 3m + 1$(say)$[m \in I^+]$

$\therefore$ 1 is the remainder, when 32^{32} is divided by 3

$\Rightarrow p = 1$ ...(ii)

Also, $32^{32^{32}} = 32^{3m+1} = 2^{15m+5} = 2^2\cdot(2^3)^{5m+1}$

$= 4[(7+1)^{5m+1}] = 4[7^{5m+1} + {}^{5m+1}C_1\cdot 7^{5m} + \ldots\ldots + {}^{5m+1}C_{5m}\cdot 7 + 1] = 4[$A multiple of 7) + 1]

$\therefore$ Clearly when $32^{32^{32}}$ is divided by 7, the remainder will be 4 $\Rightarrow m = 4$ ...(iii)

$\therefore$ (i), (ii) and (iii) $\Rightarrow$ p, a, m are in G.P.

18. (c) : Maximum value of $\cos\theta$ is 1

$\therefore\ f_{\max.} = 1$, if possible for some $x \in R$

$\because\ f(0) = \cos(2010\ \{0\}\ (2011^{[0]} + 2012 \times 0)) = \cos 0 = 1$

Hence, $f_{\max.} = 1$

19. (b) : on re-arranging, we get

$2010\sin x(\cos x + 2\sin x) - 2011(\cos x + 2\sin x) = 0$

$\Rightarrow (\cos x + 2\sin x)(2010\sin x - 2011) = 0$

$\Rightarrow \cos x + 2\sin x = 0$ or, $2010\sin x - 2011 = 0$

$\Rightarrow \dfrac{1-t^2}{1+t^2} + \dfrac{2.2t}{1+t^2} = 0$ or, $\sin x = \dfrac{2011}{2010} > 1$ (Impossible)

$\Rightarrow t^2 - 4t - 1 = 0 \Rightarrow t = 2 \pm \sqrt{5}$ $\left[\text{where } t = \tan\dfrac{x}{2}\right]$

$\therefore$ 2 values of t *i.e.* $\tan\dfrac{x}{2}$ are possible.

20. (a) : Sum of n A.M.s' between any 2 nos. in n times the single mean between the 2 nos.

$\therefore a+b+c = 3\left(\dfrac{\alpha+\beta}{2}\right) \Rightarrow \dfrac{a}{2} = \dfrac{3}{2}(\alpha+\beta)\ \therefore\ \alpha+\beta = 3$ ...(i)

and, $\dfrac{1}{x}+\dfrac{1}{y}+\dfrac{1}{z} = 3\left(\dfrac{\frac{1}{\alpha}+\frac{1}{\beta}}{2}\right) \Rightarrow \dfrac{9}{4} = \dfrac{3}{2}\left(\dfrac{1}{\alpha}+\dfrac{1}{\beta}\right)$

$\Rightarrow \dfrac{\alpha+\beta}{\alpha\beta} = \dfrac{3}{2}\ \therefore\ \alpha\beta = 2$ ...(ii)

On solving (i) and (ii) : $\alpha = 1, \beta = 2$ or $\alpha = 2, \beta = 1$

But, $2\alpha < 3\beta$ (given) $\therefore$ $\alpha = 1, \beta = 2$

$\therefore\ 1 + 10(\beta + 100\alpha) = 1 + 10(2 + 100) = 1021$

21. (a) : Rearranging the equation, we get

$$\Rightarrow \lim_{x\to 0}\frac{\left(\dfrac{2x}{\sqrt{\dfrac{2a}{3}-\dfrac{x}{2}+104}}\right)}{(e^x-1)}=\frac{1}{19}$$

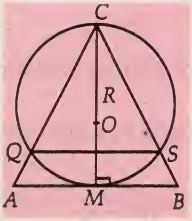
$$\Rightarrow 2\lim_{x\to 0}\frac{x}{e^x-1}\cdot\lim_{x\to 0}\frac{1}{\sqrt{\dfrac{2a}{3}-\dfrac{x}{2}+104}}=\frac{1}{19}$$

$$\Rightarrow 2(1)\cdot\frac{1}{\sqrt{\dfrac{2a}{3}+104}}=\frac{1}{19}\Rightarrow \frac{2a}{3}+104=4\times 361=1444$$

$\Rightarrow a = 2010$

22. (b) : $P(\text{exactly one}) = P(A\cap B')$ or $P(A'\cap B)$
$= \{P(A)-P(A\cap B)\}+\{P(B)-P(A\cap B)\}$
$= P(A)+P(B)-2P(A\cap B) \Rightarrow \lambda = 1, \mu = 1, \phi = -2$
$\therefore \lambda+\mu+\phi = 0$

23. (d) : $\because \dfrac{CM}{AC}=\sin A \Rightarrow CM = b\sin A$ ...(i)



In ΔCQS, $\dfrac{SQ}{\sin C}=2R$ (sine rule)
$\Rightarrow SQ = CM.\sin C$
[$\because$ $CM = 2R$, where R is radius of circumcircle)
$= b\sin A.\sin C$ [from (i)]
$= c\sin B\sin A$ [$\because b\sin C = c\sin B$)
$= c\sin 45°\sin 60° = \dfrac{c\sqrt{3}}{2\sqrt{2}}=\dfrac{c\sqrt{\lambda}}{\mu\sqrt{\mu}}$ (given)
$\therefore \lambda = 3, \mu = 2$
$\therefore 1 + 10(c^2\lambda-\mu^3)\lambda = 1 + 10(75-8)3 = 2011$

24. (b) : $I=\displaystyle\int\frac{\cos x\,dx}{\sqrt{\cos^2 x-\sin^2 x}}=\int\frac{\cos x\,dx}{\sqrt{1-2\sin^2 x}}$

Put $\sin x = z$, $\cos x\,dx = dz$

$$=\int\frac{dz}{\sqrt{1-2z^2}}==\frac{1}{\sqrt{2}}\sin^{-1}\left\{\frac{z}{(1/\sqrt{2})}\right\}+C$$

$$=\frac{1}{\sqrt{2}}\sin^{-1}\left(\sqrt{2}\sin x\right)+C=\frac{1}{\alpha}\sin^{-1}(\alpha\sin x)+C \text{ [given]}$$

$\Rightarrow \alpha=\sqrt{2}$ $\therefore \alpha^2(1+\alpha^2)(1+\alpha^4)(1+\alpha^2+\alpha^{12}) = 2010$

25. (d) : Let $A_1, A_2, A_3, ...A_n$ be the areas of n squares, which are in A.P. and $S_n = n^2$

$\because A_r = S_r - S_{r-1} = r^2-(r-1)^2 = 2r-1$

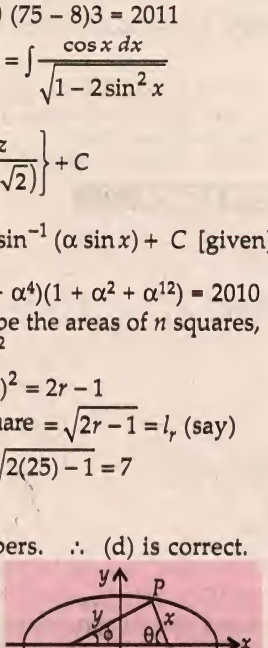
$\therefore$ length of each side of r^{th} square $=\sqrt{2r-1}=l_r$ (say)

$\therefore l_{13}=\sqrt{2(13)-1}=5;\ l_{25}=\sqrt{2(25)-1}=7$

$l_5=\sqrt{2(5)-1}=3$

$\because$ 3, 5, 7 all are prime numbers. $\therefore$ (d) is correct.

26. (c) : Here, $a=4\sqrt{3},\ b=6$



$e=\sqrt{1-\dfrac{36}{48}}=\dfrac{1}{2}$

$SS' = 2ae = 4\sqrt{3}$

Let $PS = x$ and $PS' = y$

$\therefore$ In $\Delta PSS'$, $S=\dfrac{PS+PS'+SS'}{2}=\dfrac{2a+2ae}{2}=a(1+e)$

[$\because$ In any ellipse, $PS + PS' = 2a$ (if $a > b$)]

$$\text{Now, } \tan\frac{\theta}{2}\tan\frac{\phi}{2}=\sqrt{\frac{(S-x)(S-2ae)}{S(S-y)}}\cdot\sqrt{\frac{(S-y)(S-2ae)}{S(S-x)}}$$

$$=\frac{S-2ae}{S}=\frac{a(1+e)-2ae}{a(1+e)}=\frac{1-e}{1+e}=\frac{1-1/2}{1+1/2}=\frac{1}{3}$$

$$\Rightarrow \frac{2\sin^2(\theta/2)\cdot 2\sin^2(\phi/2)}{2\cos^2(\theta/2)\cdot 2\cos^2(\phi/2)}=\frac{1}{9}$$

$$\Rightarrow \frac{(1-\cos\theta)(1-\cos\phi)}{(1+\cos\theta)(1+\cos\phi)}=\frac{1}{9}\ \therefore \frac{1-\cos\theta}{1+\cos\phi}:\frac{1+\cos\theta}{1-\cos\phi}=1:9$$

27. (a) : Let $P(a\cos\alpha, b\sin\alpha)$, $Q(a\cos\beta, b\sin\beta)$ and $S(ae, 0)$

$\because$ P, S, Q are collinear $\therefore$ Slope of PS = slope of QS

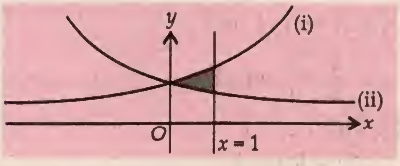
$$\Rightarrow \frac{b\sin\alpha-0}{a\cos\alpha-ae}=\frac{b\sin\beta-0}{a\cos\beta-ae}$$

$\Rightarrow \sin\alpha(\cos\beta-e)=\sin\beta(\cos\alpha-e)$

$$\Rightarrow e=\frac{\sin(\alpha-\beta)}{\sin\alpha-\sin\beta}=\frac{\cos\dfrac{\alpha-\beta}{2}}{\cos\dfrac{\alpha+\beta}{2}}$$

28. (a) : $y = 2011^x\cdot\log_e 2011$ ...(i)

$y = 2011^{-x}\cdot\log_e 2011$ ...(ii)



$\therefore$ Reqd. area $=\int_0^1(2011^x\cdot\log_e 2011-2011^{-x}\cdot\log_e 2011)dx$

$$=2009+\frac{1}{2011}=A\text{ (given)}\therefore\ [A+2]=\left[2011+\frac{1}{2011}\right]=2011$$

and $\{A\}=\dfrac{1}{2011}$ $\therefore [A+2]\{A\}=1$

29. (a) : $2\cdot\dfrac{b^2+c^2+a^2}{2bc}\cdot\dfrac{c}{2R}=\dfrac{b}{2R}$

$\Rightarrow b^2+c^2-a^2=b^2 \Rightarrow a=c$
$\therefore$ $A = C$ or ΔABC is isosceles
$\Delta = 0$ [$\because A = C$ & R_1 and R_3 are identical]

30. (a) : $f(x)=\dfrac{2011}{x}+x^{2011}\Rightarrow f'(x)=-\dfrac{2011}{x^2}+2011x^{2010}$

$f''(x)=+\dfrac{2011(2)}{x^3}+2011.2010x^{2009}$

For $f(x)$ to be max. or min. $f'(x) = 0 \Rightarrow x = \pm 1$
$\therefore f''(1) > 0 \Rightarrow f(x)$ is min. at $x = 1$
$\Rightarrow f_{\min} = f(1) = 2012 = m$
$f''(-1) < 0 \Rightarrow f(x)$ is max. at $x = -1$
$\Rightarrow f_{\max.} = f(-1) = -2012 = M$ $\therefore m = |M|$

Concept Boosters

Class XII

INDEFINITE INTEGRATION

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This column is aimed at Class XII students so that they can prepare for competitive exams such as IIT, AIEEE, etc and be also in command of what is being covered in their school as part of NCERT syllabus. The problems here are a happy blend of the straight and the twisted, the simple and the difficult and the easy and the challenging.

Substitution :

1. Evaluate the following integrals:

(i) $I=\int \frac{\sqrt{1+\ln x}}{x\ln x}dx$ (ii) $I=\int \frac{dx}{x\sqrt{x^2+4x-4}}$

(iii) $I=\int \frac{dx}{x^2\cdot\sqrt{x^2+a^2}}$ (iv) $I=\int \frac{\sqrt{1+x^2}}{x^4}dx$

(v) $\int \frac{x^2}{\sqrt{(x^6+2x^3+2)}}dx$ (vi) $I=\int \frac{dx}{(x^2+4)\sqrt{4x^2+1}}$

(vii) $I=\int \frac{dx}{\sin^2 x+\tan^2 x}$ (viii) $I=\int \frac{\sqrt{1+x^8}}{x^{13}}dx$

(ix) $I=\int \frac{dx}{\tan x\cdot\cos 2x}$ (x) $I=\int \sqrt[3]{\frac{\sin^2 x}{\cos^{14} x}}\,dx$

Integration by Parts :

2. Evaluate the following integrals:

(i) $\int \frac{\sin^{-1}x}{x^2}dx$ (ii) $\int e^{2x}\cos^3 x\,dx$

(iii) $\int e^x \sin x\sin 2x\sin 3x\,dx$

(iv) $\int x^2(e^x\sin x)dx$ (v) $\int \sin^{-1}\sqrt{\left(\frac{x}{a+x}\right)}dx$

(vi) $\int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx$ (vii) $\int x.\log(x^3+1)dx$

(viii) $\int e^x.\log(e^{2x}+5e^x+6)dx$

3. Evaluate the following integrals :

(i) $\int \sqrt{x}(\log x)^2dx$ (ii) $\int x^3\tan^{-1}x\,dx$

(iii) $\int x^2\sin x\cos x\,dx$ (iv) $\int x\sin x\sec^3 x\,dx$

(v) $\int x\cos^3 x\sin x\,dx$ (vi) $\int \frac{1+x}{(2+x)^2}e^x\,dx$

(vii) $\int \frac{\sqrt{(1-\sin x)}}{1+\cos x}\,e^{-x/2}\,dx$

Rational Functions (Partial Fractions) :

4. Evaluate the following integrals:

(i) $\int \frac{x^2dx}{(x+2)^2(x+4)^2}$ (ii) $\int \frac{x^2+1}{(x^2-1)(x^2-4)}dx$

(iii) $\int \frac{x^2+x+1}{(x+1)^2(x+2)}dx$ (iv) $\int e^{x^{1/3}}dx$

(v) $\int \frac{x+1}{(x-1)^2(x+2)^2}dx$

(vi) $\int \frac{(x^2+1)(x^2+2)(x^2+3)}{(x^2+4)(x^2+5)(x^2+6)}dx$

(vii) $\int \frac{1}{x^6+1}dx$

5. Evaluate the following integrals:

(i) $I=\int \frac{x+a}{x^2(x-a)(x^2+a^2)}dx$

(ii) $I=\int \frac{x^3-6x^2+9x+7}{(x-2)^3(x-5)}dx$

(iii) $I=\int \frac{\cos^2 x dx}{\sin x.\cos 3x}$

(iv) $I=\int \frac{1}{x^2(x^2+1)(x^2+2)^2}dx$

(v) $I=\int \frac{x^5dx}{(x-1)^2(x^2-1)}$

Irrational Functions :

6. Evaluate the following integrals:

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(i) $I=\int \frac{\sqrt{1-x^2}}{x^2}dx$ (ii) $I=\int \frac{\sqrt{x}\,dx}{x^{2/3}-x^{1/4}}$

(iii) $I=\int \frac{\sqrt{(9-x^2)}}{x^6}dx$ (iv) $I=\int \frac{dx}{x\sqrt{x^2+2x-1}}$

(v) $\int \sqrt{(1+\sec x)}\,dx$ (vi) $I=\int \frac{\sqrt{2x+x^2}}{x^2}dx$

(vii) $I=\int \frac{dx}{(x-1)\sqrt{x^2+x+1}}$

(viii) $I=\int \frac{dx}{x\sqrt{2+x-x^2}}$ (ix) $I=\int \frac{\sqrt{1+x^2}}{2+x^2}dx$

7. Evaluate the following integrals:

(i) $I=\int \frac{\ln^3 x}{x^2}dx$ (ii) $I=\int \frac{2-\sin x}{2+\cos x}dx$

(iii) $I=\int \frac{dx}{(\sin x+2\sec x)^2}$ (iv) $I=\int \sqrt{1+\operatorname{cosec} x}\,dx$

(v) $I=\int \frac{dx}{1-\sin^4 x}$ (vi) $I=\int \frac{dx}{\sin^5 x\cdot\cos^5 x}$

(vii) $I=\int \frac{dx}{\sin^2 x+\cos^4 x}$

(viii) $\int \sec^{4/7}x\operatorname{cosec}^{10/7}x\,dx$

(ix) $I=\int \sin^8 x\,dx$

(x) $I=\int \left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x\,dx$

(xi) $I=\int \frac{\cos x+2\sin x+3}{4\cos x+5\sin x+6}dx$

Integrals Involving Exponential and Logarithmic Functions :

8. Evaluate the following integrals :

(i) $I=\int (\sin^{-1}x)^2dx$ (ii) $I=\int (\tan^{-1}x)^2\cdot x dx$

(iii) $I=\int \frac{x\ln x}{\sqrt{(x^2-1)^3}}$

(iv) $I=\int (x^3-2x^2+5)e^{3x}dx$

(v) $I=\int x\cdot e^{x^{1/3}}dx$ (vi) $I=\int \frac{dx}{\sqrt{1+e^x+e^{2x}}}$

(vii) $I=\int e^{\sin x}\cdot\frac{x\cos^3 x-\sin x}{\cos^2 x}$

(viii) $I=\int \frac{dx}{e^x+e^{2x}}$

Miscellaneous :

9. Evaluate the following integrals:

(i) $I=\int \frac{x^6-2x^4+3x^3-9x^2+4}{x^5-5x^3+4x}dx$

(ii) $I=\int \frac{dx}{(x^4-1)^2}$ (iii) $I=\int \frac{\sqrt{\sin^3 2x}\,dx}{\sin^5 x}$

(iv) $I=\int \sqrt{\tan^2 x+2}\,dx$ (v) $I=\int \frac{dx}{(1-2^x)^4}$

(vi) $I=\int \frac{dx}{1-x^6}$ (vii) $I=\int \frac{x^3+2}{(x-1)(x-2)^3}dx$

(viii) $I=\int \frac{(1+x)^3}{(1-x)^3}dx$ (ix) $I=\int \frac{xdx}{(x+1)^3(x^2+1)}$

10. Evaluate the following integrals :

(i) $I=\int \frac{(e^{3x}+e^x)dx}{e^{4x}-e^{2x}+1}$ (ii) $I=\int \frac{dx}{\sqrt{1-\sin^4 x}}$

(iii) $I=\int \frac{x^2-1}{x^2+1}\cdot\frac{dx}{\sqrt{1+x^4}}$ (iv) $I=\int \sin^6 x\cos^2 x\,dx$

(v) $I=\int \frac{dx}{\sin^2\left(\frac{x}{2}\right)\cdot\sqrt{\cos^3\frac{x}{2}}}$

(vi) $I=\int \frac{x^2+3x+1}{x^4-x^2+1}dx$ (vii) $I=\int \frac{\sin x}{\sqrt{(1+\sin x)}}dx$

(viii) $I=\int \frac{(2x+3)dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}$

(ix) $I=\int \frac{3\cos x-4\sin x}{4\cos x+5\sin x}dx$

(x) Prove that $\int \frac{dx}{\sin^2 x\cos^4 x}=\frac{\sin(1+2\cos^2 x)}{3\cos^3 x}-2\cot 2x.$

SOLUTIONS

1. (i) $1+\ln x=t^2 \quad\Rightarrow\quad \frac{1}{x}dx=2t\,dt$

$$I=\int \frac{t\cdot 2t\,dt}{t^2-1}=2\int \frac{(t^2+1-1)}{t^2-1}dt$$

$$=2\int dt+2\int \frac{dt}{t^2-1}=2t+\frac{2}{2}\ln\left|\frac{t-1}{t+1}\right|$$

$$I=2\sqrt{1+\ln x}+\ln\left|\frac{\sqrt{1+\ln x}-1}{\sqrt{1+\ln x}+1}\right|+c$$

(ii) Let $x=\frac{1}{u} \quad\Rightarrow\quad dx=-\frac{1}{u^2}du$

$$I=\int \frac{-\frac{1}{u^2}du}{\frac{1}{u^2}\sqrt{1+4u-4u^2}}=\int \frac{-du}{\sqrt{1+4u-4u^2}}$$

$$=-\int \frac{du}{\sqrt{1-4(u^2-u)}}=-\int \frac{du}{\sqrt{1-4\left[\left(u-\frac{1}{2}\right)^2-\frac{1}{4}\right]}}$$

$$= -\int \frac{du}{\sqrt{2-4\left(u-\frac{1}{2}\right)^2}} = -\frac{1}{(\sqrt{2})^2}\int \frac{du}{\sqrt{\left(\frac{1}{\sqrt{2}}\right)^2-\left(u-\frac{1}{2}\right)^2}}$$

$$= -\frac{1}{(\sqrt{2})^2}\sin^{-1}\left(\frac{2-x}{\sqrt{2}x}\right)+c$$

$$I = -\frac{1}{2}\sin^{-1}\left(\frac{2-x}{\sqrt{2}x}\right)+c.$$

(iii) Let $x = \frac{1}{t} \Rightarrow \quad dx = -\frac{1}{t^2}dt$

$$I = \int \frac{-\frac{1}{t^2}dt}{\frac{1}{t^3}\sqrt{a^2t^2+1}} = -\int \frac{tdt}{\sqrt{a^2t^2+1}}$$

$$= -\frac{1}{2a^2}\int \frac{2a^2\,tdt}{\sqrt{a^2t^2+1}} = -\frac{1}{2a^2}\cdot 2\sqrt{a^2t^2+1}+c$$

$$I = -\frac{1}{a^2}\left(\frac{a^2}{x^2}+1\right)^{1/2}+c$$

(iv) Let $x = \frac{1}{t} \quad \Rightarrow \quad dx = -\frac{1}{t^2}dt$

$$I = \int \frac{\sqrt{t^2+1}}{t\times\frac{1}{t^4}}\cdot\left(-\frac{1}{t^2}dt\right) = -\int t\sqrt{t^2+1}\,dt$$

Let $t^2+1 = u \Rightarrow 2t\,dt = du$

$$I = -\frac{1}{2}\int \sqrt{u}\,du$$

$$I = -\frac{1}{2}u^{3/2}\cdot\frac{2}{3}+c = -\frac{1}{2}(t^2+1)^{3/2}\cdot\frac{2}{3}+c$$

$$= -\frac{1}{3}\left(\frac{1}{x^2}+1\right)^{3/2}+c = -\frac{1}{3x^3}(1+x^2)^{3/2}+c$$

(v) $I = \int \frac{x^2dx}{\sqrt{(x^6+2x^3+2)}}$, Let $x^3 = t \Rightarrow 3x^2dx = dt$

$$= \frac{1}{3}\int \frac{dt}{\sqrt{[(t+1)^2+1]}} = \frac{1}{3}\sinh^{-1}(t+1)+c$$

$$= \frac{1}{3}\sinh^{-1}(x^3+1)+c$$

(vi) Let $x = \frac{1}{t} \Rightarrow \quad dx = -\frac{1}{t^2}dt$

$$I = \int \frac{-\frac{1}{t^2}dt}{\frac{(1+4t^2)\sqrt{4+t^2}}{t^3}} = \int \frac{-tdt}{(1+4t^2)\sqrt{4+t^2}}$$

Let $4+t^2 = z^2 \Rightarrow 2t\,dt = 2z\,dz \Rightarrow -t\,dt = -zdz$

$$I = \int \frac{-zdz}{[1+4(z^2-4)]z} = -\int \frac{dz}{(1+4z^2-16)} = -\int \frac{dz}{(2z)^2-(\sqrt{15})^2}$$

$$= -\frac{1}{4\times 2\times\frac{\sqrt{15}}{2}}\ln\left|\frac{z-\frac{\sqrt{15}}{2}}{z+\frac{\sqrt{15}}{2}}\right|+c$$

$$= -\frac{1}{4\times\sqrt{15}}\ln\left|\frac{2z-\sqrt{15}}{2z+\sqrt{15}}\right|+c$$

$$= -\frac{1}{4\times\sqrt{15}}\ln\left|\frac{2\sqrt{4+\frac{1}{x^2}}-\sqrt{15}}{2\sqrt{4+\frac{1}{x^2}}+\sqrt{15}}\right|+c$$

$$I = -\frac{1}{4\sqrt{15}}\ln\left|\frac{2\sqrt{4x^2+1}-\sqrt{15}}{2\sqrt{4x^2+1}+\sqrt{15}}\right|+c.$$

(vii) $I = \int \frac{dx}{(\tan^2 x+\tan^2 x\cdot\sec^2 x)\cos^2 x}$

$$= \int \frac{\sec^2 x dx}{\tan^2 x(1+\sec^2 x)} = \int \frac{\sec^2 x dx}{\tan^2 x(2+\tan^2 x)}$$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$= \int \frac{dt}{t^2(2+t^2)} = \int \frac{dt}{t^2} - \int \frac{dt}{2+t^2}$$

$$= -\frac{1}{t} - \frac{1}{\sqrt{2}}\tan^{-1}\frac{t}{\sqrt{2}}+c$$

$$I = -\frac{1}{\tan x} - \frac{1}{\sqrt{2}}\tan^{-1}\left(\frac{\tan x}{\sqrt{2}}\right)+c.$$

(viii) $= \int \frac{x^4\cdot\sqrt{1+\frac{1}{x^8}}}{x^{13}}dx = \int \sqrt{1+\frac{1}{x^8}}\cdot\frac{dx}{x^9}$

Let $1+\frac{1}{x^8} = t^2 \Rightarrow \quad -\frac{8}{x^9}dx = 2t\,dt$ or $\frac{dx}{x^9} = -\frac{tdt}{4}$

$$= \frac{-1}{4}\int t\cdot(tdt) = -\frac{1}{4}\int t^2dt = -\frac{1}{4}\cdot\frac{t^3}{3}+c = \frac{-t^3}{12}+c$$

$$I = \frac{-1}{12}\left(1+\frac{1}{x^8}\right)^{3/2}+c.$$

(ix) Let $\cos 2x = t \Rightarrow -2\sin 2x\,dx = dt$

$$\Rightarrow \quad dx = \frac{dt}{-2\sin 2x}$$

$$I = \frac{1}{-4}\int \frac{dt}{\frac{\sin x}{\cos x}\cdot\sin x\cos x\cdot t} = -\frac{1}{4}\int \frac{dt}{\sin^2 x\cdot t}$$

$$= -\frac{1}{4}\int \frac{dt}{\frac{1-t}{2}\cdot t} = \frac{-1}{2}\int \frac{dt}{t(1-t)}$$

$$= \frac{-1}{2}\left[\int \frac{dt}{1-t}+\int \frac{dt}{t}\right] = \frac{1}{2}\ln|1-t| - \frac{1}{2}\ln t+c$$

$$= \frac{1}{2}\ln\left|\frac{1-t}{t}\right| + c = \frac{1}{2}\ln\left|\frac{1-\cos 2x}{\cos 2x}\right| + c$$

$$I = \ln\left|\frac{\sqrt{2}\sin x}{\sqrt{\cos 2x}}\right| + c.$$

(x) $I = \int (\tan x)^{2/3} \cdot \sec^4 x dx$

$$= \int (\tan x)^{2/3} \sec^2 x(1+\tan^2 x)dx$$

Let $\tan x = t \Rightarrow \sec^2 x dx = dt$

$$= \int t^{2/3}(1+t^2)dt = \int t^{2/3}dt + \int t^{8/3}dt$$

$$= \frac{3}{5}t^{5/3} + \frac{3}{11}t^{11/3} + c = \frac{3}{5}(\tan x)^{5/3} + \frac{3}{11}(\tan x)^{11/3} + c.$$

2. (i) Let $x = \sin\theta \Rightarrow dx = \cos\theta\, d\theta$

$$I = \int \theta \cdot \cot\theta \cdot \operatorname{cosec}\theta\, d\theta$$

$$= \theta \cdot \int \cot\theta \cdot \operatorname{cosec}\theta\, d\theta - \int\left[\frac{d}{d\theta}(\theta) \cdot \int \cot\theta \cdot \operatorname{cosec}\theta\, d\theta\right] d\theta$$

$$= -\theta\operatorname{cosec}\theta + \int \operatorname{cosec}\theta\, d\theta$$

$$= -\theta\operatorname{cosec}\theta + \log_e \tan\frac{\theta}{2} + c$$

$$I = \frac{-\sin^{-1}x}{x} + \log_e\left(\frac{1-\cos\theta}{\sin\theta}\right) + c$$

$$I = \frac{-\sin^{-1}x}{x} + \log_e\left(\frac{1-\sqrt{1-x^2}}{x}\right) + c.$$

(ii) $\int e^{2x}\cos^3 x\, dx = \frac{1}{4}\int e^{2x}(\cos 3x + 3\cos x)dx$

$\because \quad \cos 3x = 4\cos^3 x - 3\cos x$

$\therefore \quad \cos^3 x = \dfrac{\cos 3x + 3\cos x}{4}$

$$= \frac{1}{4}\int e^{2x}.\cos 3x\, dx + \frac{3}{4}\int e^{2x}\cos x\, dx$$

$$= \frac{1}{4}e^{2x}.\frac{\cos\left(3x - \tan^{-1}\frac{3}{2}\right)}{\sqrt{(3^2+2^2)}} + \frac{3}{4}\left[e^{2x}\frac{\cos\left(x - \tan^{-1}\frac{1}{2}\right)}{\sqrt{5}}\right] + c$$

(iii) $\int e^x \sin x \sin 2x \sin 3x\, dx$

$$= \frac{1}{2}\int e^x(2\sin 2x \sin x)\sin 3x\, dx$$

$$= \frac{1}{2}\int e^x(\cos x - \cos 3x)\sin 3x\, dx$$

$$= \frac{1}{4}\int e^x\{2\sin 3x\cos x - 2\sin 3x\cos 3x\}dx$$

$$= \frac{1}{4}\left[e^x\frac{\sin(4x - \tan^{-1}4)}{\sqrt{17}} + e^x\frac{\sin(2x-\tan^{-1}2)}{\sqrt{5}} - e^x\frac{\sin(6x - \tan^{-1}6)}{\sqrt{37}}\right] + c$$

(iv) $\int x^2(e^x \sin x)dx$

$$= x^2.\frac{e^x(\sin(x-\tan^{-1}1))}{\sqrt{2}} - \int 2x.\frac{e^x}{\sqrt{2}}\sin(x-\tan^{-1}1)dx$$

[($e^x \sin x$) is taken as 2nd function]

$$= \frac{x^2e^x}{\sqrt{2}}\sin\left(x-\frac{\pi}{4}\right) - \frac{2}{\sqrt{2}}\left[x.\frac{e^x}{\sqrt{2}}\sin\left(x-\frac{\pi}{2}\right) - \int\frac{e^x}{\sqrt{2}}\sin\left(x-\frac{\pi}{2}\right)dx\right]$$

$$= \frac{x^2e^x}{\sqrt{2}}\sin\left(x-\frac{\pi}{4}\right) + x\,e^x\cos x + \int e^x\sin\left(x-\frac{\pi}{2}\right)dx$$

$$= \frac{x^2\,e^x\sin\left(x-\frac{\pi}{4}\right)}{\sqrt{2}} + x\,e^x\cos x + \frac{e^x\sin\left(x-\frac{3\pi}{4}\right)}{\sqrt{2}} + c$$

(v) Set $x = a\tan^2\theta \Rightarrow dx = 2a\tan\theta\sec^2\theta\, d\theta$

$$\int \sin^{-1}\sqrt{\left(\frac{x}{a+x}\right)}dx$$

$$= \int \sin^{-1}\sqrt{\left[\left(\frac{a\tan^2\theta}{a+a\tan^2\theta}\right)\right]}2a\tan\theta\sec^2\theta\, d\theta$$

$$= 2a\int \sin^{-1}[\sin\theta]\tan\theta.\sec^2\theta\, d\theta$$

$$= 2a\int \theta.(\tan\theta.\sec^2\theta)d\theta$$

$$= 2a\left[\theta.\frac{\tan^2\theta}{2} - \int 1.\frac{\tan^2\theta}{2}.d\theta\right]$$

$$= a\theta\tan^2\theta - a\int(\sec^2\theta - 1)d\theta$$

$$= a\theta\tan^2\theta - a[\tan\theta - \theta] + c$$

$$= a.\tan^{-1}\sqrt{\left(\frac{x}{a}\right)}.\frac{x}{a} - a\sqrt{\left(\frac{x}{a}\right)} + a\tan^{-1}\sqrt{\left(\frac{x}{a}\right)} + c$$

$$= (x+a)\tan^{-1}\frac{x}{a} - \sqrt{(ax)} + c.$$

(vi) $\int \tan^{-1}\left(\frac{3x-x^3}{1-3x^2}\right)dx$

$$= \int \tan^{-1}\left[\frac{3\tan\theta - \tan^3\theta}{1-3\tan^2\theta}\right].\sec^2\theta.\, d\theta$$

$$= \int \tan^{-1}(\tan 3\theta)\sec^2\theta\, d\theta = 3\int\theta.\sec^2\theta\, d\theta$$

$$= 3[\theta.\tan\theta - \int 1.\tan\theta\, d\theta] = 3\theta.\tan\theta - 3\log\sec\theta + c$$

$$= 3x\tan^{-1}x - \frac{3}{2}\log\sec^2\theta + c$$

[$\because \sec^2\theta = 1 + \tan^2\theta = 1 + x^2$]

$$= 3x\tan^{-1}x - \frac{3}{2}\log(1+x^2) + c$$

(vii) $I = \int x.\log(x^3+1)dx$

$= \log(x^3+1).\frac{x^2}{2} - \int \frac{1}{x^3+1}.3x^2.\frac{x^2}{2}dx$ (By parts)

$= \frac{x^2}{2}.\log(x^3+1) - \frac{3}{2}\int\left(x - \frac{x}{x^3+1}\right)dx$

Dividing x^4 by x^3+1

$= \frac{x^2}{2}\log(x^3+1) - \frac{3}{2}\frac{x^2}{2} + \frac{3}{2}\int\frac{x}{x^3+1}dx$

If $\frac{x}{x^3+1} \equiv \frac{A}{x+1} + \frac{Bx+C}{x^2-x+1}$

then $x \equiv A(x^2-x+1) + (Bx+C)(x+1)$

Putting $x = -1 \Rightarrow -1 = A(3) \quad \therefore A = -\frac{1}{3}$

Putting $x^2 = x - 1 \Rightarrow x \equiv Bx^2 + Bx + Cx + C$

$\equiv B(x-1) + Bx + Cx + C \ (\because x^2 = x-1)$

$\equiv 2Bx + Cx - B + C$

Equating coefficient of x and constant terms

$2B + C = 1, C - B = 0 \ \therefore \quad B = \frac{1}{3}$ and $C = \frac{1}{3}$

$\therefore \int\frac{x}{x^3+1}dx = \int -\frac{1}{3(x+1)}dx + \int\frac{x+1}{3(x^2-x+1)}dx$

$= -\frac{1}{3}\log(x+1) + \frac{1}{6}\int\frac{2x-1}{x^2-x+1}dx + \frac{1}{2}\int\frac{dx}{x^2-x+1}$

$= -\frac{1}{3}\log(x+1) + \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2}$

$= -\frac{1}{3}\log(x+1) + \frac{1}{6}\log(x^2-x+1) + \frac{1}{2}\frac{2}{\sqrt{3}}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$

Putting them all together, we have

$$I = \frac{x^2}{2}\log(x^3+1) - \frac{3x^2}{4} - \frac{1}{2}\log(x+1)$$
$$+ \frac{1}{4}\log(x^2-x+1) + \frac{\sqrt{3}}{2}\tan^{-1}\left(\frac{2x-1}{\sqrt{3}}\right)$$

(viii) $\int e^x.\log(e^{2x}+5e^x+6)dx = \int\log[(t+2)(t+3)]dt$

$= \int\log(t+2)dt + \int\log(t+3)dt$

$= \log(t+2).t - \int\frac{1}{t+2}.t\,dt + \log(t+3).t - \int\frac{1}{(t+3)}.t\,dt$

$= t.\log(t+2) - \int\left(1 - \frac{2}{t+2}\right)dt$

$+ t.\log(t+3) - \int\left(1 - \frac{3}{t+3}\right)dt$

$= t.\log(t+2) - t + 2\log(t+2) + t\log(t+3) - t + 3\log(t+3) + c$

$= t[\log(t+2) + \log(t+3)] - 2t + [2\log(t+2) + 3\log(t+3)] + c$

$= e^x[\log(e^{2x}+5e^x+6)] - 2e^x + 2\log(e^x+2) + 3\log(e^x+3) + c.$

3. (i) $\int\sqrt{x}(\log x)^2dx$

$= (\log x)^2\frac{2}{3}x^{3/2} - \int\frac{2.(\log x)}{x}.\frac{2}{3}.x^{3/2}dx$

$= \frac{2}{3}.x^{3/2}(\log x)^2 - \frac{4}{3}\int(\log x)x^{1/2}dx$

$= \frac{2}{3}x^{3/2}(\log x)^2 - \frac{4}{3}\left[(\log x)\frac{2}{3}x^{3/2} - \int\frac{1}{x}.\frac{2}{3}x^{3/2}dx\right]$

$= \frac{2}{3}x^{3/2}(\log x)^2 - \frac{8}{9}(\log x)x^{3/2} + \frac{8}{9}.\frac{2}{3}.x^{3/2} + c$

$= \frac{1}{27}x^{3/2}[18(\log x)^2 - 24\log x + 16] + c$

(ii) $\int x^3\tan^{-1}x\,dx = \tan^{-1}x.\frac{x^4}{4} - \frac{1}{4}\int\left(x^2 - 1 + \frac{1}{x^2+1}\right)dx$

$= \frac{x^4}{4}.\tan^{-1}x - \frac{1}{4}\int\left(x^2 - 1 + \frac{1}{x^2+1}\right)dx$

$= \frac{x^4}{4}.\tan^{-1}x - \frac{x^3}{12} + \frac{x}{4} - \frac{1}{4}\tan^{-1}x + c$

(iii) $\int x^2\sin x\cos x\,dx = \frac{1}{2}\int x^2\sin 2x\,dx$

$= \frac{1}{2}x^2\left(-\frac{\cos 2x}{2}\right) + \frac{1}{2}\int 2x.\frac{\cos 2x}{2}dx$

$= -\frac{x^2\cos 2x}{4} + \frac{1}{2}\left[x.\frac{\sin 2x}{2} - \int\frac{\sin 2x}{2}dx\right]$

$= -\frac{x^2\cos 2x}{4} + \frac{x\sin 2x}{4} + \frac{\cos 2x}{8} + c$

$= \frac{1}{8}(1-2x^2)\cos 2x + \frac{x\sin 2x}{4} + c.$

(iv) $\int x\sin x\sec^3 x\,dx = \int x\sec x.\left(\frac{\sin x}{\cos x}.\sec x\right)dx$

$= \int x.\sec x.(\sec x.\tan x)dx$

[Take $\sec x$ as 2nd function]

$= x.\frac{\sec^2 x}{2} - \int\frac{\sec^2 x}{2}.dx$

$= x.\frac{\sec^2 x}{2} - \frac{\tan x}{2} + c = \frac{1}{2}(x\sec^2 x - \tan x) + c$

(v) $\int x\cos^3 x\sin x\,dx = -\int x\cos^3 x(-\sin x)dx$

$= -\left[x.\frac{\cos^4 x}{4} - \int\frac{\cos^4 x}{4}dx\right]$

$= -\frac{x}{4}.\cos^4 x + \frac{1}{4}\int(\cos^2 x)^2dx$

$$= -\frac{x\cos^4 x}{4} + \frac{1}{4}\int\left(\frac{1+\cos 2x}{2}\right)^2 dx$$

$$= -\frac{x\cos^4 x}{4} + \frac{1}{16}\int(1+\cos^2 2x + 2\cos 2x)dx$$

$$= -\frac{x\cos^4 x}{4} + \frac{1}{16}\left[x + 2.\frac{\sin 2x}{2}\right] + \frac{1}{16}\int\frac{1+\cos 4x}{2}dx$$

$$= -\frac{x\cos^4 x}{4} + \frac{x}{16} + \frac{\sin 2x}{16} + \frac{1}{32}\left(x + \frac{\sin 4x}{4}\right) + c$$

$$= -\frac{x\cos^4 x}{4} + \frac{1}{128}(12x + 8\sin 2x + \sin 4x) + c$$

(vi) $\int\frac{1+x}{(2+x)^2}e^x\,dx = \int\frac{2+x-1}{(2+x)^2}e^x\,dx$

$$= \int\left[\frac{2+x}{(2+x)^2} - \frac{1}{(2+x)^2}\right]e^x\,dx$$

$$= \int\left(\frac{1}{2+x} - \frac{1}{(2+x)^2}\right)e^x\,dx$$

$$= \int e^x.\frac{1}{2+x}\,dx - \int e^x.\frac{1}{(2+x)^2}\,dx \quad \text{...... (i)}$$

Integrating $e^x.\frac{1}{(2+x)}$ by parts, we get

$$\int e^x.\frac{1}{(2+x)}\,dx = \frac{1}{2+x}e^x + \int e^x.\frac{1}{(2+x)^2}dx + c$$

Putting this value in (i), we have

$$\int\frac{1+x}{(2+x)^2}e^x\,dx$$

$$= \frac{e^x}{2+x} + \int e^x\frac{1}{(2+x)^2}dx - \int e^x\frac{1}{(2+x)^2}dx + c$$

$$= \frac{e^x}{2+x} + c.$$

(vii) $\int\frac{\sqrt{(1-\sin x)}}{1+\cos x}e^{-x/2}\,dx$

$$= \int\frac{\sqrt{\cos^2\frac{x}{2} + \sin^2\frac{x}{2} - 2\sin\frac{x}{2}\cos\frac{x}{2}}}{2\cos^2\frac{x}{2}}e^{-x/2}\,dx$$

$$= -e^{-x/2}\sec\frac{x}{2} + c.$$

4. (i) Let $x + 2 = t \Rightarrow dx = dt$

$$I = \int\frac{(t-2)^2 dt}{t^2(t+2)^2} = \int\frac{(t+2-4)^2}{t^2(t+2)^2}dt$$

$$= \int\frac{(t+2)^2 - 8(t+2) + 16}{t^2(t+2)^2}dt$$

$$= \int\frac{dt}{t^2} - 8\int\frac{dt}{t^2(t+2)} + 16\int\frac{dt}{t^2(t+2)^2}$$

$$= -\frac{1}{t} - \frac{8}{4}\left[\int\frac{dt}{t+2} - \int\frac{(t-2)dt}{t^2}\right] + 16\int\frac{dt}{t^2(t+2)^2}$$

$$= \frac{3}{t} + 2\left[\ln\frac{t}{t+2}\right] + 16\left[\int\frac{dt}{t^2(t+2)^2}\right] + c$$

$$I = \frac{3}{x+2} + 2\ln\left|\frac{x+2}{x+4}\right|$$

$$+16\left[\frac{1}{4}\int\frac{dt}{(t+2)^2} - \frac{1}{4}\int\frac{dt}{t^2} + \frac{1}{4}\int\left(\frac{1}{t+2} - \frac{t-2}{t^2}\right)dt\right]$$

$$= \frac{3}{x+2} + 2\ln\left|\frac{x+2}{x+4}\right| + 4\left(\frac{-1}{x+4}\right) + \frac{4}{x+2} + 4\ln(x+4)$$

$$-4\ln(x+2) - \frac{8}{x+2}$$

$$= \frac{-1}{x+2} - \frac{4}{(x+4)} + \ln\left\{\left[\frac{x+2}{x+4}\right]^2\left(\frac{x+4}{x+2}\right)^4\right\}$$

$$I = -\frac{1}{x+2} - \frac{4}{(x+4)} + \ln\left(\frac{x+4}{x+2}\right)^2 + c.$$

(ii) $\int\frac{x^2+1}{(x^2-1)(x^2-4)}dx$

$$\frac{x^2+1}{(x^2-1)(x^2-4)} \equiv \frac{y+1}{(y-1)(y-4)} \quad \text{where } y = x^2$$

Now let $\frac{y+1}{(y-1)(y-4)} \equiv \frac{A}{y-1} + \frac{B}{y-4}$

or $y + 1 \equiv A(y-4) + B(y-1)$

Putting $y = 1$; $2 = -3A \Rightarrow A = -\frac{2}{3}$

Putting $y = 4$; $5 = 3B \Rightarrow B = \frac{5}{3}$

then $\frac{x^2+1}{(x^2-1)(x^2-4)} = -\frac{2}{3(x^2-1)} + \frac{5}{3(x^2-4)}$

$\therefore$ Given integral $= \int\frac{x^2+1}{(x^2-1)(x^2-4)}dx$

$$= -\frac{2}{3}\int\frac{dx}{x^2-1} + \frac{5}{3}\int\frac{dx}{x^2-4}$$

$$-\frac{2}{3}.\frac{1}{2}.\log\left(\frac{x-1}{x+1}\right) + \frac{5}{3}.\frac{1}{4}\log\left(\frac{x-2}{x+2}\right) + c$$

$$\left[\because \int\frac{dx}{x^2-a^2} = \frac{1}{2a}\log\left(\frac{x-a}{x+a}\right)\right]$$

$$= \frac{5}{12}\log\left(\frac{x-2}{x+2}\right) - \frac{1}{3}\log\frac{x-1}{x+1} + c.$$

(iii) Suppose $\frac{x^2+x+1}{(x+1)^2(x+2)} \equiv \frac{A}{x+2} + \frac{B}{x+1} + \frac{C}{(x+1)^2}$

$\therefore x^2 + x + 1 \equiv A(x+1)^2 + B(x+1)(x+2) + C(x+2)$

Putting $x = -2$, $A = 4 - 2 + 1 = 3$

Putting $x = -1$, $C = 1 - 1 + 1 = 1$

Equating constant terms on both sides

$1 = A + 2B + 2C = 3 + 2B + 2 \therefore B = -2$

Hence $I = \int\left[\frac{3}{(x+2)} - \frac{2}{(x+1)} + \frac{1}{(x+1)^2}\right]dx$

$= 3\log(x+2) - 2\log(x+1) - \frac{1}{x+1} + c$

$= -\frac{1}{x+1} + \log\frac{(x+2)^3}{(x+1)^2} + c.$

(iv) Let $x = t^3 \Rightarrow dx = 3t^2\,dt$

$= 3\int e^t\, t^2 dt$

$= 3\int\left[t^2 \cdot \int e^t\,dt - \int 2t \cdot e^t\,dt\right] + c$

$= 3t^2e^t - 6te^t + 6e^t + c$

$= e^t[3t^2 - 6t + 6] + c$

$I = e^{\sqrt[3]{x}}[3x^{2/3} - 6x^{1/3} + 6] + c.$

(v) Let $I = \int\frac{x+1}{(x-1)^2(x+2)^2}dx$

Suppose

$$\frac{x+1}{(x-1)^2(x+2)^2} \equiv \frac{A}{(x-1)} + \frac{B}{(x-1)^2} + \frac{C}{(x+2)} + \frac{D}{(x+2)^2}$$

$\therefore x + 1 \equiv A(x-1)(x+2)^2 + B(x+2)^2 + C(x-1)^2(x+2) + D(x-1)^2$ .... (i)

Putting $x = 1$, $9B = 2$ *i.e.,* $B = \frac{2}{9}$

Putting $x = -2$, $9D = -1$ *i.e.,* $D = -\frac{1}{9}$

To obtain A, C we equate the coefficient of x^3, x^2, x on both sides of (1), we have $0 = A + C$

$0 = -A + 4A + B - 2C + 2C + D$ *i.e.,* $0 = 3A + B + D$

or $0 = 3A + \frac{2}{9} - \frac{1}{9}$ $\therefore A = -\frac{1}{27}$ $\therefore C = \frac{1}{27}$

$$\therefore I = \int\left[-\frac{1}{27(x-1)} + \frac{2}{9(x-1)^2} + \frac{1}{27(x+2)} - \frac{1}{9(x+2)^2}\right]dx$$

$$= -\frac{1}{27}\log(x-1) + \frac{2}{9}\left(-\frac{1}{x-1}\right) + \frac{1}{27}\log(x+2) - \frac{1}{9}\left(-\frac{1}{x+2}\right) + c$$

$$= -\frac{1}{27}\log(x-1) - \frac{2}{9}\frac{1}{(x-1)} + \frac{1}{27}\log(x+2) + \frac{1}{9}\frac{1}{(x+2)} + c$$

$$= \frac{1}{27}\log\left(\frac{x+2}{x-1}\right) - \frac{2}{9(x-1)} + \frac{1}{9(x+2)} + c$$

(vi) $I = \int\frac{(x^2+1)(x^2+2)(x^2+3)}{(x^2+4)(x^2+5)(x^2+6)}dx$

$$\frac{(x^2+1)(x^2+2)(x^2+3)}{(x^2+4)(x^2+5)(x^2+6)} = \frac{(y+1)(y+2)(y+3)}{(y+4)(y+5)(y+6)}$$

Putting $x^2 = y$

Suppose $\frac{(y+1)(y+2)(y+3)}{(y+4)(y+5)(y+6)}$

$= 1 + \frac{A}{(y+4)} + \frac{B}{(y+5)} + \frac{C}{(y+6)}$,

$(y+1)(y+2)(y+3) = (y+4)(y+5)(y+6) + A(y+5)(y+6) + B(y+4)(y+6) + C(y+4)(y+5)$

Putting $y = -4 \Rightarrow 2A = -6 \Rightarrow A = -3$

Putting $y = -5 \Rightarrow -B = -24 \Rightarrow B = 24$

Putting $y = -6 \Rightarrow 2C = -60 \Rightarrow C = -30$

$$\therefore I = \int\left(1 - \frac{3}{x^2+4} + \frac{24}{x^2+5} - \frac{30}{x^2+6}\right)dx$$

$$= x - \frac{3}{2}\tan^{-1}\frac{x}{2} + \frac{24}{\sqrt{5}}\tan^{-1}\frac{x}{\sqrt{5}} - \frac{30}{\sqrt{6}}\tan^{-1}\frac{x}{\sqrt{6}} + c.$$

(vii) $I = \int\frac{1}{x^6+1}dx$

For P.F. only, put $x^2 = y$, then $\frac{1}{x^6+1} = \frac{1}{(y^3+1)}$

$$= \frac{1}{(y+1)(y^2-y+1)} = \frac{A}{y+1} + \frac{By+C}{y^2-y+1}$$

$1 = A(y^2 - y + 1) + (By + C)(y + 1)$

Putting $y = -1$, we get $3A = 1 \Rightarrow A = \frac{1}{3}$

Equate the coefficient of y^2 : $A + B = 0 \Rightarrow B = -\frac{1}{3}$

Equate the constant terms : $A + C = 1 \Rightarrow C = \frac{2}{3}$

$$\therefore \frac{1}{x^6+1} = \frac{1}{3(x^2+1)} - \frac{x^2-2}{3(x^4-x^2+1)}$$

$$\therefore I = \frac{1}{3}\int\frac{dx}{x^2+1} - \frac{1}{3}\int\frac{x^2-2}{x^4-x^2+1}dx$$

$$I = \frac{1}{3}\tan^{-1}x - \frac{1}{3}\underbrace{\int\frac{x^2-1}{x^4-x^2+1}}_{I_1} + \frac{1}{3}\underbrace{\int\frac{dx}{x^4-x^2+1}}_{I_2} \quad ..(1)$$

$$I_1 = \int\frac{x^2-1}{x^4-x^2+1}dx = \int\frac{\left(1-\frac{1}{x^2}\right)}{x^2-1+\frac{1}{x^2}}dx$$

$$= \int\frac{\left(1-\frac{1}{x^2}\right)}{\left(x+\frac{1}{x}\right)^2-3}dx$$

Put $x + \frac{1}{x} = \int\frac{dy}{y^2-3} = \frac{1}{2\sqrt{3}}\log\left(\frac{y-\sqrt{3}}{y+\sqrt{3}}\right) + c$

$$= \frac{1}{2\sqrt{3}}\log\left(\frac{x+\frac{1}{x}-\sqrt{3}}{x+\frac{1}{x}+\sqrt{3}}\right) + c$$

$$= \frac{1}{2\sqrt{3}}\log\left(\frac{x^2-x\sqrt{3}+1}{x^2+x\sqrt{3}+1}\right) + c \quad .... (3)$$

$$I_2 = \int \frac{dx}{x^4 - x^2 + 1} = \frac{1}{2}\int \frac{(x^2+1)-(x^2-1)}{x^2 - x + 1}dx$$

$$= \frac{1}{2}\int \frac{x^2+1}{x^4 - x^2+1}dx - \frac{1}{2}\int \frac{x^2-1}{x^4-x^2+1}dx$$

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{x^2 - 1 + \frac{1}{x^2}}dx - \frac{1}{2}\left[\frac{1}{2\sqrt{3}}\log\left(\frac{x^2 - x\sqrt{3}+1}{x^2+x\sqrt{3}+1}\right)\right] + c$$

from (3)

$$= \frac{1}{2}\int \frac{1+\frac{1}{x^2}}{\left(x - \frac{1}{x}\right)^2 + 1}dx - \frac{1}{4\sqrt{3}}\log\left(\frac{x^2 - x\sqrt{3}+1}{x^2+x\sqrt{3}+1}\right) + c$$

Putting $x - \frac{1}{x} = z$ etc.

$$= \frac{1}{2}\int \frac{dz}{z^2+1} - \frac{1}{4\sqrt{3}}\log\left(\frac{x^2 - x\sqrt{3}+1}{x^2+x\sqrt{3}+1}\right) + c$$

Putting $x - \frac{1}{x} = z$ etc.

$$= \frac{1}{2}\tan^{-1}\left(x - \frac{1}{x}\right) - \frac{1}{4\sqrt{3}}\log\left(\frac{x^2 - x\sqrt{3}+1}{x^2+x\sqrt{3}+1}\right) + c$$

Putting the values of I_1 and I_2 in (1), we have

$$I = \frac{1}{3}\tan^{-1}x - \frac{1}{6\sqrt{3}}\log\left(\frac{x^2 - x\sqrt{3}+1}{x^2+x\sqrt{3}+1}\right)$$

$$+ \frac{1}{6}\tan^{-1}\left(\frac{x^2-1}{x}\right) - \frac{1}{12\sqrt{3}}\log\left(\frac{x^2 - x\sqrt{3}+1}{x^2+x\sqrt{3}+1}\right) + c$$

$$= \frac{1}{3}\tan^{-1}x - \frac{1}{4\sqrt{3}}\log\left(\frac{x^2 - x\sqrt{3}+1}{x^2+x\sqrt{3}+1}\right) + \frac{1}{6}\tan^{-1}\left(\frac{x^2-1}{x}\right) + c$$

5. **(i)** $\displaystyle\int \frac{x+a}{x^2(x-a)(x^2+a^2)}dx$

Suppose $\displaystyle\frac{x+a}{x^2(x-a)(x^2+a^2)} \equiv \frac{A}{x} + \frac{B}{x^2}$

$$+ \frac{C}{x-a} + \frac{Dx+E}{x^2+a^2}$$

$\therefore x + a \equiv Ax(x-a)(x^2+a^2) + B(x-a)(x^2+a^2) + Cx^2(x^2+a^2) + (Dx+E)x^2(x-a)$

Putting $x = 0$, $a = -a^3B \Rightarrow B = \dfrac{-1}{a^2}$

$x = a$, $2a = 2a^4C \Rightarrow C = \dfrac{1}{a^3}$

Equating coefficients of x^4, and constant terms

$0 = A + C + D$ or $0 = A + D = \dfrac{1}{a^3}$ ....(i)

and $a = -Ba^3 - Ea$ or $-1 = Ba^2 + E$ ....(ii)

or $-1 = -\dfrac{1}{a^2}.a^2 + E \therefore E = 0$

also coeff. of x, $1 = -a^3A + a^2B = -a^3A - 1 \therefore A = -\dfrac{2}{a^3}$

$\therefore$ from (i), $-\dfrac{2}{a^3} + \dfrac{1}{a^3} + D = 0$ *i.e.*, $D = \dfrac{1}{a^3}$

$$\therefore I = \int\left[-\frac{2}{a^3x} - \frac{1}{a^2x^2} + \frac{1}{a^3(x-a)} + \frac{1}{a^3}\frac{x}{x^2+a^2}\right]dx$$

$$= -\frac{2}{a^3}\log x + \frac{1}{a^2x} + \frac{1}{a^3}\log(x-a) + \frac{1}{2a^3}\log(x^2+a^2) + c$$

(ii) Let $x - 2 = t \Rightarrow dx = dt$

$$I = \int \frac{(t+2)^3 - 6(t+2)^2 + 9(t+2) + 7}{t^3\cdot(t-3)}dt$$

$$= \int \frac{t^3 + 6t^2 + 12t + 8 - 6t^2 - 24t - 24 + 9t + 18 + 7}{t^3(t-3)}dt$$

$$= \int \frac{t^3 - 3t + 9}{t^3(t-3)}dt = \int \frac{dt}{t-3} - 3\int \frac{dt}{t^2(t-3)} + \int \frac{9dt}{t^3(t-3)}$$

$$= \ln|t-3| - \frac{1}{3}\left[\int\left(\frac{1}{t-3} - \frac{t+3}{t^2}\right)dt\right] + 9\int \frac{dt}{t^3(t-3)} + c$$

$$= \ln|t-3| - \frac{1}{3}\ln|t-3| + \frac{1}{3}\ln t - \frac{1}{t}$$

$$+ \int \frac{1}{t}\left[\frac{1}{t-3} - \frac{(t+3)}{t^2}\right]dt + c$$

$$= \frac{2}{3}\ln|t-3| + \frac{1}{3}\ln t - \frac{1}{t} + \frac{1}{3}\left[\int \frac{dt}{t-3} - \int \frac{dt}{t}\right]$$

$$+ \int -\frac{1}{t^2} - \frac{3}{t^3}dt + c$$

$$= \frac{2}{3}\ln|t-3| + \frac{1}{3}\ln t - \frac{1}{t} + \frac{1}{3}\ln|t-3| - \frac{1}{3}\ln t$$

$$+ \frac{1}{t} + \frac{3}{2}\cdot\frac{1}{t^2} + c$$

$$= \frac{3}{3}\ln|t-3| + \frac{3}{2}\cdot\frac{1}{t^2} + c$$

$$I = \ln|x-5| + \frac{3}{2}\cdot\frac{1}{(x-2)^2} + c.$$

(iii) $\displaystyle I = \int \frac{\cos x dx}{\sin x.(4\cos^2 x - 3)} = \int \frac{\cos x dx}{\sin x\,[4(1-\sin^2 x) - 3]}$

$\displaystyle = \int \frac{\cos x dx}{\sin x(1 - 4\sin^2 x)}$ Let $\sin x = t \Rightarrow \cos dx = dt$

$$= \int \frac{dt}{t(1-4t^2)} = \int \frac{dt}{t(1-2t)(1+2t)}$$

$$= \int \frac{1}{2t}\left[\frac{1}{1-2t} + \frac{1}{1+2t}\right]dt = \frac{1}{2}\left[\int \frac{dt}{t(1-2t)} + \int \frac{dt}{t(1+2t)}\right]$$

$$= \frac{1}{2}\int\left[\frac{1}{t} + \frac{2}{1-2t} + \frac{1}{t} - \frac{2}{1+2t}\right]dt$$

$$= \frac{1}{2}\left[2\ln t - \ln|1-2t| - \ln|1+2t|\right] + c$$

$$= \frac{1}{2}[2\ln t - \ln|1-4t^2|] + c = \frac{1}{2}\ln\frac{t^2}{1-4t^2} + c$$

$$I = \frac{1}{2}\ln\left|\frac{\sin^2 x}{1-4\sin^2 x}\right| + c.$$

(iv) $I = \int \frac{1}{x^2(x^2+1)(x^2+2)^2}\,dx = \frac{1}{2}\int \frac{dy}{y(y+1)(y+2)^2}$

Let $\frac{1}{y(y+1)(y+2)^2} = \frac{A}{y} + \frac{B}{y+1} + \frac{C}{y+2} + \frac{D}{(y+2)^2}$

$\Rightarrow 1 = A(y+1)(y+2)^2 + By(y+2)^2 + Cy(y+1)(y+2) + Dy(y+1)$

Putting $y = 0$ we get $4A = 1 \Rightarrow \quad A = \frac{1}{4}$

Putting $y = -1$ we get $-B = 1 \Rightarrow B = -1$

Putting $y = -2$ we get $2D = 1 \Rightarrow \quad D = \frac{1}{2}$

Equating coeff. of y on both sides

$0 = 8A + 4B + 2C + D = 8.\frac{1}{4} - 4 + 2C + \frac{1}{2} \Rightarrow \quad C = \frac{3}{4}$

$$\therefore I = \frac{1}{2}\int\left[\frac{1}{4y} - \frac{1}{y+1} + \frac{3}{4(y+2)} + \frac{1}{2(y+2)^2}\right]dy$$

$$= \frac{1}{2}\left[\frac{1}{4}\log y - \log(y+1) + \frac{3}{4}\log(y+2) - \frac{1}{2(y+2)}\right] + c$$

$$= \frac{1}{2}\left[\frac{1}{4}\log y - \frac{1}{4}\log(y+1)^4 + \frac{1}{4}\log(y+2)^3 - \frac{1}{2}\frac{1}{y+2}\right] + c$$

$$= \frac{1}{8}\log\frac{y(y+2)^3}{(y+1)^4} - \frac{1}{4(y+2)} + c$$

$$= \frac{1}{8}.\log\frac{x^2(x^2+1)^3}{(x^2+1)^4} - \frac{1}{4(x^2+2)} + c.$$

(v) $I = \int \frac{x^5\,dx}{(x-1)^3(x+1)}$. Let $x - 1 = t \Rightarrow dx = dt$

$$= \int \frac{(t+1)^5\,dt}{t^3(t+2)} = \int \frac{t^5 + 5t^4 + 10t^3 + 10t^2 + 5t + 1}{t^3(t+2)}\,dt$$

$$= \int \frac{t^2}{t+2}\,dt + 5\int \frac{t}{t+2}\,dt + 10\int \frac{dt}{t+2} + 10\int \frac{dt}{t(t+2)} + 5\int \frac{dt}{t^2(t+2)} + \int \frac{dt}{t^3(t+2)}$$

$$= \left[\int (t-2)\,dt + \int \frac{4}{t+2}\,dt\right] + \left[5\int dt - 10\int \frac{dt}{t+2}\right] + 10\int \frac{dt}{t+2} + \frac{10}{2}\left[\int \frac{dt}{t} - \int \frac{dt}{t+2}\right] + \frac{5}{4}\left[\int \frac{dt}{t+2} - \int \frac{t-2}{t^2}\,dt\right] + \int \frac{dt}{t^3(t+2)}$$

$$= \left[\int (t-2)\,dt + 4\int \frac{dt}{t+2}\right] + 5\int dt + \left[5\int \frac{dt}{t} - 5\int \frac{dt}{t+2}\right] + \left[\frac{5}{4}\int \frac{dt}{t+2} - \frac{5}{4}\int \frac{dt}{t} + \frac{10}{4}\int \frac{dt}{t^2}\right] + \int \frac{dt}{t^3(t+2)}$$

$$= \int (t-2)\,dt + \frac{1}{4}\int \frac{dt}{t+2} + \frac{15}{4}\int \frac{dt}{t} + 5\int dt + \frac{5}{2}\int \frac{dt}{t^2} + \frac{1}{4}\int\left[\frac{1}{t+2} - \frac{(t-2)}{t^2}\right]\frac{1}{t}\,dt$$

$$= \int (t-2)\,dt + \frac{1}{4}\int \frac{dt}{t+2} + \frac{15}{4}\int \frac{dt}{t} + 5\int dt + \frac{5}{2}\int \frac{dt}{t^2} + \frac{1}{8}\int\left(\frac{1}{t} - \frac{1}{t+2}\right)dt + \frac{1}{4}\int\left(\frac{-1}{t^2} + \frac{2}{t^3}\right)dt$$

$$= \int (t-2)\,dt + \frac{1}{8}\int \frac{dt}{t+2} + \frac{31}{8}\int \frac{dt}{t} + 5\int dt + \frac{9}{4}\int \frac{dt}{t^2} + \frac{1}{2}\int \frac{dt}{t^3}$$

$$= \frac{(t-2)^2}{2} + \frac{1}{8}\ln(t+2) + \frac{31}{8}\ln t + 5t - \frac{9}{4}\cdot\frac{1}{t} - \frac{1.1}{4\,t^2} + c$$

$$I = \frac{(x-3)^2}{2} + \frac{1}{8}\ln(x+1) + \frac{31}{8}\ln(x-1) + 5(x-1) - \frac{9}{4}\cdot\frac{1}{(x-1)} - \frac{1}{4(x-1)^2} + c.$$

6. (i) Let $x = \frac{1}{t} \quad \Rightarrow dx = -\frac{1}{t^2}\,dt$

$$I = \int \frac{\sqrt{t^2-1}\left(-\frac{1}{t^2}\right)\frac{1}{t}\,dt}{\frac{1}{t^2}} = -\int \frac{\sqrt{t^2-1}\,dt}{t}$$

Let $t^2 - 1 = u^2 \;\therefore\; 2t\,dt = 2u\,du$ or $dt = \frac{u\,du}{t}$

$$I = -\int \frac{u^2\,du}{u^2+1} = -\int \frac{(u^2-1+1)\,du}{u^2+1} = -\int du + \int \frac{du}{u^2+1}$$

$$= -u + \tan^{-1}u + c = -\sqrt{t^2-1} + \tan^{-1}\sqrt{t^2-1} + c$$

$$I = -\frac{1}{x}\sqrt{1-x^2} + \tan^{-1}\frac{\sqrt{1-x^2}}{x} + c.$$

(ii) $x = t^{12} \Rightarrow dx = 12t^{11}\,dt$

$$I = \int \frac{t^6 \cdot 12t^{11}\,dt}{(t^{12})^{2/3} - (t^{12})^{1/4}} = \int \frac{12t^{17}}{t^8 - t^3}\,dt = 12\int \frac{t^{14}\,dt}{t^5 - 1}$$

$$= 12\int\left[t^9 + t^4 + \frac{t^4}{t^5-1}\right]dt$$

$$= \frac{12t^{10}}{10} + \frac{12t^5}{5} + \frac{12}{5}\ln(t^5-1) + c$$

$$I = \frac{12}{10}(x)^{10/12} + \frac{12}{5}(x)^{5/12} + \frac{12}{5}\ln(x^{5/12}-1) + c.$$

(iii) Let $x = 3\sin\theta \Rightarrow dx = 3\cos\theta\,d\theta$

$$I = \frac{3^2}{3^6}\int \frac{\cos^2\theta}{\sin^6\theta}\,d\theta = \frac{1}{3^4}\int \cot^2\theta \cdot \operatorname{cosec}^2\theta(1+\cot^2\theta)\,d\theta$$

set $\cot\theta = t \Rightarrow -\operatorname{cosec}^2\theta\,d\theta = dt$

$$I = -\frac{1}{3^4}\int t^2(1+t^2)\,dt = -\frac{1}{3^4}\left(\frac{t^3}{3} + \frac{t^5}{5}\right) + c$$

$$= -\frac{1}{3^4}\left(\frac{\cot^3\theta}{3} + \frac{\cot^5\theta}{5}\right) + c$$

$$= -\frac{1}{3^4}\left[\frac{\left(\sqrt{\frac{9-x^2}{x}}\right)^3}{3} + \frac{\left(\sqrt{\frac{9-x^2}{x}}\right)^5}{5}\right] + c.$$

$$\left[x = 3\sin\theta, \cot\theta = \frac{\cos\theta}{\sin\theta} = \frac{\sqrt{1-\sin^2\theta}}{\sin\theta}\right]$$

(iv) Let $x = \frac{1}{u} \Rightarrow dx = -\frac{1}{u^2}du$

$$I = \int \frac{-\frac{1}{u^2}du}{\frac{1}{u^2}\sqrt{1+2u-u^2}} = -\int \frac{du}{\sqrt{1-(u^2-2u)}}$$

$$= -\int \frac{du}{\sqrt{2-[u-1]^2}} = -\sin^{-1}\frac{(u-1)}{\sqrt{2}} + c = \sin^{-1}\frac{\left(\frac{1}{x}-1\right)}{\sqrt{2}} + c$$

$$I = -\sin^{-1}\frac{(1-x)}{x\sqrt{2}} + c.$$

(v) $\int \sqrt{(1+\sec x)}\, dx = \int \sqrt{\left(1+\frac{1}{\cos x}\right)}\, dx$

$$= \int \sqrt{\left(\frac{1+\cos x}{\cos x}\right)}\, dx = \int \frac{\sqrt{\left(2\cos^2\frac{x}{2}\right)}}{\sqrt{\left(1-2\sin^2\frac{x}{2}\right)}}\, dx$$

$$= \sqrt{2}\int \frac{\cos\frac{x}{2}}{\sqrt{1-2\sin^2\frac{x}{2}}}\, dx = \sqrt{2}\int \frac{2dy}{\sqrt{(1-2y^2)}}$$

Put $\sin\frac{x}{2} = y, \cos\frac{x}{2}.\frac{1}{2}dx = dy$

$$I = 2\int \frac{dy}{\sqrt{\left(\frac{1}{2}-y^2\right)}} = 2\sin^{-1}\left(\frac{y}{\frac{1}{\sqrt{2}}}\right) + c$$

$$= 2\sin^{-1}\left(\sin\frac{x}{2}.\sqrt{2}\right) + c = 2\sin^{-1}\left[\sqrt{\left(2\sin^2\frac{x}{2}\right)}\right] + c$$

$$= 2\sin^{-1}\left[\sqrt{(1-\cos x)}\right] + c$$

$$= 2\sin^{-1}\left[\sqrt{\left(\frac{\sec x - 1}{\sec x}\right)}\right] + c$$

Put $\sin^{-1}\sqrt{\left(\frac{\sec x-1}{\sec x}\right)} = \theta \therefore \sin\theta = \sqrt{\left\{\left(\frac{\sec x-1}{\sec x}\right)\right\}}$

$$\Rightarrow \tan\theta = \sqrt{\left(\frac{\sec x - 1}{1}\right)} \Rightarrow \theta = \tan^{-1}\sqrt{(\sec x - 1)}$$

$\therefore$ Given integral $= 2\tan^{-1}\sqrt{(\sec x - 1)}$.

(vi) $I = \int \frac{\sqrt{2x+x^2}}{x^2}dx$ Let $x = \frac{1}{u} \Rightarrow dx = -\frac{1}{u^2}du$

$$I = \int \frac{\sqrt{\frac{2}{u}+\frac{1}{u^2}}\cdot\left(-\frac{1}{u^2}\right)}{\left(\frac{1}{u^2}\right)}du = \int \frac{1}{u}\sqrt{2u+1}\, du$$

Let $2u + 1 = X^2 \Rightarrow 2du = 2X\, dX$ or $du = X\, dX$

$$I = -\int \frac{2X^2dx}{X^2-1} = -2\int \frac{(X^2-1+1)}{X^2-1}dx$$

$$= -2X - 2\ln\left|\frac{X-1}{X+1}\right| + c$$

$$= -2\sqrt{2u+1} - \frac{2}{2}\ln\left|\frac{\sqrt{2u+1}-1}{\sqrt{2u+1}+1}\right| + c$$

$$= -2\sqrt{\frac{2}{x}+1} - \ln\left|\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}+\sqrt{x}}\right| + c$$

$$I = \frac{-2\sqrt{2+x}}{\sqrt{x}} - \ln\left|\frac{\sqrt{2+x}-\sqrt{x}}{\sqrt{2+x}+\sqrt{x}}\right| + c.$$

(vii) $I = \int \frac{dx}{(x-1)\sqrt{(x-1)^2+3x}}$

Let $(x-1) = t \Rightarrow dx = dt$

$$I = \int \frac{dt}{t\sqrt{t^2+3(t+1)}} = \int \frac{dt}{t.\sqrt{t^2+3t+3}}$$

Let $t = \frac{1}{u} \Rightarrow dt = -\frac{1}{u^2}du = -\int \frac{du}{\sqrt{1+3u+3u^2}}$

$$= -\int \frac{du}{\sqrt{1+3\left(u+\frac{1}{2}\right)^2-\frac{3}{4}}} = -\int \frac{du}{\sqrt{\frac{1}{4}+3\left(u+\frac{1}{2}\right)^2}}$$

$$= -\frac{1}{\sqrt{3}}\int \frac{du}{\sqrt{\frac{1}{12}+\left(u+\frac{1}{2}\right)^2}}$$

$$= -\frac{1}{\sqrt{3}}\ln\left|u+\frac{1}{2}+\sqrt{\left(u+\frac{1}{2}\right)^2+\frac{1}{12}}\right| + c$$

$$= -\frac{1}{\sqrt{3}}\ln\left|\frac{1}{t}+\frac{1}{2}+\sqrt{\left(\frac{1}{t}+\frac{1}{2}\right)^2+\frac{1}{12}}\right| + c$$

$$I = -\frac{1}{\sqrt{3}}\ln\left|\frac{2+x-1}{2(x-1)}+\sqrt{\left(\frac{x+1}{2(x-1)}\right)^2+\frac{1}{12}}\right| + c.$$

(viii) Let $x = \frac{1}{u} \Rightarrow dx = -\frac{1}{u^2}du$

$$I = \int \frac{-\frac{1}{u^2}du}{\frac{1}{u^2}\sqrt{2u^2+u-1}} = \frac{-1}{\sqrt{2}}\int \frac{du}{\sqrt{u^2+\frac{u}{2}-\frac{1}{2}}}$$

$$= -\frac{1}{\sqrt{2}}\int \frac{du}{\sqrt{\left(u+\frac{1}{4}\right)^2 - \frac{1}{16} - \frac{1}{2}}}$$

$$I = -\frac{1}{\sqrt{2}}\int \frac{du}{\sqrt{\left(u+\frac{1}{4}\right)^2 - \frac{9}{16}}}$$

$$= -\frac{1}{\sqrt{2}}\ln\left|u + \frac{1}{4} + \sqrt{2u^2 + u - 1}\right| + c$$

$$= -\frac{1}{\sqrt{2}}\ln\left|\frac{1}{x} + \frac{1}{4} + \sqrt{\frac{2}{x^2} + \frac{1}{x} - 1}\right| + c$$

$$= -\frac{1}{\sqrt{2}}\ln\left|\frac{4 + x + 4\sqrt{2 + x - x^2}}{x}\right| + c$$

$$I = \frac{1}{\sqrt{2}}\ln\left|\frac{x}{4 + x + 4\sqrt{2 + x - x^2}}\right| + c.$$

(ix) $I = \int \frac{(1+x^2)dx}{(2+x^2)\sqrt{1+x^2}}$

$$= \int \frac{dx}{(2+x^2)\sqrt{1+x^2}} + \int \frac{x^2 dx}{(2+x^2)\sqrt{1+x^2}}$$

$$= \int \frac{dx}{(2+x^2)\sqrt{1+x^2}} + \int \frac{dx}{\sqrt{1+x^2}} - 2\int \frac{dx}{(2+x^2)\sqrt{1+x^2}}$$

$$= -\underbrace{\int \frac{dx}{(2+x^2)\sqrt{1+x^2}}}_{I_1} + \ln(x + \sqrt{1+x^2}) + c_1$$

Let $x = \frac{1}{u} \Rightarrow dx = -\frac{1}{u^2}du$

$$I_1 = -\int \frac{\frac{-1}{u^2}du}{\frac{(2u^2+1)\sqrt{u^2+1}}{u^3}} = \int \frac{u\,du}{(2u^2+1)\sqrt{u^2+1}}$$

$$= \int \frac{u\,du}{(2u^2+1)\sqrt{1+u^2}}$$ Let $1 + u^2 = z^2 \Rightarrow 2u\,du = 2z\,dz$

$$I_1 = \int \frac{z\,dz}{[2(z^2-1)+1]z} = \int \frac{dz}{2z^2 - 1}$$

$$= \frac{\sqrt{2}}{4}\ln\left|\frac{z - \frac{1}{\sqrt{2}}}{z + \frac{1}{\sqrt{2}}}\right| + c_2$$

$$\therefore I = \frac{1}{2\sqrt{2}}\ln\left|\frac{\sqrt{x^2+1} - \frac{1}{\sqrt{2}}x}{\sqrt{x^2+1} + \frac{1}{\sqrt{2}}x}\right| + \ln(x + \sqrt{1+x^2}) + c.$$

7. (i) $I = \ln^3 x \cdot \int \frac{1}{x^2}dx - \int\left[\frac{d}{dx}(\ln^3 x)\cdot\int\frac{1}{x^2}dx\right]dx$

$$= \frac{-\ln^3 x}{x} - \int \frac{3\ln^2 x}{x}\cdot\left(-\frac{1}{x}\right)dx = \frac{-\ln^3 x}{x} + 3\int\frac{\ln^2 x}{x^2}dx$$

$$= \frac{-\ln^3 x}{x} - \frac{3\ln^2 x}{x} + 3\int\left[\frac{d}{dx}(\ln^2 x)\cdot\int\frac{dx}{x^2}\right]dx$$

$$= \frac{-\ln^3 x}{x} - \frac{3\ln^2 x}{x} + 3\int\frac{2\ln x}{x}\cdot\left(-\frac{1}{x}\right)dx$$

$$= \frac{-\ln^3 x}{x} - \frac{3\ln^2 x}{x} - \frac{6\ln x}{x} + 6\int\left[\frac{d}{dx}(\ln x)\cdot\int\frac{dx}{x^2}\right]dx$$

$$= -\frac{\ln^3 x}{x} - \frac{3\ln^2 x}{x} + \frac{6\ln x}{x} + \frac{6}{x} + c$$

$$= -\frac{1}{x}(\ln^3 x + 3\ln^2 x - 6\ln x - 6) + c$$

(ii) $I = \int \frac{2}{2+\cos x}dx - \int\frac{\sin x}{2+\cos x}dx$

$$= \ln(2+\cos x) + \int\frac{2}{2+\cos x}dx$$

$$= \ln(2+\cos x) + \int\frac{2\left(\sin^2\frac{x}{2} + \cos^2\frac{x}{2}\right)}{2 + 2\cos^2\frac{x}{2} - 1}dx$$

$$= \ln(2+\cos x) + \int\frac{2\cdot\sec^2\frac{x}{2}}{1 + \tan^2\frac{x}{2} + 2}dx$$

$$= \ln(2+\cos x) + 4\int\frac{dt}{t^2 + (\sqrt{3})^2}$$

$$= \ln(2+\cos x) + \frac{4}{\sqrt{3}}\tan^{-1}\frac{t}{\sqrt{3}} + c$$

$$= \ln(2+\cos x) + \frac{4}{\sqrt{3}}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + c$$

$$I = \ln(2+\cos x) + \frac{4}{3}\tan^{-1}\left(\frac{\tan\frac{x}{2}}{\sqrt{3}}\right) + c.$$

(iii) $I = \int\frac{4\cos^2 x\,dx}{(\sin 2x + 4)^2} = 2\int\frac{2\cos^2 x\,dx}{(\sin 2x+4)^2}$

$$= 2\int\frac{(1+\cos 2x)dx}{(\sin 2x+4)^2} = 2\int\frac{dx}{(\sin 2x+4)^2} + 2\int\frac{\cos 2x\,dx}{(\sin 2x+4)^2}$$

Let $\sin 2x + 4 = t \Rightarrow 2\cos 2x\,dx = dt$ or $dx = \frac{dt}{2\cos 2x}$

$$I = \int\frac{dt}{t^2\cdot\sqrt{1-(t-4)^2}} + \int\frac{dt}{t^2}$$

$$= \int\frac{dt}{t^2\cdot\sqrt{1-(t-4)^2}} - \frac{1}{(\sin 2x + 4)} + c$$

$t-4=X \Rightarrow dt=dX$

$$I=\underbrace{\int\frac{dX}{(X+4)^2\sqrt{1-X^2}}}_{I_1}-\frac{1}{(\sin 2X+4)}+c$$

Let $X=\frac{1}{u} \Rightarrow dX=-\frac{1}{u^2}du$

$$I_1=\int\frac{-\frac{1}{u^2}du}{\frac{1}{u^2}(1+4u^2)\cdot\frac{1}{u}\sqrt{u^2-1}}=\int\frac{-udu}{(1+4u^2)\sqrt{u^2-1}}$$

Let $u^2-1=z^2 \Rightarrow udu=zdz$

$$=\int\frac{-zdz}{[1+4(z^2+1)]\cdot z}=-\int\frac{dz}{4z^2+5}=-\frac{1}{4}\int\frac{dz}{z^2+\frac{5}{4}}$$

$$=\frac{-1}{2\sqrt5}\tan^{-1}\frac{2\sqrt{u^2-1}}{\sqrt5}+c_2$$

$$=-\frac{1}{2\sqrt5}\tan^{-1}\frac{2}{\sqrt5}\frac{\sqrt{1-X^2}}{X}+c_2$$

$$=-\frac{1}{2\sqrt5}\tan^{-1}\frac{2}{\sqrt5}\frac{\sqrt{1-(t-4)^2}}{t-4}+c$$

$$=-\frac{1}{2\sqrt5}\tan^{-1}\frac{2}{\sqrt5}\cdot\frac{\sqrt{1-\sin^2 2x}}{\sin 2x}+c_2$$

$$=-\frac{1}{2\sqrt5}\tan^{-1}\frac{2}{\sqrt5}\cdot\frac{\cos 2x}{\sin 2x}+c_2$$

$$\therefore I=-\frac{1}{2\sqrt5}\tan^{-1}\frac{2}{\sqrt5}\cot 2x-\frac{1}{(\sin 2x+4)}+c.$$

(iv) $I=\int\frac{\sqrt{1+\sin x}}{\sqrt{\sin x}}dx=\int\frac{\sqrt{1+\sin x}\cdot\sqrt{1-\sin x}}{\sqrt{\sin x}\cdot\sqrt{1-\sin x}}dx$

$=\int\frac{\cos x dx}{\sqrt{\sin x}\cdot\sqrt{1-\sin x}}$ Let $\sin x=t \Rightarrow \cos x\,dx=dt$

$$=\int\frac{dt}{\sqrt{t(1-t)}}=\int\frac{dt}{\sqrt{t-t^2}}=\int\frac{dt}{\sqrt{\frac{1}{4}-\left(t-\frac{1}{2}\right)^2}}$$

$$=\int\frac{dt}{\sqrt{\left(\frac{1}{2}\right)^2-\left(t-\frac{1}{2}\right)^2}}=\sin^{-1}\frac{\left(t-\frac{1}{2}\right)}{\frac{1}{2}}+c$$

$$=\sin^{-1}2\left(\sin x-\frac{1}{2}\right)+c \Rightarrow I=\sin^{-1}(2\sin x-1)+c.$$

(v) $I=\int\frac{dx}{\cos^2 x(1+\sin^2 x)}=\int\frac{\sec^2 xdx}{1+\sin^2 x}$

$$=\int\frac{\sec^4 xdx}{2\tan^2 x+1}=\int\frac{(1+\tan^2 x)\sec^2 xdx}{2\tan^2 x+1}$$

Let $\tan x=t \Rightarrow \sec^2 x\,dx=dt$

$$=\int\frac{1+t^2}{2t^2+1}dt=\frac{1}{2}\int\frac{dt}{t^2+\frac{1}{2}}+\int\frac{t^2dt}{2t^2+1}$$

$$=\frac{1}{2\cdot\frac{1}{\sqrt2}}\tan^{-1}\sqrt2 t+\frac{1}{2}\int\frac{2t^2+1-1}{2t^2+1}dt$$

$$=\frac{1}{\sqrt2}\tan^{-1}\sqrt{2}t+\frac{1}{2}t-\frac{1}{2}\int\frac{dt}{2t^2+1}+c$$

$$=\frac{1}{2}t+\frac{1}{\sqrt2}\tan^{-1}\sqrt2 t-\frac{1}{2\sqrt2}\tan^{-1}\sqrt2 t+c$$

$$=\frac{1}{2}t+\frac{1}{2\sqrt2}\tan^{-1}\sqrt2 t+c$$

$$I=\frac{1}{2}\tan x+\frac{1}{2\sqrt2}\tan^{-1}\sqrt2\tan x+c.$$

(vi) $I=\int\frac{dx}{\tan^5 x\cdot\cos^{10}x}=\int\frac{\sec^8 x\cdot\sec^2 xdx}{\tan^5 x}$

$$=\int\frac{(1+\tan^2 x)^4}{\tan^5 x}\sec^2 x\,dx$$

$=\int\frac{(1+t^2)^4}{t^5}dt$ $\quad\tan x=t \Rightarrow \sec^2 xdx=dt$

$$=\int\frac{(1+4t^2+6t^4+4t^6+t^8)}{t^5}dt$$

$$=-\frac{1}{4\tan^4 x}-\frac{2}{\tan^2 x}+6\ln|\tan x|+\frac{4\tan^2 x}{2}+\frac{\tan^4 x}{4}+c.$$

(vii) $I=\int\frac{\sec^4 xdx}{1+\tan^4 x}=\int\frac{(1+\tan^2 x)\sec^2 xdx}{1+\tan^4 x}$

$$=\int\frac{(1+t^2)dt}{1+t^4}=\int\frac{\left(1+\frac{1}{t^2}\right)dt}{t^2+\frac{1}{t^2}}$$

$=\int\frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+(\sqrt2)^2}$, set $t-\frac{1}{t}=X$ $=\int\frac{dX}{X^2+(\sqrt2)^2}$

$$I=\frac{1}{\sqrt2}\tan^{-1}\frac{X}{\sqrt2}+c=\frac{1}{\sqrt2}\tan^{-1}\frac{t^2-1}{t\sqrt2}+c$$

$$=\frac{1}{\sqrt2}\tan^{-1}\left(\frac{\tan^2 x-1}{\sqrt2\tan x}\right)+c.$$

(viii) $I=\int\sec^{4/7}x\,\mathrm{cosec}^{10/7}x\,dx$

Put $\tan x=t \Rightarrow \sec^2 x\,dx=dt \Rightarrow dx=\frac{dt}{1+t^2}$

$$=\int\frac{dx}{\sin^{10/7}x\cos^{4/7}x}$$

$$=\int\frac{1}{\left(\frac{t^{10/7}}{(1+t^2)^{5/7}}\right)\left(\frac{1}{(1+t^2)^{2/7}}\right)}\frac{dt}{1+t^2}$$

$$\int \frac{dt}{t^{10/7}} = -\frac{7}{3}\left(\frac{1}{t^{3/7}}\right) + c = -\frac{7}{3}(\cot x)^{3/7} + c$$

(ix) Let $\cos x + i\sin x = y$; then

$$2\cos x = y + \frac{1}{y} \Rightarrow 2\cos nx = y^n + \frac{1}{y^n},$$

$$2i\sin x = y - \frac{1}{y} \Rightarrow 2i\sin nx = y^n - \frac{1}{y^n}.$$

Thus $2^8 i^8 \sin^8 x = \left(y - \frac{1}{y}\right)^8$

$$= \left(y^8 + \frac{1}{y^8}\right) - 8\left(y^6 + \frac{1}{y^6}\right) + 28\left(y^4 + \frac{1}{y^4}\right) - 56\left(y^2 + \frac{1}{y^2}\right) + 70$$

$= 2\cos 8x - 16\cos 6x + 56\cos 4x - 112\cos 2x + 70.$

Thus,

$$\sin^8 x = \frac{1}{2^7}(\cos 8x - 8\cos 6x + 28\cos 4x - 56\cos 2x + 35),$$

$$\int \sin^8 x\,dx$$

$$= \frac{1}{2^7}\left[\frac{\sin 8x}{8} - 8\frac{\sin 6x}{6} + 28\frac{\sin 4x}{4} - 56\frac{\sin 2x}{2} + 35x\right] + c.$$

(x) $$\int\left(\frac{2+\sin 2x}{1+\cos 2x}\right)e^x\,dx = \int\left(\frac{2}{1+\cos 2x} + \frac{\sin 2x}{1+\cos 2x}\right)e^x dx$$

$$= \int\left(\frac{2}{2\cos^2 x} + \frac{\sin x}{\cos x}\right)e^x dx$$

$$= \int(\tan x + \sec^2 x)e^x dx = e^x.\tan x + c.$$

(xi) Let $I = \int \frac{\cos x + 2\sin x + 3}{4\cos x + 5\sin x + 6}dx$

Suppose $(\cos x + 2\sin x + 3) \equiv A(4\cos x + 5\sin x + 6) + B(-4\sin x + 5\cos x) + C$

Equating coefficients of $\cos x$, $\sin x$ and constant terms.

$\therefore 4A + 5B = 1,\ 5A - 4B = 2$ and $6A + C = 3.$

Solving $A = \frac{14}{41}, B = -\frac{3}{41}, C = \frac{39}{41}$

$$\therefore I = \int \frac{A(4\cos x + 5\sin x + 6) + B(-4\sin x + 5\cos x) + C}{(4\cos x + 5\sin x + 6)}dx$$

$$= \int \frac{A(4\cos x + 5\sin x + 6) + B(-4\sin x + 5\cos)}{(4\cos x + 5\sin x + 6)}dx + C\int \frac{dx}{4\cos x + 5\sin x + 6}$$

$= Ax + B\log(4\cos x + 5\sin x + 6)$

$$+ C\int \frac{dx}{4\left(\cos^2\frac{x}{2} - \sin^2\frac{x}{2}\right) + 10\sin\frac{x}{2}\cos\frac{x}{2} + 6\left(\cos^2\frac{x}{2} + \sin^2\frac{x}{2}\right)}$$

$$= \frac{14}{41}x - \frac{3}{41}\log(4\cos x + 5\sin x + 6)$$

$$+ \frac{39}{41}\int \frac{dx}{10\cos^2\frac{x}{2} + 2\sin^2\frac{x}{2} + 10\sin\frac{x}{2}\cos\frac{x}{2}}$$

Now, $\int \frac{dx}{10\cos^2\frac{x}{2} + 2\sin^2\frac{x}{2} + 10\sin\frac{x}{2}\cos\frac{x}{2}}$

$$= \int \frac{\sec^2\frac{x}{2}\,dx}{10 + 2\tan^2\frac{x}{2} + 10\tan\frac{x}{2}}$$

Put $\tan\frac{x}{2} = t \Rightarrow \sec^2\frac{x}{2}.\frac{1}{2}dx = dt$

$$= \int \frac{2\,dt}{2.t^2 + 10t + 10} = \int \frac{dt}{t^2 + 5t + 5}$$

$$= \int \frac{dt}{\left(t + \frac{5}{2}\right)^2 - \left(\frac{\sqrt{5}}{2}\right)^2} = \frac{2}{2\sqrt{5}}.\log\frac{t + \frac{5}{2} - \frac{\sqrt{5}}{2}}{t + \frac{5}{2} + \frac{\sqrt{5}}{2}} + c$$

$$= \frac{1}{\sqrt{5}}\log\frac{2\tan\frac{x}{2} + (5 - \sqrt{5})}{2\tan\frac{x}{2} + (5 + \sqrt{5})} + c$$

$$\therefore I = \frac{14}{41}x - \frac{3}{41}(4\cos x + 5\sin x + 6)$$

$$+ \frac{39}{41\sqrt{5}}\log\frac{2\tan\frac{x}{2} + (5 - \sqrt{5})}{2\tan\frac{x}{2} + (5 + \sqrt{5})} + c.$$

8. (i) $I = (\sin^{-1}x)^2 \cdot \int 1\cdot dx - \int\left[\frac{d}{dx}(\sin^{-1}x)^2 \cdot \int 1\cdot dx\right]dx$

$$= x(\sin^{-1}x)^2 - \int \frac{2\cdot\sin^{-1}x}{\sqrt{1-x^2}}\cdot x\,dx$$

$$= x(\sin^{-1}x)^2 - \sin^{-1}x \cdot \int \frac{2x}{\sqrt{1-x^2}}dx$$

$$+ \int\left[\frac{d}{dx}(\sin^{-1}x)\cdot\int\frac{2x}{\sqrt{1-x^2}}dx\right]dx + c$$

$$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\cdot\sin^{-1}x$$

$$- \int \frac{2}{\sqrt{1-x^2}}\cdot\sqrt{1-x^2}\,dx + c$$

$$= x(\sin^{-1}x)^2 + 2\sqrt{1-x^2}\sin^{-1}x - 2x + c.$$

(ii) $I = (\tan^{-1}x)^2 \cdot \int x\,dx - \int\left[\frac{d}{dx}(\tan^{-1}x)^2 \cdot \int x\cdot dx\right]dx$

$$= \frac{x^2}{2}(\tan^{-1}x)^2 - \int \frac{2\tan^{-1}x}{1+x^2}\cdot\frac{x^2}{2}dx$$

$$= \frac{x^2}{2}(\tan^{-1}x)^2 - \int \frac{x^2\tan^{-1}x}{1+x^2}dx$$

$$= \frac{x^2}{2}(\tan^{-1}x)^2 - \underbrace{\int\left[\tan^{-1}x - \frac{\tan^{-1}x}{1+x^2}\right]dx}_{I_1}$$

where $I_1 = x\cdot\tan^{-1}x - \int\frac{x}{1+x^2}dx - \frac{(\tan^{-1}x)^2}{2} + c$

$$\therefore I = \frac{x^2}{2}(\tan^{-1}x)^2 - x\tan^{-1}x + \frac{1}{2}\ln(1+x^2) + \frac{(\tan^{-1}x)^2}{2} + c.$$

(iii) $I = \ln x\cdot\int\frac{xdx}{\sqrt{(x^2-1)^3}} - \int\left[\frac{d}{dx}(x\ln x)\cdot\int\frac{xdx}{\sqrt{(x^2-1)^3}}\right]dx$

$$= \frac{\ln x}{2}\int\frac{2x\,dx}{\sqrt{(x^2-1)^3}} - \underbrace{\int\frac{1}{x}\cdot\frac{1}{2}\cdot\frac{-2}{\sqrt{x^2-1}}dx}_{I_1}$$

$$= -\frac{\ln x}{\sqrt{x^2-1}} + \int\frac{dx}{x\sqrt{x^2-1}} + \int\frac{tdt}{(t^2+1)t}$$

For I_1, Let $x^2 - 1 = t^2$ $\quad\therefore 2x\,dx = 2t\,dt$

$$I = -\frac{\ln x}{\sqrt{x^2-1}} + \tan^{-1}\sqrt{x^2-1} + c.$$

(iv) $I = x^3\cdot\int e^{3x}dx - 3x^2\int\frac{e^{3x}}{3}dx + 6x\cdot\int\frac{e^{3x}}{9}dx - 6\int\frac{e^{3x}}{27}dx$

$$-2x^2\cdot\int e^{3x}dx + 4x\int\frac{e^{3x}}{3}dx - 4\int\frac{e^{3x}}{9}dx + 5\frac{e^{3x}}{3} + c$$

$$= e^{3x}\left[\frac{x^3}{3} - \frac{x^2}{3} + \frac{2x}{9} - \frac{6}{81} - \frac{2x^2}{3} + \frac{4x}{9} - \frac{4}{27} + \frac{5}{3}\right] + c$$

$$I = e^{3x}\left[\frac{x^3}{3} - x^2 + \frac{2x}{3} + \frac{13}{9}\right] + c.$$

(v) Let $x = t^3 \Rightarrow dx = 3t^2dt \quad\therefore I = 3\int t^5e^t\,dt$

$= 3t^5e^t - 15t^4e^t + 60t^3e^t - 180t^2e^t + 360te^t - 360e^t + c$

$I = e^{x^{1/3}}[3(x)^{5/3} - 15(x)^{4/3} + 60x - 180(x)^{2/3} + 360x^{1/3} - 360] + c.$

(vi) $e^x = t \Rightarrow \frac{dt}{e^x}.\quad I = \int\frac{dt}{t\cdot\sqrt{1+t+t^2}}$

Let $t = \frac{1}{u} \Rightarrow dt = -\frac{1}{u^2}du$

$$= -\int\frac{du}{\sqrt{1+u+u^2}} = -\int\frac{du}{\sqrt{\left(u+\frac{1}{2}\right)^2 + \frac{3}{4}}}$$

$$= -\ln\left|\left(u+\frac{1}{2}\right) + \sqrt{1+u+u^2}\right| + c$$

$$I = -\ln\left|\frac{2+e^x+\sqrt{1+e^x+e^{2x}}}{2e^x}\right| + c.$$

(vii) $I = \int xe^{\sin x}\cdot\cos x dx - \int\frac{e^{\sin x}\cdot\sin x}{\cos^2 x}dx$

$$= xe^{\sin x} - \int e^{\sin x}dx - \underbrace{\int\frac{e^{\sin x}\cdot\sin x}{\cos^2 x}dx}_{I_1}$$

Now $I_1 = e^{\sin x}\cdot\int\frac{\sin x}{\cos^2 x}dx - \int\frac{de^{\sin x}}{dx}\cdot\int\frac{\sin x}{\cos^2 x}dx$

$$\therefore I = xe^{\sin x} - \int e^{\sin x}dx - e^{\sin x}\cdot\sec x + \int\frac{e^{\sin x}\cdot\cos x}{\cos x}dx$$

$I = e^{\sin x}(x - \sec x) + c.$

(viii) 1st solution :

$$\int\frac{dx}{e^x + e^{2x}} = \int\frac{e^{-2x}}{e^{-x}+1}dx$$

Multiplying Nr. and Den. by e^{-2x}

$$= \int\frac{e^{-x}.e^{-x}\,dx}{e^{-x}+1}$$

Put $e^{-x} + 1 = t \Rightarrow -e^{-x}dx = dt$ so that $e^{-x} = t - 1$

$$= -\int\frac{(t-1)}{t}dt = -\int\left(1 - \frac{1}{t}\right)dt = -[t - \log t] + c$$

$= \log t - t + c = \log(1 + e^{-x}) - (e^{-x} + 1) + c$

2nd solution :

$\int\frac{dx}{e^x + e^{2x}}$ Put $e^x = y \Rightarrow e^x\,dx = dy \Rightarrow dx = \frac{dy}{y}$

$$\int\frac{dx}{e^x+e^{2x}} = \int\frac{dy}{y(y+y^2)}$$

$$= \int\frac{dy}{y^2(1+y)} = \int\left(\frac{1}{1+y} - \frac{1}{y} + \frac{1}{y^2}\right)dy$$

$$= \log(1+y) - \log y - \frac{1}{y} + c$$

$$= \log\frac{1+y}{y} - \frac{1}{y} + c$$

$$= \log\frac{1+e^x}{e^x} - e^{-x} + c = \log(e^{-x}+1) - e^{-x} + c$$

9. (i) $I = \int\frac{x^6 - 2x^4 + 3x^3 - 9x^2 + 4}{x(x^4 - 5x^2 + 4)}dx$

$$= \int\frac{x^6 - 2x^4 + 3x^3 - 9x^2 + 4}{x(x^2-4)(x^2-1)}dx$$

$$= \int\frac{dx}{x} + \int\frac{(-3x^4 + 3x^3 + x^6 - 4x^2)}{x(x^2-4)(x^2-1)}dx$$

$$= \int\frac{dx}{x} - \int\frac{x(3x^2 - 3x - x^4 + 4)}{(x^2-4)(x^2-1)}dx$$

$$= \ln x - \frac{1}{3}\underbrace{\int\frac{x(3x^2 - 3x - x^4 + 4)}{x^2-4}}_{I_1}$$

$$+\frac{1}{3}\underbrace{\int\frac{x(3x^2-3x-x^4+4)}{x^2-1}dx}_{I_2}$$

$$I_1=\frac{1}{3}\int\frac{3x^3-3x^2-x^5+4x}{x^2-4}dx$$

$$=\frac{1}{3}\int\frac{3x\cdot x^2}{x^2-4}dx-\int\frac{x^2dx}{x^2-4}-\frac{1}{3}\int\frac{x^4\cdot xdx}{x^2-4}+\frac{4}{3}\int\frac{xdx}{x^2-4}$$

$$=\ln x\frac{1}{2}(x^2-4)+2\ln|x^2-4|-x-\frac{4}{4}\ln\left|\frac{x-2}{x+2}\right|$$

Similarly I_2, $=-\frac{1}{6}\cdot\frac{(x^2-4)^2}{2}-\frac{8}{6}(x^2-4)$

$$-\frac{16}{6}\ln|x^2-4|\cdot\frac{2}{3}\ln|x^2-4|+c_1$$

Rearranging, $I=I_1+I_2$

$$I=\frac{x^2}{2}+\ln\left|\frac{x(x-2)\sqrt{(x-1)(x+1)^3}}{x+2}\right|+c$$

(ii) $I=\int\frac{dx}{(x^4-1)(x^4-1)}=\int\frac{1}{(1-x^4)}\cdot\frac{1}{2}\left[\frac{1}{1-x^2}+\frac{1}{1+x^2}\right]dx$

$$=\frac{1}{2}\int\frac{dx}{(1-x^4)(1-x^2)}+\frac{1}{2}\int\frac{dx}{(1-x^4)(1+x^2)}$$

$$=\frac{1}{2}\int\frac{1}{2}\left[\frac{1}{(1-x^2)^2}+\frac{1}{1-x^4}\right]dx$$

$$+\frac{1}{2}\int\frac{1}{2}\left[\frac{1}{(1+x^2)^2}+\frac{1}{1-x^4}\right]dx$$

$$=\underbrace{\frac{1}{4}\int\frac{dx}{(1-x^2)^2}}_{I_1}+\underbrace{\frac{1}{4}\int\frac{dx}{(1+x^2)^2}}_{I_2}+\underbrace{\frac{1}{2}\int\frac{dx}{1-x^4}}_{I_3}$$

put $x=\sin\theta$ in I_1 and $x=\tan\theta$ in I_2

$\Rightarrow dx=\cos\theta\,d\theta$, and $dx=\sec^2\theta\,d\theta$

$$=\frac{1}{4}\int\sec^3\theta\,d\theta+\frac{1}{4}\int\cos^2\theta\,d\theta$$

$$+\frac{1}{8}\ln\left|\frac{1+x}{1-x}\right|+\frac{1}{4}\tan^{-1}x+c$$

$$=\frac{1}{8}\cdot\sec\theta\tan\theta+\frac{1}{8}\ln(\sec\theta+\tan\theta)+\frac{1}{4}\int\frac{1+\cos2\theta}{2}d\theta$$

$$+\frac{1}{8}\ln\left|\frac{1+x}{1-x}\right|+\frac{1}{4}\tan^{-1}x+c$$

$$=\frac{1}{8}\cdot\frac{x}{1-x^2}+\frac{1}{8}\ln\left(\frac{1+x}{\sqrt{1-x^2}}\right)+\frac{1}{8}\tan^{-1}x+\frac{2}{16}\cdot\frac{x}{1+x^2}$$

$$+\frac{1}{8}\ln\left|\frac{1+x}{1-x}\right|+\frac{1}{4}\tan^{-1}x+c$$

$$=\frac{1}{8}.\frac{x}{1-x^2}+\frac{1}{8}.\frac{x}{1+x^2}+\frac{1}{16}\ln\left|\frac{1+x}{1-x}\right|$$

$$+\frac{1}{8}\ln\left|\frac{1+x}{1-x}\right|+\frac{3}{8}\tan^{-1}x+c$$

$$=\frac{1}{8}\cdot\frac{x}{1-x^2}+\frac{1}{8}\cdot\frac{x}{1+x^2}+\frac{3}{16}\ln\left|\frac{1+x}{1-x}\right|+\frac{3}{8}\tan^{-1}x+c$$

$$I=\frac{1}{8}x\left[\frac{1}{1-x^2}+\frac{1}{1+x^2}\right]+\frac{3}{16}\ln\left|\frac{1+x}{1-x}\right|+\frac{3}{8}\tan^{-1}x+c$$

(iii) $I=\int\frac{2\cos x\cdot\sqrt{2\sin x\cos x}}{\sin^4 x}dx$

$$=\int\frac{2\cos x}{\frac{\sin^4 x}{\cos^4 x}\cdot\cos^4 x}\sqrt{\frac{2\sin x}{\cos x}\cdot\cos^2 x}\,dx$$

$$=\int\frac{2\cos^2 x\cdot\sqrt{2\tan x}}{\tan^4 x\cdot\cos^4 x}dx=2\int\frac{\sqrt{2\tan x}\cdot\sec^2 x}{\tan^4 x}dx$$

$$=2\sqrt{2}\int(\tan x)^{(1/2)-4}\sec^2 xdx$$

$$=2\sqrt{2}\int(\tan x)^{-7/2}\sec^2 xdx$$

Let $\tan x=t\Rightarrow\sec^2 xdx=dt\Rightarrow I=2\sqrt{2}\int t^{-7/2}dt$

$$\Rightarrow I=-2\sqrt{2}\left(\frac{2}{5}\right)(\tan x)^{-5/2}+c.$$

(iv) Let $2+\tan^2x=t^2\Rightarrow 2\tan x.\sec^2xdx=2tdt;$

$$dx=\frac{tdt}{\sqrt{t^2-2(t^2-1)}}\quad\Rightarrow I=\int\frac{t^2dt}{\sqrt{t^2-2(t^2-1)}}$$

$$=\int\frac{dt}{\sqrt{t^2-2}}+\int\frac{dt}{(t^2-1)\sqrt{t^2-2}}$$

$$=\ln(t+\sqrt{t^2-2})+\int\frac{dt}{(t^2-1)\sqrt{t^2-2}}$$

$$I=\ln(\sqrt{2+\tan^2 x}+\tan x)+\int\frac{dt}{(t^2-1)\sqrt{t^2-2}}$$

$$=\ln(\sqrt{2+\tan^2 x}+\tan x)$$

$$I_1=\int\frac{dt}{\sqrt{t^2-2(t^2-1)}},\text{ let }t=\frac{1}{u}\Rightarrow dt=\frac{-1}{u^2}du$$

$$=\int\frac{-\frac{1}{u^2}du}{\sqrt{1-2u^2}(1-u^2)\frac{1}{u^3}}=-\int\frac{udu}{\sqrt{1-2u^2}(1-u^2)}$$

$$1-2u^2=t_1^2\Rightarrow-4udu=2t_1dt_1\Rightarrow udu=-\frac{1}{2}t_1dt_1$$

$$I_1=-\int\frac{dt_1}{1-t_1^2}$$

$$=\frac{-1}{2}\log\left|\frac{1+t_1}{1-t_1}\right|+c=\frac{-1}{2}\log\left|\frac{1+\sqrt{1-2u^2}}{1-\sqrt{1-2u^2}}\right|+c$$

$$=\frac{-1}{2}\log\left|\frac{t+\sqrt{t^2-2}}{t-\sqrt{t^2-2}}\right|+c$$

$$= \frac{-1}{2}\log\left|\frac{\sqrt{2+\tan^2 x}+\sqrt{\tan^2 x}}{\sqrt{2+\tan^2 x}-\sqrt{\tan^2 x}}\right| + c$$

$$= \frac{-1}{2}\log\left|\frac{\sqrt{2+\tan^2 x}+\tan x}{\sqrt{2+\tan^2 x}-\tan x}\right| + c$$

(v) Let $1-2^x = t \Rightarrow -2^x \ln 2dx = dt$

$$= \frac{1}{\ln 2}\int \frac{dt}{t^4(t-1)} = \frac{1}{\ln 2}\left[\int\left[\frac{1}{t-1}-\frac{1}{t}\right]\frac{1}{t^3}\,dt\right]$$

$$= \frac{1}{\ln 2}\left[\int\frac{dt}{t-1}-\int\frac{dt}{t}-\int\frac{dt}{t^2}-\int\frac{dt}{t^3}-\int\frac{dt}{t^4}\right]$$

$$I = \frac{1}{\ln 2}\left[\ln(-2^x)-\ln(1-2^x)+\frac{1}{\ln(1-2^x)}\right.$$
$$\left.+\frac{1}{2[\ln(1-2^x)]^2}+\frac{1}{3[\ln(1-2x)]^3}\right]+c.$$

(vi) Let $I = \int\frac{dx}{1-x^6} = \int\frac{dx}{(1-x)^3(1+x^3)}$

$$= \frac{1}{2}\int\left[\frac{1}{1-x^3}+\frac{1}{1+x^3}\right]dx$$

$$= \frac{1}{2}\int\frac{dx}{(1-x)(1+x+x^2)}+\frac{1}{2}\int\frac{dx}{(1+x)(1-x+x^2)}$$

Suppose $\frac{1}{(1-x)(1+x+x^2)} \equiv \frac{A}{1-x}+\frac{Bx+C}{1+x+x^2}$

$\therefore 1 \equiv A(1+x+x^2)+(Bx+C)(1-x)$

Putting $x=1$, $3A=1 \Rightarrow A=\frac{1}{3}$

Equating coefficients of x^2 and x

$0 = A-B \Rightarrow A = B \Rightarrow B = \frac{1}{3}$

$0 = A+B-C \Rightarrow C = \frac{2}{3}$

Again suppose

$$\frac{1}{(1+x)(1-x+x^2)} \equiv \frac{L}{1+x}+\frac{Mx+N}{1-x+x^2}$$

$\therefore 1\ (1-x+x^2)+(Mx+N)(1+x)$

Putting $x=-1$, $1 = L(3) \Rightarrow L = \frac{1}{3}$

Equating coefficients of x^2, x, we have

$0 = L+M \Rightarrow M = -\frac{1}{3}$ $0 = -L+M+N \Rightarrow N = \frac{2}{3}$

$$\therefore I = \frac{1}{2}\int\left[\frac{1}{3(1+x)}+\frac{1}{3}\left(\frac{-x+2}{1-x+x^2}\right)\right]dx$$

$$+\frac{1}{2}\int\left[\frac{1}{3(1-x)}+\frac{1}{3}\left(\frac{x+2}{(1+x+x^2)}\right)\right]dx$$

$$= \frac{1}{6}\log(1+x)-\frac{1}{12}\int\frac{2x-1}{x^2-x+1}dx+\frac{1}{4}\int\frac{dx}{\left(x-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$+\frac{1}{6}\int\frac{dx}{1-x}+\frac{1}{12}\int\frac{2x+1}{x^2+x+1}dx+\frac{1}{4}\int\frac{dx}{\left(x+\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \frac{1}{6}\log(1+x)-\frac{1}{12}\log(x^2-x+1)+\frac{1}{2\sqrt{3}}\tan^{-1}\frac{\left(x-\frac{1}{2}\right)}{\frac{\sqrt{3}}{2}}$$

$$-\frac{1}{6}\log(1-x)+\frac{1}{12}\log(x^2+x+1)+\frac{1}{2\sqrt{3}}\tan^{-1}\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}+c$$

$$= \frac{1}{6}\log\frac{1+x}{1-x}+\frac{1}{12}\log\frac{x^2+x+1}{x^2-x+1}$$
$$+\frac{1}{2\sqrt{3}}\left[\tan^{-1}\frac{2x+1}{\sqrt{3}}+\tan^{-1}\frac{2x-1}{\sqrt{3}}\right]+c$$

(vii) $\int\frac{x^3+2}{(x-1)(x-2)^3}dx = \int\frac{(y+2)^3+2}{y^3(y+1)}dy$

Put $x-2=y$, then $x-1=y+1$

$$= \int\frac{2+8+12y+6y^2+y^3}{y^2(1+y)}dy$$

$$= \int\frac{10+12y+6y^2+y^3}{y^3(1+y)}dy$$

$$= \int\frac{1}{y^3}\left[10+2y+4y^2-\frac{3y^3}{1+y}\right]dy$$

$$= \int\frac{10}{y^3}+\frac{2}{y^2}+\frac{4}{y}-\frac{3}{1+y}dy$$

$$= \int\frac{10}{(x-2)^3}+\frac{2}{(x-2)^2}+\frac{4}{(x-2)}-\frac{3}{x-1}dx$$

$$= \frac{-5}{(x-2)^2}-\frac{2}{(x-2)}+4\log(x-2)-3\log(x-1)+c.$$

(viii) $I = \int\frac{(1+x)^3}{(1-x)^3}dx$

Put $1-x=y$, $x=1-y$ $\therefore 1+x=2-y$

$$\frac{(1+x)^3}{(1-x)^3} = \frac{(2-y)^3}{y^3} = \frac{8-12y+6y^2-y^3}{y^3}$$

$$= \frac{8}{y^3}-\frac{12}{y^2}+\frac{6}{y}-1 = -1+\frac{6}{1-x}-\frac{12}{(1-x)^2}+\frac{8}{(1-x)^3}$$

$$\therefore I = \int\left[-1+\frac{6}{(1-x)}-\frac{12}{(1-x)^2}+\frac{8}{(1-x)^3}\right]dx$$

$$= -x-6\log(1-x)-\frac{12}{(1-x)}+\frac{4}{(1-x)^2}+c$$

$$= -x-12(1-x)^{-1}+4(1-x)^{-2}-6\log(1-x)+c$$

(ix) $I = \int\frac{xdx}{(x+1)^3(x^2+1)}$

Let $\dfrac{x}{(x+1)^3(x^2+1)} \equiv \dfrac{A}{x+1} + \dfrac{B}{(x+1)^2} + \dfrac{C}{(x+1)^3} + \dfrac{Dx+E}{x^2+1}$

$\therefore x \equiv A(x+1)^2(x^2+1)+B(x+1)(x^2+1)+C(x^2+1)+(Dx+E)(x+1)^3$

Putting $x=-1 \Rightarrow -1 = 2C \Rightarrow C = -\dfrac{1}{2}$

Equating coefficients of x^4, x^3, x^2, x and constant terms

$0 = A + D$ .... (i)

$0 = 2A + B + E + 3D$ .... (ii)

$0 = 2A + B + C + 3D + 3E$ .... (iii)

$1 = 2A + B + D + 3E$ .... (iv)

$0 = A + B + C + E$ .... (v)

Solving $A = \dfrac{1}{4}, B = 0, B = -\dfrac{1}{4}, E = \dfrac{1}{4}$

$$\therefore I = \int\left[\frac{1}{4(x+1)} - \frac{1}{2(x+1)^3} - \frac{1}{4}\,\frac{x-1}{x^2+1}\right]dx$$

$$I = \frac{1}{4}\int\frac{1}{x+1}dx - \frac{1}{2}\int\frac{dx}{(x+1)^3} - \frac{1}{8}\int\frac{2x}{x^2+1}dx + \frac{1}{4}\int\frac{dx}{x^2+1}$$

$$= \frac{1}{4}\log(x+1) + \frac{1}{4}\,\frac{1}{(x+1)^2} - \frac{1}{8}\log(x^2+1) + \frac{1}{4}\tan^{-1}x + c$$

10. (i)

$$I = \int\frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t^2+\frac{1}{t^2}-1\right)} = \int\frac{\left(1+\frac{1}{t^2}\right)dt}{\left(t-\frac{1}{t}\right)^2+1} = \tan^{-1}\left(t-\frac{1}{t}\right)+c$$

$I = \tan^{-1}(e^x - e^{-x}) + c.$

(ii) $I = \dfrac{dx}{\cos x\cdot\sqrt{1+\sin^2 x}}$

$$= \int\frac{dx}{\cos x\sqrt{(\sec^2 x + \tan^2 x)\cos^2 x}}$$

$$= \int\frac{dx}{\cos^2 x\sqrt{1+2\tan^2 x}} = \int\frac{\sec^2 x\,dx}{\sqrt{1+2\tan^2 x}}$$

Let $\tan x = t \Rightarrow \sec^2 x\,dx = dt$

$$I = \int\frac{dt}{\sqrt{2t^2+1}} = \frac{1}{\sqrt{2}}\int\frac{dt}{\sqrt{t^2+\left(\frac{1}{\sqrt{2}}\right)^2}}$$

$$= \frac{1}{\sqrt{2}}\ln\left(t+\sqrt{t^2+\left(\frac{1}{\sqrt{2}}\right)^2}\right)+c$$

$$I = \frac{1}{\sqrt{2}}\ln\left(\tan x+\sqrt{\tan^2 x+\frac{1}{2}}\right)+c.$$

(iii) $I = \displaystyle\int\frac{1-\frac{1}{x^2}}{\left(1+\frac{1}{x^2}\right)}\cdot\frac{dx}{x\sqrt{x^2+\frac{1}{x^2}}}$, set $x+\dfrac{1}{x}=t$

$$= \int\frac{dt}{t\cdot\sqrt{t^2-2}} \quad \text{Now } t^2-2=z^2 \Rightarrow 2t\,dt = 2z\,dz$$

$$= \int\frac{dz}{z^2+2} = \frac{1}{\sqrt{2}}\tan^{-1}\frac{z}{\sqrt{2}}+c$$

$$= \frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{t^2-2}}{\sqrt{2}}+c$$

$$I = \frac{1}{\sqrt{2}}\tan^{-1}\frac{\sqrt{\left(x+\frac{1}{x}\right)^2-2}}{\sqrt{2}}+c.$$

(iv) Put $\cos x + i\sin x = y$; then $2^6 i^6 \sin^6 x\cdot 2^2\cos^2 x$

$$= \left(y-\frac{1}{y}\right)^6\left(y+\frac{1}{y}\right)^2$$

$$= y^8+\frac{1}{y^8}-4\left(y^6+\frac{1}{y^6}\right)+4\left(y^4+\frac{1}{y^4}\right)+4\left(y^2+\frac{1}{y^2}\right)-10$$

$= 2\cos 8x - 8\cos 6x + 8\cos 4x + 8\cos 2x - 10,$

and $\sin^6 x\cos^2 x = \dfrac{1}{2^7}\{-\cos 8x + 4\cos 6x - 4\cos 4x - 4\cos 2x + 5\},$

whence $\displaystyle\int\sin^6 x\cos^2 x\,dx$

$$= \frac{1}{2^7}\left\{-\frac{\sin 8x}{8}+4\frac{\sin 6x}{6}-4\frac{\sin 4x}{4}-4\frac{\sin 2x}{2}+5x\right\}.$$

(v) Let $\cos\dfrac{x}{2} = t^2; -\dfrac{1}{2}\sin\dfrac{x}{2}dx = 2tdt \Rightarrow dx = \dfrac{-4tdt}{\sin\frac{x}{2}}$

$$I = \int\frac{-4tdt}{\sin^2\frac{x}{2}\cdot t^3} = \int\frac{-4tdt}{(1-t^4)t^3}$$

$$= -4\int\frac{dt}{t^2(1-t^4)} = -4\int\frac{dt}{t^2(1-t^2)(1+t^2)}$$

$$= -4\int\frac{1}{2t^2}\left[\frac{1}{1-t^2}+\frac{1}{1+t^2}\right]dt = -2\int\frac{1}{t^2}\left[\frac{1}{1-t^2}+\frac{1}{1+t^2}\right]dt$$

$$= -2\int\left(\frac{1}{t^2}+\frac{1}{1-t^2}\right)dt - 2\int\left(\frac{1}{t^2}-\frac{1}{1+t^2}\right)dt$$

$$= -4\int\frac{1}{t^2}dt - 2\int\frac{dt}{1-t^2}+2\int\frac{dt}{1+t^2}$$

$$= \frac{4}{t} - 2\ln\left|\frac{1-t}{1+t}\right| + 2\tan^{-1}t + c$$

$$I = \frac{4}{\sqrt{\cos\frac{x}{2}}} - 2\ln\left|\frac{1-\sqrt{\cos\frac{x}{2}}}{1+\sqrt{\cos\frac{x}{2}}}\right| + 2\tan^{-1}\sqrt{\cos\frac{x}{2}} + c.$$

(vi) $I = \displaystyle\int\frac{x^2+3x+1}{x^4-x^2+1}dx = \int\frac{x^2+1}{x^4-x^2+1}dx + \int\frac{3x}{x^4-x^2+1}dx$

$$= \int \frac{1+\frac{1}{x^2}}{x^2-1+\frac{1}{x^2}}\,dx + \frac{3}{2}\int \frac{2x\,dx}{x^4-x^2+1}$$

$$= \int \frac{1+\frac{1}{x^2}}{\left(x-\frac{1}{x}\right)^2+1}\,dx + \frac{3}{2}\int \frac{2x\,dx}{\left(x^2-\frac{1}{2}\right)^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

Putting $x-\frac{1}{x}=y$ and $x^2-\frac{1}{2}=t$

$$= \int \frac{dy}{y^2+1} + \frac{3}{2}\int \frac{dt}{t^2+\left(\frac{\sqrt{3}}{2}\right)^2}$$

$$= \tan^{-1} y + \frac{3}{2}\cdot\frac{1}{\sqrt{3}}.\tan^{-1}\left(\frac{2t}{\sqrt{3}}\right)+c$$

$$= \tan^{-1}\left(\frac{x-\frac{1}{x}}{1}\right)+\frac{\sqrt{3}}{2}\tan^{-1}\left\{\frac{2\left(x^2-\frac{1}{2}\right)}{\sqrt{3}}\right\}+c$$

$$= \tan^{-1}\left(\frac{x^2-1}{x}\right)+\frac{\sqrt{3}}{2}\tan^{-1}\frac{(2x^2-1)}{\sqrt{3}}+c.$$

(vii) $I = \int \frac{\sin x}{\sqrt{(1+\sin x)}}\,dx = \int \frac{1+\sin x-1}{\sqrt{(1+\sin x)}}\,dx$

$$= \int \sqrt{(1+\sin x)}\,dx - \int \frac{dx}{\sqrt{(1+\sin x)}}$$

$$= -2\sqrt{(1-\sin x)} - \int \frac{dx}{\sqrt{\left\{\left(\cos^2\frac{x}{2}+\sin^2\frac{x}{2}+2\sin\frac{x}{2}\cos\frac{x}{2}\right)\right\}}}$$

$$= -2\sqrt{(1-\sin x)} - \int \frac{dx}{\left(\cos\frac{x}{2}+\sin\frac{x}{2}\right)}$$

$$= -2\sqrt{(1-\sin x)} - \frac{1}{\sqrt{2}}\int \frac{dx}{\sin\frac{x}{2}.\frac{1}{\sqrt{2}}+\cos\frac{x}{2}.\frac{1}{\sqrt{2}}}$$

$$\left[\because \frac{1}{\sqrt{2}}=\cos\frac{\pi}{4}=\sin\frac{\pi}{4}\right]$$

$$= -2\sqrt{(1-\sin x)} - \frac{1}{\sqrt{2}}\int \frac{dx}{\sin\left(\frac{x}{2}+\frac{\pi}{4}\right)}$$

$$= -2\sqrt{(1-\sin x)} - \frac{1}{\sqrt{2}}2.\log\tan\left(\frac{x}{4}+\frac{\pi}{8}\right)+c$$

$$= -2\sqrt{(1-\sin x)} - \sqrt{2}\log\tan\left(\frac{x}{4}+\frac{\pi}{8}\right)+c.$$

(viii) $I = \int \frac{(2x+3)dx}{(x^2+2x+3)\sqrt{x^2+2x+4}}$

$$= \int \frac{(2x+3)dx}{[(x+1)^2+2]\sqrt{(x+1)^2+3}}$$

Let $x+1=t \Rightarrow dx=dt$

$$I = \int \frac{[2(t-1)+3]dt}{[t^2+2]\sqrt{t^2+3}}$$

$$= \underbrace{\int \frac{2t\,dt}{(t^2+2)\sqrt{t^2+3}}}_{I_1} + \underbrace{\int \frac{dt}{(t^2+2)\sqrt{t^2+3}}}_{I_2}$$

$t^2+3=X^2$; $2t\,dt = 2X\,dX$

Now $I_1 = \int \frac{2X\,dX}{(X^2-1)X} = \ln\left|\frac{X-1}{X+1}\right|+c$

For I_2

Let $t=\frac{1}{u} \Rightarrow dt = \frac{-du}{u^2}$

$$I_2 = \int \frac{-du.u^3}{u^2(1+2u^2)\sqrt{1+3u^2}}$$

Let $1+3u^2=z^2 \Rightarrow 6udu = 2zdz$ or $udu = \frac{zdz}{3}$

$$I_2 = \int \frac{-\frac{1}{3}zdz}{\left[1+2\left(\frac{z^2-1}{3}\right)\right]z} = \frac{-1}{3}\int \frac{3dz}{3+2z^2-2}$$

$$= -\int \frac{dz}{2z^2+1} = -\frac{1}{2}\int \frac{dz}{z^2+\left(\frac{1}{\sqrt{2}}\right)^2}$$

$$= -\frac{1}{2}\times\frac{1}{\left(\frac{1}{\sqrt{2}}\right)}\tan^{-1}\frac{z}{\frac{1}{\sqrt{2}}}+c = \frac{-\sqrt{2}}{2}\tan^{-1}\sqrt{2}\sqrt{1+3u^2}+c$$

$$= \frac{-1}{2\sqrt{2}}\tan^{-1}\frac{\sqrt{2}\sqrt{t^2+3}}{t}+c$$

$$= \frac{-1}{2\sqrt{2}}\tan^{-1}\frac{\sqrt{2(x^2+1+2x+3)}}{x+1}+c$$

$$= \frac{-1}{2\sqrt{2}}\tan^{-1}\frac{\sqrt{2(x^2+2x+4)}}{x+1}+c.$$

(ix) $I = \int \frac{3\cos x-4\sin x}{4\cos x+5\sin x}\,dx$

Suppose, $3\cos x - 4\sin x = \lambda(4\cos x + 5\sin x) + \mu(-4\sin x + 5\cos x)$

Equating coeff. of $\sin x$ and $\cos x$ on both sides $4\lambda + 5\mu = 3$, $5\lambda - 4\mu = 4$.

Solving $\lambda = \frac{32}{41}$, $\mu = -\frac{1}{41}$

Hence,

$$I = \int \frac{\lambda(4\cos x+5\sin x)+\mu(-4\sin x+5\cos x)}{(4\cos x+5\sin x)}\,dx$$

$$= \lambda\int dx + \mu\int\left(\frac{-4\sin x + 5\cos x}{4\cos x + 5\sin x}\right)dx$$

$$= \lambda x + \mu\log(4\cos x + 5\sin x) + c$$

$$= \frac{32}{41}x - \frac{1}{41}\log(4\cos x + 5\sin x) + c.$$

(x) $I = \int\frac{dx}{\sin^2 x\cos^4 x}$

Put $\tan x = t \Rightarrow \sec^2 x\, dx = dt \Rightarrow dx = \frac{dt}{1+t^2}$

$$\text{L.H.S.} = \int\frac{1}{\frac{t^2}{1+t^2}\cdot\frac{1}{(1+t^2)^2}}\,\frac{dt}{1+t^2}$$

$$= \int\frac{(1+t^2)^2}{t^2}\left(\because \sin x = \frac{1}{\sqrt{(1+t^2)^2}},\ \cos x = \frac{1}{\sqrt{(1+t^2)}}\right)$$

$$= \int\frac{1+2t^2+t^4}{t^2}\,dt = \int\left(\frac{1}{t^2} + 2 + t^2\right)dt$$

$$= \left(-\frac{1}{t} + 2t + \frac{t^3}{3} + c\right) = -\frac{1}{\tan x} + 2\tan x + \frac{\tan^3 x}{3} + c$$

$$= -\cot x + 2\tan x + \frac{\tan^3 x}{3} + c$$

$$\text{R.H.S.} = \frac{\sin x(1+2\cos^2 x)}{3\cos^3 x} - 2\cot 2x$$

$$= \frac{\sin x}{3\cos^2 x} + \frac{2\sin x\cos^2 x}{3\cos^3 x} - 2\left(\frac{1-\tan^2 x}{2\tan x}\right)$$

$$= \frac{1}{3}\tan x\sec^2 x + \frac{2}{3}\tan x - \cot x + \tan x$$

$$= -\cot x + \frac{1}{3}\tan x(1+\tan^2 x) + \frac{2}{3}\tan x + \tan x$$

$$= -\cot x + \frac{1}{3}\tan x + \frac{\tan^3 x}{3} + \frac{2}{3}\tan x + \tan x$$

$$= -\cot x + 2\tan x + \frac{\tan^3 x}{3}$$

$$\therefore \int\frac{dx}{\sin^2 x\cos^4 x} = \frac{\sin x(1+2\cos^2 x)}{3\cos^2 x} - 2\cot 2x.$$

2010 : SOLVED PAPER

Regional Mathematical Olympiad

MTG presents incisive, insightful and instructive solutions to the Regional Mathematical Olympiad (RMO) that was held nationwide on December 5.

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1. Let $ABCDEF$ be a convex hexagon in which the diagonals AD,. BE, CF are concurrent at O. Suppose the area of triangle OAF is the geometric mean of those of OAB and OEF; and the area of triangle OBC is the geometric mean of those of OAB and OCD. Prove that the area of triangle OED is the geometric mean of those of OCD and OEF.

2. Let $P_1(x) = ax^2 - bx - c$, $P_2(x) = bx^2 - cx - a$, $P_3(x) = cx^2 - ax - b$ be three quadratic polynomials where a, b, c are non-zero real numbers. Suppose there exists a real number α such that $P_1(\alpha) = P_2(\alpha) = P_3(\alpha)$. Prove that $a = b = c$.

3. Find the number of 4-digit numbers (in base 10) having non-zero digits and which are divisible by 4 but not by 8.

4. Find three distinct positive integers with the least possible sum such that the sum of the reciprocals of any two integers among them is an integral multiple of the reciprocal of the third integer.

5. Let ABC be a triangle in which $\angle A = 60°$. Let BE and CF be the bisectors of the angles $\angle B$ and $\angle C$ with E on AC and F on AB. Let M be the reflection of A in the line EF. Prove that M lies on BC.

6. For each integer $n \ge 1$, define $a_n = \left[\dfrac{n}{[\sqrt{n}]}\right]$, where $[x]$ denotes the largest integer not exceeding x, for any real number x. Find the number of all n in the set $\{1, 2, 3, ..., 2010\}$ for which $a_n > a_{n+1}$.

SOLUTIONS

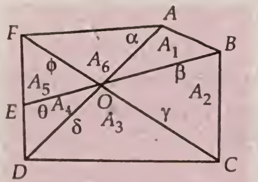
1. Let α be length of OA, β that of OB, γ that of OC, etc.

Let area of OAB, OBC,.... be respectively, $A_1, A_2...., A_6$

We have $\dfrac{(OAB)}{(OED)}$ i.e. $\dfrac{A_1}{A_4} = \dfrac{\alpha\beta}{\theta\delta}$

Similarly $\dfrac{A_2}{A_5} = \dfrac{\beta\gamma}{\theta\phi}$

and $\dfrac{A_3}{A_6} = \dfrac{\gamma\delta}{\alpha\phi}$



We have $\dfrac{A_1}{A_4} \cdot \dfrac{A_5}{A_2} \cdot \dfrac{A_3}{A_6}$

$= \dfrac{\alpha\beta}{\theta\delta} \cdot \dfrac{\theta\phi}{\beta\gamma} \cdot \dfrac{\gamma\delta}{\alpha\phi} = 1$

i.e $A_1 A_3 A_5 = A_2 A_4 A_6$

$\Rightarrow \quad A_1^2 A_3^2 A_5^2 = A_2^2 A_4^2 A_6^2$

Now $A_6^2 = A_1 A_5$, $A_2^2 = A_1 A_3$

Now $A_1^2 A_3^2 A_5^2 = (A_1A_3)(A_4{}^2)(A_1A_5)$

$\Rightarrow \quad A_3 A_5 = A_4{}^2$

Thus (OED) is the geometric mean of (OCD) and (OEF).

2. Ist Solution :

Let $a\alpha^2 - b\alpha - c = k$

$b\alpha^2 - c\alpha - a = k$

$c\alpha^2 - a\alpha - b = k$

We have

$$a\alpha^2 - b\alpha - (c+k) = 0$$
$$b\alpha^2 - c\alpha - (a+k) = 0$$
$$c\alpha^2 - a\alpha - (b+k) = 0$$

Eliminating $\alpha^2, -\alpha, -1$ from the system, we obtain

$$\begin{vmatrix} a & b & c+k \\ b & c & a+k \\ c & a & b+k \end{vmatrix} = 0$$

$$\Rightarrow (a+b+c+k)\begin{vmatrix} a & b & 1 \\ b & c & 1 \\ c & a & 1 \end{vmatrix} = 0$$

$\Rightarrow \quad (a + b + c + k)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$

$a^2 + b^2 + c^2 - ab - bc - ca = 0 \Rightarrow a = b = c.$

$a + b + c + k = 0$ also gives the same result.

2nd Solution :

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$$a\alpha^2 - b\alpha - c = k$$
$$b\alpha^2 - c\alpha - a = k$$
$$c\alpha^2 - a\alpha - b = k$$

Eliminating α^2, we have

$$(ab - c^2)\alpha - (ca - b^2) = k(c - b)$$
$$(bc - a^2)\alpha - (ab - c^2) = k(a - c)$$

and $(ca - b^2)\alpha - (bc - a^2) = k(b - a)$

Adding them all, we have

$(\alpha - 1)(ab + bc + ca - a^2 - b^2 - c^2) = 0$

$ab + bc + ca - a^2 - b^2 - c^2 = 0 \Rightarrow a = b = c$

Also $\alpha = 1$ gives

$a - b - c = b - c - a = c - a - b$

and we again obtain $a = b = c$.

3rd Solution :

$$a\alpha^2 - b\alpha - c = k \quad \text{...(i)}$$
$$b\alpha^2 - c\alpha - a = k \quad \text{...(ii)}$$
$$c\alpha^2 - a\alpha - b = k \quad \text{...(iii)}$$

Multiplying eqn. (i) by $(b - c)$, eqn. (ii) by $(c - a)$ and eqn. (iii) by $(a - b)$ we get

$\Sigma a(b - c)\alpha^2 - \{b(b - c) + c(c - a) + a(a - b)\}\alpha - c(b - c) - a(c - a) - b(a - b) = k\,\Sigma(b - c)$

$\Rightarrow \quad 0 + (a^2 + b^2 + c^2 - ab - bc - ca)(\alpha - 1) = 0$

$\therefore \quad (\alpha - 1)(a^2 + b^2 + c^2 - ab - bc - ca) = 0$, as earlier.

Transform the equation as

$$(a - b)\alpha^2 - (b - c)\alpha - (c - a) = 0$$
$$(b - c)\alpha^2 - (c - a)\alpha - (a - b) = 0$$
$$(c - a)\alpha^2 - (a - b)\alpha - (b - c) = 0$$

Take $b - c = u$, $c - a = v$, $a - b = w$, we have

$$w\alpha^2 + u\alpha - v = 0 \quad \text{... (iv)}$$
$$u\alpha^2 - v\alpha - w = 0 \quad \text{... (v)}$$
$$v\alpha^2 - w\alpha - u = 0 \quad \text{... (vi)}$$

Eliminating α from (v) and (vi), we have

$$(-v^2 + uw)\alpha = vw - u^2$$

As $u + v + w = 0$ we have $uw - v^2 = uw + v(u + w)$
$= (uv + vw + wu)$

Thus $(uv + vw + wu)\alpha = (uv + vw + wu)$

$\Rightarrow \quad (uv + vw + wu)(\alpha - 1) = 0$

Thus $\alpha = 1$, as earlier

or $uv + vw + wu = 0$, which together with $u + v + w = 0$ yields, $u = 0$, $v = 0$, $w = 0$, *i.e.* $a = b = c$

5th Solution :

$$a\alpha^2 - b\alpha - (c + k) = 0$$
$$b\alpha^2 - c\alpha - (a + k) = 0$$
$$c\alpha^2 - a\alpha - (b + k) = 0$$

One can use the idea of non-negativity of discriminant to compute the solution.

3. Ist Solution :

Here we do it by case work. The even four digit number will be grouped into 4 cases.

(i) Let the number end in 2. Then the last two digits can be 12, 32, 52, 72 or 92. But we must focus on third digit from right for it to be divisible by 8. There are $9 \times 4 \times 3 + 9 \times 5 \times 2 = 108 + 90 = 198$ such numbers. Recall that first digit can be filled in 9 ways.

(ii) Let the number end in 4. Then the last two digits must be of the form 24, 44, 64, 84. But we must focus on third digit from right for it to be divisible by 8. There are $9 \times 5 \times 2 + 9 \times 4 \times 2 = 90 + 72 = 162$ such numbers.

(iii) Similarly the number ending in 6 are $9 \times 5 \times 3 + 9 \times 4 \times 2 = 135 + 72 = 207$ in number.

(iv) And finally the number ending in 8 are $9 \times 4 \times 2 + 9 \times 5 \times 2 = 72 + 90 = 162$ in number.

The total count is $198 + 162 + 207 + 162 = 729$.

2nd Solution :

In every block of 4 consecutive even numbers there is precisely one number which is divisible by 4 but not by 8. Consider 4 consecutive even numbers.

$\alpha\beta\gamma 2, \alpha\beta\gamma 4, \alpha\beta\gamma 6, \alpha\beta\gamma 8$

of them there is first one number meeting our requirements. As $\alpha, \beta, \gamma \in \{1, 2, \ldots\ldots, 9\}$, the total number of numbers $= 9 \times 9 \times 9 = 729$.

4. Let x, y, z be the distinct numbers, Now

$$\frac{1}{y}+\frac{1}{z}=\frac{\alpha}{x};\quad \frac{1}{z}+\frac{1}{x}=\frac{\beta}{y};\quad \frac{1}{x}+\frac{1}{y}=\frac{\gamma}{z}$$

for some positive integers α, β, γ.

Rewrite them as

$$\frac{x}{y}+\frac{x}{z}=\alpha;\quad \frac{y}{z}+\frac{y}{x}=\beta;\quad \frac{z}{x}+\frac{z}{y}=\gamma$$

we will eliminate x, y, z from the system.

Adding we have

$$\left(\frac{x}{y}+\frac{y}{x}\right)+\left(\frac{x}{z}+\frac{z}{x}\right)+\left(\frac{y}{z}+\frac{z}{y}\right)=\alpha+\beta+\gamma$$

Multiplying we have

$$\left(\frac{x}{y}+\frac{x}{z}\right)\left(\frac{y}{z}+\frac{y}{x}\right)\left(\frac{z}{x}+\frac{z}{y}\right)=\alpha\beta\gamma$$

$$\Rightarrow \quad \left(\frac{x}{y}+\frac{x}{z}\right)\left(\frac{y}{x}+1+\frac{yz}{x}+\frac{z}{x}\right)=\alpha\beta\gamma$$

$$\Rightarrow \quad 1+\frac{x}{y}+\frac{z}{x}+\frac{z}{y}+\frac{y}{z}+\frac{x}{z}+\frac{y}{x}+1=\alpha\beta\gamma$$

$$\Rightarrow \quad 2+\alpha+\beta+\gamma=\alpha\beta\gamma$$

Set $\alpha + 1 = u$, $\beta + 1 = v$, $\gamma + 1 = w$ to obtain

$$\frac{1}{u}+\frac{1}{v}+\frac{1}{w}=1,\ (u, v, w \geq 2)$$

Now, let $x < y < z$, which would mean $\alpha < \beta < \gamma$ i.e. $u < v < w$

Now $\quad \frac{1}{u}+\frac{1}{v}+\frac{1}{w}<\frac{3}{u}$

$$\Rightarrow 1 < \frac{3}{u} \Rightarrow u < 3 \quad \text{so } u = 2$$

This could in turn gives $v = 3$, $w = 6$. Thus x, y, z will be 2, 3, 6

2nd Solution :

Adding the equations

$$\frac{1}{x}+\frac{1}{y}+\frac{1}{z}=\frac{\alpha+1}{x}=\frac{\beta+1}{y}=\frac{\gamma+1}{z}$$

$$=\frac{\frac{1}{x}}{\frac{1}{\alpha+1}}=\frac{\frac{1}{y}}{\frac{1}{\beta+1}}=\frac{\frac{1}{z}}{\frac{1}{\gamma+1}}=\frac{\frac{1}{x}+\frac{1}{y}+\frac{1}{z}}{\frac{1}{\alpha+1}+\frac{1}{\beta+1}+\frac{1}{\gamma+1}}$$

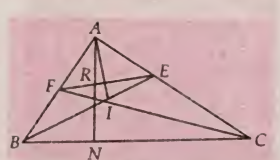
Again we have

$$\frac{1}{\alpha+1}+\frac{1}{\beta+1}+\frac{1}{\gamma+1}=1$$

Also x, y, z are proportional to $\alpha + 1, \beta + 1, \gamma + 1$. From here, the solution can be completed along the lines of previous solution.

5. Construct AR perpendicular to EF and extend it to meet BC in N.

$$\angle BIC = 90° + A/2 = 90° + \frac{60°}{2} = 120°$$



Thus $AFIE$ is concyclic, giving $\angle BEF = A/2$

$$\angle FAN = 90° - \frac{B}{2} - \frac{A}{2} = \frac{C}{2}$$

$= \angle FCN$

Thus F, A, C, N are concyclic.

Now $\angle FNA = \angle FCA = \frac{C}{2} = \angle FAN$

We note that ΔFAN is an isosceles triangle giving $AR = RN$.

6. We claim that $a_n < a_{n+1}$ happens for only then value of n for which $(n + 1)$ is perfect square.

Consider the numbers $n = k^2 + r, 0 \le r \le 2k$.

For the number $k^2, k^2 + 1, k^2 + 2, \ldots\ldots k^2 + 2m$

We have $\left[\frac{n}{(\sqrt{n})}\right]=\left[\frac{k^2+r}{k}\right]=k+\left[\frac{r}{k}\right]$

Which realizes the values

$k, k, k \ldots\ldots k$; (k times), $k + 1, k + 1, \ldots\ldots k + 1$ (k terms) are $k + z$.

So the value drops down when $n + 1 = 2^2, 3^2, \ldots\ldots 44^2$

The corresponding values of n are $2^2 - 1, 3^2 - 1, . 44^2 - 1$. So we have 43 values of n for which the hypothesis is fulfilled.

Things you will not learn in school

Bill Gates recently gave a speech at a High School about 11 things they did not teach him and you will not learn in school. He talked about how feel-good, politically correct teaching has created a generation of kids with no concept of reality and how this concept has set them up for failure in the real world.

Rule 1 : Life is not fair - get used to it.

Rule 2 : The world doesn't care about your self-esteem. The world will expect you to accomplish something before you feel good about yourself.

Rule 3 : You will not make $60,000 a year right out of high school. You won't be a vice-president with a car phone until you earn both.

Rule 4 : If you think your teacher is tough, wait till you get a boss.

Rule 5 : Flipping burgers is not beneath your dignity. Your grandparents had a different word for burger flipping: they called it opportunity.

Rule 6 : If you mess up, it's not your parents' fault, so don't whine about your mistakes, learn from them.

Rule 7 : Before you were born, your parents weren't as boring as they are now. They got that way from paying your bills, cleaning your clothes and listening to you talk about how cool you thought you were. So before you save the rainforest from the parasites of your parent's generation, try delousing the closet in your own room.

Rule 8 : Your school may have done away with winners and losers, but life has not. In some schools, they have abolished failing grades and they'll give you as many times as you want to get the right answer. This doesn't bear the slightest resemblance to anything in real life.

Rule 9 : Life is not divided into semesters. You don't get summers off and very few employers are interested in helping you find yourself. Do that on your own time.

Rule 10 : Television is not real life. In real life people, actually have to leave the coffee shop and go to jobs.

Rule 11 : Be nice to nerds. Chances are you'll end up working for one.

Mock Test **PAPER** 4

West Bengal JEE 2011

The entire syllabus of Mathematics of WB-JEE is being divided into six modules, on each module there will be a Mock Test Paper (MTP) of 30 marks and two MTP on the whole syllabus each of 100 marks which will be published in subsequent issues.

The syllabus for module break-up is given below :

Module I : Algebra - I, **Module - II :** Trigonometry, **Module - III :** Co-ordinate geometry of two dimensions, **Module - IV :** Differential calculus and its applications, **Module - V :** Algebra - II, **Module - VI :** Integral calculus and its applications, Differential equations.

Module – IV : DIFFERENTIAL CALCULUS AND ITS APPLICATIONS

CONTENTS :

Differential Calculus : Functions, composition of two functions and inverse of a function, limit, continuity, derivative, chain rule, derivatives of implicit functions and derivatives of functions defined parametrically.

Rolle's Theorem and Lagrange's mean value theorem (statement only), their geometric interpretation and elementary application, L'Hospital's rule (statement only) and applications, second order derivative.

Application of Calculus : Tangents and normals, conditions of tangency. Determination of monotonicity, maxima and minima. Differential coefficient as a rate of measure.

This paper is useful for AIEEE | IIT-JEE and other Engineering Entrance Exams.

MODULE - IV : Differential Calculus And Its Applications

Time : 40 min **SECTION - I** **Marks : 30**

This section contains 24 multiple choice questions numbered 1 to 24. Each question has four choices out of which one is correct. Each question carries + 1 mark for correct answer and –1/3 for wrong answer.

1. The value of c in $(0, 2)$ satisfying the mean value theroem for the function $f(x) = x(x-1)^2$, $x \in [0, 2]$ is equal to

(a) $\frac{3}{4}$ (b) $\frac{4}{3}$ (c) $\frac{1}{3}$ (d) $\frac{2}{3}$

2. Let $f(x) = \dfrac{(e^x-1)^2}{\sin\left(\dfrac{x}{a}\right)\log\left(1+\dfrac{x}{4}\right)}$ for $x \neq 0$ and $f(0) = 12$. If $f(x)$ is continuous at $x = 0$, then the value of a is equal to

(a) 1 (b) –1 (c) 2 (d) 3

3. If $f(x) = (a - x^n)^{1/n}$, $a > 0$, $n \in N$. Then $f(f(x)) =$

(a) 1 (b) n (c) x (d) nx

4. If $x = a(1 + \cos\theta)$, $y = a(\theta + \sin\theta)$, then $\dfrac{d^2y}{dx^2}$ at $\theta = \dfrac{\pi}{2}$ is

(a) $-\frac{1}{a}$ (b) $\frac{1}{a}$ (c) –1 (d) –2

5. If $2^x + 2^y = 2^{x+y}$ then the value of $\dfrac{dy}{dx}$ at $x = y = 1$ is

(a) 0 (b) –1 (c) 1 (d) 2

6. An edge of a variable cube is increasing at the rate of 10 cm/sec. How fast the volume of the cube will increase when the edge is 5 cm long ?

(a) 750 cm^3/sec (b) 75 cm^3/sec
(c) 300 cm^3/sec (d) 150 cm^3/sec

7. $\underset{x\to 0}{\text{Lt}} \dfrac{\sqrt{1-\cos 2x}}{\sqrt{2}x} =$

(a) 1 (b) –1
(c) 0 (d) does not exist

8. The domain of $f(x) = \sqrt{\log_{10}\left(\dfrac{5x-x^2}{4}\right)}$ is

(a) (0, 5) (b) (–5, –4) (c) [1, 4] (d) [0, 5]

9. The derivative of $\sin^{-1}(2x\sqrt{1-x^2})$ with respect to $\sin^{-1}(3x - 4x^3)$ is

(a) $\frac{2}{3}$ (b) $\frac{3}{2}$ (c) $\frac{1}{2}$ (d) 1

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